



Fermion number $1/2$ of sphalerons and spectral mirror symmetry

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Abstract We present a rederivation of the baryon and lepton numbers $\frac{1}{2}$ of the $SU(2)_L$ S sphaleron of the standard electroweak theory based on spectral mirror symmetry. We explore the properties of a fermionic Hamiltonian under discrete transformations along a noncontractible loop of field configurations that passes through the sphaleron and whose endpoints are the vacuum. As is well known, CP transformation is not a symmetry of the system anywhere on the loop, except at the endpoints. By augmenting CP with a chirality transformation, we observe that the Dirac Hamiltonian is odd under the new transformation precisely at the sphaleron, and this ensures the mirror symmetry of the spectrum, including the continua. As a consistency check, we show that the fermionic zero mode presented by Ringwald in the sphaleron background is invariant under the new transformation. The spectral mirror symmetry which we establish here, together with the presence of the zero mode, are the two necessary conditions whence the fermion number $\frac{1}{2}$ of the sphaleron can be inferred using the reasoning presented by Jackiw and Rebbi or, equivalently, using the spectral deficiency $\frac{1}{2}$ of the Dirac sea. The relevance of this analysis to other solutions is also discussed.

1 Introduction

In their seminal paper on the subject of charge fractionalization, Jackiw and Rebbi studied the Dirac equation in classical bosonic backgrounds for a number of field theories [1]. Their key discovery was the existence of states with half-integer fermion numbers in theories where all the fundamental fields have integer fermion numbers. As was pointed out in [1–4], in order for a bosonic configuration to have half-integer fermion numbers, the following two conditions must be simultaneously satisfied:

1. The existence of a normalizable fermionic zero mode precisely at the bosonic configuration;
2. Mirror symmetry of the entire fermionic spectrum, consisting of the bound and continuum states, at the bosonic configuration, or equivalently, the fermion number conjugation invariance of the system.

Since then, charge fractionalization has been widely studied and has found many applications in different areas, such as particle physics [2–8], condensed matter physics [9–11], polymer physics [12–14], quantum wires [15] and topological insulators [16, 17].

One class of solutions that can be found in certain field theories are sphalerons, which are saddle-point solutions in field configuration space [18, 19]. An important member of this class of solutions is the ‘S’ sphaleron [20] of the standard electroweak theory. Its importance is due to the role that it is believed to play in the early Universe, including the generation of the matter-antimatter asymmetry of the Universe [21–23]. Following the discovery of this solution in hadronic models [24, 25], it was rediscovered [18] in $SU(2)_L$ theory and its properties and implications for cosmology were detailed in [21].

In 1974 Dashen et al. not only constructed a sphaleron solution as an extended model of hadrons, but also presented a framework for calculating the bound state energies of fermions which couple to the $SU(2)$ gauge field component of the sphalerons [24]. The coupling of the fermions to the Higgs component was represented as an explicit fermionic mass term. Based on their work, the author of [26] showed that in the classical $SU(2)$ gauge field of the sphaleron, a fermion has a single bound-state solution with zero energy. In the standard electroweak theory, a single normalizable zero energy solution of the Dirac equation in the sphaleron background was shown to exist for massless fermions in [27]. Shortly after, this result was extended by Ringwald to the case of massive fermions [28].

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Later on, in the level-crossing picture for the $SU(2)_L$ theory, the change in the bound state energy of fermions coupled to the bosonic fields of the noncontractible loop (NCL) was numerically determined [29]. There, it was shown that as one traverses the NCL passing through the sphaleron, a single negative eigenvalue of the Dirac Hamiltonian arises from the Dirac sea, crosses the zero energy level precisely at the sphaleron and enters the positive energy continuum as one returns to the vacuum configuration. The numerical study of [29] thus reconfirmed the existence of a zero energy bound state in the sphaleron background.

