



Maximal efficiency of the collisional Penrose process with spinning particles in Kerr-Sen black hole

Yunlong Liu, Xiangdong Zhang^a

Department of Physics, South China University of Technology, Guangzhou 510641, China

Received: 24 November 2019 / Accepted: 31 December 2019 / Published online: 13 January 2020
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Abstract We study the collision of two uncharged spinning particles around an extreme Kerr-Sen black hole and calculate the maximal efficiency of the energy extraction from the Kerr-Sen black hole via super Penrose process. We consider the collision of two massive particles as well as collision of a massless particle with a massive particle. We calculate the maximum efficiency for all the cases, and found that the efficiency increases as the Kerr-Sen black hole's parameter ($b = 1 - a$) decreases.

1 Introduction

Penrose process, a mechanism to extract rotational energy from black hole, was first discovered by Penrose in 1969 with Kerr black hole [1]. The original version of Penrose process happened in the ergosphere, an object splits into two parts while it falls toward Kerr black hole. The one falls into the black hole with negative energy, while the other escapes to infinity. The energy of escaped part is larger than the original one. Therefore, rotational energy can be extracted from Kerr black hole. For such a process, Wald obtain the maximum efficiency $\eta_{max} = (\text{output energy})/(\text{input energy}) \approx 1.21$ [2]. After that, Piran et al. consider a different type of collision Penrose process which two particles collide inside the ergosphere, and found to be have similar energy contracting efficiency with the original Penrose process [3].

In 2009, Bañados, Silk and West(BSW) proposed that a rotating black hole can act as accelerators for non-spin particles [4]. They show that the collision center-of-mass energy can be arbitrarily high for extremal Kerr black hole [4]. Inspired by this work, some authors suggest to construct Penrose process based on the BSW mechanism [5–7]. These collision process are called as super Penrose process since it usually has far more higher energy contraction efficiency. For example, Schnittman obtain the maximal efficiency is about

13.92 when a massless and a massive particle collide near the horizon [7]. Along this line, the super Penrose process have been extended to various black holes [8–11].

Recently, the BSW mechanism has been generalized to include the spinning particles [12–22]. It has been shown [23–27] that the trajectory of a spinning test particle is no longer a geodesic and therefore is more close to the real particle. The corresponding super Penrose process also have been investigated in many cases [8–11]. It worth to note that in [28], the authors obtain some general result on the energy in the center of mass frame for BSW mechanism. However, this paper is devoted to the efficiency of Penrose process that was not studied in [28].

On the other hand, the Kerr-Sen black hole is a rotating and charged solution of the low-energy effective field theory for heterotic string theory [29]. After it proposed, many aspects of Kerr-Sen solution has been investigated [30]. This black hole solution characterized by three parameters, which are mass M , angular momentum a , and charge $Q(b = Q^2/2M)$. It reduces to the Kerr black hole when the parameter $b = 0$. As a grand unified theory, string theory is the most promising candidate of unified all the interactions, to this sense, the expected rotating and charged black hole solution would be the Kerr-Sen black hole rather than the Kerr-Newman one. Therefore in this paper we investigate the issue of the maximal energy contraction efficiency of the super Penrose process for spin particles in Kerr-Sen background. We provide a deep analysis of the super Penrose process for spinning particles and investigate the dependence of the maximal energy contraction efficiency with the Kerr-Sen black hole's parameter.

This paper is organized as follows: after an introduction, we discuss the equations of motion for spinning particles in Kerr-Sen black hole in Sect. 2. While in Sect. 3, we study the super Penrose collision of spinning particles in extreme Kerr-Sen background. This section is divided into three cases and calculate the the maximal efficiency with different param-

^a e-mail: scxdzhang@scut.edu.cn

ters of extreme Kerr-Sen black hole. The summary and conclusion was given in Sect. 4. Through out the paper, we adopt the geometrical unit ($c = G = 1$).

2 Basic equation

2.1 Equations of motion of a spinning particle

The equations of motion for spin particle in the curved space-time can be described by Mathisson–Papapetrou–Dixon (MPD) equations [23–25]

$$\frac{Dp^a}{D\tau} = -\frac{1}{2}R^a{}_{bcd}v^b S^{cd} \tag{2.1}$$

$$\frac{DS^{ab}}{D\tau} = p^a v^b - p^b v^a \tag{2.2}$$

where

$$v^a = \left(\frac{\partial}{\partial\tau}\right)^a \tag{2.3}$$

is the tangent vector of the center-of-mass world line, $\frac{D}{D\tau}$ is the covariant derivative along worldline, and $p^a = mu^a$ is the canonical 4-momentum of the spinning particles which satisfy

$$p^a p_a = -m^2. \tag{2.4}$$

Moreover S^{ab} is the particle’s antisymmetric spin tensor, and its square turns out to be the spin of the particle as follows,

$$S^{ab} S_{ab} = 2S^2 = 2m^2 s^2 \tag{2.5}$$

where s and m are the spin and mass of the given particle respectively. In the following, for the convenient of the calculation, we add a supplementary conditions between S^{ab} and P^a as follows

$$S^{ab} p_a = 0 \tag{2.6}$$

Furthermore, we also normalize the affine parameter τ through

$$u^a v_a = -1 \tag{2.7}$$

A detailed calculation shows a relation between v^a and u^a as

$$v^a - u^a = \frac{S^{ab} R_{bcde} u^c S^{de}}{2(m^2 + \frac{1}{4} R_{bcde} S^{bc} S^{de})} \tag{2.8}$$

The Eq. (2.8) means that the 4-velocity and 4-momentum are not always parallel. In addition, we can obtain the conserved quantities for spin particles with Killing vector fields ξ_a as follows:

$$Q_\xi = p^a \xi_a + \frac{1}{2} S^{ab} \nabla_a \xi_b \tag{2.9}$$

2.2 Conserved quantities in the Kerr-Sen black hole

Here we will consider the Kerr-Sen background and we can calculate the conserved quantities explicitly. In the Boyer–Lindquist coordinates (t, r, θ, ϕ) , the Kerr-Sen metric can be written as

$$ds^2 = -\frac{\Delta - a^2 \sin^2 \theta}{\Sigma} dt^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \frac{\Xi \sin^2(\theta)}{\Sigma} d\phi^2 - \frac{4Mr a \sin^2 \theta}{\Sigma} dt d\phi \tag{2.10}$$

where $\Sigma = r(r+2b)+a^2 \cos^2 \theta$, $\Delta = r(r+2b)-2Mr+a^2$, $\Xi = (a^2 + r(2b+r))^2 - \Delta a^2 \sin^2 \theta$ and $b = Q^2/2M$.

