



F-theory and heterotic duality, Weierstrass models from Wilson lines

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Abstract We present how to construct elliptically fibered K3 surfaces via Weierstrass models which can be parametrized in terms of Wilson lines in the dual heterotic string theory. We work with a subset of reflexive polyhedra that admit two fibers whose moduli spaces contain the ones of the $E_8 \times E_8$ or $\frac{Spin(32)}{\mathbb{Z}_2}$ heterotic theory compactified on a two torus without Wilson lines. One can then interpret the additional moduli as a particular Wilson line content in the heterotic strings. A convenient way to find such polytopes is to use graphs of polytopes where links are related to inclusion relations of moduli spaces of different fibers. We are then able to map monomials in the defining equations of particular K3 surfaces to Wilson line moduli in the dual theories. Graphs were constructed developing three different programs which give the gauge group for a generic point in the moduli space, the Weierstrass model as well as basic enhancements of the gauge group obtained by sending coefficients of the hypersurface equation defining the K3 surface to zero.

1 Introduction

F-theory compactified on elliptically fibered K3 surfaces is believed to be dual at the quantum level to the heterotic string compactified on a two torus with Wilson lines [1–6]. In particular, one should be able to relate the complex parameters of the moduli space on the F-theory side to the ones on the heterotic one as their moduli space are the same, namely the Narain space [7, 8]

$$\mathcal{M} = (O(2) \times O(18)) \backslash O(2, 18) / O(2, 18, \mathbb{Z}). \quad (1)$$

The study of the full moduli space is however a tedious exercise and one wants to focus on subspaces with fewer complex modular parameters. In heterotic string one can consider for example compactifications on a two torus with Wilson lines parametrized by few moduli. In F-theory, one can choose an

algebraic K3 with a large Picard number p , as its modular space is parametrized by $20 - p$ complex variables [9]. Now, a particularly interesting way to construct K3 surfaces is to use lattice polarized K3 obtained by considering reflexive polyhedra in 3 dimensions that define hypersurfaces in toric varieties [10, 11]. Thanks to Kreuzer and Skarke [12] it is possible to have access to the totality of the 4319 different reflexive polyhedra in 3 dimensions. One can then first construct K3 surfaces, and afterwards elliptically fibered K3s by considering a particular subdivision of the fan in the dual polytope, dictated by a choice of a two dimensional reflexive subpolytope which plays the role of the fiber. In particular, it is possible to classify the different reflexive polyhedra with respect to their Picard number p , and therefore focus on moduli spaces associated with specific K3 surfaces with a low number of moduli.

The duality between F-theory and heterotic string has been written explicitly for only two of the 4319 different K3 surfaces one can construct via reflexive polytopes. First the duality between the parameter of a Weierstrass model presenting a particular $E_8 \times E_8$ singularity and the complex structure and Kahler moduli of the two torus on which the $E_8 \times E_8$ heterotic string is compactified was constructed in [13]. Later it was found that a specific reflexive polyhedra admitting two fibrations has for gauge groups $E_8 \times E_8$ and $\frac{Spin(32)}{\mathbb{Z}_2}$ [14]. In a more general case with three moduli, Malmendier and Morrison showed that a polytope with again two fibers with gauge group $E_7 \times E_8$ and $\frac{Spin(28) \times SU(2)}{\mathbb{Z}_2}$ is related to compactifications of heterotic strings with one Wilson line moduli [15]. All of this suggests that compactifying F-theory on elliptically fibered K3 surfaces described by polytopes with two fibers seems to be related in some cases to compactifications of both heterotic strings with Wilson line moduli.

Here we show that if we focus on particular reflexive polyhedra that are linked in some way to the $E_8 \times E_8 / \frac{Spin(32)}{\mathbb{Z}_2}$ polytope, we can understand the Wilson line structure of the dual heterotic strings. This is due to the fact that we

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can recover the torus on which we compactify the heterotic strings as a particular subspace of the moduli spaces of the elliptically fibered K3s. To find these polytopes we construct graphs. The nodes of the graphs are three dimensional reflexive polyhedra, or equivalently the K3 surfaces which are constructed using these polyhedra. Consider now two polyhedra which we write M^+ and M^- , where the K3 surface associated with M^+ has more moduli than the one associated with M^- . We draw a link between the nodes M^+ and M^- if, for every elliptically fibered K3 surface obtained via M^+ , there exists a limit in the moduli space where one obtains elliptically fibered K3 surfaces of the other polytope M^- . In particular, we will consider the limit where we send monomials of the hypersurface equation defining the K3 surface associated with M^+ to zero, which is equivalent to removing a point in M^+ . This can be seen as an extension of the notion of chains presented by Kreuzer and Skarke in [12]. Focusing on polytopes that have two fibers, links between polytopes then correspond to inclusion relations between the moduli spaces of elliptically fibered K3s. Considering polytopes that are linked to $E_8 \times E_8 / \frac{Spin(32)}{\mathbb{Z}_2}$, we show that additional monomials in the hypersurface equation which defines the elliptically fibered K3s on which we compactify on correspond to additional Wilson line moduli in both the $E_8 \times E_8$ and $\frac{Spin(32)}{\mathbb{Z}_2}$ heterotic strings. Using this Wilson line/monomial duality we can construct Weierstrass models of elliptically fibered K3s which are not directly obtained from reflexive polyhedra. They can then be interpreted as a certain Wilson line content in the dual heterotic theories. Finally, we show that in some cases, this notion of *Wilson line description of K3 surfaces* can be extended to polytopes with more than two fibers. This should be helpful to explicitly understand the duality between F-theory compactified on K3s and heterotic string on a two torus, and eventually in compactifications to lower dimensions involving K3 surfaces.

The paper is organised as follows: in Sect. 2 we present some basic properties of reflexive polyhedra and how they define elliptically fibered K3 surfaces. In Sect. 3, we present several computer programs which are helpful for constructing graphs of polytopes. We wrote the programs on SageMath and with the help of the package PALP [16–18]. The first program uses the extended Dynkin diagram structure of reflexive polyhedra with fibers in order to construct tables of gauge groups for each fibration of every reflexive polytope. The second program gives the corresponding Weierstrass model for every fiber of reflexive polytopes. The third one uses this Weierstrass model and finds the enhancements one can obtain by simply sending the coefficients which parametrize the hypersurface equation of the K3 in some toric varieties to zero. This can be particularly useful to construct graphs of polytopes and we show how one can link polytopes up to three moduli. In the appendix we present typical outputs of the programs and explain how to use them. To our knowledge

these functionalities were not available on any software. We are making the computer programs available on GitHub at <https://github.com/lilianChabrol/Reflexivepolyhedras>.

To summarize, here are the three SageMath programs available online

- Program 1 (Typical output in Appendix A): Gauge groups from the extended Dynkin diagram structure in the N lattice.
- Program 2 (Typical output in Appendix B): Determination of the Weierstrass model of the corresponding elliptically fibered K3.
- Program 3 (Typical output in Appendix C): Possible enhancements of the gauge groups for each fibers by sending defining coefficients of the hypersurface to zero.

Finally in Sect. 4 we present a *Wilson line description of K3 surfaces* by considering a particular graph of polytope which goes up to 6 moduli, or equivalently in this case four Wilson line moduli on the heterotic side. We then show how to construct Weierstrass models of elliptically fibered K3s which one can directly interpret in the dual theory as a particular Wilson line content.

2 Reflexive polyhedra and elliptically fibered K3s

2.1 Hypersurfaces on projective spaces and Fano variety

Here we introduce various notations about reflexive polyhedra and present briefly results about toric Fano varieties. Detailed constructions of toric Fano varieties have been widely discussed in the literature (see e.g. [19,20]). A pedagogical introduction to toric geometry can be found in [21].

Let us consider two dual lattices M (Monomials) and N (faN) in \mathbb{Z}^n with real extension $M_{\mathbb{R}}$ and $N_{\mathbb{R}}$ and an associated product $(*, *) : M \times N \rightarrow \mathbb{Z}$. We note Δ an integral convex polytope whose vertices are in M and which contains only the origin as an interior point. We then define the dual of Δ as

$$\nabla \equiv \{v \in N_{\mathbb{R}} : \langle w, v \rangle \geq -1 \text{ for all } w \in \Delta\}. \tag{2}$$

As usual we consider Δ to be reflexive, meaning that ∇ is also convex, only contains the origin and has its vertices $\{v_i, i = 1, \dots, k\}$ in N . With this we define *strongly convex rational polyhedral cone*, which we simply call cone therefor for simplicity, as well as *fans* [21,22]. A cone $\sigma \in N_{\mathbb{R}}$ is a set

$$\sigma = \left\{ \sum_i a_i v_i / a_i \geq 0, \quad i \leq k \right\} \tag{3}$$

such that $\sigma \cap (-\sigma) = \{0\}$. A fan is then defined as a collection Σ of cones such that each face of a cone in Σ is also a cone in Σ and the intersection of two cones is a face of each. The one dimensional cones of a fan are usually called *rays*. The normal fan of the polytope Δ whose rays are the vertex of ∇ then defines a projective toric variety P_Δ (which is Fano if and only if Δ is reflexive, which it is in this paper). Explicitly, one associates a variable x_i to each of the vertices v_i of the polytope ∇ in N which therefore defines \mathbb{C}^k . Then one has to remove the sets

$$Z_\Sigma = \bigcup_I \{(x_1, \dots, x_k) / x_i = 0 \quad \forall i \in I\} \tag{4}$$

with I subsets of $[1, k]$ such that $\{v_i, i \in I\}$ is not included in a cone. Finally one has to quotient this space by an abelian group G as well as $(\mathbb{C} \setminus \{0\})^{k-n}$ acting as

$$(x_1, \dots, x_k) \sim (\lambda^{q_j^1} x_1, \dots, \lambda^{q_j^k} x_k) \tag{5}$$

if $\sum_{i=1}^k q_j^i v_j = 0, \quad \lambda \in \mathbb{C}.$

j goes from 1 to $(k - n)$ as we can find $(k - n)$ independent relations such as these in the polytope N . Moreover, one chooses integer q_j^i s such that for each linear relation one coefficient is equal to 1. P_Δ is then

$$P_\Delta = \frac{\mathbb{C}^k \setminus Z_\Sigma}{G \times \mathbb{C}^{*(k-n)}}. \tag{6}$$

Now that we constructed this Fano toric variety using the pair of polytopes (Δ, ∇) it is possible when $n = 3$ to construct a K3 surface X_Δ as the locus in P_Δ of

$$\sum_{m \in \Delta \cap M} c_m \prod_{i=1}^k x_i^{(m, v_i)+1} = 0 \tag{7}$$

with $c_m \in \mathbb{C}$.