It is well known that the baryon and lepton numbers of the S sphaleron are $\frac{1}{2}$ [21]. This can be seen by using the chiral anomaly and integrating the temporal component of the Chern-Simons current over one-point compactified 3-space and obtaining the resultant Chern-Simons charge for the sphaleron configuration, which is half-integer due to $SO(3)$ and reflection symmetries in the bosonic sector [19]. However, the use of the one-point compactification scheme corresponds to a gauge that breaks the reflection symmetry of the bosonic fields of the NCL about the sphaleron [19, 21]. On the other hand, the spectral flow for the Dirac Hamiltonian along the NCL and symmetries of the fermionic sector are independent of the gauge, and constitute the core of this paper.

The spectra of fermions coupled to the $SU(2)_L$ gauge-Higgs fields of the NCL passing through the S sphaleron have been studied in great detail over the past three decades by various authors. In doing so, many of the symmetries of the spectra have been pointed out and explored [19, 21, 29–32]. However, a symmetry that seems to have not been fully elucidated hitherto in the literature is a certain conjugation symmetry precisely at the sphaleron. This manifests itself as the mirror symmetry of the fermionic spectrum about $E = 0$. Following the path of the bound state as the NCL is traversed, this symmetry can be obviously seen to exist for the bound state precisely at the sphaleron, where it crosses $E = 0$. Now, the question is whether the entire fermionic spectrum, including the continuum states, has mirror symmetry at the sphaleron. As we shall show, this is indeed the case, and this has an important implication which brings us to the subject of this paper.

The main goal of this paper is to present a rederivation of the half-integer fermion numbers of $SU(2)_L$ S sphalerons by adopting an approach that is based on discrete symmetries. To do this, we explicitly construct the transformation operator, which consists of the chiral and CP transformations, under which the Dirac Hamiltonian at the sphaleron is odd. Hence we show that the entire spectrum of the Dirac Hamiltonian has mirror symmetry in the presence of the sphaleron. We then use the results presented by Jackiw and Rebbi [1] to argue that the presence of the zero mode mandates half-integer fermion numbers for the sphaleron.

We should mention that the relation between the results of [1] and half-integer fermion numbers of sphalerons have been claimed without proof in works such as [20, 21, 27, 33]. However, as mentioned before, the necessary conditions to make such a connection are the mirror symmetry of the entire spectrum along with the invariance of the zero mode, both of which we establish here, thereby completing the proof. Furthermore, whereas some works have based their arguments on a fermionic zero mode in the limit of vanishing fermion mass [27, 33], in this work we have used the Ansatz given by [28], which is an extension to massive fermions within the standard electroweak theory. In this case, the Higgs component of the sphaleron plays a nontrivial and essential role, which is beyond an explicit mass term.

The outline of this paper is as follows. In Sects. 2 and 3, we briefly review the bosonic sector of the standard electroweak theory and the sphaleron Ansatz of $SU(2)_L$ Yang-Mills-Higgs theory in the limit of vanishing weak mixing angle. We also revisit the standard derivation of the sphaleron's fermion numbers based on the chiral anomaly. In Sect. 4, we analyze the behavior of the Dirac Hamiltonian operator under a CP transformation for all configurations along the NCL. Then, we augment CP to arrive at a suitable choice of conjugation operator which reveals the spectral mirror symmetry at the sphaleron. Then, in Sect. 5, we perform the symmetry transformation on the zero mode given by Ringwald [28] in the sphaleron background. In Sect. 6, we summarize our results and present an outlook.