The nonvanishing components of the inverse metric $g^{\mu\nu}$ read

$$\begin{aligned} g^{tt} &= -\frac{\Xi \Sigma}{a^2 \sin^2(\theta)(4M^2 r^2 - \Xi) + \Delta \Xi} \\ g^{rr} &= g^{\theta\theta} \Delta = \frac{\Delta}{\Sigma} \\ g^{\phi\phi} &= \frac{\Sigma \csc^2(\theta)(\Delta \csc^2(\theta) - a^2)}{a^2(4M^2 r^2 - \Xi) + \Delta \Xi \csc^2(\theta)} \\ g^{t\phi} &= -\frac{2aMr \Sigma}{a^2 \sin^2(\theta)(4M^2 r^2 - \Xi) + \Delta \Xi} \end{aligned} \tag{2.11}$$

In order to simplify the equation, we introduce a tetrad basis as

$$\begin{aligned} e_a^{(0)} &= \sqrt{\frac{\Delta}{\Sigma}} (dt_a - a \sin^2 \theta d\phi_a) \\ e_a^{(1)} &= \sqrt{\frac{\Delta}{\Sigma}} dr_a \\ e_a^{(2)} &= \sqrt{\Sigma} d\theta_a \\ e_a^{(3)} &= \frac{\sin \theta}{\sqrt{\Sigma}} (-adt_a + (r^2 + 2br + a^2)d\phi_a) \end{aligned} \tag{2.12}$$

There exist two Killing vectors in the Kerr-Sen geometry:

$$\xi^a = \left(\frac{\partial}{\partial t}\right)^a ; \quad \phi^a = \left(\frac{\partial}{\partial \phi}\right)^a \tag{2.13}$$

Then the conserved quantities in Kerr-Sen background associated to the above two Killing vectors can be written as

$$\begin{aligned} E = -Q_{\xi^t} &= \sqrt{\frac{\Delta}{\Sigma}} p^{(0)} + \frac{a \sin \theta}{\sqrt{\Sigma}} p^{(3)} \\ &\quad - \frac{M(r^2 - a^2 \cos^2 \theta)}{\Sigma^2} S^{(01)} + \frac{2aMr \cos \theta}{\Sigma^2} S^{(23)} \end{aligned} \tag{2.14}$$

$$\begin{aligned} J = Q_{\xi^\phi} &= a \sin^2 \theta \sqrt{\frac{\Delta}{\Sigma}} p^{(0)} + \frac{(a^2 + r(2b+r)) \sin \theta}{\sqrt{\Sigma}} p^{(3)} \\ &\quad - \frac{a \sin^2 \theta}{\Sigma^2} (b(2Mr + \Sigma) + 2Mr^2 + (r - M)\Sigma) S^{(01)} \end{aligned}$$

$$-\frac{a\sqrt{\Delta}\cos\theta\sin\theta}{\Sigma}S^{(02)} + \frac{(b+r)\sqrt{\Delta}\sin\theta}{\Sigma}S^{13} + \frac{\cos\theta}{\Sigma^2}\left(\left(a^2+r(2b+r)\right)^2 - a^2\Delta\sin^2\theta\right)S^{(23)} \tag{2.15}$$

where E and J are the energy and angular momentum of the particle respectively.

2.3 Equations of motion on the equatorial plane

When the particle’s spin is aligned with the spin of the black hole, the spin $s^{(a)}$ can be show as follow:

$$s^{(a)} = -\frac{1}{2m}\varepsilon^{(a)}{}_{(b)(c)(d)}u^{(b)}S^{(c)(d)} \tag{2.16}$$

equivalently

$$S^{(a)(b)} = m\varepsilon^{(a)(b)}{}_{(c)(d)}u^{(c)}s^{(d)} \tag{2.17}$$

where $\varepsilon_{(a)(b)(c)(d)}$ is the completely antisymmetric tensor, with component $\varepsilon_{(0)(1)(2)(3)} = 1$. Furthermore, we consider that the particle was confined in the equatorial plane($\theta = \pi/2$) [31]. The non-zero components of spin tensor read

$$S^{(0)(1)} = -sp^{(3)}, \quad S^{(0)(3)} = sp^{(1)}, \quad S^{(1)(3)} = sp^{(0)} \tag{2.18}$$

Combining Eqs. (2.14), (2.15), and (2.18), the equations of momentum can be written as

$$p^{(0)} = \frac{\Sigma}{\mathcal{D}_1}\left(-J\left(a\Sigma\sqrt{\Sigma} + Mr^2s\right) + E\left(\Sigma\sqrt{\Sigma}\left(a^2+r(2b+r)\right) + as\Sigma(b-M+r) + 2aMrs(b+r)\right)\right) \tag{2.19}$$

$$p^{(3)} = \frac{\Sigma^2\sqrt{\Delta}}{\mathcal{D}_1}\left(J\sqrt{\Sigma} - E\left(s(b+r) + a\sqrt{\Sigma}\right)\right) \tag{2.20}$$

where

$$\mathcal{D}_1 = aMs\sqrt{\Delta\Sigma}\left(2br+r^2-\Sigma\right) + r\sqrt{\Delta}\left(\Sigma^2(2b+r) - Mrs^2(b+r)\right) \tag{2.21}$$

there are a normalization condition of the 4-momentum as [7]

$$p^{(a)}p_{(a)} = k \tag{2.22}$$

where $k = -m^2$ for the massive and $k = 0$ for massless particles. As for massive particles, we defined a specific 4-momentum $u^{(a)}$, by $u^{(a)} = p^{(a)}/m$. Hence, with Eq. (2.22) in hand, for the massive particles, we have

$$u^{(1)} = \sigma\sqrt{(u^{(0)})^2 - (u^{(3)})^2 - 1} \tag{2.23}$$

Here $\sigma = \pm 1$ denote the outgoing and ingoing motions respectively. Moreover, combining Eqs. (2.8), (2.18) and (2.10), the expressions of the 4-velocity read

$$v^{(0)} = \frac{1}{\mathcal{D}_2}\left(u^{(0)}(\mathcal{F}_1 + f_{v1}) + u^{(3)}\mathcal{F}_2\right) \tag{2.24}$$

$$v^{(1)} = \frac{1}{\mathcal{D}_2}u^{(1)}\mathcal{F}_1 \tag{2.25}$$

$$v^{(3)} = \frac{1}{\mathcal{D}_2}\left(u^{(3)}(\mathcal{F}_1 + f_{v2}) - u^{(0)}\mathcal{F}_2\right) \tag{2.26}$$

where

$$f_{v1} = b^2s^2\Delta; \quad f_{v2} = s^2(Mr^2(b+3r) - a^2b^2); \tag{2.27}$$

$$\mathcal{F}_1 = r^2(r(2b+r)^3 - M(b+r)s^2);$$

$$\mathcal{F}_2 = ab^2\sqrt{\Delta}s^2; \tag{2.28}$$

$$\begin{aligned} \mathcal{D}_2 = & r^6 + r^2\left(-bMs^2\left((u^{(3)})^2 + 1\right) + 6br^3 - Mr s^2\left(3(u^{(3)})^2 + 1\right)\right) + 2b^3\left(4r^3 + rs^2(u^{(0)})^2\right) \\ & + b^2\left(as^2\left(a\left((u^{(0)})^2 + (u^{(3)})^2\right) + 2\sqrt{\Delta}u^{(0)}u^{(3)}\right) - 2Mrs^2(u^{(0)})^2 + 12r^4 + r^2s^2(u^{(0)})^2\right) \end{aligned} \tag{2.29}$$

By employing the tetrad basis (2.12), 4-velocity can be rewritten as

$$\frac{dt}{d\tau} = \frac{(a^2 + \Sigma)v^{(0)} + a\sqrt{\Delta}v^{(3)}}{\sqrt{\Delta\Sigma}} \tag{2.30}$$

$$\frac{dr}{d\tau} = \frac{\sqrt{\Delta}v^{(1)}}{\sqrt{\Sigma}} \tag{2.31}$$

$$\frac{d\varphi}{d\tau} = \frac{av^{(0)} + \sqrt{\Delta}v^{(3)}}{\sqrt{\Delta\Sigma}} \tag{2.32}$$

By plugging (2.25) to Eq. (2.31), the radial equation of motion for spin particle gives rise to

$$\frac{dr}{d\tau} = \frac{\mathcal{F}_2\sqrt{\Delta}}{\mathcal{D}_2\sqrt{\Sigma}}u^{(1)} \tag{2.33}$$

In order to facilitate the numerical calculation and without loss generality, we simply set the variables to the dimensionless variables as

$$\begin{aligned} \tilde{E} = \frac{E}{m}, \quad \tilde{J} = \frac{J}{mM}, \quad \tilde{s} = \frac{s}{M}, \quad \tilde{t} = \frac{t}{M}, \\ \tilde{r} = \frac{r}{M}, \quad \tilde{a} = \frac{a}{M}, \quad \tilde{Q} = \frac{Q}{M}, \quad \tilde{\tau} = \frac{\tau}{M} \end{aligned} \tag{2.34}$$

This is equivalent to discuss the energy and other quantity with unity mass. In the following, we omit the $\tilde{\sim}$ for simplicity. For example, E in the following text actually means \tilde{E} .

