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T-folds as Poisson–Lie plurals

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Abstract In previous papers we have presented many purely bosonic solutions of Generalized Supergravity Equations obtained by Poisson–Lie T-duality and plurality of flat and Bianchi cosmologies. In this paper we focus on their compactifications and identify solutions that can be interpreted as T-folds. To recognize T-folds we adopt the language of Double Field Theory and discuss how Poisson–Lie T-duality/plurality fits into this framework. As a special case we confirm that all non-Abelian T-duals can be compactified as T-folds.

Contents

1 Introduction

Dualities in string theory relate apparently different physical models and allow to address issues that are otherwise hard to tackle. T-duality [1] connects models in backgrounds with distinct curvature properties. Together with its non-Abelian generalization [2] it was extended to RR fields [3,4] and can be used as solution generating technique in supergravity [5–8] and generalized supergravity [9–11]. It also often contributes to the study of integrable models [12,13]. However, most of the papers deal with local aspects of non-Abelian duals, and global properties remain unclear, see e.g. [14]. The same holds for Poisson–Lie T-duality [15,16] or plurality [17] that introduce Drinfel'd double as the underlying algebraic structure of T-duality allowing us to treat both original and dual/plural models equally.

Recently several papers [18,19] appeared that describe compactifications of Yang–Baxter deformations of Minkowski and $AdS_5 \times S^5$ backgrounds in terms of T-folds. T-folds represent a special class of nongeometric backgrounds that appeared in string theory in an attempt to accommodate T-duality as symmetry of some models [20–25]. T-folds generalize the notion of manifold by allowing not only diffeomorphisms but also T-duality transformations as transition functions between local charts. Natural language for their description has become Double Field Theory [26–29]. It turns out that Double Field Theory can describe not only Abelian T-duality but also Poisson–Lie T-duality [30–33] and may help investigate quantum aspects of Poisson–Lie T-duality [34] or its extension to U-duality [35,36].

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In [37,38] we have presented many purely bosonic solutions of Generalized Supergravity Equations that were obtained by Poisson–Lie T-duality or plurality transformation of flat and Bianchi cosmologies [39,40]. The purpose of this paper is to present solutions that can be interpreted as T-folds. We follow the idea that T-folds can be identified using non-commutative structure Θ in the open string picture [41]. We give the argument both in terms of Poisson–Lie T-plurality and Double Field Theory to show the interplay between these two formalisms. From the structure of Drinfel'd double underlying non-Abelian T-duality one finds that all non-Abelian T-duals can be compactified as T-folds (as noticed e.g. in [42]). In the case of general plurality transformation additional conditions have to be satisfied.

The paper is organized as follows. In Sects. 2.1 and 2.2 we briefly recapitulate Poisson–Lie T-plurality and Generalized Supergravity Equations. Elements of Double Field Theory, T-folds, and the method that we use to identify T-folds are explained in Sect. 2.3. In Sect. 3 we present backgrounds obtained as Poisson–Lie plurals of flat background that can be interpreted as T-folds. Examples of T-folds obtained as Poisson–Lie plurals of curved Bianchi cosmologies are presented in Sect. 4.

2 Preliminaries

In this Section we will summarize main features of Poisson–Lie T-duality and plurality [15–17,43], Generalized Supergravity Equations [9–11] and T-folds [19,21,27–29,44]. For detailed information see the original papers.

2.1 Poisson–Lie T-duality/plurality

A convenient way to describe Poisson–Lie T-plurality is in terms of a Drinfel'd double. As this has been done in many preceeding papers, e.g. [38], we shall not go into details and restrict to a summary of necessary formulas.

Let $\mathscr{G}$ be a d-dimensional Lie group with free action on manifold $\mathscr{M}$ of dimension $M = n + d$. Since the action of $\mathscr{G}$ is transitive on its orbits, we may locally consider $\mathscr{M} \approx (\mathscr{M}/\mathscr{G}) \times \mathscr{G} = \mathscr{N} \times \mathscr{G}$. This allows us to introduce the so-called adapted coordinates[1]

$$\{s_\alpha, x^a\}, \quad \alpha = 1, \ldots, n = \dim \mathscr{N}, \quad a = 1, \ldots, d = \dim \mathscr{G}$$

where x^a denote group coordinates while s_α label the orbits of $\mathscr{G}$ and will be treated as spectators in the duality/plurality transformation.

We shall consider sigma models on $\mathscr{N} \times \mathscr{G}$ given by covariant tensor field $\mathcal{F}$ invariant with respect to the action of group $\mathscr{G}$. Such $\mathcal{F}$ is defined by spectator-dependent $(n + d) \times (n + d)$ matrix $E(s)$ and group dependent $\mathcal{E}(x)$ as

$$\mathcal{F}(s, x) = \mathcal{E}(x) \cdot E(s) \cdot \mathcal{E}^T(x), \qquad \mathcal{E}(x) = \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & e(x) \end{pmatrix}. \tag{1}$$

The $d \times d$ matrix $e(x)$ contains components of right-invariant Maurer–Cartan form $(dg)g^{-1}$ on $\mathscr{G}$. The dynamics of sigma model on $\mathscr{M}$ follows from Lagrangian

$$\mathcal{L} = \partial_- \phi^\mu \mathcal{F}_{\mu\nu}(\phi) \partial_+ \phi^\nu, \qquad \phi^\mu = \phi^\mu(\sigma_+, \sigma_-),$$
$$\mu = 1, \ldots, M = n + d \tag{2}$$

where tensor field $\mathcal{F} = \mathcal{G} + \mathcal{B}$ on $\mathscr{M}$ contains metric $\mathcal{G}$ and torsion potential (Kalb–Ramond field) $\mathcal{B}$.

To find Poisson–Lie dual model [15] on $\mathscr{N} \times \widetilde{\mathscr{G}}$ we embed group $\mathscr{G}$ into Drinfel'd double, i.e. $2d$-dimensional Lie group $\mathscr{D} = (\mathscr{G}|\widetilde{\mathscr{G}})$ formed by a pair of Lie subgroups $\mathscr{G}$ and $\widetilde{\mathscr{G}}$. The Lie algebra $\mathfrak{d}$ of the Drinfel'd double is endowed with an ad-invariant non-degenerate symmetric bilinear form $\langle .,. \rangle$. $\mathfrak{d}$ can be written as a double cross sum $\mathfrak{g} \bowtie \tilde{\mathfrak{g}}$ of subalgebras $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ [46] corresponding to $\mathscr{G}$ and $\widetilde{\mathscr{G}}$. $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ are maximally isotropic with respect to the form $\langle .,. \rangle$. The resulting algebraic structure $(\mathfrak{d}, \mathfrak{g}, \tilde{\mathfrak{g}})$ is called Manin triple. As noted already in [15], for a particular Drinfel'd double $\mathscr{D} = (\mathscr{G}|\widetilde{\mathscr{G}})$ there may exist various Manin triples. If there is another pair of subgroups $\widehat{\mathscr{G}}$ and $\bar{\mathscr{G}}$ (with corresponding Manin triple $(\mathfrak{d}, \hat{\mathfrak{g}}, \bar{\mathfrak{g}})$) that form the same Drinfel'd double $\mathscr{D}$, we can find Poisson–Lie plural models [17] on $\mathscr{N} \times \widehat{\mathscr{G}}$ or $\mathscr{N} \times \bar{\mathscr{G}}$. The Poisson–Lie T-plural sigma model on $\mathscr{N} \times \widehat{\mathscr{G}}$ is specified by tensor field

$$\widehat{\mathcal{F}}(s, \hat{x}) = \widehat{\mathcal{E}}(\hat{x}) \cdot \widehat{E}(s, \hat{x}) \cdot \widehat{\mathcal{E}}^T(\hat{x}), \qquad \widehat{\mathcal{E}}(\hat{x}) = \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & \widehat{e}(\hat{x}) \end{pmatrix} \tag{3}$$

where $\widehat{e}(\hat{x})$ contains components of $(d\hat{g})\hat{g}^{-1}$ on $\widehat{\mathscr{G}}$,

$$\widehat{E}(s, \hat{x}) = \left(\mathbf{1}_{n+d} + \widehat{E}(s) \cdot \widehat{\Pi}(\hat{x})\right)^{-1} \cdot \widehat{E}(s)$$
$$= \left(\widehat{E}^{-1}(s) + \widehat{\Pi}(\hat{x})\right)^{-1}, \tag{4}$$

$$\widehat{\Pi}(\hat{x}) = \begin{pmatrix} \mathbf{0}_n & 0 \\ 0 & \widehat{b}(\hat{x}) \cdot \widehat{a}^{-1}(\hat{x}) \end{pmatrix},$$

and[2] $\widehat{b}(\hat{x}), \widehat{a}(\hat{x})$ denote submatrices of the adjoint representation of $\widehat{\mathscr{G}}$ on algebra $\mathfrak{d}$ given by

$$ad_{\hat{g}^{-1}}(\bar{T}) = \widehat{b}(\hat{x}) \cdot \widehat{T} + \widehat{a}^{-1}(\hat{x}) \cdot \bar{T}.$$