2 $SU(2)_L$ sphaleron

Consider the bosonic sector of the well-established electroweak Lagrangian

$$\mathcal{L} = -\frac{1}{4}G_{\mu\nu}^a G_a^{\mu\nu} - \frac{1}{4}F_{\mu\nu} F^{\mu\nu} + (D_\mu \Phi)^\dagger D^\mu \Phi - \lambda (\Phi^\dagger \Phi - \eta^2)^2, \quad (2.1)$$

where the $U(1)$ field strength tensor is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad (2.2)$$

the $SU(2)$ field strength tensor is given by

$$G_{\mu\nu}^a = \partial_\mu B_\nu^a - \partial_\nu B_\mu^a + g\epsilon^{abc} B_\mu^b B_\nu^c, \quad (2.3)$$

and the covariant derivative of the Higgs field is

$$D_\mu \Phi = \left(\partial_\mu - ig \frac{\tau^a}{2} B_\mu^a - ig' Y A_\mu \right) \Phi. \quad (2.4)$$

In the limit of vanishing mixing angle, the $U(1)$ field decouples and this allows for a spherically symmetric Ansatz for the gauge and Higgs fields of the NCL. To this end, consider the following map:

$$U : S^1 \wedge S^2 \sim S^3 \rightarrow SU(2),$$

$$(\mu, \theta, \phi) \mapsto U(\mu, \theta, \phi), \tag{2.5}$$

where \wedge is the smash product¹ and μ is the loop parameter. A suitable representation is [18,20]

$$U(\mu, \theta, \phi) = -iy^1\tau_1 - iy^2\tau_2 - iy^3\tau_3 + y^4\mathbb{1}, \tag{2.6}$$

where

$$\begin{pmatrix} y^1 \\ y^2 \\ y^3 \\ y^4 \end{pmatrix} = \begin{pmatrix} -\sin\mu \sin\theta \sin\phi \\ -\sin\mu \sin\theta \cos\phi \\ \sin\mu \cos\mu (\cos\theta - 1) \\ \cos^2\mu + \sin^2\mu \cos\theta \end{pmatrix}, \tag{2.7}$$

and τ^i are chosen to be the usual Pauli matrices while $\tau^i/2$ are the generators in weak isospace. Using the above map, the Ansatz² for the static gauge and Higgs fields of the $SU(2)_L$ sphaleron barrier becomes [18]

$$B(\mu, r, \theta, \phi) = -\frac{f(r)}{g} dU(\mu, \theta, \phi) U^{-1}(\mu, \theta, \phi),$$

$$\Phi(\mu, r, \theta, \phi) = \eta h(r) U(\mu, \theta, \phi) \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \eta [1 - h(r)] \begin{pmatrix} 0 \\ e^{-i\mu} \cos\mu \end{pmatrix}, \tag{2.8}$$

where the radial functions have the following boundary conditions:

$$\lim_{r \rightarrow 0} \frac{f(r)}{r} = 0, \quad \lim_{r \rightarrow \infty} f(r) = 1,$$

$$\lim_{r \rightarrow 0} h(r) = 0, \quad \lim_{r \rightarrow \infty} h(r) = 1. \tag{2.9}$$

The field B is an $SU(2)$ -valued one-form,

$$B(\mu, r, \theta, \phi) = B_r dr + B_\theta d\theta + B_\phi d\phi = B_i dx^i, \tag{2.10}$$

for which we impose the radial gauge condition $B_r = 0$ [18]. We assume that in the radial gauge there exists a limiting field

$$\Phi^\infty(\theta, \phi) \equiv \lim_{r \rightarrow \infty} \Phi(r, \theta, \phi), \tag{2.11}$$

¹ For a definition, see [20].

² It can be shown that, even when the Ansatz is not manifestly spherically symmetric, it can always be transformed to one that is [34,35].

such that $|\Phi^\infty| = 1$ and

$$\Phi^\infty(\theta = 0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{2.12}$$

3 Standard derivation of Fermion number

In order to put our rederivation of the fermion number of the sphaleron configuration in proper context, it is useful to first review briefly its standard derivation based on anomalies. In particular, this makes the motivation for our proposed spectral symmetry operator, as well as the connection between the two derivations, more clear. Consider the baryon current for a single generation of quarks

$$j_B^\mu = \frac{1}{3} (\bar{u}_\alpha \gamma^\mu u_\alpha + \bar{d}_\alpha \gamma^\mu d_\alpha), \tag{3.1}$$

where α is the $SU(3)$ color index. As a result of the Abelian anomaly in the Standard Model [36], the non-vanishing divergence of this current has an $SU(2)$ contribution given by [21]

$$\partial_\mu j_B^\mu = \frac{g^2}{64\pi^2} \epsilon_{\mu\nu\rho\sigma} G_a^{\mu\nu} G_a^{\rho\sigma}. \tag{3.2}$$