We can then construct in some cases an elliptically fibered K3 as X_Δ together with a surjective morphism $\pi : X_\Delta \rightarrow \mathbb{P}_1$ such that generic fibers are genus one elliptic curves. They can be constructed by considering the K3 surface we just described, as well as finding a subpolytope $\nabla^{(2)}$ of ∇ in the N lattice which plays the role of the fiber of the elliptic K3.¹ There are 16 reflexive polyhedra for $n = 2$, and 4319 reflexive polytopes for $n = 3$ [12]. It is then possible to obtain Weierstrass models of elliptically fibered K3 surfaces upon a choice of fans that contain as rays points of the fiber $\nabla^{(2)}$. Quite amazingly, and upon a particular choice of a fan which will be described in Sect. 3.2, the gauge groups associated with singularities of the elliptically fibered K3s can be read off directly once one chooses a particular subpolytope

$\nabla^{(2)}$ [23]. Indeed, Candelas and Font noticed that the points located on both sides of the fiber of the polytope ∇ in the N lattice are exactly the extended Dynkin diagrams which correspond to the gauge groups associated with singularities appearing in the Weierstrass model via the Kodaira and Néron classification [24, 25] (see Fig. 1). This was later explained by Perevalov and Skarke [26]. Depending on which of the 16 two dimensional reflexive polyhedra is the fiber, additional contribution coming from the Mordell–Weil group of rational sections of the elliptic fibration can occur [27–32]. In particular the fibers F1, F2 or F4 give additional discrete symmetries $\mathbb{Z}_\#$ and fibers F13, F15 and F16 quotient by discrete symmetries $\frac{1}{\mathbb{Z}_\#}$.² Finally, additional contribution of $U(1)$ s or $SU(\#)$ factors can appear, depending on how the polytope $\nabla^{(2)}$ intersects with ∇ .

2.2 Invariant parameters of the moduli space

The number of complex moduli for a K3 surface with Picard number p is $20 - p$. Previously we defined an algebraic K3 as an hypersurface (7) in the toric variety P_Δ whose number of parameters is a priori given by the number of points in $\Delta \cap M$. However different sets of those parameters correspond to the same point in the moduli space. For example several of the coefficients can be put to 1 by a reparametrization of the coordinates in the projective space. In order to properly define complex parameters on the moduli space of the K3 surface we use the construction developed in [34]. It was shown there that monomials defined by points interior to facets in $\Delta \cap M$ can be removed by an appropriate change of coordinates for the different reflexive polyhedra they considered. We therefore restrict the hypersurface equation (7) to the integral points $m \in \text{Edges}(\Delta \cap M) \equiv \text{Edg}(\Delta)$ as well as the origin. The hypersurface equation can then be written as

$$H = -c_0 \prod_{i=1}^k x_i + \sum_{m \in \text{Edg}(\Delta)} c_m \prod_{i=1}^k x_i^{(m, v_i)+1} = 0 \tag{8}$$

with v_i rays of the normal fan P_Δ . Due to the strong link between the period map of K3 surfaces and their moduli spaces [9], one can seek for parameters of the moduli space by considering the fundamental period of the holomorphic two-forms which can be written in our case as [35]

$$\bar{w}_{00} = -\frac{c_0}{(2\pi i)^k} \oint_{\mathcal{C}} \frac{dx_1 \wedge \dots \wedge dx_k}{H} \tag{9}$$

¹ Such subpolytope cannot be found sometimes, in particular for small Picard numbers.

² Using the notations of [29]. The polytopes are i1,i2,i4 and i9,i7,i6 using notations of [33].

with \mathcal{C} a product of cycles that enclose the hypersurface defined by $x_i = 0$ [34]. This can be recast as

$$\tilde{w}_{00} = \frac{1}{(2\pi i)^k} \oint_{\mathcal{C}} \frac{dx_1 \wedge \dots \wedge dx_k}{\prod_{i=1}^k x_i} \sum_{l=0}^{\infty} \tilde{H}^l \tag{10}$$

with

$$\tilde{H} = \sum_{m \in \text{Edg}(\Delta)} \frac{c_m \prod_{i=1}^k x_i^{< m, v_i > + 1}}{c_0 \prod_{i=1}^k x_i} \tag{11}$$

The only non zero terms in (10) are the constant terms in the development of \tilde{H}^l by the residue theorem. The fundamental period of the holomorphic two-forms can therefore be parametrized by the following invariants

$$\text{Moduli} \sim \frac{l!}{c_0^l} \prod_{m \in \text{Edg}(\Delta)} \frac{c_m^{l_m}}{l_m!} \tag{12}$$

such that

$$\sum_{m \in \text{Edg}(\Delta)} l_m = l \tag{13}$$

and

$$\forall i \sum_{m \in \text{Edg}(\Delta)} l_m (< m, v_i > + 1) = l. \tag{14}$$

Taking the second equation one can then simply look for inequivalent linear relations in the M lattice such that $\sum l_m \cdot m = (0, 0, 0)$ with l_m s positive and minimal. By a change of variables of these invariants one can in fact look for inequivalent linear relations between points in the edges of Δ such that (13) and (14) are verified but this time with l_m in \mathbb{Z} and $|l_m|$ minimal. The complex parameters can then be taken to have the following form

$$\text{Moduli} \sim \left(\prod_{m \in \text{Edg}(\Delta)} c_m^{l_m} \right) c_0^{-l}. \tag{15}$$

As an example let us take the polytope $M476$, with Picard number equal to 16 i.e. 4 moduli. Its vertices are given by

$$M476 : \underbrace{(1, 0, 0)}_{(1)_M}, \underbrace{(0, 1, 0)}_{(2)_M}, \underbrace{(0, 0, 1)}_{(3)_M}, \underbrace{(-4, -2, -1)}_{(4)_M}, \\ \underbrace{(-5, -3, -1)}_{(5)_M}, \underbrace{(-1, -1, 1)}_{(6)_M}. \tag{16}$$

An additional point, $(7)_M = (-3, -2, 0)$, is situated on the edges of the polytope. We can thus consider four inequivalent linear relations between these points, a possibility being

$$(5)_M + (6)_M - 2 \cdot (7)_M, \quad (7)_M + 2 \cdot (2)_M + 3 \cdot (1)_M \tag{17}$$

$$(3)_M - (1)_M - (2)_M - (6)_M, \quad (4)_M - (1)_M - (2)_M - (5)_M \tag{18}$$

which leads, using (15), to the complex parameters of the moduli space

$$M476 \text{ (4 Moduli): } \frac{c_5 c_6}{c_7^2}, \quad \frac{c_7 c_2^2 c_1^3}{c_0^6}, \quad \frac{c_3 c_0^2}{c_1 c_2 c_6}, \quad \frac{c_4 c_0^2}{c_1 c_2 c_5}. \tag{19}$$

3 Obtaining data on elliptically fibered K3s

We now present three computer programs that allow to obtain different information about elliptically fibered K3 surfaces automatically. To our knowledge the results we obtain were not available on any software directly. We note $M\#$ the polytope Δ in the M lattice corresponding to *Reflexive Polytope*(3, #) in SageMath.³ We write schematically what is done in Sagemath, more accurate descriptions of the computer programs as well as the programs themselves are available on GitHub at

<https://github.com/lilianChabrol/Reflexivepolyhedras>.

3.1 Extended Dynkin diagram from polyhedra

As discussed in the first part of this paper it is possible to have access to the gauge structure of an elliptically fibered K3 upon a choice of reflexive polytope (Δ, ∇) , and a choice of fiber $\nabla^{(2)}$. We now present a generic way to find the gauge group associated with each fiber of every reflexive polytope in the Kreuzer–Skarke classification of reflexive polyhedra in 3 dimensions.

We first find all two dimensional reflexive polyhedra $\nabla^{(2)}$ which are subpolytopes of ∇ modulo $SL(3, \mathbb{Z})$ transformations in the N lattice. Then we identify which of the 16 possible two dimensional reflexive polytope corresponds to each of the fibers $\nabla^{(2)}$. This permits in particular to know if the fiber contains product or quotient by discrete symmetry group [29]: F1, F2 and F3 contribute to a product by $\mathbb{Z}_3, \mathbb{Z}_2$ and \mathbb{Z}_4 respectively while fibers F13, F15 contribute by $\frac{1}{\mathbb{Z}_2}$ and F16 by $\frac{1}{\mathbb{Z}_3}$. We do not write the additional contributions of $U(1)$ factors coming from the Mordell Weil group as in the end the gauge group must be of rank $p-2$, where p is the Picard number of the K3 surface. We however look for additional $SU(\#)$ contribution from the fiber: if polytopes $\nabla^{(2)}$ and ∇ have a common edge with n points, then there appears an additional $SU(n-1)$ part in the final gauge group.⁴ Finally, the fiber $\nabla^{(2)}$ dividing ∇ into two parts, we look at points “above” and “below” the fiber and read off the extended Dynkin diagrams.

³ The vertices of each of the polytopes presented in this paper are written in the Appendix D.

⁴ Equivalently the rank of the $SU(\#)$ can be seen by looking at the number of interior points in the common edge. See the red points in Fig. 1.