The matrix $\widehat{E}(s)$ is obtained as follows. Let C be an invertible $2d \times 2d$ matrix relating bases of Manin triples $\mathfrak{g} \bowtie \tilde{\mathfrak{g}}$ and

[1] For a thorough description of the process of finding adapted coordinates see [45].

[2] The second identity in (4) holds for invertible $\widehat{E}(s)$.

$\widehat{\mathfrak{g}} \bowtie \bar{\mathfrak{g}}$ as

$$\begin{pmatrix}\widehat{T}\\ \bar{T}\end{pmatrix} = C \cdot \begin{pmatrix}T\\ \widetilde{T}\end{pmatrix}$$
$$T_a \in \mathfrak{g},\ \widetilde{T}^a \in \tilde{\mathfrak{g}},\ \widehat{T}_a \in \hat{\mathfrak{g}},\ \bar{T}^a \in \bar{\mathfrak{g}},\ a = 1,\ldots,d. \tag{5}$$

We denote the $d \times d$ blocks of C^{-1} as P, Q, R, S, i.e.

$$\begin{pmatrix}T\\ \widetilde{T}\end{pmatrix} = C^{-1} \cdot \begin{pmatrix}\widehat{T}\\ \bar{T}\end{pmatrix} = \begin{pmatrix}P & Q\\ R & S\end{pmatrix} \cdot \begin{pmatrix}\widehat{T}\\ \bar{T}\end{pmatrix}. \tag{6}$$

To account for the spectator fields we form $(n+d)\times(n+d)$ matrices

$$\mathcal{P} = \begin{pmatrix}\mathbf{1}_n & 0\\ 0 & P\end{pmatrix},\quad \mathcal{Q} = \begin{pmatrix}\mathbf{0}_n & 0\\ 0 & Q\end{pmatrix},\quad \mathcal{R} = \begin{pmatrix}\mathbf{0}_n & 0\\ 0 & R\end{pmatrix},$$
$$\mathcal{S} = \begin{pmatrix}\mathbf{1}_n & 0\\ 0 & S\end{pmatrix}. \tag{7}$$

$\widehat{E}(s)$ then reads

$$\widehat{E}(s) = (\mathcal{P} + E(s)\cdot\mathcal{R})^{-1}\cdot(\mathcal{Q} + E(s)\cdot\mathcal{S}). \tag{8}$$

This procedure was used in [37,38] to construct new solutions of Generalized Supergravity Equations from Bianchi cosmologies.

Matrices C relating Manin triples $\mathfrak{g} \bowtie \tilde{\mathfrak{g}}$ and $\widehat{\mathfrak{g}} \bowtie \bar{\mathfrak{g}}$ are not unique and different choices may lead to backgrounds with different curvature or torsion properties [47]. However, in [37,38] we have shown that many parameters appearing in general C are irrelevant as the resulting backgrounds differ only by a coordinate or gauge transformation. In such case we choose a representative of the class of C matrices leading to such "equivalent" backgrounds.

In this paper we deal with particular C's leading to backgrounds that can be interpreted as T-folds. Important special cases include $P = S = \mathbf{0}_d,\ Q = R = \mathbf{1}_d$ in which case plurality reduces to Poisson–Lie T-duality. For $P = S = \mathbf{1}_d,\ R = \mathbf{0}_d,\ Q = B$ where $B = -B^T$ we obtain the so-called $\mathcal{B}$-shifts, while the so-called β-shifts are given by $P = S = \mathbf{1}_d,\ Q = \mathbf{0}_d$ and $R = \beta,\ \beta = -\beta^T$.

Throughout this paper we deal with non-semisimple Bianchi groups $\mathscr{G}$, while $\widetilde{\mathscr{G}}$ is three-dimensional Abelian group $\mathscr{A}$. We parametrize group elements as $g = e^{x^1 T_1} e^{x^2 T_2} e^{x^3 T_3}$ where $e^{x^2 T_2} e^{x^3 T_3}$ and $e^{x^3 T_3}$ parametrize normal subgroups of $\mathscr{G}$. Poisson–Lie T-plurality transformation between models on groups $\mathscr{G}, \widetilde{\mathscr{G}}$ and $\widehat{\mathscr{G}}, \bar{\mathscr{G}}$ is specified by mapping (5), but the relation can be also formulated in terms of group elements as

$$l = g(y)\tilde{h}(\tilde{y}) = \widehat{g}(\hat{x})\bar{h}(\bar{x}),\ l \in \mathscr{D},$$
$$g \in \mathscr{G},\ \tilde{h} \in \widetilde{\mathscr{G}},\ \widehat{g} \in \widehat{\mathscr{G}},\ \bar{h} \in \bar{\mathscr{G}}. \tag{9}$$

In this paper we consider Bianchi cosmologies on four-dimensional manifolds, thus $\dim \mathscr{N} = 1$. For simplicity we denote the spectator coordinate s_1 by t.

2.2 Generalized supergravity equations

Adopting the convention used in [40] we write the Generalized Supergravity Equations of Motion [9,10] as[3]

$$0 = R_{\mu\nu} - \frac{1}{4}H_{\mu\rho\sigma}H_\nu{}^{\rho\sigma} + \nabla_\mu X_\nu + \nabla_\nu X_\mu, \tag{10}$$
$$0 = -\frac{1}{2}\nabla^\rho H_{\rho\mu\nu} + X^\rho H_{\rho\mu\nu} + \nabla_\mu X_\nu - \nabla_\nu X_\mu, \tag{11}$$
$$0 = R - \frac{1}{12}H_{\rho\sigma\tau}H^{\rho\sigma\tau} + 4\nabla_\mu X^\mu - 4X_\mu X^\mu. \tag{12}$$

$H_{\rho\mu\nu}$ are components of torsion

$$H_{\rho\mu\nu} = \partial_\rho \mathcal{B}_{\mu\nu} + \partial_\mu \mathcal{B}_{\nu\rho} + \partial_\nu \mathcal{B}_{\rho\mu},$$

∇_μ are covariant derivatives with respect to metric $\mathcal{G}$, and X is given by

$$X_\mu = \partial_\mu \Phi + \mathcal{J}^\nu \mathcal{F}_{\nu\mu}, \tag{13}$$

where Φ is the dilaton and $\mathcal{J}$ is Killing vector of the background $\mathcal{F}$ [10], i.e.

$$\mathcal{L}_{\mathcal{J}}\mathcal{F} = 0.$$

When vector field $\mathcal{J}$ vanishes the usual beta function equations are recovered.