Furthermore, the right-hand side of Eq. (3.2) can be rewritten as the total divergence of the gauge variant Chern-Simons current given by

$$K_\mu = \frac{g^2}{16\pi^2} \epsilon_{\mu\nu\rho\sigma} \text{Tr} \left(G^{\nu\rho} B^\sigma + \frac{2}{3} ig B^\nu B^\rho B^\sigma \right), \tag{3.3}$$

where

$$G_{\nu\rho} = \frac{1}{2} \tau^a G_{\nu\rho}^a, \quad B_\nu = \frac{1}{2} \tau^a B_\nu^a. \tag{3.4}$$

For gauge field configurations that tend to pure gauge at infinity, the spatial integral of the Pontryagin density $\mathcal{P}(x) = \partial_\mu K^\mu$ measures the difference between Chern-Simons charges of different vacua

$$\Delta Q_{CS} = \int d^3r K^0. \tag{3.5}$$

When calculated in the correct gauge, namely one in which the integral of $\vec{K} \cdot d\vec{S}$ over the surface of a sphere S at spatial infinity vanishes, Eqs. (3.2–3.5) imply that the Chern-Simons charge of the field configuration is equal to its baryon number Q_B . Thus, for the $SU(2)_L$ sphaleron of YMH theory, if one uses a one-point compactification for 3-space, starts from a vacuum configuration with Q_{CS} set to zero and traverses the NCL through the sphaleron, one finds that the sphaleron will

have $Q_B = Q_{CS} = \frac{1}{2}$ [21], while the neighboring vacuum will have $Q_{CS} = 1$. Since the above argument also applies to leptonic currents, the electroweak sphaleron’s lepton number is also $\frac{1}{2}$.

A point that is worth mentioning here is that the Pontryagin index is odd under CP. Thus, as in the case of Yang–Mills instantons, in which a $\theta \int d^4x \mathcal{P}(x)$ term in the effective action signals CP violation, one can infer that CP alone is not the proper symmetry transformation to be used when applying the argument of Jackiw and Rebbi to the sphaleron. Furthermore, the relation of the derivation given above to the chiral anomaly suggests that the extension of CP that is necessary in order to satisfy the conditions of Jackiw and Rebbi should include a chiral transformation. In the next section, we present the necessary augmentation of CP, thereby providing an alternative derivation of the fermion numbers of the S sphaleron based on discrete symmetries reflected in the spectral mirror symmetry of the Dirac Hamiltonian.

4 CP transformation along NCL

In this section we study the behavior of the Dirac Hamiltonian under discrete transformations including C and P in a sphaleron background. When the weak mixing angle goes to zero, we perform our analysis for arbitrary loop parameter μ . For the $SU(2)_L \times U(1)_Y$ sphaleron, only the sphaleron Ansatz has been constructed and not the full barrier. This restricts our analysis to the sphaleron when $\theta_w = 0$. Nevertheless, this strategy can be readily extended to the full barrier once it is constructed.

Consider the Dirac Hamiltonian operator of the standard electroweak theory at $\theta_w = 0$ [20]

$$\hat{\mathcal{H}} = -i\gamma^0\gamma^j D_j + k\gamma^0 (\Phi_M^\dagger P_L + \Phi_M P_R), \tag{4.1}$$

where the matrix Φ_M contains the scalar fields of the Higgs doublet and its charge-conjugated doublet and is given by

$$\Phi_M = \begin{pmatrix} \phi_2^* & \phi_1 \\ -\phi_1^* & \phi_2 \end{pmatrix}, \tag{4.2}$$

and the projection operators are defined as

$$P_L = \frac{1}{2}(1 - \gamma^5), \quad P_R = \frac{1}{2}(1 + \gamma^5). \tag{4.3}$$

We now use the Ansatz given in Eq. (2.8) to construct Φ_M shown in Eq. (4.2) and insert it into Eq. (4.1) to obtain the expression for $\hat{\mathcal{H}}$ along the NCL. The final expression for $\hat{\mathcal{H}}$ is shown in the appendix.