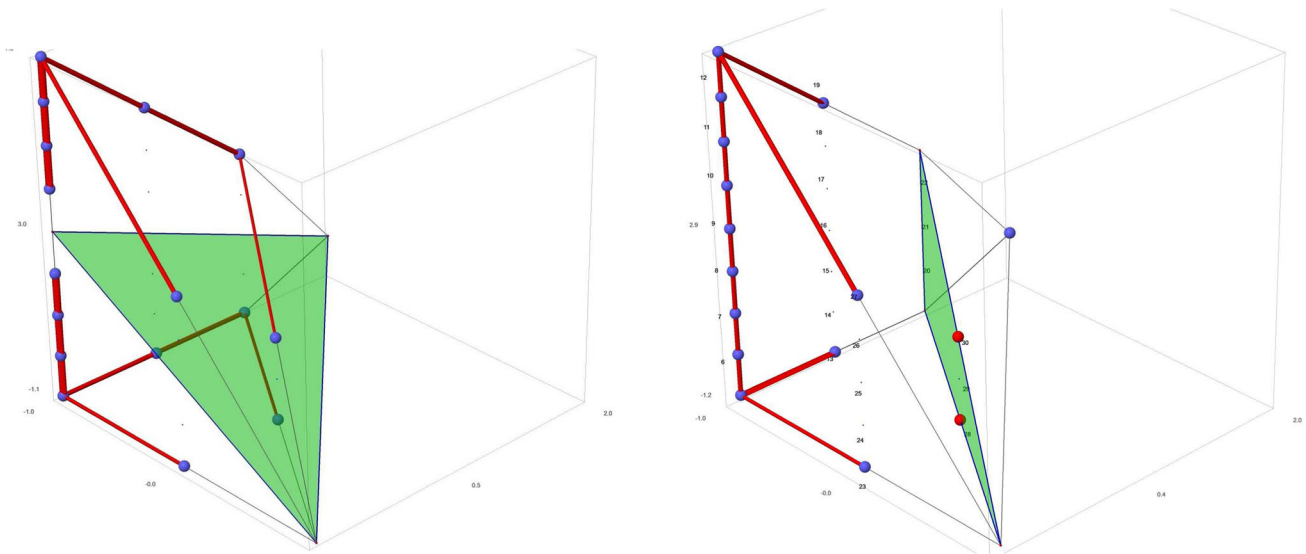


Fig. 1 $E_7 \times E_7$ and $\frac{SO(24) \times SU(2)^2}{\mathbb{Z}_2}$ fiber of the polytope $M476$. The points in blue draw the extended Dynkin diagram of E_7 s on the left, $SO(24)$ on the right. The contribution of $SU(2)$ s from the fiber are symbolised by red points. The fiber being $F13$ there is an additional contribution of $\frac{1}{\mathbb{Z}_2}$

Table 1 Gauge groups for polytopes with Picard 19. Columns represent the inequivalent fibers $\nabla^{(2)}$ dividing the dual $N\#$ of $M\#$ into two parts. Additional contributions of $U(1)$ factors should be considered so that the gauge group is of rank $p - 2$ where p is the Picard number

M0	$\frac{SO(16) \times SO(16)}{\mathbb{Z}_2}$	$\frac{SU(12) \times E_6}{\mathbb{Z}_3}$	$E_8 \times E_8$	$\frac{E_7 \times E_7 \times SU(4)}{\mathbb{Z}_2}$
M2	$\frac{E_7 \times SO(20)}{\mathbb{Z}_2}$	$\frac{SU(18)}{\mathbb{Z}_3}$	$E_8 \times E_8 \times SU(2)$	

As an example let us consider the polytope $M476$. In Fig. 1 we represent the dual polytope $N476$ of $M476$ for the two inequivalent fibrations $\nabla^{(2)}$ it contains. On the left one can read off two extended Dynkin diagrams of E_7 . On the right there is a $SO(24)$ as well as $\frac{1}{\mathbb{Z}_2}$ coming from the fiber $F13$, and $SU(2) \times SU(2)$ contribution due to the intersection of $\nabla^{(2)}$ and ∇ symbolised by red points.

The results for K3 surfaces with Picard number 19 and 18 i.e. one and two complex parameters respectively, are presented in the Tables 1 and 2. They were compared with results of an unpublished paper [36] presented at a seminar at CERN [37]. The result with Picard 17 and 3 complex parameters is in the Appendix A (Table 3). Tables with complex parameters up to 5 moduli are available on GitHub, and up to 10 moduli for elliptically fibered K3s admitting only two inequivalent fibrations.

3.2 Weierstrass model, gauge groups and basic enhancements

The computer program introduced in Sect. 3 is particularly interesting to determine the gauge group at a generic point in the moduli space associated with any fiber of any reflexive polytope in three dimensions. It would however be interesting to get the Weierstrass model which corresponds to these

gauge groups in order to find their enhancements for particular values of the moduli. Some of the enhancements can then be found quite easily by removing points in the polytope Δ in the M lattice which amounts to sending coefficients to zero in the hypersurface equation that defines the K3 surface. This is what the second and third program do: find the Weierstrass model, and the enhancements described above.⁵

We first look at the polytope ∇ in the N lattice. As explained in the introduction we then find inequivalent subpolytopes $\nabla^{(2)}$ of dimension 2 in ∇ . For each of this 2 dimensional polytope we want to associate homogenous coordinates such that they describe the fiber. For most cases one can just associate one of them to each vertices of the subpolytope and obtain later the gauge groups expected from reading the extended Dynkin diagrams directly on the polytope ∇ . However in 3 cases (F13, F15 and F16) out of the 16 possible two dimensional reflexive polytopes, considering the vertices will not lead to these groups. This is due to the fact that for these particular polytopes the fibrations admit more than one section [14]. Using a similar construction to the one of [37] and in an upcoming paper [36], we then consider the homogeneous coordinates x_i of the fiber to be associated with the points as described in Fig. 2.

⁵ See Appendix B and C for the output of the programs.

Table 2 Gauge groups for polytopes with Picard 18. Columns represent the inequivalent fibers $\nabla^{(2)}$ dividing the dual $N\#$ of $M\#$ into two parts. Additional contributions of $U(1)$ factors should be considered so that the gauge group is of rank $p - 2$ where p is the Picard number

M3	$SO(14) \times E_7$	$SO(14) \times SU(9)$	$\frac{SU(12) \times SO(8)}{Z_2}$	$\frac{E_6 \times E_6 \times SU(3) \times SU(3)}{Z_3}$	$E_8 \times E_8 \times Z_3$
M4	$E_8 \times E_8 \times Z_3$	$E_6 \times SO(14) \times SU(3)$	$E_7 \times E_7$	$\frac{SU(10) \times SO(12)}{Z_2}$	$\frac{SU(9) \times SU(9)}{Z_3}$
M5	$E_7 \times E_7 \times SU(2)$	$SU(10) \times E_6$	$\frac{SO(16) \times SO(12) \times SU(2)}{Z_2}$	$\frac{E_7 \times SO(12) \times SU(4)}{Z_2}$	$\frac{SU(6) \times SU(12)}{Z_3}$
M6	$E_6 \times E_7 \times SU(3)$	$E_7 \times E_8$	$E_8 \times E_8 \times Z_3$	$SO(14) \times SO(14)$	$\frac{E_6 \times SU(9) \times SU(3)}{Z_3}$
M7	$E_7 \times E_8$	$\frac{SU(10) \times E_7}{Z_2}$	$\frac{SU(3) \times SU(15)}{Z_3}$	$E_6 \times SO(18)$	
M10	$\frac{SO(16) \times SO(16)}{Z_2}$	$\frac{E_7 \times E_7 \times SU(2) \times SU(2)}{Z_2}$	$E_8 \times E_8 \times Z_4$	$\frac{SU(16)}{Z_2}$	
M11	$\frac{SO(16) \times E_7 \times SU(2)}{Z_2}$	$E_8 \times E_8 \times Z_4$	$E_8 \times E_7 \times SU(2)$	$\frac{SO(12) \times SO(20)}{Z_2}$	$SU(16)$
M16	$SO(18) \times E_6$	$\frac{SU(15) \times SU(3)}{Z_3}$	$E_7 \times E_8$	$\frac{SU(10) \times E_7}{Z_2}$	
M88	$E_8 \times E_8$	$\frac{SO(32)}{Z_2}$			

To define coordinates (s, t) on the base space $\mathbb{P}^{(1)}$ we seek for two vectors v_s and v_t , “above” and “below” the fiber. A fast way to obtain the appropriate Weierstrass model with correct ADE singularities, which correspond to the extended Dynkin diagrams seen in ∇ , is then to seek for the closest vectors to the fiber in $\nabla \cap N$.

Finally, we write the hypersurface equation by considering the points on the edges of Δ and using Eq. (7). To each of these points corresponds a monomial in the hypersurface equation to which we associate a parameter $c_i \in \mathbb{C}$. Using SageMath we can finally recast this equation into the Weierstrass form

$$y^2 = x^3 + f(s, t)xz^4 + g(s, t)z^6 \tag{20}$$

where the homogeneous coordinates of the fiber are now $[x, y, z]$ in $\mathbb{P}^{(2,3,1)}$, f and g are respectively polynomials of degree 8 and 12 in (s, t) . The discriminant of (20) is then $\Delta_{(f,g)} = 4f^3 + 27g^2$ and vanishes at 24 points that are the locations of 7-branes.

Once one has the Weierstrass form of the elliptically fibered K3, one finds the ADE groups associated with the various singularities using Kodaira and Neron classification [24,25]. Moreover, the moduli can be expressed via the parameters c_i as shown in Sect. 2.2. Sending those parameters to zero, we can therefore find possible enhancements of the group associated with a generic point in the moduli space. The third SageMath program we wrote then gives all possible enhancements obtained by sending all possible combinations of parameters c_i to zero, when the hypersurface still defines an elliptically fibered K3.