Transformation formulas for Φ and $\mathcal{J}$ in the context of Poisson–Lie T-plurality were given in [17,31,33,38]. In this paper we shall use these formulas in the form that take into account possible non-locality of the resulting dilaton $\widehat{\Phi}$. We set

$$\Phi^0(y) = \Phi(y) + \frac{1}{2}\ln\left|\det\left[(\mathbf{1} + \Pi(y)E(s))\,a(y)\right]\right|, \tag{14}$$

and express y (coordinates of $\mathscr{G}$) from (9) in terms of $\hat{x}$ and $\bar{x}$. If the dependence is linear

$$y^k = \hat{d}^k_m \hat{x}^m + \bar{d}^{km}\bar{x}_m,$$

the transformed dilaton reads

$$\widehat{\Phi}(\hat{x}) = \Phi^0(\hat{d}^k_m \hat{x}^m) - \frac{1}{2}\ln\left|\det\left[\left(N + \widehat{\Pi}(\hat{x})M\right)\widehat{a}(\hat{x})\right]\right| \tag{15}$$

where

$$M = \mathcal{S}^T E(s) - \mathcal{Q}^T,\qquad N = \mathcal{P}^T - \mathcal{R}^T E(s).$$

Components of vectors $\widehat{\mathcal{J}}$ for backgrounds on $\mathscr{N} \times \widehat{\mathscr{G}}$ are

$$\widehat{\mathcal{J}}^\alpha = 0,\qquad \alpha = 1,\ldots,\dim\mathscr{N},$$
$$\widehat{\mathcal{J}}^{\dim\mathscr{N}+m} = \left(\frac{1}{2}\bar{f}^{ab}{}_b - \frac{\partial\Phi^0}{\partial y^k}\bar{d}^{ka}\right)\widehat{V}^m_a,$$
$$a,b,k,m = 1,\ldots,\dim\mathscr{G}, \tag{16}$$

[3] We restrict to bosonic fields in the NS–NS sector.

where $\widehat{V}_a$ are left-invariant fields on $\widehat{\mathscr{G}}$ and $\bar{f}^{ab}{}_c$ are structure constants of $\bar{\mathscr{G}}$.

2.3 Short review of double field theory and T-folds

The presence of dualities in string theory suggests that it might be convenient to generalize the standard concept of manifold. T-folds do so by allowing T-dualities beside diffeomorphisms as transition functions between charts. To describe T-folds, we shall use the framework of Double Field Theory (DFT).

While T-folds are examples of the so-called nongeometric backgrounds [44], in DFT T-duality transformations become diffeomorphisms of a manifold with doubled dimension. In this construction all the local patches are geometric.

The central concepts in DFT are the generalized metric $\mathcal{H} \in \mathcal{O}(M, M)$ and DFT dilaton $\mathcal{D}$ that are defined using the initial background fields $\mathcal{G}, \mathcal{B}, \Phi$ as

$$\mathcal{H} = \begin{pmatrix} \mathcal{G} - \mathcal{B}\cdot\mathcal{G}^{-1}\cdot\mathcal{B} & \mathcal{B}\cdot\mathcal{G}^{-1} \\ -\mathcal{G}^{-1}\cdot\mathcal{B} & \mathcal{G}^{-1} \end{pmatrix} \tag{17}$$

and

$$\mathcal{D} = \Phi - \frac{1}{4}\ln(\det\mathcal{G}). \tag{18}$$

The possibility to compactify both initial and extended $2M$-dimensional manifold in the direction of the vector field α is conditioned by existence of a monodromy matrix Ω whose action is equivalent to action of the vector field α

$$(e^{\mathcal{L}_\alpha} \triangleright \mathcal{H})(x^\mu) = \Omega \triangleright \mathcal{H}(x^\mu) := \Omega\cdot\mathcal{H}(x^\mu)\cdot\Omega^T, \tag{19}$$

and invariance of DFT dilaton

$$(e^{\mathcal{L}_\alpha} \triangleright \mathcal{D})(x^\mu) = \mathcal{D}(x^\mu), \tag{20}$$

where $\mathcal{L}_\alpha$ is the Lie derivative in the direction of α. In the adapted coordinates for α, which we assume here, this action is a shift of coordinates in direction α^μ, i.e.

$$\mathcal{H}(x^\mu + \alpha^\mu) = \Omega\cdot\mathcal{H}(x^\mu)\cdot\Omega^T, \tag{21}$$
$$\mathcal{D}(x^\mu + \alpha^\mu) = \mathcal{D}(x^\mu) \tag{22}$$

where Ω is a constant matrix.

If the transformation by Ω can be reinterpreted as transformation of the background $\mathcal{F} = \mathcal{G} + \mathcal{B}$ induced by coordinate or gauge transformation ($\mathcal{B}$-shift), then $\mathcal{F}$ can be considered as background on compactified manifold $\mathscr{M}$. Such cases will be considered trivial in the rest of the paper and we shall not discuss them further. However, monodromy matrix of the form

$$\Omega_\beta = \begin{pmatrix} \mathbf{1}_M & 0 \\ \beta & \mathbf{1}_M \end{pmatrix}, \qquad \beta^T = -\beta \tag{23}$$

where β is constant antisymmetric $M \times M$ matrix cannot be obtained neither by transformation of coordinates on $\mathscr{M}$ nor gauge transformation of $\mathcal{B}$ and tensor $\mathcal{F}$ must be interpreted as background on a T-fold.

Authors of paper [19] give simple procedure for T-fold identification via open[4] background [41]. To understand the action of Ω_β we rewrite the generalized metric (17) in terms of open fields G and Θ as

$$\mathcal{H} = \begin{pmatrix} G & -G\cdot\Theta \\ \Theta\cdot G & G^{-1} - \Theta\cdot G\cdot\Theta \end{pmatrix}. \tag{24}$$

where Θ is the antisymmetric part of $\mathcal{F}^{-1}(s, x)$, i.e.

$$\Theta^{\mu\nu} = -((\mathcal{G}+\mathcal{B})^{-1}\cdot\mathcal{B}\cdot(\mathcal{G}-\mathcal{B})^{-1})^{\mu\nu} = -\Theta^{\nu\mu},$$

and

$$G_{\mu\nu} = (\mathcal{G} - \mathcal{B}\cdot\mathcal{G}^{-1}\cdot\mathcal{B})_{\mu\nu} = G_{\nu\mu}.$$

These two tensors form the so-called open background [41] and the form of bivector Θ is important for identification of T-folds.

It is easy to verify that action of Ω_β given by (23) does not change G, but shifts Θ by β. Necessary condition (21) then reads

$$\Theta(x^\mu + \alpha^\mu) = \Theta(x^\mu) + \beta, \quad G(x^\mu + \alpha^\mu) = G(x^\mu). \tag{25}$$

It can be satisfied only for backgrounds $\mathcal{F}$ where Θ are linear in a coordinate x^μ and suitable matrices β can be obtained from Θ as linear combinations of the so-called Q-fluxes [32]

$$\beta^{\mu\nu} = \alpha^\lambda\,\partial_\lambda\Theta^{\mu\nu} = \alpha^\lambda\,Q_\lambda{}^{\mu\nu}. \tag{26}$$

Here, the sum (26) runs only over λ for which the open metric G is invariant with respect to $x^\lambda \to x^\lambda + \alpha^\lambda$. Conclusion then is that backgrounds with constant nonvanishing Q-fluxes can be globally defined as T-folds.

In terms of the background on $\mathscr{M}$ the condition (21) is then equivalent to

$$\mathcal{F}^{-1}(x^\mu + \alpha^\mu) = \mathcal{F}^{-1}(x^\mu) + \beta. \tag{27}$$

The right-hand side can be considered as Poisson–Lie transformation (8) by β-shift. For quantized strings one needs $\Omega_\beta \in \mathcal{O}(M, M, \mathbb{Z})$, which is satisfied if entries of the antisymmetric matrix β are integers.