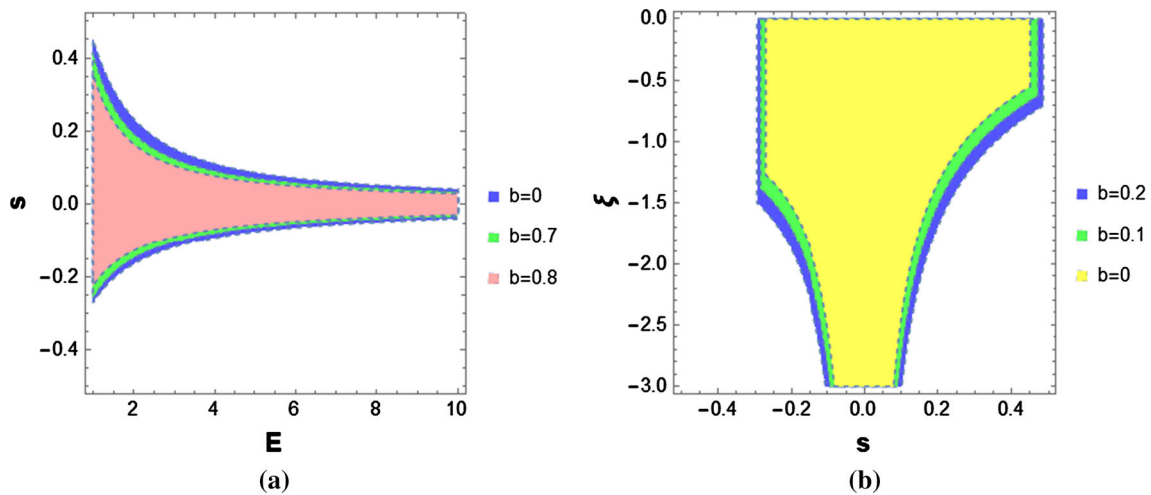


Fig. 1 **a** The condition of the spin s and the energy E that the particles can reach the event horizon for different value of b . **b** The condition of the spin s and the ξ that the particles can fall into the horizon for different value of b

2.4 Constraints on the orbits

In this part, we devoted to find the admissible trajectory of the spin particle which can approach to the horizon $r_H = \sqrt{(1-b)^2 - a^2} - b + 1$, this means that the equation (2.33) must be have real solutions. Combining this fact with the Eq. (2.23) gives us

$$(u^{(0)})^2 - (u^{(3)})^2 - 1 \geq 0 \tag{2.35}$$

when $r \geq r_H$. By plugging Eqs. (2.19) and (2.20) into the above equation, we get a constraint on the orbits

$$E^2 \left(\Sigma^2 \left(\Sigma^{3/2} \left(a^2 - aB_r + r(2b+r) \right) + rs(2a(b+r) - B_r r) + as\Sigma(b-1+r) \right)^2 - \Sigma^4 \left((a - B_r)\sqrt{\Delta\Sigma} + \sqrt{\Delta}s(b+r) \right)^2 \right) \geq \mathcal{D}_1^2 \tag{2.36}$$

$$E^2 < \frac{((b-1)(b+1)^3 + s^2)^4}{(b-1)(b+1)^4(b+3)s^2 \left((b+1)\sqrt{1-b^2} - s \right)^2 \left((b(b+2) - 1)s^2 + 2(b-1)(b+1)^3 \right)} \tag{2.40}$$

where $B_r = J/E$ and \mathcal{D}_1 is given at Eq. (2.21). For the orbits which can reach the horizon, note that Δ and \mathcal{D}_1 vanished at the horizon($r = r_H$), Eq. (2.36) gives a critical value of B_r

$$B_{cr} = \frac{-a^2 r_H(2b+r_H)^2 - as\sqrt{r_H(2b+r_H)}((b+r_H)(2b+r_H) + r_H) - r_H^2(2b+r_H)^3}{-ar_H(2b+r_H)^2 - sr_H\sqrt{r_H(2b+r_H)}} \tag{2.37}$$

Hence the condition that the orbit can reach the horizon equal to $B_r \leq B_{cr}$. On the other hand, we know that for a massive particle, the 4-velocity along the admissible trajectory must be timelike as

$$v_{(a)}v^{(a)} = -(v^{(0)})^2 + (v^{(1)})^2 + (v^{(3)})^2 < 0 \tag{2.38}$$

Along the same line of [10], the above timelike condition is equivalent to the following constraint

$$\mathcal{U}E^2 < \mathcal{C}_a^2 r^2 \mathcal{F}_1^2 \tag{2.39}$$

where $\mathcal{C}_a = r(2b+r)^3 - s^2(b+r)$, \mathcal{F}_1 is at Eq. (2.28), and the detailed expression for \mathcal{U} can be found in APPENDIX.

Since we consider the maximal energy contraction efficiency from black hole, in the following we only focus on the extreme Kerr-Sen black hole($b = 1 - a$). In this situation, if one of the collision particle possess the critical angular momentum, it is easy to see that $B_{cr} = 2$ from Eq. (2.37). Then from Eq. (2.39), we have

which gives us a constraint on energy E for different values of spin s and b and is showed in the Fig. 1a. The figure shows that when b increase, the admissible range of spin s shrinks

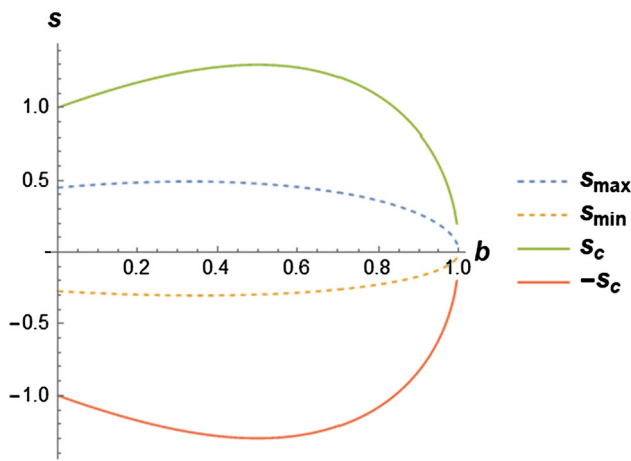


Fig. 2 The maximal and minimum value of spin s of the particles in the extreme case for different values of b as well as its comparison with s_c

for a given energy E . If the particle falling from infinity, that is $E \geq 1$, combining this fact with Eq. (2.40), the spin s will be restricted to $s_{min} < s < s_{max}$ for a given value of b . For example, when $b = 0.1$, we can obtain $s_{min} \approx -0.285$ and $s_{max} \approx 0.471$. More information of s_{min} and s_{max} for different value of b can be found in Fig. 2. Moreover, it worth to note that, the authors of Ref. [28] point out that when the particle process critical spin $s = -s_c = -a^2 \left(\frac{2}{a} - 1\right)^{\frac{3}{2}}$, the timelike condition is violated. We show in Fig. 2, our admissible spin s corresponding to the maximum of efficiency is always bigger than critical value ($s > -s_c$), and therefore the timelike condition is satisfied in our case.