3.3 Graphs of polytopes

Using this we construct *graphs* of K3 surfaces, generalising the “chains” defined by Kreuzer and Skarke in [12]. Nodes on a graph correspond to polytopes, or equivalently their associated K3 surface. We then link two polytopes if, by sending the same coefficient c_i of (8) in every hypersurface equation for every possible fibration, we obtain the Weierstrass models of fibers of the other polytope.⁶ Some of these graphs are represented in Figs. 3, 4 and 5 and are discussed below. A less trivial case will be discussed in Sect. 4.

Let us consider Fig. 3: $M0$ is linked to both $M5$ and $M6$ by which we mean that if one removes a particular point in the polytopes $M5$ and $M6$, one recovers the Weierstrass models corresponding to fibers of $M0$. This means that the moduli spaces of elliptically fibered K3s corresponding to the fibrations of the polytopes $M5$ and $M6$ contains the moduli spaces of fibers of the polytope $M0$.

⁶ One does not necessarily obtain all the fibers of the polytope with fewer moduli. This is however the case for Fig. 6.

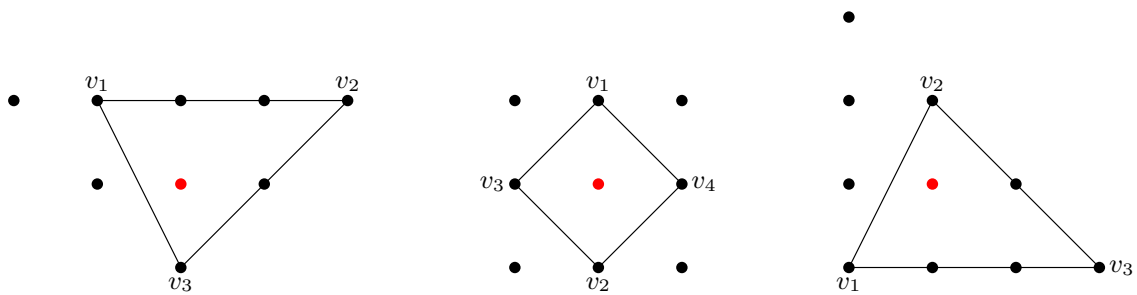


Fig. 2 In order to obtain the groups associated with the extended Dynkin diagrams on the N lattice we consider the following rays when the two dimensional subpolytopes are F13, F15 and F16

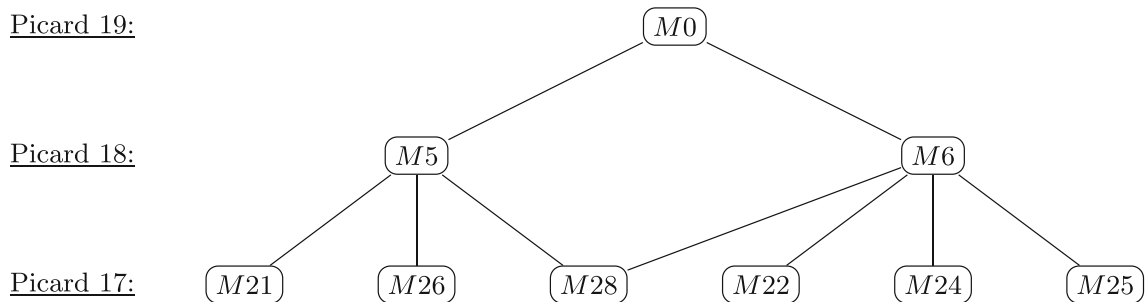


Fig. 3 Polytopes up to 3 complex parameters that are linked to $M0$ by removing points in their M lattice (i.e. a monomial in the hypersurface equation)

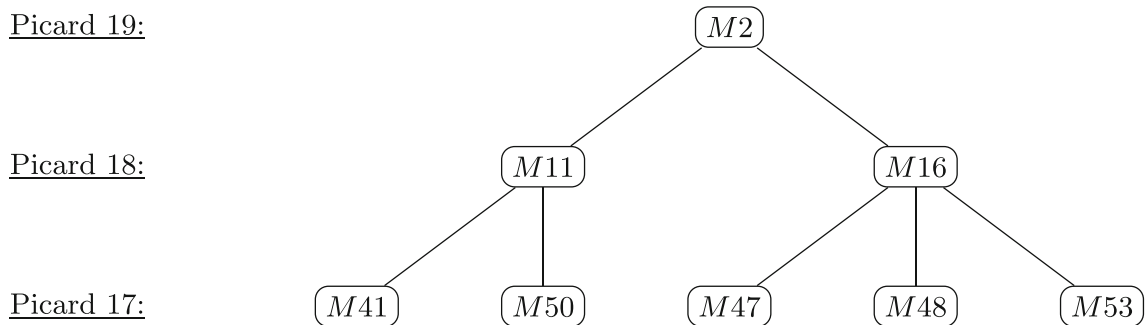


Fig. 4 Polytopes up to 3 complex parameters that are linked to $M2$ by removing points in their M lattice (i.e. a monomial in the hypersurface equation)

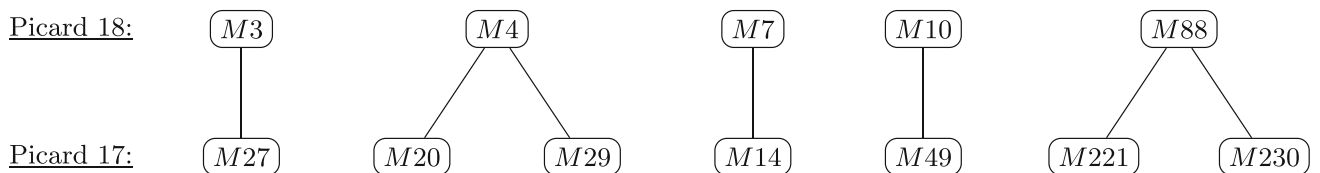


Fig. 5 Links between polytopes with Picard 18 and 17. Going from Picard 17 to 18 amounts to removing a point the in the polytope in the M lattice (i.e. a monomial in the hypersurface equation which defines the K3 surface)

Figures 3, 4 and 5, combined with the polytopes $M15$, $M30$, $M38$, $M104$ and $M117$ with Picard 17 which are not linked to any polytope with higher Picard number, i.e. polytope with a lower number of moduli, describe all reflexive polyhedra up to 3 complex parameters.

**4 F-theory/heterotic string duality in 8 dimensions:
Wilson lines from reflexive polyhedra?**

4.1 Basic aspects of heterotic string

Here we introduce basic notions concerning the compactifications of heterotic strings on a two torus, based on [38,39]. As was presented in the beginning of the introduction, the moduli space of the heterotic string compactified on a two torus is given by the Narain space (1). This is due to the fact that the twenty dimensional internal momentum

$$\mathbf{P} = (\mathbf{p_R}, \mathbf{p_L}) = (p_{Ra}, (p_{La}, p_L^A)) \tag{21}$$

with $a = 1, 2$ an index corresponding to directions along the two torus and $A = 1, \dots, 16$ an index along the 16 dimensional torus of heterotic string transforms as a vector under $O(2, 18, \mathbb{R})$. One then obtains that $\mathbf{P} \cdot \mathbf{P} \in 2\mathbb{Z}$ and \mathbf{P} forms a lorentzian lattice of signature (2, 18). The momentum (21) can be written in terms of the metric G of the two torus, the two form field B , $U(1)^{16}$ gauge field A_a^A as well as winding w_a and momentum number n_a . When the Wilson lines A_a^A are null, one obtains that admissible states verify $|p_L^A|^2 \in 2\mathbb{Z}$, meaning that p_L^A is lying in a 16 dimensional even self dual lattice,⁷ i.e. corresponds to a root vector of $E_8 \times E_8$ or the weight of $\frac{Spin(32)}{\mathbb{Z}_2}$. On the other hand, considering generic values for the two Wilson lines A_1^A and A_2^A completely breaks the gauge symmetry from $E_8 \times E_8$ or $\frac{Spin(32)}{\mathbb{Z}_2}$ to $U(1)^{16}$ due to the quantization condition of the momentum number.

In [39] is presented the compactifications of heterotic strings on a circle, with one Wilson line parametrized by one parameter. This leads to a wide variety of possible gauge groups one can obtain from heterotic strings. As an example one can consider one Wilson line of the form $A = (a_8, 0_8)$ with $a \in \mathbb{R}$ breaking $SO(32)$ to $SO(16) \times SO(16)$ and $E_8 \times E_8$ to $E_7 \times E_8$ due to the quantization condition of the momentum number. Here we will write the Wilson lines A_1 and A_2 in a complex form $A = A_1 + iA_2$. We then seek for possible Wilson lines parametrizations which will correspond to the gauge groups we obtain from elliptic fibrations of K3 surfaces.

⁷ Self duality is due to modular invariance.

4.2 Graphs of polytopes: from monomials to Wilson lines...

The duality map between F-theory on K3 and heterotic strings on a two torus has been explicitly written for two polytopes having each two inequivalent fibrations. The gauge groups associated with these fibers amazingly are $E_8 \times E_8$ and $SO(32)$ for the first polytope ($M88$ using our notation) and $E_7 \times E_8$ and $SO(28) \times SU(2)$ ($M221$). Using [39], we can see that adding a Wilson line of the form $A = (a_2, 0_{14})$ with $a \in \mathbb{C}$, using the notation of the previous Sect. 4.1 breaks $E_8 \times E_8$ to $E_7 \times E_8$, and $SO(32)$ to $SO(28) \times SU(2)$ for a generic value of a . On the heterotic side one might be able to interpret the polytope $M221$ as a compactification on a two torus, together with one Wilson line of the form $(a_2, 0_{14})$. In fact considering this particular parametrization of Wilson line, the enhancements one finds on both heterotic strings and F-theory exactly match, as was presented by Anamaria Font at CERN [37] and studied with more details in an upcoming paper [36].