In [37,38] we investigated Poisson–Lie duals and plurals of Bianchi cosmologies. To find backgrounds that can be interpreted as T-folds we will present in the rest of this paper dual or plural backgrounds whose bivector Θ is linear in coordinates x^μ. As the plural backgrounds are given by (3), (4), (8), bivector $\widehat{\Theta}$ can be expressed as

$$\widehat{\Theta}(\hat{x}) = \widehat{\mathcal{V}}(\hat{x})\cdot\left(\frac{1}{2}\left(\widehat{E}^{-1}(s) - \widehat{E}^{-T}(s)\right) + \widehat{\Pi}(\hat{x})\right)\cdot\widehat{\mathcal{V}}(\hat{x})^T,$$
$$\widehat{\mathcal{V}}(\hat{x}) = \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & \widehat{v}(\hat{x}) \end{pmatrix} \tag{28}$$

[4] Called dual in [19].

where $\widehat{v}(\hat{x})$ is $d \times d$ matrix of components of right-invariant vector fields of the group $\widehat{\mathscr{G}}$. From the formula (28) it is clear that bivectors $\widehat{\Theta}$ linear in $\hat{x}$ occur beside others for backgrounds on Abelian groups $\widehat{\mathscr{G}} = \mathscr{A}$. For such groups $\widehat{\mathcal{V}}(\hat{x}) = \mathbf{1}_M$ and $\widehat{\Pi}$ is linear in $\hat{x}$ since

$$\widehat{\Pi}(\widehat{x}) = \begin{pmatrix} \mathbf{0}_n & 0 \\ 0 & \widehat{b}(\widehat{x}) \end{pmatrix}, \quad \widehat{b}_{ab}(\widehat{x}) = \bar{f}_{ab}^{c}\, \widehat{x}_c. \tag{29}$$

Moreover, the open metric $\widehat{G}$ is completely independent of the group coordinates $\hat{x}$. It means that all backgrounds obtained by non-Abelian T-duality, i.e. on a semi-Abelian Drinfel'd double $\mathscr{D} = (\mathscr{G}|\mathscr{A})$, can be compactified as T-folds in coordinates $\hat{x}$. In a different way this has been shown also in [42].

3 Transformations of flat metric

In this section we investigate T-folds obtained from Poisson–Lie T-pluralities of the Minkowski metric following from its invariance with respect to Bianchi groups [48].

3.1 T-folds obtained by Poisson–Lie transformations given by Bianchi V isometry

The flat Minkowski metric is invariant with respect to the action of Bianchi V group. In adapted coordinates it has the form[5]

$$\mathcal{F}(t, y_1) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & t^2 & 0 & 0 \\ 0 & 0 & e^{2y_1} t^2 & 0 \\ 0 & 0 & 0 & e^{2y_1} t^2 \end{pmatrix}. \tag{30}$$

Together with vanishing dilaton $\Phi = 0$ the background satisfies beta function equations, i.e. Eqs. (10)–(12) with $\mathcal{J} = 0$. Its non-Abelian dual with respect to non-semisimple Bianchi V group appears repeatedly in the literature as it is not conformal [49,50] but satisfies Generalized Supergravity Equations [19]. The flat background in the form (30) can be obtained from (1) by virtue of six-dimensional semi-Abelian Drinfel'd double[6] $\mathscr{D} = (\mathscr{B}_V|\mathscr{A})$ with corresponding Manin triple $\mathfrak{d} = \mathfrak{b}_V \bowtie \mathfrak{a}$.

3.1.1 *Transformation of* $\mathfrak{b}_V \bowtie \mathfrak{a}$ *to* $\mathfrak{b}_{VI_{-1}} \bowtie \mathfrak{b}_{Vii}$

The algebra $\mathfrak{d} = \mathfrak{b}_V \bowtie \mathfrak{a}$ allows several other decompositions into Manin triples [51], such as $\mathfrak{d} = \mathfrak{b}_{VI_{-1}} \bowtie \mathfrak{b}_{Vii}$. Interested reader may find its commutation relations in [38]. Two matrices C transforming Manin triple $\mathfrak{b}_V \bowtie \mathfrak{a}$ to $\mathfrak{b}_{VI_{-1}} \bowtie \mathfrak{b}_{Vii}$ and producing geometrically different backgrounds are

$$C_1 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \qquad C_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Using C_1 we get background tensor

$$\widehat{\mathcal{F}}(t, \hat{x}_1, \hat{x}_3) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{e^{2\hat{x}_1}\left(\hat{x}_3^2+1\right)t^2}{t^4+e^{2\hat{x}_1}\left(\hat{x}_3^2+1\right)} & \frac{t^4}{t^4+e^{2\hat{x}_1}\left(\hat{x}_3^2+1\right)} & \frac{e^{2\hat{x}_1}\hat{x}_3 t^2}{t^4+e^{2\hat{x}_1}\left(\hat{x}_3^2+1\right)} \\ 0 & -\frac{t^4}{t^4+e^{2\hat{x}_1}\left(\hat{x}_3^2+1\right)} & \frac{t^2}{t^4+e^{2\hat{x}_1}\left(\hat{x}_3^2+1\right)} & \frac{e^{2\hat{x}_1}\hat{x}_3}{t^4+e^{2\hat{x}_1}\left(\hat{x}_3^2+1\right)} \\ 0 & \frac{e^{2\hat{x}_1}\hat{x}_3 t^2}{t^4+e^{2\hat{x}_1}\left(\hat{x}_3^2+1\right)} & -\frac{e^{2\hat{x}_1}\hat{x}_3}{t^4+e^{2\hat{x}_1}\left(\hat{x}_3^2+1\right)} & \frac{e^{2\hat{x}_1}\left(t^4+e^{2\hat{x}_1}\right)}{t^2\left(t^4+e^{2\hat{x}_1}\left(\hat{x}_3^2+1\right)\right)} \end{pmatrix} \tag{31}$$

whose metric is curved and torsion does not vanish. Supported by dilaton

$$\widehat{\Phi}(t, \hat{x}_1, \hat{x}_3) = -\frac{1}{2}\ln\left(t^6 e^{-4\hat{x}_1} + t^2 e^{-2\hat{x}_1}\left(\hat{x}_3^2+1\right)\right)$$

the background satisfies Generalized Supergravity Equations with constant vector $\widehat{\mathcal{J}} = (0, 0, 2, 0)$.

DFT metric then has the form

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & t^2 & 0 & 0 & 0 & 0 & t^2 & 0 \\ 0 & 0 & e^{-2\hat{x}_1}t^2 & 0 & 0 & -e^{-2\hat{x}_1}t^2 & 0 & e^{-2\hat{x}_1}t^2\hat{x}_3 \\ 0 & 0 & 0 & \frac{e^{2\hat{x}_1}}{t^2} & 0 & 0 & -\frac{e^{2\hat{x}_1}\hat{x}_3}{t^2} & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -e^{-2\hat{x}_1}t^2 & 0 & 0 & e^{-2\hat{x}_1}t^2+\frac{1}{t^2} & 0 & -e^{-2\hat{x}_1}t^2\hat{x}_3 \\ 0 & t^2 & 0 & -\frac{e^{2\hat{x}_1}\hat{x}_3}{t^2} & 0 & 0 & t^2+\frac{e^{2\hat{x}_1}\left(\hat{x}_3^2+1\right)}{t^2} & 0 \\ 0 & 0 & e^{-2\hat{x}_1}t^2\hat{x}_3 & 0 & 0 & -e^{-2\hat{x}_1}t^2\hat{x}_3 & 0 & e^{-2\hat{x}_1}t^2\left(\hat{x}_3^2+1\right) \end{pmatrix}$$

and DFT dilaton is

$$\mathcal{D}(t, \hat{x}_1) = -\frac{1}{4}\ln(t^6) + \hat{x}_1.$$

Bivector $\widehat{\Theta}$ for the background above is of the form

$$\widehat{\Theta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -\hat{x}_3 \\ 0 & 0 & \hat{x}_3 & 0 \end{pmatrix} \tag{32}$$

so that the Q-flux is constant. Its nonvanishing components are

$$Q_4{}^{34} = -Q_4{}^{43} = -1 \tag{33}$$

[5] To get matrix $E(s)$ one has to set $y_1 = 0$ in $\mathcal{F}$.