If the particle’s angular momentum is deviate from critical value, we set $B_r = 2(1 + \xi)$ with ξ being a negative number.

$$p^{(1)} = \sigma \frac{(b + 1)\sqrt{2\sqrt{1 - b^2}(b + 1)s + (1 - b)(b + 1)^3 + s^2} |J - 2E|}{\epsilon \left((1 - b)(b + 1)^3 - s^2 \right)} + O(\epsilon^0) \tag{3.7}$$

From Eq. (2.39), the energy E now is a function of the s , b and ξ and is showed as the Fig. 1b. This figure shows that the allowed range of ξ increase when b increases.

We assume the particles are freely falling from infinity. If $B_r > B_{rc}$, such a particle falling from infinity will find a turning point away from horizon, and then bounce back to infinity. So if $B_r = B_{rc} + \delta (\delta \rightarrow 0^+)$, the turning point of the particle can very close to the horizon. Then, these particles will moving outward. Therefore this situation should also need to be taken into account.

3 Collision of spinning particles

In this section, we consider the collision of two spin particles that are freely falling from infinity, and find the formula of the

efficiency of the energy extraction from the extreme Kerr-Sen black hole.

We denotes that the 4-momentum of particle 1 and particle 2 are p_1^μ and p_2^μ . Our picture is the following: The particles collide outside the horizon. After collision, particle 3, whose 4-momenta is p_3^μ , will move to infinity, while the particle 4 with p_4^μ falls into the Kerr-Sen black hole. We assume that the sum of initial spins and 4-momenta are conserved throughout the collision process. That is,

$$S_1^{\mu\nu} + S_2^{\mu\nu} = S_3^{\mu\nu} + S_4^{\mu\nu} \tag{3.1}$$

$$p_1^\mu + p_2^\mu = p_3^\mu + p_4^\mu \tag{3.2}$$

Since the Kerr-Sen spacetime exists two Killing vectors, contracting these two Killing vector with the above equation gives the conservation of the energy E and angular momentum J as follows

$$E_1 + E_2 = E_3 + E_4 \tag{3.3}$$

$$J_1 + J_2 = J_3 + J_4 \tag{3.4}$$

From the Eqs. (3.1) and (3.2), we can also obtain the conservation of particle’s spin and the radial components of 4-momentum throughout the collision

$$m_1 s_1 + m_2 s_2 = m_3 s_3 + m_4 s_4 \tag{3.5}$$

$$p_1^{(1)} + p_2^{(1)} = p_3^{(1)} + p_4^{(1)} \tag{3.6}$$

Now we assume that particle 1 and particle 2 collide near the horizon of extreme Kerr-Sen black hole, the radial position of collision point r_c is very close to extreme Kerr-Sen black hole’s horizon $r_H (r_H = a = 1 - b)$, so that we can assume ($r_c = a/(1 - \epsilon)$) with $\epsilon \rightarrow 0^+$. Then, we expand the particles’ radial 4-momentum in terms of ϵ as follows:

In the following analysis, without loss of generality, along the same line of [10], we doing calculation in case that particle 1 is critical ($J_1 = 2E_1$), while particle 3 is near-critical ($J_3 = 2E_3 + O(\epsilon)$) and particle 2 is non-critical ($J_2 < 2E_2$) [10].

Then the total angular momentum of the particle can relate to the energy as follows:

$$J_1 = 2E_1 \tag{3.8}$$

$$J_3 = 2E_3 \left(1 + \alpha_3 \epsilon + \beta_3 \epsilon^2 + \dots \right) \tag{3.9}$$

where α_3 and β_3 are expansion parameters of $O(\epsilon^0)$.

For particle 2, since it is non-critical, we assume that:

$$J_2 = 2E_2 (1 + \xi) \tag{3.10}$$

where $\xi < 0$ and $\xi = O(\epsilon^0)$

From the conservation law (3.3) and (3.4), we get the following equations

$$E_4 = E_1 + E_2 - E_3; \quad J_4 = J_1 + J_2 - J_3; \quad (3.11)$$

which give us:

$$J_4 = 2E_4 \left(1 + \frac{E_2}{E_4} \xi + \dots \right) \quad (3.12)$$

Since we consider the collision of the particle 1 and particle 2, the particle 2 must be ingoing ($\sigma_2 = -1$) because the particle 2 is noncritical [10]. Combine Eq. (3.7) with the conservation of 4-momentum (3.6), we can get the equation as follows:

$$|J_2 - 2E_2| \left(\frac{\sigma_4 \sqrt{C_a(s_4, b) + C_b(s_4, b)}}{C_a(s_4, b)} - \frac{\sigma_2 \sqrt{C_a(s_2, b) + C_b(s_2, b)}}{C_a(s_2, b)} \right) = O(\epsilon^1) \quad (3.13)$$

where $C_a(s, b) = (1 - b)(1 + b)^3 - s^2$ is the critical case ($r_H = 1 - b$) of the C_a in Eq. (2.39) and $C_b(s, b) = 2(b + 1)\sqrt{1 - b^2}s + 2s^2$. From Eq. (3.13), we find that $\sigma_4 = \sigma_2$ and $s_4 = s_2$. Then Eq. (3.5) further forces us to impose $s_3 = s_1$.

In the following section, we will consider three different types of collision. The first case is the collision of two massive particles (MMM). The second type is the collision of one massless particle with another massive particle, which is called as compton scattering (PMP) [10] and third type is the inverse compton scattering (MPM) [10], which is the inverse process of type two case.

Now, we come to calculate E_2 and E_3 for the cases [A] (MMM), [B] (MPM), and [C] (PMP).

3.1 Maximal efficiency in case [A] MMM

For the case[A], to simplify the discussion, we just assume that the mass of collision particles are all equal to m , i.e. $m_1 = m_2 = m_3 = m_4 = m$. With this in hand, the equations of conservation laws (3.3)–(3.6) can be simplified as

$$E_1 + E_2 = E_3 + E_4 \quad (3.14)$$

$$J_1 + J_2 = J_3 + J_4 \quad (3.15)$$

$$s_1 + s_2 = s_3 + s_4 \quad (3.16)$$

$$u_1^{(1)} + u_2^{(1)} = u_3^{(1)} + u_4^{(1)} \quad (3.17)$$

The radial component of the 4-momentum of massive particle can be calculated from the Eqs. (2.19)–(2.21), and (2.23). With the help of Eqs. (3.8)–(3.10), and (3.12), we expand the particles' radial 4-momentum in terms of ϵ as follows:

$$u_1^{(1)} = f_{11}\epsilon^{-1} + f_{12}\epsilon^0 + f_{13}\epsilon^1 + \dots \quad (3.18)$$

$$u_2^{(1)} = f_{21}\epsilon^{-1} + f_{22}\epsilon^0 + f_{23}\epsilon^1 + \dots \quad (3.19)$$

$$u_3^{(1)} = f_{31}\epsilon^{-1} + f_{32}\epsilon^0 + f_{33}\epsilon^1 + \dots \quad (3.20)$$

$$u_4^{(1)} = f_{41}\epsilon^{-1} + f_{42}\epsilon^0 + f_{43}\epsilon^1 + \dots \quad (3.21)$$