Now we want to see if we can make similar interpretations by considering polytopes which admit only two fibrations. Following the construction we presented in Sect. 3.3 we seek a graph of polytopes with two fibers in its dual lattice and which contains $M88$. As an example let us consider the polytope $M1328$ which has Picard number 14. Its moduli space is parametrized by 6 complex parameters. We write the hypersurfaces equations $P_G = 0$ of its two fibers below, where G is the group associated with its ADE singularities

$$\begin{aligned} P_{E_6 \times SO(10)} = & -\underline{c_0 x_0 x_1 x_2 x_3 s t} + \underline{c_1 x_2 x_3^2 s} + c_2 x_0^2 x_3 s \\ & + \underline{c_3 x_0^3 x_1 s t} + c_4 x_0 x_1^3 x_2^2 s^2 t^2 \\ & + c_5 x_1^2 x_2^2 x_3 t^3 + \underline{c_6 x_1^4 x_2^3 s^2 t^3} \\ & + c_7 x_0 x_1^3 x_2^2 t^4 + \underline{c_8 x_1^4 x_2^3 t^5} + \underline{c_9 x_1^4 x_2^3 s t^4}, \end{aligned} \tag{22}$$

$$\begin{aligned} P_{SU(11) \times SU(2)} = & -\underline{c_0 x_0 x_1 x_2 x_3 s t} + \underline{c_1 x_1 x_2^2 t} \\ & + c_2 x_0 x_1 x_2 x_3 s^2 + \underline{c_3 x_0^2 x_1 x_3^2 s^3} \\ & + c_4 x_1^2 x_3^3 s t^4 + c_5 x_0^3 x_2 + \underline{c_6 x_1^2 x_3^3 t^5} \\ & + c_7 x_0^4 x_3 s + \underline{c_8 x_0^4 x_3 t} + \underline{c_9 x_0^2 x_1 x_3^2 t^3} \end{aligned} \tag{23}$$

with x_i homogeneous coordinates of the fiber, and (s, t) coordinates on the base. These hypersurfaces can then be recast into a Weierstrass form where (s, t) correspond to coordinates on the base \mathbb{P}_1 . We do not write however the full Weierstrass form of Eqs. (22) and (23) due to the size of the expressions. Instead, let us focus on the case where we consider $c_2 = c_4 = c_5 = c_7 = 0$. Considering only the underlined monomials to be non-zero in the Eqs. (22) and (23) then gives the Weierstrass models

$$\begin{aligned}
 f &= \left(-\frac{1}{48}\right) \cdot t^4 \cdot s^4 \cdot c_0^4 \\
 g &= \left(-\frac{1}{864}\right) \cdot t^5 \cdot s^5 \cdot \left(864c_1^3c_3^2c_6s^2 - c_0^6st \right. \\
 &\quad \left. + 864c_1^3c_3^2c_9st + 864c_1^3c_3^2c_8t^2\right) \\
 \Delta_{(f,g)} &= \left(\frac{1}{16}\right) \cdot c_3^2 \cdot c_1^3 \cdot t^{10} \cdot s^{10} \cdot (c_6s^2 + c_9st + c_8t^2) \\
 &\quad \left(432c_1^3c_3^2c_6s^2 - c_0^6st \right. \\
 &\quad \left. + 432c_1^3c_3^2c_9st + 432c_1^3c_3^2c_8t^2\right) \tag{24}
 \end{aligned}$$

and

$$\begin{aligned}
 f &= \left(-\frac{1}{48}\right) \cdot t^2 \cdot \left(16c_1^2c_3^2s^6 - 8c_0^2c_1c_3s^5t + c_0^4s^4t^2 \right. \\
 &\quad \left. + 32c_1^2c_3c_9s^3t^3 - 8c_0^2c_1c_9s^2t^4 \right. \\
 &\quad \left. - 48c_1^2c_6c_8t^6 + 16c_1^2c_3^2t^6\right) \\
 g &= \left(-\frac{1}{864}\right) \cdot t^3 \cdot (4c_1c_3s^3 - c_0^2s^2t + 4c_1c_9t^3) \\
 &\quad \left(16c_1^2c_3^2s^6 - 8c_0^2c_1c_3s^5t + c_0^4s^4t^2 + 32c_1^2c_3c_9s^3t^3 \right. \\
 &\quad \left. - 8c_0^2c_1c_9s^2t^4 - 72c_1^2c_6c_8t^6 + 16c_1^2c_3^2t^6\right) \\
 \Delta_{(f,g)} &= \left(-\frac{1}{16}\right) \cdot c_8^2 \cdot c_6^2 \cdot c_1^4 \cdot t^{18} \\
 &\quad \cdot \left(16c_1^2c_3^2s^6 - 8c_0^2c_1c_3s^5t \right. \\
 &\quad \left. + c_0^4s^4t^2 + 32c_1^2c_3c_9s^3t^3 - 8c_0^2c_1c_9s^2t^4 \right. \\
 &\quad \left. - 64c_1^2c_6c_8t^6 + 16c_1^2c_3^2t^6\right) \tag{25}
 \end{aligned}$$

respectively. They are the parameters of the Weierstrass models obtained from the polytope $M88$ and thus correspond to the heterotic strings without Wilson lines with gauge groups $E_8 \times E_8$ for (24) and $SO(32)$ for (25). We can then define two moduli ξ and ρ

$$\xi = \frac{c_8c_6}{c_9^2}, \quad \eta = \frac{c_9c_3^2c_1^3}{c_0^6} \tag{26}$$

They parametrize the two dimensional moduli space and are found by considering linear relations on the edges of the polytope $M1328$ (see Eq. (15)).⁸ Now, we know that adding

⁸ They correspond to the parameter u and v in [34].

monomials corresponds to adding complex parameters in the Wilson lines. As we have to add four complex parameters which correspond to the four additional monomials c_2, c_4, c_5 and c_7 in (22) and (23), we use the full graph which links $M1328$ to $M88$ represented in Fig. 6.⁹ Going down in the graph, we define four additional complex parameters as

$$\begin{aligned}
 A_{c_7} &= \frac{c_7c_0^2}{c_1c_3c_8}, & A_{c_2} &= \frac{c_2c_0}{c_1c_3}, \\
 A_{c_4} &= \frac{c_4c_0^2}{c_1c_3c_6}, & A_{c_5} &= \frac{c_5c_0^3}{c_3c_8c_1^2}. \tag{27}
 \end{aligned}$$

We already know that the polytope $M221$ is obtained on the heterotic side by adding a Wilson line $a(1_2, 0_{14})$ therefore the monomial “ c_7 ” is associated with this Wilson line. Looking at all the gauge groups in the graph and using results on the compactifications of heterotic strings on a circle [39] we find that a possibility for the Wilson lines associated with each monomial is

$$\begin{aligned}
 A_{c_7} &\sim a(1_2, 0_{14}) & A_{c_2} &\sim b(1_{16}) & A_{c_4} &\sim c(0_{14}, 1_2) \\
 A_{c_5} &\sim d[(1_2, 0_{14}) + i(0, 1_2, 0_{13})] \tag{28}
 \end{aligned}$$

with a, b, c and d in \mathbb{C} parametrizing the moduli on the heterotic side. We can see that A_{c_7} and A_{c_4} are linked to the same Wilson line content if it were not for the symmetry breaking of A_{c_5} .¹⁰ Indeed if one does not add the monomial c_5 , or the Wilson line A_{c_5} in the dual theory, one can interchange c_7 and c_4 and obtain the same Weierstrass models obtained from $M221$. Moreover, due to the symmetry of the two parameters A_{c_7} and A_{c_4} , if $A_{c_7} = A_{c_4}$ i.e $a = c$ in (28), we obtain what we expect on the heterotic side, namely $SO(24) \times SU(2)^2 \rightarrow SO(24) \times SU(4)$ for the polytope $M476$ while $E_7 \times E_7$ is not enhanced.

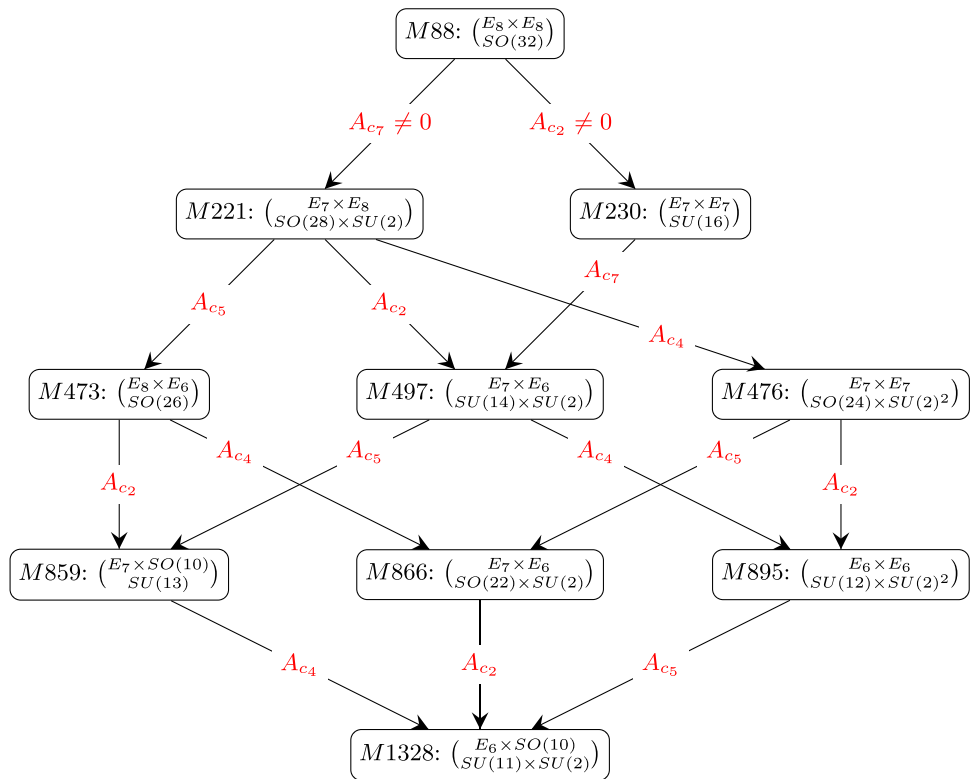
4.3 ... and back to monomials

We are now able for particular polytopes $M\#$ whose dual $N\#$ contain two fibers to describe K3s as parametrizations of Wilson lines of its dual theory (both for $E_8 \times E_8$ and $SO(32)$). Rather we linked *monomials* in the defining hypersurface equation of K3s to parameters in the Wilson lines. This means that we can construct Weierstrass models of elliptically fibered K3s which are not per say described by reflexive polyhedra, and directly interpret them as a particular Wilson lines content on the $E_8 \times E_8$ and $SO(32)$ heterotic strings. Indeed let us go back to the graph of Fig. 6: adding the monomial c_4 to the underlined terms of (22) and (23) gives the

⁹ This graph was found using the third program presented in this paper applied to the polytope $M1328$.