[6] We use $\mathscr{B}_V$ and $\mathfrak{b}_V$ to denote the group Bianchi V and its Lie algebra. Other Bianchi groups and algebras are denoted similarly. The three-dimensional Abelian group and its Lie algebra are written as $\mathscr{A}$ and $\mathfrak{a}$.

and β-shift matrix is

$$\beta = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\alpha \\ 0 & 0 & \alpha & 0 \end{pmatrix}. \tag{34}$$

β-shift (27) of $\widehat{\mathcal{F}}$ is then equivalent to the coordinate shift $\hat{x}_3 \mapsto \hat{x}_3 + \alpha$. DFT dilaton is independent of $\hat{x}_3$ so the condition (22) is satisfied. Therefore, the background (31) can be compactified as a T-fold as

$$(t, \hat{x}_1, \hat{x}_2, \hat{x}_3) \sim (t, \hat{x}_1, \hat{x}_2, \hat{x}_3 + \alpha).$$

Note that none of the relevant fields depend on $\hat{x}_2$ and it is possible to compactify also in $\hat{x}_2$ to obtain a T^2-fold. Such possibilities are disregarded in the rest of the paper to emphasize nontrivial compactifications.

Using C_2 we obtain plural background that reads

$$\widehat{\mathcal{F}}(t, \hat{x}_1, \hat{x}_3) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{e^{2\hat{x}_1}(\hat{x}_3^2+1)t^2}{e^{2\hat{x}_1}(\hat{x}_3^2+1)+1} & \frac{1}{e^{2\hat{x}_1}(\hat{x}_3^2+1)+1} & \frac{e^{2\hat{x}_1}\hat{x}_3 t^2}{e^{2\hat{x}_1}(\hat{x}_3^2+1)+1} \\ 0 & -\frac{1}{e^{2\hat{x}_1}(\hat{x}_3^2+1)+1} & \frac{1}{\left(e^{2\hat{x}_1}(\hat{x}_3^2+1)+1\right)t^2} & \frac{e^{2\hat{x}_1}\hat{x}_3}{e^{2\hat{x}_1}(\hat{x}_3^2+1)+1} \\ 0 & \frac{e^{2\hat{x}_1}\hat{x}_3 t^2}{e^{2\hat{x}_1}(\hat{x}_3^2+1)+1} & -\frac{e^{2\hat{x}_1}\hat{x}_3}{e^{2\hat{x}_1}(\hat{x}_3^2+1)+1} & \frac{e^{2\hat{x}_1}\left(1+e^{2\hat{x}_1}\right)t^2}{e^{2\hat{x}_1}(\hat{x}_3^2+1)+1} \end{pmatrix}. \tag{35}$$

Its metric is curved but torsion vanishes. The beta function equations are satisfied for dilaton

$$\widehat{\Phi}(t, \hat{x}_1, \hat{x}_3) = -\frac{1}{2} \ln \left(t^2 \left(1 + e^{2\hat{x}_1} \left(\hat{x}_3^2 + 1 \right) \right) \right).$$

Bivector $\widehat{\Theta}$ is the same as in the preceding case and DFT dilaton is $\mathcal{D}(t, \hat{x}_1) = -\frac{1}{4} \ln(t^6) - \hat{x}_1$. The background (35) can be again compactified as T-fold by

$$(t, \hat{x}_1, \hat{x}_2, \hat{x}_3) \sim (t, \hat{x}_1, \hat{x}_2, \hat{x}_3 + \alpha).$$

3.1.2 *Transformation of* $\mathfrak{b}_V \bowtie \mathfrak{a}$ *to* $\mathfrak{b}_{II} \bowtie \mathfrak{b}_V$

Examples of mappings producing geometrically different backgrounds that transform the algebraic structure of Manin triple $\mathfrak{b}_V \bowtie \mathfrak{a}$ to $\mathfrak{b}_{II} \bowtie \mathfrak{b}_V$ are

$$C_1 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & -\frac{1}{2} & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Background obtained by C_1 is given by tensor

$$\widehat{\mathcal{F}}(t, \hat{x}_2, \hat{x}_3) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{t^2}{t^4+\hat{x}_2^2+\hat{x}_3^2} & \frac{\hat{x}_3 t^2+2\hat{x}_2}{2\left(t^4+\hat{x}_2^2+\hat{x}_3^2\right)} & \frac{\hat{x}_2 t^2+2\hat{x}_3}{2\left(t^4+\hat{x}_2^2+\hat{x}_3^2\right)} \\ 0 & \frac{t^2\hat{x}_3-2\hat{x}_2}{2\left(t^4+\hat{x}_2^2+\hat{x}_3^2\right)} & \frac{\left(\hat{x}_3^2+4\right)t^4+4\hat{x}_3^2}{4t^2\left(t^4+\hat{x}_2^2+\hat{x}_3^2\right)} & \frac{-2t^6+\hat{x}_2\hat{x}_3 t^4-4\hat{x}_2^2 t^2-4\hat{x}_2\hat{x}_3}{4t^2\left(t^4+\hat{x}_2^2+\hat{x}_3^2\right)} \\ 0 & \frac{t^2\hat{x}_2-2\hat{x}_3}{2\left(t^4+\hat{x}_2^2+\hat{x}_3^2\right)} & \frac{2t^6+\hat{x}_2\hat{x}_3 t^4+4\hat{x}_2^2 t^2-4\hat{x}_2\hat{x}_3}{4t^2\left(t^4+\hat{x}_2^2+\hat{x}_3^2\right)} & \frac{\left(\hat{x}_2^2+4\right)t^4+4\hat{x}_2^2}{4t^2\left(t^4+\hat{x}_2^2+\hat{x}_3^2\right)} \end{pmatrix} \tag{36}$$

with curved metric and nonvanishing torsion. Together with

$$\widehat{\Phi}(t, \hat{x}_2, \hat{x}_3) = -\frac{1}{2} \ln \left(t^2 \left(t^4 + \hat{x}_2^2 + \hat{x}_3^2 \right) \right),$$
$$\widehat{\mathcal{J}} = (0, 2, 0, 0)$$

the background satisfies Generalized Supergravity Equations.

Bivector $\widehat{\Theta}$ has the form

$$\widehat{\Theta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{(t^4-4)\hat{x}_2}{t^4+4} & -\hat{x}_3 \\ 0 & -\frac{(t^4-4)\hat{x}_2}{t^4+4} & 0 & \frac{2t^4}{t^4+4} \\ 0 & \hat{x}_3 & -\frac{2t^4}{t^4+4} & 0 \end{pmatrix}$$

and is clearly linear in both $\hat{x}_2$ and $\hat{x}_3$. However, components of Q-flux corresponding to shift in $\hat{x}_2$ depend on t and the open metric $\widehat{G}$ depends on $\hat{x}_2$ so the above given procedure does not apply. Constant nonvanishing components of Q-flux are

$$Q_4{}^{24} = -Q_4{}^{42} = -1, \tag{37}$$

and β-shift (27) of $\widehat{\mathcal{F}}$ given by

$$\beta = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\alpha \\ 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \end{pmatrix} \tag{38}$$

is equivalent to the coordinate shift $\hat{x}_3 \mapsto \hat{x}_3 + \alpha$. DFT dilaton

$$\mathcal{D}(t) = -\frac{3}{4} \ln \left(t^2 \right) \tag{39}$$

is independent of $\hat{x}_3$. Therefore, the background (36) can be compactified as T-fold by

$$(t, \hat{x}_1, \hat{x}_2, \hat{x}_3) \sim (t, \hat{x}_1, \hat{x}_2, \hat{x}_3 + \alpha).$$

Background obtained by C_2 is given by tensor

$$\widehat{\mathcal{F}}(t,\hat{x}_2,\hat{x}_3)=$$
$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{1}{t^2\left(\hat{x}_2^2+\hat{x}_3^2+1\right)} & \frac{2\hat{x}_2t^2+\hat{x}_3}{2t^2\left(\hat{x}_2^2+\hat{x}_3^2+1\right)} & \frac{2\hat{x}_3t^2+\hat{x}_2}{2t^2\left(\hat{x}_2^2+\hat{x}_3^2+1\right)} \\ 0 & \frac{\hat{x}_3-2t^2\hat{x}_2}{2t^2\left(\hat{x}_2^2+\hat{x}_3^2+1\right)} & \frac{4\left(\hat{x}_3^2+1\right)t^4+\hat{x}_3^2}{4t^2\left(\hat{x}_2^2+\hat{x}_3^2+1\right)} & \frac{-4\hat{x}_2x_3t^4-2\left(2\hat{x}_2^2+1\right)t^2+\hat{x}_2\hat{x}_3}{4t^2\left(\hat{x}_2^2+\hat{x}_3^2+1\right)} \\ 0 & \frac{\hat{x}_2-2t^2\hat{x}_3}{2t^2\left(\hat{x}_2^2+\hat{x}_3^2+1\right)} & \frac{-4\hat{x}_2\hat{x}_3t^4+\left(4\hat{x}_2^2+2\right)t^2+\hat{x}_2\hat{x}_3}{4t^2\left(\hat{x}_2^2+\hat{x}_3^2+1\right)} & \frac{4\left(\hat{x}_2^2+1\right)t^4+\hat{x}_2^2}{4t^2\left(\hat{x}_2^2+\hat{x}_3^2+1\right)} \end{pmatrix} \tag{40}$$

with curved metric and vanishing torsion. Together with dilaton

$$\widehat{\Phi}(t,\hat{x}_2,\hat{x}_3)=-\frac{1}{2}\ln\left(t^2\left(\hat{x}_2^2+\hat{x}_3^2+1\right)\right)$$

they satisfy beta function equations.