We use the following choice of Weyl representation for our gamma matrices

$$\gamma^0 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} -I_2 & 0 \\ 0 & I_2 \end{pmatrix}. \tag{4.4}$$

In this representation, charge conjugation acts non-trivially on scalars and spinors, both of which transform in the fundamental representation of $SU(2)$, as

$$\begin{aligned} \Phi(\vec{x}, t) &\xrightarrow{C} i\tau^2 \Phi^*(\vec{x}, t), \\ \Psi(\vec{x}, t) &\xrightarrow{C} i\tau^2 \gamma^2 \Psi^*(\vec{x}, t), \end{aligned} \tag{4.5}$$

while under the combined transformation of C and P,

$$\begin{aligned} \Phi(\vec{x}, t) &\xrightarrow{CP} i\tau^2 \Phi^*(-\vec{x}, t), \\ \Psi(\vec{x}, t) &\xrightarrow{CP} i\tau^2 \gamma^2 \gamma^0 \Psi^*(-\vec{x}, t). \end{aligned} \tag{4.6}$$

Therefore, the charge-conjugated, parity-inverted Hamiltonian becomes

$$\hat{\mathcal{H}}^{CP}(\vec{x}, t, \mu) = \gamma^2 \gamma^0 \begin{pmatrix} -\hat{\mathcal{H}}_{22}^* & \hat{\mathcal{H}}_{21}^* \\ \hat{\mathcal{H}}_{12}^* & -\hat{\mathcal{H}}_{11}^* \end{pmatrix}_{(-\vec{x}, t, \mu)} \gamma^0 \gamma^2. \tag{4.7}$$

After inserting the explicit expressions for the matrix elements of $\hat{\mathcal{H}}$ given in Eq. (A.2) into Eq. (4.7), we observe that nowhere along the NCL is $\hat{\mathcal{H}}$ odd under CP, except at the trivial vacuum ($\mu = 0, \pi$). However, at $\mu = \frac{\pi}{2}$, there are many cancellations and the even part reduces to

$$\begin{aligned} \hat{\mathcal{H}}^{CP}(\vec{x}, t, \mu = \frac{\pi}{2}) + \hat{\mathcal{H}}(\vec{x}, t, \mu = \frac{\pi}{2}) &= 2k\eta h(r)\gamma^0 \\ &\begin{pmatrix} \cos\theta (P_L + P_R) & -\sin\theta e^{i\phi} (P_L - P_R) \\ \sin\theta e^{-i\phi} (P_L - P_R) & \cos\theta (P_L + P_R) \end{pmatrix}. \end{aligned} \tag{4.8}$$

This reflects the fact that the pseudoscalar sphaleron configuration breaks the CP invariance of the one-generation electroweak theory that we have been considering. We now define a new conjugation transformation, \widetilde{CP} , which consists of CP and is augmented by an additional operation as follows

$$\widetilde{CP} \equiv CP\mathcal{X}, \tag{4.9}$$

where $\mathcal{X} = e^{-i\mu\gamma^5}$. By repeating the calculation leading to Eq. (4.8) for the new operation, Eq. (4.9), we see that

$$\hat{\mathcal{H}}^{\widetilde{CP}}(\vec{x}, t, \mu = \frac{\pi}{2}) = -\hat{\mathcal{H}}(\vec{x}, t, \mu = \frac{\pi}{2}). \tag{4.10}$$