¹⁰ In the $E_8 \times E_8$ heterotic string one can just interchange the E_8 s.

Fig. 6 Links between various reflexive polyhedra. Going upward from $M1328$ amounts to removing points in the polytope $M1328$, or equivalently monomials in the hypersurface equations (22) and (23). Going downward corresponds to adding a complex modulus $A_{c\#}$



Weierstrass models one gets from $M221$ as explained previously. Adding only c_5 however, we cannot obtain a polytope with 3 moduli which will give the same Weierstrass models. Thus let us write the Weierstrass models of the polytope $M88$, together with the additional monomial c_5 in (22) and (23). For the first fiber we find

$$f = \left(-\frac{1}{48}\right) \cdot c_0 \cdot s^3 \cdot t^4 \cdot (c_0^3 s - 24c_1 c_3 c_5 t) \tag{29}$$

$$g = \left(-\frac{1}{864}\right) \cdot s^4 \cdot t^5 \cdot \left(864c_1^3 c_3^2 c_6 s^3 - c_0^6 s^2 t + 864c_1^3 c_3^2 c_9 s^2 t + 36c_0^3 c_1 c_3 c_5 s t^2 + 864c_1^3 c_3^2 c_8 s t^2 - 216c_1^2 c_3^2 c_5^2 t^3\right) \tag{30}$$

$$\Delta_{(f,g)} = \left(\frac{1}{16}\right) \cdot c_3^2 \cdot c_1^3 \cdot s^8 \cdot t^{10} \cdot \left(432c_1^3 c_3^2 c_6^2 s^6 - c_0^6 c_6 s^5 t + 864c_1^3 c_3^2 c_6 c_9 s^5 t + 36c_0^3 c_1 c_3 c_5 c_6 s^4 t^2 + 864c_1^3 c_3^2 c_6 c_8 s^4 t^2 - c_0^6 c_9 s^4 t^2 + 432c_1^3 c_3^2 c_9^2 s^4 t^2 - 216c_1^2 c_3^2 c_5^2 c_6 s^3 t^3 - c_0^6 c_8 s^3 t^3 + 36c_0^3 c_1 c_3 c_5 c_9 s^3 t^3 + 864c_1^3 c_3^2 c_8 c_9 s^3 t^3 + 36c_0^3 c_1 c_3 c_5 c_8 s^2 t^4 + 432c_1^3 c_3^2 c_8^2 s^2 t^4 - 216c_1^2 c_3^2 c_5^2 c_9 s^2 t^4 - c_0^3 c_3^3 s t^5 - 216c_1^2 c_3^2 c_5^2 c_8 s t^5 + 27c_1 c_3^2 c_5^4 t^6\right) \tag{31}$$

$$f = \left(-\frac{1}{48}\right) \cdot t^2 \cdot \left(16c_1^2 c_3^2 s^6 - 8c_0^2 c_1 c_3 s^5 t + c_0^4 s^4 t^2 + 32c_1^2 c_3 c_9 s^3 t^3 - 8c_0^2 c_1 c_9 s^2 t^4 - 24c_0 c_1 c_5 c_6 s t^5 - 48c_1^2 c_6 c_8 t^6 + 16c_1^2 c_9^2 t^6\right) \tag{32}$$

$$g = \left(-\frac{1}{864}\right) \cdot t^3 \cdot \left(64c_1^3 c_3^3 s^9 - 48c_0^2 c_1^2 c_3^2 s^8 t + 12c_0^4 c_1 c_3 s^7 t^2 - c_0^6 s^6 t^3 + 192c_1^3 c_3^2 c_9 s^6 t^3 - 96c_0^2 c_1^2 c_3 c_9 s^5 t^4 - 144c_0 c_1^2 c_3 c_5 c_6 s^4 t^5 + 12c_0^4 c_1 c_9 s^4 t^5 + 36c_0^3 c_1 c_5 c_6 s^3 t^6 - 288c_1^3 c_3 c_6 c_8 s^3 t^6 + 192c_1^3 c_3 c_9^2 s^3 t^6 + 72c_0^2 c_1^2 c_6 c_8 s^2 t^7 - 48c_0^2 c_1^2 c_9^2 s^2 t^7 - 144c_0 c_1^2 c_5 c_6 c_9 s t^8 - 216c_1^2 c_3^2 c_6^2 t^9 - 288c_1^3 c_6 c_8 c_9 t^9 + 64c_1^3 c_9^3 t^9\right) \tag{33}$$

$$\Delta_{(f,g)} = \left(-\frac{1}{16}\right) \cdot c_6^2 \cdot c_1^3 \cdot t^{15} \cdot \left(16c_1^2 c_3^2 c_5^2 s^9 - 8c_0^2 c_1 c_3^2 c_5^2 s^8 t + c_0^4 c_3^2 c_5^2 s^7 t^2 + 16c_0 c_1^2 c_3^2 c_5 c_8 s^7 t^2\right)$$

and for the second

$$\begin{aligned}
 & -8c_0^3c_1c_3c_5c_8s^6t^3 + 16c_1^3c_3^2c_8^2s^6t^3 \\
 & +48c_1^2c_3^2c_5^2c_9s^6t^3 + c_0^5c_5c_8s^5t^4 - 8c_0^2c_1^2c_3c_8^2s^5t^4 \\
 & -16c_0^2c_1c_3c_5^2c_9s^5t^4 - 36c_0c_1c_3c_5^3c_6s^4t^5 \\
 & +c_0^4c_1c_8^2s^4t^5 + c_0^4c_5^2c_9s^4t^5 + 32c_0c_1^2c_3c_5c_8c_9s^4t^5 \\
 & +c_0^3c_5^3c_6s^3t^6 - 72c_1^2c_3c_5^2c_6c_8s^3t^6 \\
 & -8c_0^3c_1c_5c_8c_9s^3t^6 + 32c_1^3c_3c_8^2c_9s^3t^6 \\
 & +48c_1^2c_3^2c_5^2c_9^3t^6 - 30c_0^2c_1c_5^2c_6c_8s^2t^7 \\
 & -8c_0^2c_1^2c_8^2c_9s^2t^7 - 8c_0^2c_1c_5^2c_9^2s^2t^7 \\
 & -96c_0c_1^2c_5c_6c_8^2st^8 - 36c_0c_1c_5^3c_6c_9st^8 \\
 & +16c_0c_1^2c_5c_8c_9^2st^8 - 27c_1c_5^4c_6^2t^9 - 64c_1^3c_6c_8^3t^9 \\
 & -72c_1^2c_5^2c_6c_8c_9t^9 + 16c_1^3c_8^2c_9^2t^9 + 16c_1^2c_5^3c_9^3t^9
 \end{aligned} \tag{34}$$

The gauge groups associated with the singularities of these Weierstrass models are $E_6 \times E_8$ and $SO(26)$ respectively. They are exactly what we expect from heterotic string theories with one Wilson line A_{c_5} in Eq. (28). This means that if we compactify F-theory on these elliptically fibered K3s, we know that the Wilson line content on the dual heterotic strings should be of a similar kind as A_{c_5} . Using this it is then possible to restrict the study of the duality map between the two theories to a three dimensional moduli space to verify that the enhancements on both F-theory and heterotic sides match.

4.4 Wilson line interpretation for polytope with more than two fibers

The Wilson line description of reflexive polyhedra can be extended to K3 surfaces that have more than two inequivalent elliptic fibrations. Indeed let us consider the polytope $M2$ with three fibers presented in the Fig. 1. The fiber $E_8 \times E_8 \times SU(2)$ is obtained via the fiber $E_8 \times E_8$ of the polytope $M88$ with $\xi = \frac{1}{4}$ [34]. This in fact corresponds to taking the complex structure and Kahler moduli equal when compactifying on the two torus on the heterotic string. The two remaining fibers ($E_7 \times SO(20)$ and $SU(18)$) of $M2$ can be obtained by considering $M1328$ with $c_3 = c_4 = c_7 = c_8 = c_9 = 0$: $E_6 \times SO(10)$ is enhanced to $E_7 \times SO(20)$ while $SU(11) \times SU(2)$ to $SU(18)$. From our construction, the Wilson lines on the dual theory are therefore parametrized by A_{c_5} and A_{c_2} . The moduli spaces for the fibers $E_7 \times SO(20)$ and $SU(18)$ are thus contained in the moduli spaces of the heterotic strings $E_8 \times E_8$ and $SO(32)$ with this particular Wilson lines parametrization respectively.