Bivector $\widehat{\Theta}$ for this background is of the form

$$\widehat{\Theta}=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{(1-4t^4)\hat{x}_2}{4t^4+1} & -\hat{x}_3 \\ 0 & \frac{(4t^4-1)\hat{x}_2}{4t^4+1} & 0 & \frac{2}{4t^4+1} \\ 0 & \hat{x}_3 & -\frac{2}{4t^4+1} & 0 \end{pmatrix}.$$

It is slightly different from the previous case, however, the constant nonvanishing components of Q-flux, β-shift matrix and DFT dilaton are the same as in (37), (38) and (39), and the background (40) can be compactified as T-fold by

$$(t,\hat{x}_1,\hat{x}_2,\hat{x}_3)\sim(t,\hat{x}_1,\hat{x}_2,\hat{x}_3+\alpha).$$

3.2 T-folds obtained by Poisson–Lie transformations given by Bianchi IV isometry

Next we investigate backgrounds that follow from Poisson–Lie T-pluralities obtained from the invariance of Minkowski metric with respect to the action of Bianchi IV group. The group $\mathscr{B}_{IV}$ is not semisimple and trace of its structure constants does not vanish. The metric in adapted coordinates reads

$$\mathcal{F}(t,y_1)=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & e^{-y_1}y_1 & e^{-y_1} \\ 0 & e^{-y_1}y_1 & e^{-2y_1} & 0 \\ 0 & e^{-y_1} & 0 & 0 \end{pmatrix}. \tag{41}$$

Since the background is flat and torsionless, the beta function equations are satisfied if we choose vanishing dilaton $\Phi=0$. In this form the background can be obtained by formula (1) if we consider six-dimensional semi-Abelian Drinfel'd double $\mathscr{D}=(\mathscr{B}_{IV}|\mathscr{A})$ and Manin triple $\mathfrak{d}=\mathfrak{b}_{IV}\bowtie\mathfrak{a}$ spanned by basis $(T_1,T_2,T_3,\widetilde{T}^1,\widetilde{T}^2,\widetilde{T}^3)$ with non-trivial comutation relations

$$[T_1,T_2]=-T_2+T_3,\quad [T_1,T_3]=-T_3,\quad [T_1,\widetilde{T}^2]=\widetilde{T}^2,$$
$$[T_1,\widetilde{T}^3]=-\widetilde{T}^2+\widetilde{T}^3,\quad [T_2,\widetilde{T}^2]=-\widetilde{T}^1,\quad [T_2,\widetilde{T}^3]=\widetilde{T}^1,$$
$$[T_3,\widetilde{T}^3]=-\widetilde{T}^1. \tag{42}$$

The algebra $\mathfrak{d}$ allows several other decompositions into Manin triples.

3.2.1 *Transformation of $\mathfrak{b}_{IV}\bowtie\mathfrak{a}$ to $\mathfrak{b}_{VI_{-1}}\bowtie\mathfrak{b}_{II}$*

Lie algebra $\mathfrak{d}=\mathfrak{b}_{VI_{-1}}\bowtie\mathfrak{b}_{II}$ is spanned by basis $(\widehat{T}_1,\widehat{T}_2,\widehat{T}_3,\bar{T}^1,\bar{T}^2,\bar{T}^3)$ with algebraic relations

$$[\widehat{T}_1,\widehat{T}_2]=-\widehat{T}_2,\qquad [\widehat{T}_1,\widehat{T}_3]=\widehat{T}_3,$$
$$[\widehat{T}_1,\bar{T}^2]=\widehat{T}_3+\bar{T}^2,\quad [\widehat{T}_1,\bar{T}^3]=-\widehat{T}_2-\bar{T}^3,\quad [\widehat{T}_2,\bar{T}^2]=-\bar{T}^1,$$
$$[\widehat{T}_3,\bar{T}^3]=\bar{T}_1,\qquad [\bar{T}^2,\bar{T}^3]=\bar{T}^1. \tag{43}$$

Mapping that transforms the algebraic structure of Manin triple $\mathfrak{b}_{IV}\bowtie\mathfrak{a}$ to $\mathfrak{b}_{VI_{-1}}\bowtie\mathfrak{b}_{II}$ is given by matrix

$$C_1=\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}. \tag{44}$$

Resulting background tensor reads

$$\widehat{\mathcal{F}}(t,\hat{x}_1)=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\hat{x}_1^2 & -e^{-\hat{x}_1}\hat{x}_1 & e^{\hat{x}_1} \\ 0 & e^{-\hat{x}_1}\hat{x}_1 & e^{-2\hat{x}_1} & 0 \\ 0 & e^{\hat{x}_1} & 0 & 0 \end{pmatrix}. \tag{45}$$

The background has vanishing torsion. Its metric is curved with vanishing scalar curvature. Simple coordinate transformation brings the metric to the Brinkmann form of a plane-parallel wave

$$ds^2=\frac{2z_3^2}{u^2}du^2+2du\,dv+dz_3^2+dz_4^2. \tag{46}$$

Together with the dilaton

$$\widehat{\Phi}(\hat{x}_1)=-\hat{x}_1,$$

the background satisfies beta function equations.

Bivector $\widehat{\Theta}$ for this background is of the form

$$\widehat{\Theta}=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\hat{x}_1 \\ 0 & 0 & \hat{x}_1 & 0 \end{pmatrix}.$$

Nonvanishing components of constant Q-flux are

$$Q_2{}^{34}=-Q_2{}^{43}=-1, \tag{47}$$

so the β-shift matrix is

$$\beta = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\alpha \\ 0 & 0 & \alpha & 0 \end{pmatrix}. \tag{48}$$

Unfortunately, DFT dilaton $\mathcal{D}$ and the open metric depend on $\hat{x}_1$, i.e. the condition (22) and second part of (25) are not satisfied. Therefore, the background (45) cannot be compactified as T-fold. This is also true for background obtained by the other matrix

$$C_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

that transforms $\mathfrak{b}_{IV} \bowtie \mathfrak{a}$ to $\mathfrak{b}_{VI_{-1}} \bowtie \mathfrak{b}_{II}$ since it only gives the background (45) in different coordinates. A bit different situation holds in the following case.

3.2.2 *Transformation of* $\mathfrak{b}_{IV} \bowtie \mathfrak{a}$ *to* $\mathfrak{b}_{II} \bowtie \mathfrak{b}_{VI_{-1}}$

Manin triple $\mathfrak{b}_{II} \bowtie \mathfrak{b}_{VI_{-1}}$ is dual to that of preceding section. Background given by matrix

$$C_1 = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}$$

that is dual to (44) has the form

$$\bar{\mathcal{F}}(t, \bar{x}_2, \bar{x}_3) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\bar{x}_3+1} \\ 0 & 0 & 1 & \frac{\bar{x}_2}{\bar{x}_3+1} \\ 0 & \frac{1}{1-\bar{x}_3} & \frac{\bar{x}_2}{\bar{x}_3-1} & \frac{(\bar{x}_2-2)\bar{x}_2}{\bar{x}_3^2-1} \end{pmatrix}. \tag{49}$$

Formulas (15), (16) for transformation of dilaton and supplementary Killing vector $\bar{\mathcal{J}}$ give

$$\bar{\Phi}(t, \bar{x}_3) = -\frac{1}{2} \ln\left(-1 + \bar{x}_3^2\right) \qquad \bar{\mathcal{J}} = (0, 1, 0, 0). \tag{50}$$

Together with this nontrivial dilaton and $\bar{\mathcal{J}}$ the background satisfies Generalized Supergravity Equations. The background has flat metric and vanishing torsion so it also satisfies beta function equations with vanishing dilaton. This is rather interesting situation. Dilaton $\bar{\Phi}$ and $\bar{\mathcal{J}}$ enter Generalized Supergravity Equations combined into one-form X and there might be more suitable combinations.