The existence of a transformation under which $\hat{\mathcal{H}}$ is odd ascertains the spectral mirror symmetry. That is, under such a transformation every eigenstate of $\hat{\mathcal{H}}$ with energy E is transformed into one with energy $-E$, the only exception being a zero energy bound state which must then be invariant under such a transformation. From the viewpoint of spectral deficiency [7], this implies that as the NCL is traversed, spectral deficiency \mathcal{D} in the positive continuum caused by the bound state starts to replenish, while deficiency starts to build up in the Dirac sea. At $\mu = \frac{\pi}{2}$ the fermionic bound state is at $E = 0$, and the spectral deficiencies in both continua are [7]

$$\mathcal{D} = \frac{\mu}{\pi}. \tag{4.11}$$

Therefore, at $\mu = \frac{\pi}{2}$, the quantum field theoretic expectation value of the fermion number operator is [4]

$$|\langle N \rangle| = \frac{1}{2}. \tag{4.12}$$

This number can be interpreted as the fermion number of the bosonic configuration.

In the next section, we check the invariance of the zero mode presented by Ringwald (which is the one that is relevant to our model) [28] under \widetilde{CP} . The fermion numbers $\frac{1}{2}$ of the sphaleron then follow immediately from the reasoning presented by Jackiw and Rebbi or, equivalently, from the spectral deficiency $\frac{1}{2}$ of the Dirac sea. It is worth mentioning that at the trivial vacuum ($\mu = 0, \pi$), $\hat{\mathcal{H}}$ is odd under CP, showing that the spectrum has mirror symmetry there. However, there are no bound states in the trivial vacuum.

5 The zero mode

Recall that in the original analysis of Jackiw and Rebbi, the fermionic zero mode in the soliton background was fermion number self-conjugate [1]. Thus, an important consistency check on our symmetry transformation would be to operate it on the fermionic zero mode that was given by Ringwald at the electroweak S sphaleron [28]. To this end, consider the zero-energy solution of the Dirac equation in the sphaleron background. The Ansatz for the left-handed isodoublet of the zero mode is given by [28]

$$\Psi_{0,L}^{i\alpha}(\vec{x}, t) = \epsilon^{i\alpha} z(r), \tag{5.1}$$

where $i = 1, 2$ is the weak isospin index, $\alpha = 1, 2$ is the spinor index and ϵ^{ij} is the Levi-Civita symbol ($\epsilon^{12} = +1$). The functional form of $z(r)$ is obtained by solving the radial part of the Dirac equation. Depending on whether the fermions are massive or massless, $z(r)$ will take on a different form [28]. For a single generation of left-handed quarks,

denoted by

$$\Psi_{0,L}^\alpha = \begin{pmatrix} u_{0,L}^\alpha \\ d_{0,L}^\alpha \end{pmatrix}, \tag{5.2}$$

this implies that

$$\begin{aligned} u_{0,L}(\vec{x}, t) &= z(r) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \equiv z(r) | \downarrow \rangle, \\ d_{0,L}(\vec{x}, t) &= z(r) \begin{pmatrix} -1 \\ 0 \end{pmatrix} \equiv -z(r) | \uparrow \rangle. \end{aligned} \tag{5.3}$$

Thus, Eq. (5.1) can also be written as

$$\Psi_{0,L}(\vec{x}, t) = z(r) \begin{pmatrix} | \downarrow \rangle \\ - | \uparrow \rangle \end{pmatrix}. \tag{5.4}$$

A CP transformation on Eq. (5.1) or, equivalently, Eq. (5.4) yields

$$\Psi_{0,L}^{CP}(\vec{x}, t) = i\gamma^5 \Psi_{0,L}(\vec{x}, t), \tag{5.5}$$

which shows that, as expected, the zero mode is not CP-invariant. By noting that we are performing the symmetry transformation at the sphaleron ($\mu = \frac{\pi}{2}$), implementing the additional factor of $-i\gamma^5$ required by a \widetilde{CP} transformation, we obtain

$$\Psi_{0,L}^{\widetilde{CP}}(\vec{x}, t) = \Psi_{0,L}(\vec{x}, t). \tag{5.6}$$

Thus, we observe that the fermionic zero mode of Ringwald in the sphaleron background is \widetilde{CP} -invariant.