5 Conclusion

Here we showed how to interpret particular elliptically fibered K3 surfaces directly from a Wilson line parametrization of the dual $E_8 \times E_8$ and $SO(32)$ heterotic strings. We constructed a graph of polytopes with two fibers where the links can be considered in this specific case as inclusion relations between the moduli spaces of the elliptically fibered K3s associated with each fiber. Because in some limit of the moduli space we now recover the compactifications of F-theory dual to the ones of heterotic strings on a two torus with no Wilson lines, we can interpret the additional complex parameters in the moduli spaces as being dual to Wilson line moduli. They appear in the compactifications of F-theory as additional monomials in the hypersurface equation defining the K3 surface. Therefore there seem to be a close link between monomials on the F-theory side, and Wilson line moduli in the heterotic theories. Adding only one monomial to the hypersurface equations which correspond to the $E_8 \times E_8$ and $SO(32)$ compactifications, we can restrict the number of complex parameters in the moduli space to three i.e. one Wilson line modulus for the heterotic strings. This makes finding possible enhancements more convenient, and a study similar to the ones presented in [37,40] and upcoming paper [36] should give more insights on the Wilson line description of elliptically fibered K3 surfaces. Moreover, we also showed that this construction can be adapted to polytopes with more than two fibers. We were able to find heterotic duals to the three fibers of the polytope $M2$ with one modulus. Using the various SageMath programs we developed as well as graphs of polytopes, we hope to find more insights on K3s with three fibrations and more. Finally, here we focused on compactifications of F-theory and heterotic strings to eight dimensions. We expect that the monomial and Wilson line moduli duality should be valuable in studying compactifications to lower dimensions, both involving K3 surfaces and more generally Calabi-Yau threefolds.

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Appendix A Program 1: Dynkin diagram from reflexive polyhedra

Here we present how to use the first program. The first line is simply *reflexivePolytopes = []*. Just enter a list of number between 0 and 4318 to consider the reflexive polytopes *ReflexivePolytope(3,#)* of this list into Sagemath. The program then returns a table containing all gauge groups for all the fibers of any reflexive polytope. The table is written in latex format on a text file.

Figure 3 shows the output of this first program where we gave as an entry the reflexive polytope with Picard number 17 (i.e. 3 complex parameters) and no correction term.

Appendix B Program 2: Weierstrass models

Here we present again the typical output of the second computer program. Again on the first line one just specifies in a list the reflexive polytopes # (associated with *ReflexivePolytope(3,#)*). The output is the hypersurface equation for every fibration of the K3 surface as well as the corresponding Weierstrass models (upon a choice fiber described in Fig. 2 for F13, F15 and F16. In another file are saved all the hypersurface equations in Sagemath form.

The following is the typical Latex output when putting as an input “[476]”.

Polytope M476

Number of different Fiber is 2

Fiber 1

The hypersurface equation is:

$$p = -c_0x_0x_1x_2st + c_1x_0^2 + c_2x_2^3 + c_3x_1^4x_2s^3t^5 + c_4x_1^4x_2s^5t^3 + c_5x_1^6s^7t^5 + c_6x_1^6s^5t^7 + c_7x_1^6s^6t^6 + (35)$$

Data of the Weierstrass model:

$$f = \left(\frac{1}{48}\right) \cdot t^3 \cdot s^3 \cdot (48c_1^2c_2c_4s^2 - c_0^4st + 48c_1^2c_2c_3t^2)(36)$$

$$g = \left(-\frac{1}{864}\right) \cdot t^5 \cdot s^5 \cdot \left(72c_0^2c_1^2c_2c_4s^2 + 864c_1^3c_2^2c_5s^2 - c_0^6st + 864c_1^3c_2^2c_7st + 72c_0^2c_1^2c_2c_3t^2 + 864c_1^3c_2^2c_6t^2\right) \quad (37)$$

$$\Delta_{(f,g)} = \left(\frac{1}{16}\right) \cdot c_2^2 \cdot c_1^3 \cdot t^9 \cdot s^9$$

$$\cdot \left(64c_1^3c_2c_3s^6 - c_0^4c_1c_2^4s^5t + 72c_0^2c_1^2c_2c_4c_5s^5t + 432c_1^3c_2^2c_3^2s^5t + 192c_1^3c_2c_3c_4^2s^4t^2 - c_0^6c_5s^4t^2 + 72c_0^2c_1^2c_2c_4c_7s^4t^2 + 864c_1^3c_2^2c_5c_7s^4t^2 - 2c_0^4c_1c_3c_4s^3t^3 + 72c_0^2c_1^2c_2c_3c_5s^3t^3 + 72c_0^2c_1^2c_2c_4c_6s^3t^3 + 864c_1^3c_2^2c_5c_6s^3t^3 - c_0^6c_7s^3t^3 + 432c_1^3c_2^2c_7^2s^3t^3 + 192c_1^3c_2c_3^2c_4s^2t^4 - c_0^6c_6s^2t^4 + 72c_0^2c_1^2c_2c_3c_7s^2t^4 + 864c_1^3c_2^2c_6c_7s^2t^4 - c_0^4c_1c_2^3s^2t^4 + 72c_0^2c_1^2c_2c_3c_6st^5 + 432c_1^3c_2^2c_6^2st^5 + 64c_1^3c_2^3t^6\right) \quad (38)$$

Fiber 2

The hypersurface equation is:

$$p = -c_0x_0x_1x_2st + c_1x_0^2 + c_2x_1^2x_2^2st^3 + c_3x_1^4x_2t + c_4x_2^3s^6t + c_5x_2^3s^7 + c_6x_1^4x_2s + c_7x_1^2x_2^2s^4 + (39)$$

Data of the Weierstrass model:

$$f = \left(-\frac{1}{48}\right) \cdot s^2 \cdot \left(-48c_1^2c_5c_6s^6 + 16c_1^2c_7^2s^6 - 48c_1^2c_3c_5s^5t - 48c_1^2c_4c_6s^5t - 48c_1^2c_3c_4s^4t^2 - 8c_0^2c_1c_7s^4t^2 + 32c_1^2c_2c_7s^3t^3 + c_0^4s^2t^4 - 8c_0^2c_1c_2st^5 + 16c_1^2c_2^2t^6\right) \quad (40)$$

$$g = \left(-\frac{1}{864}\right) \cdot s^3 \cdot (4c_1c_7s^3 - c_0^2st^2 + 4c_1c_2t^3)$$

$$\cdot \left(-72c_1^2c_5c_6s^6 + 16c_1^2c_7^2s^6 - 72c_1^2c_3c_5s^5t - 72c_1^2c_4c_6s^5t - 72c_1^2c_3c_4s^4t^2 - 8c_0^2c_1c_7s^4t^2 + 32c_1^2c_2c_7s^3t^3 + c_0^4s^2t^4 - 8c_0^2c_1c_2st^5 + 16c_1^2c_2^2t^6\right) \quad (41)$$

Table 3 Gauge groups for polytopes with Picard 17. Columns represent the inequivalent fibers $\nabla^{(2)}$ dividing the dual $N\#$ of $M\#$ into two parts. Additional contributions of $U(1)$ factors should be considered so that the gauge group is of rank $p - 2$ where p is the Picard number

M14	$\frac{E_7 \times SU(8) \times SU(2)}{Z_2}$	$SO(16) \times E_6$	$E_8 \times E_7 \times Z_4$	$SU(14) \times SU(2)$	$E_8 \times E_6$	$SO(10) \times SO(18)$
M15	$E_7 \times E_6 \times SU(2)$	$SO(14) \times SO(14)$	$\frac{SO(16) \times SU(8)}{Z_2}$	$E_7 \times E_8 \times Z_4$	$\frac{SU(14) \times SU(2)}{Z_2}$	
M20	$E_7 \times E_8 \times Z_3$	$E_6 \times SO(14)$	$E_6 \times SO(10) \times SU(3)SU(2)$	$SU(6) \times SO(14) \times SU(3)$	$E_7 \times SO(12)$	$SO(12) \times SU(8)$
M21	$E_7 \times SO(12) \times SU(2)$	$SU(6) \times SU(10)$	$E_7 \times E_7$	$\frac{SO(12) \times SO(12) \times SU(2)SU(2)}{Z_2}$	$E_7 \times E_7 \times Z_2$	$\frac{SU(10) \times SO(8) \times SU(2)}{Z_2}$ $SO(12) \times SO(12) \times SU(4)$ $\frac{SU(9) \times SU(7)}{Z_2}$
M22	$SO(10) \times SU(9) \times SU(2)$	$SO(14) \times SU(7)$	$E_8 \times E_7 \times Z_3$	$SO(14) \times SO(12)$	$E_6 \times E_6 \times SU(3)$	$E_7 \times SO(10) \times SU(2)$ $\frac{E_6 \times SU(6) \times SU(3)SU(3)}{Z_3}$
M23	$\frac{SU(10) \times SO(12)}{Z_2}$	$E_6 \times SO(14) \times SU(2)$	$E_7 \times SU(8)$	$SO(12) \times E_8$	$E_7 \times E_7$	$SO(18) \times SU(6)$
M24	$E_7 \times E_7$	$SO(10) \times SU(8) \times SU(3)$	$E_6 \times E_6 \times SU(2)$	$SO(12) \times SO(12)$	$\frac{SU(8) \times SU(8)}{Z_2}$	
M25	$SO(10) \times SO(14) \times SU(2)$	$SO(10) \times SU(9)$	$E_6 \times E_7$	$SO(12) \times E_6 \times SU(3)$	$E_7 \times E_7$	$SU(7) \times E_6 \times SU(3)$ $\frac{SU(6) \times SU(9) \times SU(3)}{Z_3}$
M26	$SU(9) \times E_6$	$E_6 \times SU(9)$	$E_7 \times E_7$	$SO(10) \times SO(16)$	$E_6 \times E_8$	$\frac{SU(6) \times SU(12) \times SU(3)}{Z_3}$
M27	$SU(8) \times SO(14)$	$\frac{SU(4) \times SU(12)}{Z_2}$	$E_6 \times E_6 \times SU(2)SU(2)$	$E_7 \times SO(12)$		
M28	$E_6 \times E_7 \times SU(2)$	$SU(10) \times SO(10)$	$SO(10) \times E_7 \times SU(3)$	$SU(5) \times SU(11)$	$E_6 \times E_8$	$E_6 \times SU(8) \times SU(2)$ $\frac{SU(6) \times SO(16) \times SU(2)}{Z_2}$
M29	$E_7 \times E_6$	$E_7 \times E_7 \times Z_2$	$\frac{SO(12) \times SU(8) \times SU(2)}{Z_2}$	$\frac{SU(6) \times SU(10)}{Z_2}$		
M30	$E_6 \times E_6$	$E_7 \times E_7 \times Z_2$	$\frac{SO(12) \times SO(12) \times SU(2)SU(2)}{Z_2}$	$\frac{SU(8) \times SU(8)}{Z_2}$		
M38	$E_7 \times E_7$	$SO(14) \times E_8$	$\frac{SU(4) \times SO(24)}{Z_2}$	$\frac{SU(16)}{Z_2}$		
M41	$\frac{SO(12) \times SO(16) \times SU(2)}{Z_2}$	$E_7 \times E_8 \times Z_4$	$\frac{SO(12) \times E_7 \times SU(2)SU(2)}{Z_2}$	$E_7 \times E_7 \times SU(2)$	$SU(14)$	
M47	$SO(18) \times SO(10)$	$E_8 \times E_7 \times Z_4$	$SU(14) \times SU(2)$	$E_8 \times E_6$	$\frac{SU(8) \times E_7 \times SU(2)}{Z_2}$	
M48	$E_7 \times E_7$	$SO(14) \times E_6 \times SU(2)$	$SU(13) \times SU(3)$	$SO(14) \times SO(14)$		
M49	$E_7 \times E_6 \times SU(2)$	$\frac{SU(8) \times SO(16)}{Z_2}$	$E_7 \times E_8 \times Z_4$	$SO(14) \times SO(14)$		
M50	$SU(15)$	$\frac{SU(6) \times SO(20)}{Z_2}$	$SO(14) \times E_7$	$E_8 \times E_6 \times SU(2)$		
M53	$E_6 \times SU(9)$	$E_7 \times E_7$	$SO(16) \times SO(10)$	$\frac{SU(10) \times SO(12)}{Z_2}$	$SO(16) \times E_6$	
M104	$E_7 \times E_8$	$\frac{SO(28) \times SU(2)}{Z_2}$			$\frac{SU(14) \times SU(2)}{Z_2}$	
M117	$\frac{SO(24) \times SU(4)}{Z_2}$	$\frac{SU(16)}{Z_2}$	$E_7 \times E_7$			
M221	$E_7 \times E_8$	$\frac{SO(28) \times SU(2)}{Z_2}$				
M230	$\frac{SU(16)}{Z_2}$	$E_7 \times E_7$				