Bivector $\bar{\Theta}$ is of the form

$$\bar{\Theta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \bar{x}_2 & -\bar{x}_3 \\ 0 & -\bar{x}_2 & 0 & 0 \\ 0 & \bar{x}_3 & 0 & 0 \end{pmatrix}$$

and DFT dilaton $\mathcal{D}$ vanishes. Nonvanishing components of Q-flux are

$$Q_3{}^{23} = -Q_3{}^{32} = 1, \quad Q_4{}^{24} = -Q_4{}^{42} = -1 \tag{51}$$

and it may seem that β-shift matrices can be arbitrary linear combinations

$$\beta = \alpha_2 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{52}$$

However, the open metric $\bar{G}$ depends linearly on $\bar{x}_2$. Therefore, background (49) can be compactified as T-fold only by

$$(t, \bar{x}_1, \bar{x}_2, \bar{x}_3) \sim (t, \bar{x}_1, \bar{x}_2, \bar{x}_3 + \alpha_3).$$

This also happens for the other matrix

$$C_2 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

that transforms $\mathfrak{b}_{IV} \bowtie \mathfrak{a}$ to $\mathfrak{b}_{II} \bowtie \mathfrak{b}_{VI_{-1}}$ giving background (49) in different coordinates.

4 Transformations of curved cosmologies

4.1 T-folds obtained by transformation of Bianchi VI_{-1} cosmology

Let us now focus on the curved background with metric [40]

$$\mathcal{F}(t, y_1) = \begin{pmatrix} -e^{-4\Phi(t)} a_1(t)^2 a_2(t)^4 & 0 & 0 & 0 \\ 0 & a_1(t)^2 & 0 & 0 \\ 0 & 0 & e^{-2y_1} a_2(t)^2 & 0 \\ 0 & 0 & 0 & e^{2y_1} a_2(t)^2 \end{pmatrix} \tag{53}$$

that is invariant with respect to the action of Bianchi VI_{-1} group. Dilaton

$$\Phi(t) = \beta t$$

and functions $a_i(t)$ are given as in [39] by

$$a_1(t) = \sqrt{p_1}\exp\left(\frac{1}{2}e^{2p_2 t} + \frac{p_1 t}{2} + \Phi(t)\right),$$
$$a_2(t) = \sqrt{p_2}e^{\frac{p_2 t}{2}+\Phi(t)}.$$

The beta function equations are satisfied if parameters p_1, p_2 and β fulfill condition

$$\beta^2 = \frac{1}{4}(2p_1p_2 + p_2^2).$$

To obtain metric (53) via (1) we shall consider Manin triple $\mathfrak{d} = \mathfrak{b}_{VI_{-1}} \bowtie \mathfrak{a}$ that corresponds to the same Drinfel'd double that we discussed in Sect. 3.1. Although the Drinfel'd double is the same, the background in this Section is different as we use different matrix $E(t) = \mathcal{F}(t, y_1 = 0)$.

4.1.1 Transformation of $\mathfrak{b}_{VI_{-1}} \bowtie \mathfrak{a}$ to $\mathfrak{a} \bowtie \mathfrak{b}_V$

Example of mappings that transform the algebraic structure of Manin triple $\mathfrak{b}_{VI_{-1}} \bowtie \mathfrak{a}$ to $\mathfrak{a} \bowtie \mathfrak{b}_V$ reads

$$C = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{54}$$

and background obtained by Poisson–Lie T-plurality with this matrix is given by tensor

$$\widehat{\mathcal{F}}(t, \hat{x}_2, \hat{x}_3) = \begin{pmatrix} -e^{-4t\beta}a_1^2a_2^4 & 0 & 0 & 0 \\ 0 & \frac{a_2^2}{\Delta} & \frac{\hat{x}_2 a_2^4}{\Delta} & \frac{\hat{x}_3}{\Delta} \\ 0 & -\frac{\hat{x}_2 a_2^4}{\Delta} & \frac{a_2^2(a_1^2a_2^2+\hat{x}_3^2)}{\Delta} & -\frac{\hat{x}_2\hat{x}_3a_2^2}{\Delta} \\ 0 & -\frac{\hat{x}_3}{\Delta} & -\frac{\hat{x}_2\hat{x}_3a_2^2}{\Delta} & \frac{a_1^2+\hat{x}_2^2a_2^2}{\Delta} \end{pmatrix} \tag{55}$$

where

$$\Delta = a_1(t)^2a_2(t)^2 + \hat{x}_2^2a_2(t)^4 + \hat{x}_3^2.$$

The background is curved and has nontrivial torsion. Together with dilaton

$$\widehat{\Phi}(t, \hat{x}_2, \hat{x}_3) = \beta t - \frac{1}{2}\ln\Delta$$

they satisfy Generalized Supergravity Equations with $\widehat{\mathcal{J}} = (0, 1, 0, 0)$.

Bivector $\widehat{\Theta}$ for the background (55) is of the form

$$\widehat{\Theta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\hat{x}_2 & -\hat{x}_3 \\ 0 & \hat{x}_2 & 0 & 0 \\ 0 & \hat{x}_3 & 0 & 0 \end{pmatrix}$$

and the Q-flux is constant. Its nonvanishing components are

$$Q_3{}^{23} = -Q_3{}^{32} = -1, \quad Q_4{}^{24} = -Q_4{}^{42} = -1 \tag{56}$$

and β-shift matrices are arbitrary linear combinations

$$\beta = \alpha_2\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \alpha_3\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{57}$$

DFT dilaton is

$$\mathcal{D}(t) = -\ln(a_1(t)a_2(t)^2) + 2\beta t.$$

β-shift (27) of $\widehat{\mathcal{F}}$ is then equivalent to the coordinate shift $(\hat{x}_2, \hat{x}_3) \mapsto (\hat{x}_2 + \alpha_2, \hat{x}_3 + \alpha_3)$. Therefore, the background (55) can be compactified as T^2-fold by

$$(t, \hat{x}_1, \hat{x}_2, \hat{x}_3) \sim (t, \hat{x}_1, \hat{x}_2 + \alpha_2, \hat{x}_3 + \alpha_3).$$

For $\alpha_2, \alpha_3 \in \mathbb{Z}$ we have $\Omega_\beta \in \mathcal{O}(M, M, \mathbb{Z})$.