6 Summary and discussion

In this paper, we have studied the behavior of fermions under discrete transformations in a sphaleron background. For the fields of the NCL passing through the S sphaleron, it is well known that the system is not CP-invariant except at the vacua. However, we have constructed a new transformation, denoted by \widetilde{CP} , by augmenting a CP transformation with an additional operation that acts nontrivially in the Yukawa sector and has the following important property. We see that for field configurations along the NCL, the Dirac Hamiltonian is odd under \widetilde{CP} precisely at the S sphaleron sitting at the top of the barrier that begins and ends at the trivial vacuum. This ensures that the spectrum has mirror symmetry. That is, for every positive energy eigenstate there is a corresponding negative energy one and the zero mode, if any, is self-conjugate.

As an important consistency check, by performing the symmetry transformation on the fermionic zero mode given by Ringwald [28] in the sphaleron background, we observe

that the zero mode is indeed \widetilde{CP} -invariant. This is closely analogous to the analyses of [1,4]. There, fermion number conjugation symmetry of the spectrum including the zero mode in the background of the classical solution was an important condition that led to the derivation of the half-integer fermion numbers of the background bosonic fields. In the analyses of [1,4], fermion number conjugation was charge conjugation. Our transformation operator is \widetilde{CP} which reveals the spectral mirror symmetry at the sphaleron. In this configuration, the spectral deficiency in the Dirac sea is exactly $\frac{1}{2}$ and one associates this to the fermion number of the background field which is the sphaleron. This spectral deficiency $\frac{1}{2}$ is equivalent to the integrated anomaly in the standard derivation.

Overall, this analysis offers a number of other potential advantages. At a basic level, it can provide a useful consistency check for the numerous fractionally-charged sphaleron Ansätze that have been discovered so far [37–44], and helps place constraints on their functional forms. An example of this can be seen in the Ansatz for the axially symmetric sphaleron, where the arbitrary functions acquire a z-dependence [45]. Furthermore, the analyses of [1,4] required C-invariance, while the present analysis led to \widetilde{CP} -invariance. It may be that other solutions require other novel symmetry transformations for the fermionic sector to correctly explain their fractional charges. An important issue that our study has not addressed is what happens when one considers three generations of fermions, where CP symmetry is violated through the CKM and PMNS mixing matrices in the background of the even-parity Higgs field vacuum.

Finally, from a more practical perspective, one should bear in mind that sphalerons currently play an omnipresent role in physics and show up in many field theories, such as gravitation, electroweak theory and quantum chromodynamics. Thus, it seems worthy to delve even deeper into their structure to see if new symmetries emerge. It may be that studying these symmetries paves the way for a more systematic understanding of the topological properties of unstable solutions in gauge field theories and their physical applications.

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7 Dirac Hamiltonian along NCL

In this section we give the explicit functional form of the components of Eq. (4.1). As an $SU(2)$ -valued 2×2 matrix, $\hat{\mathcal{H}}$ is

$$\hat{\mathcal{H}} = \begin{pmatrix} \hat{\mathcal{H}}_{11} & \hat{\mathcal{H}}_{12} \\ \hat{\mathcal{H}}_{21} & \hat{\mathcal{H}}_{22} \end{pmatrix}. \tag{A.1}$$