From Eqs. (3.18)–(3.21), we can easily obtain corresponding equations for different order of ϵ . Note that the leading order equation of ϵ^{-1} has already been discussed in the Eqs. (3.13) and we found some constraints have to be satisfied under Eq. (3.13). So we further discuss the Eq. (3.17) from the next leading order of ϵ^0 and ϵ^1 as follows

$$f_{12} + f_{22} = f_{32} + f_{42} \quad (3.22)$$

$$f_{13} + f_{23} = f_{33} + f_{43} \quad (3.23)$$

where

$$f_{12} = \frac{\sigma_1 k_1(E_1, s_1, b, 0)}{f_1(s_1, b)} \quad (3.24)$$

$$f_{13} = -\frac{E_1^2 \sigma_1 k_3(s_1, b)}{f_1(s_1, b)^2 k_1(E_1, s_1, b, 0)} \quad (3.25)$$

$$f_{22} = \frac{E_2 k_2(s_2, b, \xi)}{f_1(s_2, b)^2} \quad (3.26)$$

$$f_{23} = \frac{(1 - b)f_1(s_2, b)^4 - E_2^2 h_{41}(s_2, b, \xi)}{4\sqrt{1 - b^2} E_2 \xi f_1(s_2, b)^3 f_2(s_2, b)} \quad (3.27)$$

$$f_{32} = \frac{\sigma_3 k_1(E_3, s_1, b, \alpha_3)}{f_1(s_1, b)} \quad (3.28)$$

$$f_{33} = -\frac{E_3^2 \sigma_3 (\beta_3 h_{61}(s_1, b, \alpha_3) + h_{62}(s_1, b, \alpha_3) + k_3(s_1, b))}{f_1(s_1, b)^2 k_1(E_3, s_1, b, \alpha_3)} \quad (3.29)$$

$$f_{42} = \frac{E_2 k_2(s_2, b, \xi) - f_1(s_2, b)(E_1 h_{71}(s_2, b) + E_3 h_{72}(s_2, b, \alpha_3))}{f_1(s_2, b)^2} \quad (3.30)$$

$$f_{43} = -\frac{1}{4(b + 1)\sqrt{1 - b^2} E_2 \xi f_1(s_2, b)^3 f_2(s_2, b)} \times \left(2(b + 1)E_2 f_1(s_2, b)(4E_3 \xi h_{85}(s_2, b, \alpha_3, \beta_3) - (E_1 - E_3)((b - 1)f_1(s_2, b)h_{81}(s_2, b) + 2\xi h_{83}(s_2, b))) + (b - 1)(b + 1)f_1(s_2, b)^2 \times (f_1(s_2, b)^2 - (E_1 - E_3)^2 h_{81}(s_2, b)) + E_2^2 (-4(b + 1)\xi f_1(s_2, b)h_{83}(s_2, b) + h_{82}(s_2, b) + \xi^2 h_{84}(s_2, b)) \right) \quad (3.31)$$

where $k_1(E_1, s_1, b, 0)$, $k_2(s_2, b, \xi)$, $h_1(s_1, b)$, $h_2(s_2, b)$, $h_{71}(s_1, b)$, $g_3(b, s_2, \xi)$ and so on are the functions of different parameters and we will show them in the appendix.

From the Eq. (3.22), with the detailed expressions given by the above, we obtain the equation of E_3 as follow

$$\mathcal{A}_1 E_3^2 - 2\mathcal{B}_1 E_3 + \mathcal{C}_1 = 0 \quad (3.32)$$

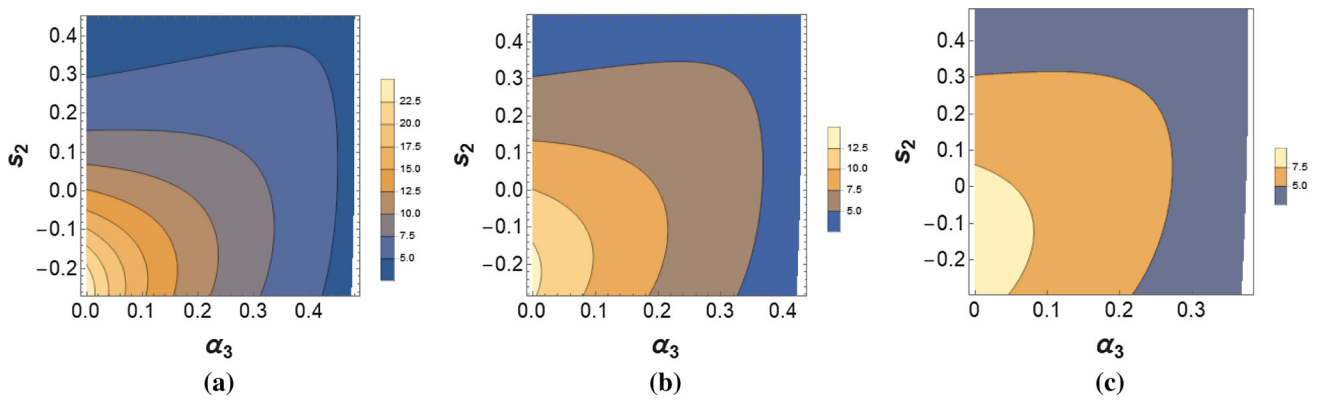


Fig. 3 The contour maps of E_3 in terms of α_3 and s_2 with $s_1 = 0$. **a** ($b = 0$), **b** ($b = 0.1$) and **c** ($b = 0.2$) show that $\alpha_3 \rightarrow 0$ give larger value of E_3 when s_2 is small enough

where

$$\begin{aligned}
 A_1 &= \frac{f_1(s_1, b)^2 h_{72}(s_2, b, \alpha_3)^2}{f_1(s_2, b)^2} - k_{12}(s_1, b, \alpha_3) \\
 B_1 &= -\frac{h_{72}(s_2, b, \alpha_3)}{f_1(s_2, b)^2} (\sigma_1 f_1(s_1, b) f_1(s_2, b) k_1(E_1, s_1, b, 0) \\
 &\quad + E_1 f_1(s_1, b)^2 h_{71}(s_2, b)) \\
 C_1 &= \frac{2E_1 \sigma_1 f_1(s_1, b) h_{71}(s_2, b) k_1(E_1, s_1, b, 0)}{f_1(s_2, b)} \\
 &\quad + E_1^2 \left(k_{12}(s_1, b, 0) + \frac{h_{71}(s_2, b)^2 f_1(s_1, b)^2}{f_1(s_2, b)^2} \right) \quad (3.33)
 \end{aligned}$$

From Eqs. (3.32) and (3.33), we find that σ_3 is decoupled. So the sign of σ_3 will not affect the value of E_3 . Since the quadratic equation of E_3 (3.32) has two solutions. The larger solution of $E_3 = E_{3,+}$ gives larger efficiency because the efficiency depends on the value of E_3 that will become explicit in following parts. Therefore, it is sufficient to consider the case of $\sigma_3 = -1$ with the larger solution of $E_3 = E_{3,+}$. In conclusion, we can get the expression of E_3 and E_2 from the Eqs. (3.22) and (3.23).