4.2 T-folds obtained by transformation of Bianchi VI_κ cosmology

Finally we are going discuss Poisson–Lie T-pluralities of curved cosmology invariant with respect to Bianchi VI_κ group. Its metric is [40]

$$\mathcal{F}(t, y_1) = \begin{pmatrix} -e^{-4\Phi(t)}a_1(t)^2a_2(t)^2a_3(t)^2 & 0 & 0 & 0 \\ 0 & a_1(t)^2 & 0 & 0 \\ 0 & 0 & e^{2\kappa y_1}a_2(t)^2 & 0 \\ 0 & 0 & 0 & e^{2y_1}a_3(t)^2 \end{pmatrix} \tag{58}$$

and dilaton reads

$$\Phi(t) = \beta t.$$

The functions $a_i(t)$ are given as in [39] by

$$a_1(t) = e^{\Phi(t)}\left(\frac{p_1}{\kappa+1}\right)^{\frac{\kappa^2+1}{(\kappa+1)^2}} e^{\frac{(\kappa-1)p_2 t}{2(\kappa+1)}} \sinh^{-\frac{\kappa^2+1}{(\kappa+1)^2}}(p_1 t),$$
$$a_2(t) = e^{\Phi(t)}\left(\frac{p_1}{\kappa+1}\right)^{\frac{\kappa}{\kappa+1}} e^{\frac{p_2 t}{2}} \sinh^{-\frac{\kappa}{\kappa+1}}(p_1 t),$$
$$a_3(t) = e^{\Phi(t)}\left(\frac{p_1}{\kappa+1}\right)^{\frac{1}{\kappa+1}} e^{-\frac{p_2 t}{2}} \sinh^{-\frac{\kappa}{\kappa+1}}(p_1 t). \tag{59}$$

The beta function equations are satisfied provided constants p_1, p_2, β and κ fulfill condition

$$\beta^2 = \frac{(\kappa^2+\kappa+1)\,p_1^2}{(\kappa+1)^2} - \frac{p_2^2}{4}. \tag{60}$$

4.2.1 Transformation of $\mathfrak{b}_{VI_\kappa} \bowtie \mathfrak{a}$ to $\mathfrak{b}_{VI_\kappa} \bowtie \mathfrak{b}_{VI_{-\kappa}.iii}$

The background (58) can be obtained from (1) if we consider six-dimensional semi-Abelian Drinfel'd double $\mathscr{D} = (\mathscr{B}_{VI_\kappa}|\mathscr{A})$. Among the decompositions of $\mathfrak{d} = \mathfrak{b}_{VI_\kappa} \bowtie \mathfrak{a}$ given in [51] is also Manin triple $\mathfrak{d} = \mathfrak{b}_{VI_\kappa} \bowtie \mathfrak{b}_{VI_{-\kappa}.iii}$. For its commutation relations see [38] where general forms of mappings C relating $\mathfrak{b}_{VI_\kappa} \bowtie \mathfrak{a}$ and $\mathfrak{b}_{VI_\kappa} \bowtie \mathfrak{b}_{VI_{-\kappa}.iii}$ were found as well. In particular we shall consider matrices

$$C_1 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \qquad C_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Using matrix C_1 we get background

$$\widehat{\mathcal{F}}(t,\hat{x}_1,\hat{x}_2) = \begin{pmatrix} -e^{-4\beta t}a_1^2a_2^2a_3^2 & 0 & 0 & 0 \\ 0 & \frac{a_1^2\left(a_2^2a_3^2+e^{2(\kappa+1)\hat{x}_1}\kappa^2\hat{x}_2^2\right)}{\Delta} & \frac{e^{2(\kappa+1)\hat{x}_1}\kappa a_1^2\hat{x}_2}{\Delta} & \frac{e^{2\hat{x}_1}a_1^2a_2^2}{\Delta} \\ 0 & \frac{e^{2(\kappa+1)\hat{x}_1}\kappa a_1^2\hat{x}_2}{\Delta} & \frac{e^{2\kappa\hat{x}_1}\left(e^{2\hat{x}_1}a_1^2+a_3^2\right)}{\Delta} & -\frac{e^{2(\kappa+1)\hat{x}_1}\kappa\hat{x}_2}{\Delta} \\ 0 & -\frac{e^{2\hat{x}_1}a_1^2a_2^2}{\Delta} & \frac{e^{2(\kappa+1)\hat{x}_1}\kappa\hat{x}_2}{\Delta} & \frac{e^{2\hat{x}_1}a_2^2}{\Delta} \end{pmatrix} \tag{61}$$

where

$$\Delta = e^{2\hat{x}_1}a_1^2a_2^2 + a_3^2a_2^2 + e^{2(\kappa+1)\hat{x}_1}\kappa^2\hat{x}_2^2.$$

For constant vector $\widehat{\mathcal{J}} = (0,0,0,\kappa)$ and dilaton

$$\widehat{\Phi}(t,\hat{x}_1,\hat{x}_2) = \beta t - \frac{1}{2}\ln(\Delta) + (\kappa+1)\hat{x}_1$$

the Generalized Supergravity Equations are satisfied provided condition (60) holds.

From the matrix C_2 we get

$$\widehat{\mathcal{F}}(t,\hat{x}_1,\hat{x}_2) = \begin{pmatrix} -e^{-4t\beta}a_1^2a_2^2a_3^2 & 0 & 0 & 0 \\ 0 & \frac{a_1^2\left(\hat{x}_2^2\kappa^2a_2^2a_3^2e^{2\hat{x}_1(\kappa+1)}+1\right)}{\Delta} & \frac{\hat{x}_2\kappa a_1^2a_2^2a_3^2e^{2\hat{x}_1(\kappa+1)}}{\Delta} & \frac{e^{2\hat{x}_1}a_1^2a_3^2}{\Delta} \\ 0 & \frac{\hat{x}_2\kappa a_1^2a_2^2a_3^2e^{2\hat{x}_1(\kappa+1)}}{\Delta} & \frac{a_2^2e^{2\hat{x}_1\kappa}\left(e^{2\hat{x}_1}a_1^2a_3^2+1\right)}{\Delta} & -\frac{\hat{x}_2\kappa a_2^2a_3^2e^{2\hat{x}_1(\kappa+1)}}{\Delta} \\ 0 & -\frac{e^{2\hat{x}_1}a_1^2a_3^2}{\Delta} & \frac{\hat{x}_2\kappa a_2^2a_3^2e^{2\hat{x}_1(\kappa+1)}}{\Delta} & \frac{e^{2\hat{x}_1}a_3^2}{\Delta} \end{pmatrix} \tag{62}$$

where

$$\Delta = e^{2\hat{x}_1}a_1^2a_3^2 + \hat{x}_2^2\kappa^2a_2^2a_3^2e^{2\hat{x}_1(\kappa+1)} + 1.$$

Generalized Supergravity Equations with $\widehat{\mathcal{J}} = (0,0,0,-1)$ are satisfied by dilaton

$$\widehat{\Phi}(t,\hat{x}_1,\hat{x}_2) = \beta t - \frac{1}{2}\ln(\Delta)$$

under the condition (60).

Bivector $\widehat{\Theta}$ and matrix β for both backgrounds read

$$\widehat{\Theta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & \kappa\hat{x}_2 \\ 0 & 1 & -\kappa\hat{x}_2 & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha\kappa \\ 0 & 0 & -\alpha\kappa & 0 \end{pmatrix},$$

and nonvanishing components of Q-flux are

$$Q_3{}^{34} = -Q_3{}^{43} = \kappa. \tag{63}$$

Since the corresponding DFT dilatons

$$\mathcal{D}(t,\hat{x}_1) = -\ln(a_1(t)a_2(t)a_3(t)) + 2\beta t \pm \frac{(\kappa+1)\hat{x}_1}{2}$$

do not depend on $\hat{x}_2$, β-shifts (27) of $\widehat{\mathcal{F}}$ are equivalent to coordinate shift $\hat{x}_2 \mapsto \hat{x}_2 + \alpha$ and backgrounds (61), (62) can be compactified as T-folds by

$$(t,\hat{x}_1,\hat{x}_2,\hat{x}_3) \sim (t,\hat{x}_1,\hat{x}_2+\alpha,\hat{x}_3).$$

5 Conclusions

In this paper we discussed the connection between Poisson–Lie T-plurality and Double Field Theory. Using tensors Θ and G constituting the open background one obtains conditions (25) and (27) that need to be satisfied in order to identify background as T-fold, namely that Θ given by the formula (28) must be linear in a coordinate x^μ. The shift $x^\mu \mapsto x^\mu + \alpha^\mu$ then is equivalent to a β-shift (27) provided G and DFT dilaton $\mathcal{D}$ are independent of x^μ. The formula (28) also implies that backgrounds obtained by non-Abelian T-duality can be compactified as T-folds. In Sects. 3 and 4 we have tested conditions (25) and (27) for Poisson–Lie plurals of flat and Bianchi cosmologies obtained in [38]. We have shown that in spite of their rather complicated forms, many Poisson–Lie plurals can be considered as T-folds.

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