In the background of the gauge and Higgs fields of the NCL, Eq. (2.8), the components of $\hat{\mathcal{H}}$ are

$$\begin{aligned} \hat{\mathcal{H}}_{11} = & -i\gamma^0\gamma^j\partial_j + f(r)r\gamma^0\gamma^3P_L\sin\mu\cos\mu\sin^2\theta \\ & - f(r)r\gamma^0\gamma^1P_L\sin\mu\sin\theta \\ & \times (\cos\mu\cos\theta\cos\phi - \sin\mu\sin^2\theta\sin\phi) \\ & - f(r)r\gamma^0\gamma^2P_L\sin\mu\sin\theta \\ & \times (\cos\mu\cos\theta\sin\phi + \sin\mu\sin^2\theta\cos\phi) \\ & + k\eta h(r)\gamma^0 \left[e^{-i\mu} \left(\frac{\cos\mu}{h(r)} \right. \right. \\ & \left. \left. + i\sin\mu\cos\theta \right) P_L + e^{i\mu} \left(\frac{\cos\mu}{h(r)} - i\sin\mu\cos\theta \right) P_R \right], \end{aligned} \tag{A.2a}$$

$$\begin{aligned} \hat{\mathcal{H}}_{12} = & if(r)r\gamma^0\gamma^1P_L e^{i(\mu+\phi)}\sin\mu \\ & \times [\cos\theta\cos\phi(\cos\mu\cos\theta \\ & - i\sin\mu) - i\sin^2\theta\sin\phi(\cos\mu - i\sin\mu\cos\theta)] \\ & + if(r)r\gamma^0\gamma^2P_L e^{i(\mu+\phi)}\sin\mu \\ & \times [\cos\theta\sin\phi(\cos\mu\cos\theta - i\sin\mu) \\ & + i\sin^2\theta\cos\phi(\cos\mu - i\sin\mu\cos\theta)] \\ & - if(r)r\gamma^0\gamma^3P_L e^{i(\mu+\phi)}\sin\mu\sin\theta(\cos\mu\cos\theta - i\sin\mu) \\ & - k\eta h(r)\gamma^0\sin\mu\sin\theta e^{i\phi}(P_L - P_R), \end{aligned} \tag{A.2b}$$

$$\begin{aligned} \hat{\mathcal{H}}_{21} = & -if(r)r\gamma^0\gamma^1P_L e^{-i(\mu+\phi)}\sin\mu \\ & \times [\cos\theta\cos\phi(\cos\mu\cos\theta + i\sin\mu) \\ & + i\sin^2\theta\sin\phi(\cos\mu + i\sin\mu\cos\theta)] \\ & - if(r)r\gamma^0\gamma^2P_L e^{-i(\mu+\phi)}\sin\mu \\ & \times [\cos\theta\sin\phi(\cos\mu\cos\theta + i\sin\mu) \\ & - i\sin^2\theta\cos\phi(\cos\mu + i\sin\mu\cos\theta)] \\ & + if(r)r\gamma^0\gamma^3P_L e^{-i(\mu+\phi)}\sin\mu\sin\theta(\cos\mu\cos\theta + i\sin\mu) \\ & + k\eta h(r)\gamma^0\sin\mu\sin\theta e^{-i\phi}(P_L - P_R), \end{aligned} \tag{A.2c}$$

$$\begin{aligned} \hat{\mathcal{H}}_{22} = & -i\gamma^0\gamma^j\partial_j - f(r)r\gamma^0\gamma^3P_L\sin\mu\cos\mu\sin^2\theta \\ & + f(r)r\gamma^0\gamma^1P_L\sin\mu\sin\theta \\ & \times (\cos\mu\cos\theta\cos\phi - \sin\mu\sin^2\theta\sin\phi) \\ & + f(r)r\gamma^0\gamma^2P_L\sin\mu\sin\theta \end{aligned}$$

$$\begin{aligned} & \times (\cos \mu \cos \theta \sin \phi + \sin \mu \sin^2 \theta \cos \phi) \\ & + k\eta h(r) \gamma^0 \left[e^{+i\mu} \left(\frac{\cos \mu}{h(r)} - i \sin \mu \cos \theta \right) P_L \right. \\ & \left. + e^{-i\mu} \left(\frac{\cos \mu}{h(r)} + i \sin \mu \cos \theta \right) P_R \right]. \end{aligned} \quad (\text{A.2d})$$

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