$$E_{3,+} = \frac{\sqrt{B_1^2 - A_1 C_1} + B_1}{A_1} \quad (3.34)$$

and

$$E_2 = \frac{(b-1)(E_1 - E_3)^2}{P_1} \quad (3.35)$$

where

$$\begin{aligned}
 P_1 &= \frac{2}{h_1(s_1, b)^2 h_{41}(s_2, b)} (-4E_3 \xi h_{85}(s_2, b, \alpha_3, \beta_3) \\
 &\quad + 2\sqrt{1 - b^2} \xi (f_{13} + f_{33}) f_1(s_2, b)^2 f_2(s_2, b) \\
 &\quad + (b-1)(E_1 - E_3) f_1(s_2, b) h_{81}(s_2, b) \\
 &\quad + 2\xi (E_1 - E_3) h_{83}(s_2, b)). \quad (3.36)
 \end{aligned}$$

With all those ingredients, the efficiency can be calculated through the following expression:

$$\eta = \frac{E_3}{E_1 + E_2} \quad (3.37)$$

3.1.1 Efficiency

With the detailed expressions of E_3 and E_2 above. We have three different types of parameters involved in the calculation of the efficiency η . The first type is the charge of extreme Kerr-Sen black hole ($b = 1 - a$). Second type is particles spins (s_1 and s_2), the third type is orbit parameters of the particles such as (α_3, β_3 and ξ) and direction of the particles' motion ($\sigma_1, \sigma_2, \sigma_3$ and σ_4).

Note that we already fix the value of $\sigma_2, \sigma_3, \sigma_4$ as $\sigma_2 = \sigma_4 = -1$ and $\sigma_3 = -1$ in the last section. So the only remaining parameter for the direction of the particles' motion is σ_1 . However, a good efficiency can't be found for $\sigma_1 = -1$ [10], so we set that $\sigma_1 = 1$.

Then, for a given value of E_1 , the maximal efficiency η_{max} would be reached with the minimum value of E_2 and the maximal value of E_3 . Without loss of generality, we just normalize the ingoing energy E_1 as $E_1 = 1$.

From the Eq. (3.34), we find that the expression of E_3 decoupled with the parameters ξ and β_3 . So we analyze the maximal value of E_3 with the remaining parameters for different values of b . Note that Fig. 1a shows that the spin magnitude s_1 close to zero for larger value of E_3 . So we first assume $s_1 = 0$ in order to find the relation of E_3 and α_3 . The contour maps of E_3 in terms of α_3 and s_2 showed in Fig. 3. From the Fig. 3, we know that the largest efficiency can found with $\alpha_3 \rightarrow 0^+$. Therefore, we set $\alpha_3 = 0^+$ to calculate the corresponding maximal efficiency.

In Fig. 4, the contour map of E_3 in terms of s_1 and s_2 is showed. The maximal value of E_3 is labeled with the red point.

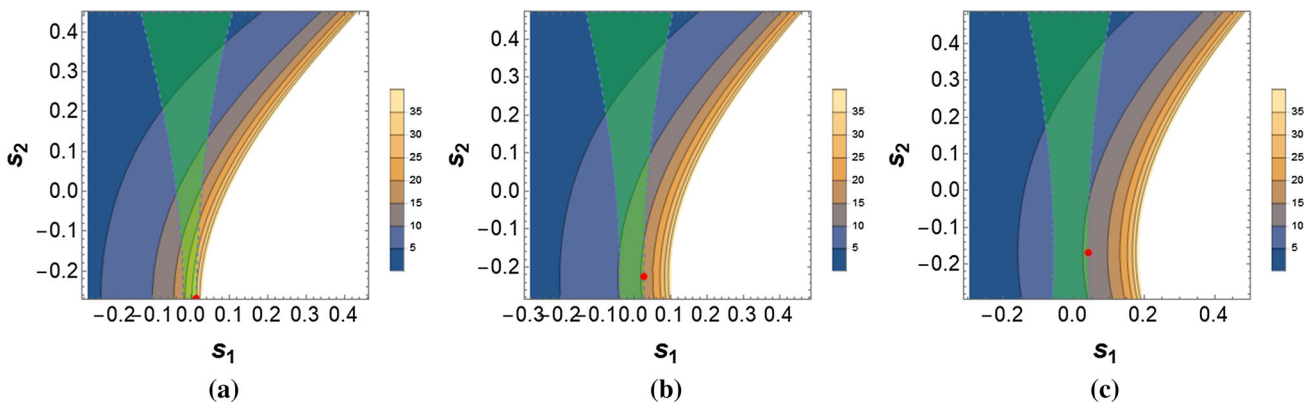
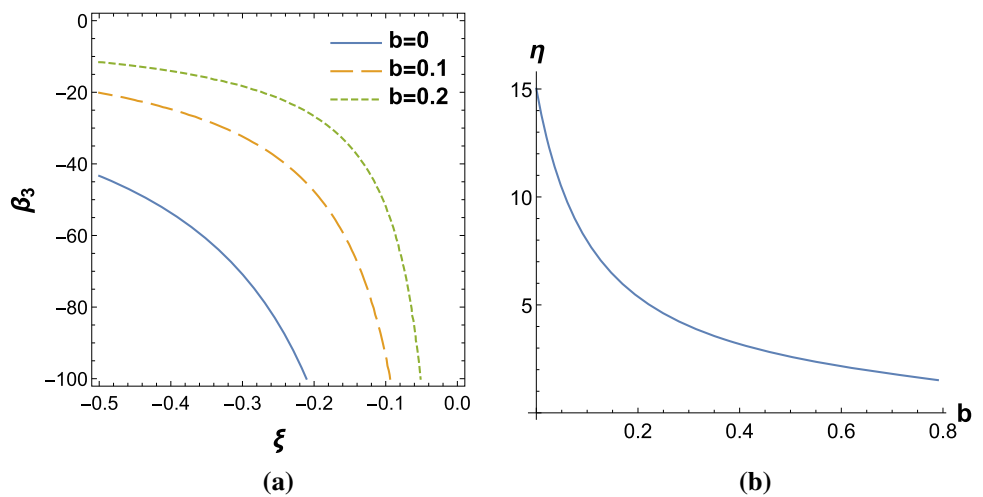


Fig. 4 The contour map of E_3 in terms of s_1 and s_2 . The time-like condition for the particle 3 orbit is satisfied in the green shaded region. The maximum value of E_3 is obtained at the red point. **a** When $b = 0$, $\eta_{max} = E_{3max}/2 \approx 15.01$ at $(s_1 = 0.01378, s_2 = -0.2679)$; **b** when $b = 0.1$, $\eta_{max} = E_{3max}/2 \approx 7.964$ at $(s_1 = 0.02694, s_2 = -0.2253)$; **c** when $b = 0.2$, $\eta_{max} = E_{3max}/2 \approx 5.378$ at $(s_1 = 0.04076, s_2 = -0.1680)$

Fig. 5 a The relation between ξ and β_3 for different value of b when $E_2 = 1$. The other parameters are chosen for giving the maximal value of E_3 . When $b = 0$, $s_1 = 0.01378$ and $s_2 = -0.2679$; When $b = 0.1$, $s_1 = 0.0269$ and $s_2 = -0.225$; When $b = 0.2$, $s_1 = 0.0408$ and $s_2 = -0.168$; **b** the relation between maximum efficiency $\eta_{max} = E_{3max}/2$ and b



Note that $E_2 \geq 1$ if the particle 2 falling from infinity, if $E_2 = 1$ is possible, we find that the maximal value of E_3 gives the maximal efficiency. Note that E_3 is decoupled with parameters β_3 and ξ . So our target is equivalent to find $E_2 = 1$ with some admissible values of β_3 and ξ . In Fig. 1b, we already have the constraint on ξ , that is, $0 > \xi \geq -0.5 > \xi_{min}$ for different values of s and b . For such constrained ξ , the relation between β_3 and ξ which gives $E_2 = 1$ can be found in Fig. 5a.