$$\Delta_{(f,g)} = \left(-\frac{1}{16}\right) \cdot c_1^4 \cdot s^{14} \cdot (c_6s + c_3t)^2 \cdot (c_5s + c_4t)^2 \cdot \left(-64c_1^2c_5c_6s^6 + 16c_1^2c_7s^6 - 64c_1^2c_3c_5s^5t - 64c_1^2c_4c_6s^5t - 64c_1^2c_3c_4s^4t^2 - 8c_0^2c_1c_7s^4t^2 + 32c_1^2c_2c_7s^3t^3 + c_0^4s^2t^4 - 8c_0^2c_1c_2st^5 + 16c_1^2c_2^2t^6\right) \tag{42}$$

Appendix C Program 3: Finding basic enhancements and constructing graphs

In the following we see the enhancement for the input [476] for the third program.

M476

fiber 1

- () – E7xE7
- (0')- E7xE7
- (3')- E8xE7
- (4')- E7xE8
- (5')- E7xE7
- (6')- E7xE7
- (7')- E7xE7
- (0'3')- E8xE7
- (0'4')- E7xE8
- (0'5')- E7xE7
- (0'6')- E7xE7
- (0'7')- E7xE7
- (3'4')- E8xE8
- (3'5')- E8xE7
- (3'7')- E8xE7
- (4'6')- E7xE8
- (4'7')- E7xE8
- (5'6')- E7xE7
- (5'7')- E7xE7
- (6'7')- E7xE7
- (0'3'4')- E8xE8
- (0'3'5')- E8xE7
- (0'3'7')- E8xE7
- (0'4'6')- E7xE8
- (0'4'7')- E7xE8
- (0'5'6')- E7xE7
- (0'5'7')- E7xE7
- (0'6'7')- E7xE7
- (3'4'7')- E8xE8
- (3'5'7')- E8xE7
- (4'6'7')- E7xE8
- (5'6'7')- E7xE7

- (0'3'4'7')- E8xE8
- (0'3'5'7')- E8xE7
- (0'4'6'7')- E7xE8
- (0'5'6'7')- E7xE7

fiber 2

- ()- SO(24)xSU(2)xSU(2)
- (0')- SO(24)xSU(2)xSU(2)
- (3')- SO(28)xSU(2)
- (4')- SO(28)xSU(2)
- (5')- SO(24)xSU(2)xSU(2)
- (6')- SO(24)xSU(2)xSU(2)
- (7')- SO(24)xSU(2)xSU(2)
- (0'3')- SO(28)xSU(2)
- (0'4')- SO(28)xSU(2)
- (0'5')- SO(24)xSU(2)xSU(2)
- (0'6')- SO(24)xSU(2)xSU(2)
- (0'7')- SO(24)xSU(2)xSU(2)
- (3'4')- SO(32)
- (3'5')- SO(28)xSU(2)
- (3'7')- SO(28)xSU(2)
- (4'6')- SO(28)xSU(2)
- (4'7')- SO(28)xSU(2)
- (5'6')- SO(24)xSU(4)
- (5'7')- SO(24)xSU(2)xSU(2)
- (6'7')- SO(24)xSU(2)xSU(2)
- (0'3'4')- SO(32)
- (0'3'5')- SO(28)xSU(2)
- (0'3'7')- SO(28)xSU(2)
- (0'4'6')- SO(28)xSU(2)
- (0'4'7')- SO(28)xSU(2)
- (0'5'6')- SO(24)xSU(4)
- (0'5'7')- SO(24)xSU(2)xSU(2)
- (0'6'7')- SO(24)xSU(2)xSU(2)
- (3'4'7')- SO(32)
- (3'5'7')- SO(28)xSU(2)
- (4'6'7')- SO(28)xSU(2)
- (5'6'7')- SO(24)xSO(8)
- (0'3'4'7')- SO(32)
- (0'3'5'7')- SO(28)xSU(2)
- (0'4'6'7')- SO(28)xSU(2)
- (0'5'6'7') = 0: SO(24)xSO(8)

Appendix D: Vertices of the polytopes presented in this paper

- M0 : ((1, 0, 0), (0, 1, 0), (0, 0, 1), (-1, -1, -1))
- M2 : ((1, 0, 0), (0, 1, 0), (0, 0, 1), (-3, -1, -1))
- M3 : ((1, 0, 0), (0, 1, 0), (-1, -1, 0), (0, 0, 1), (-1, 0, -1))
- M4 : ((1, 0, 0), (-1, 0, 0),

$(0, 1, 0), (0, 0, 1), (0, -1, -1))$
 $M5 : ((1, 0, 0), (-1, 0, 0),$
 $(0, 1, 0), (0, 0, 1), (1, -1, -1))$
 $M6 : ((1, 0, 0), (0, 1, 0),$
 $(-1, -1, 0), (0, 0, 1), (1, 0, -1))$
 $M7 : ((1, 0, 0), (-1, 0, 0),$
 $(0, 1, 0), (0, 0, 1), (2, -1, -1))$
 $M10 : ((1, 0, 0), (0, 1, 0),$
 $(-2, -1, 0), (0, 0, 1), (-2, 0, -1))$
 $M11 : ((1, 0, 0), (0, 1, 0),$
 $(-2, -1, 0), (0, 0, 1), (-1, 1, -1))$
 $M16 : ((1, 0, 0), (0, 1, 0),$
 $(0, 0, 1), (-2, -1, -1), (-1, 1, 0))$
 $M88 : ((1, 0, 0), (0, 1, 0), (0, 0, 1), (-6, -4, -1))$
 $M221 : ((1, 0, 0), (0, 1, 0),$
 $(0, 0, 1), (-5, -3, -1), (-1, -1, 1))$
 $M230 : ((1, 0, 0), (0, 1, 0),$
 $(1, -1, 0), (0, 0, 1), (-4, -2, -1))$
 $M473 : ((1, 0, 0), (0, 1, 0),$
 $(0, 0, 1), (-4, -3, -1), (-1, 0, 1),$
 $(-2, -1, 1))$
 $M476 : ((1, 0, 0), (0, 1, 0),$
 $(0, 0, 1), (-4, -2, -1),$
 $(-5, -3, -1), (-1, -1, 1))$
 $M497 : ((1, 0, 0), (0, 1, 0),$
 $(-1, 1, 0), (0, 0, 1), (-2, -3, -1),$
 $(0, -1, 1))$
 $M859 : ((1, 0, 0), (0, 1, 0),$
 $(1, -1, 0), (0, 0, 1), (-3, -1, -1),$
 $(0, -1, 1), (-1, -1, 1))$
 $M866 : ((1, 0, 0), (0, 1, 0),$
 $(0, 0, 1), (-3, -2, -1), (-1, 0, 1),$
 $(-4, -3, -1), (-2, -1, 1))$
 $M895 : ((1, 0, 0), (0, 1, 0),$
 $(-1, 1, 0), (0, 0, 1), (-2, -2, -1),$
 $(-2, -3, -1), (0, -1, 1))$
 $M1328 : ((1, 0, 0), (0, 1, 0),$
 $(-1, 1, 0), (0, 0, 1), (-1, -2, -1),$
 $(0, -1, 1), (-2, -2, -1), (-2, -3, -1))$

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