Hence the maximum efficiency is given by $\eta_{max} = E_{3max}/2$. Figure 5b shows the maximum efficiency η_{max} with different b . We found that the efficiency η_{max} decreases with the increase of b . While when $b = 0$ which corresponds the Kerr case, our results is the same as the previous results [10].

3.2 Maximal efficiency in case [B] MPM

For the case[B], as the same with section, we assume that the mass of massive particles are all equal to m and the massless particles are nonspinning. The equations of conservation law (3.5) and (3.6) reduce to

$$s_1 = s_3 \tag{3.38}$$

$$p_1^{(1)} + p_2^{(1)} = p_3^{(1)} + p_4^{(1)} \tag{3.39}$$

The radial component of the 4-momentum of massless particle can be calculated from the Eqs. (2.19)–(2.21), and (2.22) as follows

$$p^{(1)} = \sigma \sqrt{\frac{(E(b+r+1) - J)(E((3b-1)r + 2(b-1)^2 + r^2) + J(2b+r-2))}{(b+r-1)^2(2b+r)}} \tag{3.40}$$

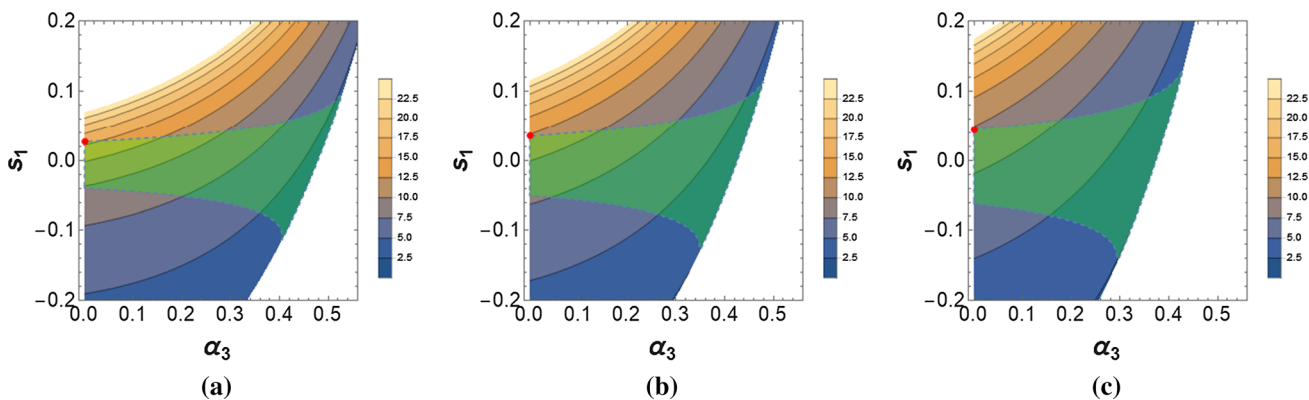


Fig. 6 The contour map of E_3 in terms of α_3 and s_1 . The time-like condition for the particle 3 is satisfied in the light-green shaded region with different b . The maximum value of E_3 is obtained at the red point.

a When $b = 0$, $E_{3max} \approx 15.6350$; **b** when $b = 0.1$, $E_{3max} \approx 12.2977$; **c** when $b = 0.2$, $E_{3max} \approx 9.7977$

So the expression of f_{22} , f_{23} , f_{42} , and f_{43} can write in an explicit way:

$$f_{22} = -\frac{2E_2(b\xi + b + 2\xi + 1)}{(b + 1)\sqrt{1 - b^2}} \tag{3.41}$$

$$f_{23} = -\frac{E_2(4(2b - 1)\xi^2 + 8b(b + 1)\xi + (b + 1)^4)}{4(b + 1)^2\sqrt{1 - b^2}\xi} \tag{3.42}$$

$$f_{42} = -\frac{2(b + 1)(E_1 + E_2 + (\alpha_3 - 1)E_3) + 2(b + 2)E_2\xi}{(b + 1)\sqrt{1 - b^2}} \tag{3.43}$$

$$f_{43} = -\frac{1}{4(b + 1)^2\sqrt{1 - b^2}E_2\xi} \left(8(b + 1)E_2\xi(bE_1 + E_2 + E_3(-\alpha_3 + \beta_3 - 1)) + E_3(\beta_3 - 2\alpha_3) + (b + 1)^4(E_1 + E_2 - E_3)^2 + 4(2b - 1)E_2^2\xi^2 \right) \tag{3.44}$$

Note that the radial component of the 4-momentum of massive particle do not change through the collision process. As in case [A], we finally get the detail expression for E_3 and E_2 respectively.

$$E_3 = \frac{\sqrt{\mathcal{B}_1^2 - \mathcal{A}_1\mathcal{C}_1 + \mathcal{B}_1}}{\mathcal{A}_1} \Big|_{s_2=0} \tag{3.45}$$

$$E_2 = \frac{(b - 1)(E_1 - E_3)^2}{\mathcal{P}_1} \Big|_{s_2=0} \tag{3.46}$$

where \mathcal{A}_1 , \mathcal{B}_1 , \mathcal{C}_1 and \mathcal{P}_1 are given by Eqs. (3.33) and (3.36) with $s_2 = 0$.

3.2.1 Efficiency

On the one hand, when the value of E_1 is given, the maximal efficiency η_{max} would be reached with minimum value of E_2 and maximal value of E_3 . On the other hand, we consider

particle 1 and particle 2 falling from infinity, we obtain the constrains of $E_1 \geq 1$ and $E_2 \geq 0$. Without loss generality, we again normalize E_1 to unity ($E_1 = 1$) as last subsection and then analyze E_3 and E_2 which are directly associated to the maximal efficiency.

Figure 6 shows the contour map of E_3 in terms of α_3 and s_1 . The maximum value of E_3 is given at the red point.

If $E_2 \rightarrow 0$ can be achieved, it certainly gives the minimal value of E_2 and thus the maximal efficiency can be simply given by $\eta_{max} = E_{3max}$. Hence it is important to analyze whether $E_2 \rightarrow 0$ is possible or not. From Eq. (3.36), we obtain the asymptotic expression of \mathcal{P} as

$$\mathcal{P} \approx 4(b + 1)E_3\beta_3\xi \left(\frac{\sqrt{1 - b^2}E_3h_{61}(s_1, b, 0)}{f_1(s_1, b)^2k_1(E_3, s_1, b, 0)} - 2 \right) \tag{3.47}$$

Equation (3.47) tells us that $E_2 \rightarrow 0^+$, if $\beta_3\xi \rightarrow +\infty$. For example, the value of parameters at red point in Fig. 6b are $b = 0.1$, $\sigma_1 = 1$, $\sigma_3 = -1$, $\alpha_3 = 0$, $s_1 = 0.03513$, $s_2 = 0$, $E_1 = 1$, $E_3 = 12.2977$. So the detail expression of E_2 can be rewritten as:

$$E_2 = \frac{7.27275}{\beta_3\xi + 1.74978\xi + 1.28748}. \tag{3.48}$$

which means $E_2 \rightarrow 0^+$ can be realized $\beta_3\xi \rightarrow \infty$ for the case of $b = 0.1$.

Hence by employing formula $\eta_{max} = E_{3max}$, we found that the efficiency η_{max} decreases with the increase of b . While $b = 0$ which corresponds the Kerr case, our results is again the same as the previous results [10].

3.3 Maximal efficiency in case [C] PMP

Now we come to the last case, which is the Compton scattering. The radial components of 4-momenta of massless particles have already been given in Eq. (3.40). So we can

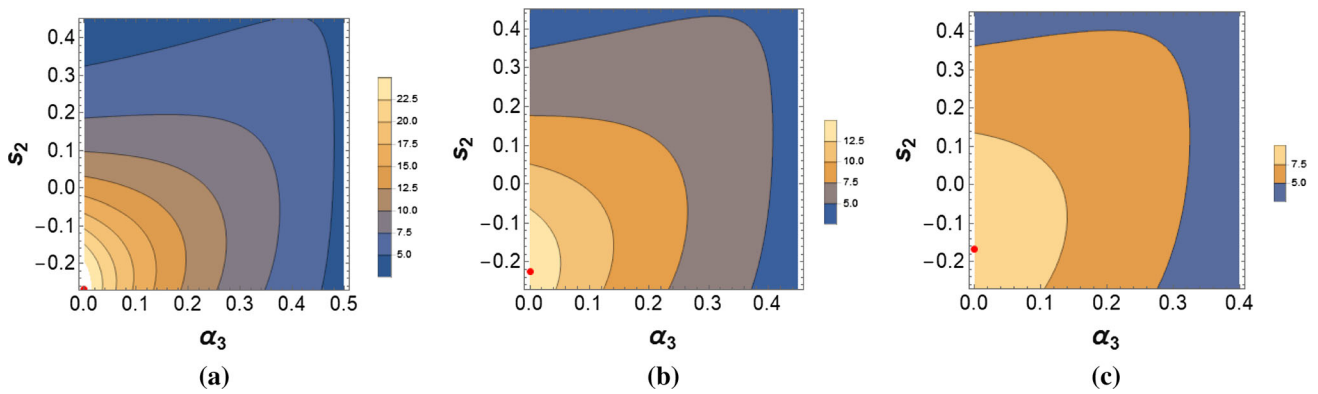


Fig. 7 The contour map of S in terms of α_3 and s_2 for different values of b . The maximum value of E_3 is labeled by the red point. **a** When $b = 0$, $\eta_{max} = S_{max} \approx 26.8564$ with $s_2 = -0.2679$; **b** when $b = 0.1$, $\eta_{max} = S_{max} \approx 14.4513$ with $s_2 = -0.2253$; **c** when $b = 0.2$, $\eta_{max} = S_{max} \approx 9.7977$ with $s_2 = -0.1680$

write the coefficients f_{12} , f_{13} , f_{32} and f_{33} in terms of energy as:

$$f_{12} = \sqrt{\frac{b+3}{b+1}} E_1 \sigma_1 \tag{3.49}$$

$$f_{13} = \frac{(b-1)E_1\sigma_1}{(b+1)^{3/2}\sqrt{b+3}} \tag{3.50}$$

$$f_{32} = E_3\sigma_3\sqrt{\frac{-4\alpha_3^2 + 8\alpha_3 + b^2 + 2b - 3}{b^2 - 1}} \tag{3.51}$$

$$f_{33} = \frac{E_3\sigma_3(8\alpha_3^2 - 8\alpha_3 + b^2 + 4\alpha_3^2b - 4(\alpha_3 - 1)(b + 1)\beta_3 - 2b + 1)}{(b + 1)\sqrt{(1 - b^2)(3 - 2\alpha_3 + b)(1 - 2\alpha_3 - b)}} \tag{3.52}$$

$$\frac{E_2}{E_1} \approx \frac{(b-1)(S-1)^2 h_{81}(s_2, b) \sqrt{(-2\alpha_3 + b + 3)(1 - b - 2\alpha_3)}}{8\beta_3 \xi S f_2(s_2, b) (2(\alpha_3 - 1) f_1(s_2, b) - (b + 1) f_2(s_2, b)) \sqrt{(-2\alpha_3 + b + 3)(1 - b - 2\alpha_3)}} \tag{3.57}$$

From the conservation of the radial components of the 4-momenta, we find

$$E_3 = S E_1 \tag{3.53}$$

where the amplification factor S is given by

$$S = \frac{\sqrt{1-b}(\sqrt{b+3}\sigma_1 f_1(s_2, b) + \sqrt{b+1}h_{71}(s_2, b))}{\sigma_3 \sqrt{(3-2\alpha_3+b)(1-2\alpha_3-b)} f_1(s_2, b) - \sqrt{1-b^2} h_{72}(s_2, b, \alpha_3)} \tag{3.54}$$

and

$$E_2 = -\frac{(b-1)(E_1 - E_3)^2 f_1(s_2, b) h_{81}(s_2, b)}{\mathcal{P}_3}; \tag{3.55}$$

where \mathcal{P}_3 keeps the same form of \mathcal{P}_1 given by Eq. (3.36) by replacing f_{13} and f_{33} with Eqs. (3.50) and (3.52)

3.3.1 Efficiency

It is easy to see in Compton scattering, the efficiency η is defined as:

$$\eta = -\frac{S}{1 + E_2/E_1}; \tag{3.56}$$

again, we consider massless particle 1 and massive particle 2 falling from infinity, we assume the constrains of $E_1 \geq 0$ and $E_2 \geq 1$ and obtain that the maximal value of S and the minimal value of E_2/E_1 gives the maximal efficiency. First, we can easily find that the ratio E_2/E_1 doesn't depends on the E_1 and E_2 , but rather depends on the parameters α_3 , β_3 , ξ , s_2 and b . From Eq. (3.55), the asymptotic expression of E_2/E_1 behaves

Note that the particle 2 is massive and can reach the horizon, therefore the constraint on ξ keeps the same form as in previous section, namely, $\xi_{min} < \xi < 0$. With this parameter space, a direct calculation shows that $S \neq 0$. From Eq. (3.57), we can see that if denominator of the equation is not equal to zero, the condition $E_2/E_1 \rightarrow 0$ can be archived when $\beta_3 \xi \rightarrow \infty$. Thus the maximal energy contraction efficiency is $\eta_{max} = S_{max}$.

Figure 7 shows the maximum value of E_3 with the red point in the contour map of S in terms of α_3 and s_2 for different values of b . The figure shows the maximum efficiency $\eta_{max} = S_{max}$ decreases when b increases.

4 Conclusions

In this paper, we study the collision of two uncharged spinning particles around an extreme Kerr-Sen black hole and cal-

culate the maximal efficiency of the energy extraction from the black hole. We consider the particles freely falling from infinity to the Kerr-Sen black hole. The Kerr-Sen spacetime is determined by three parameters, which are mass M , angular momentum a , and charge Q ($b = Q^2/2M$). It reduces to a Kerr black hole when the parameter $b = 0$ and all our results coming back to the Kerr case [10] when $b = 0$. We viewed this as a consistent check.

In this paper, we consider three types of collision, the first one is the MMM case[A], we obtain that the maximum efficiency is given by $\eta_{max} = E_{3max}/2$ and decreases monotonously with the increase of b . Then, in the MPM case[B], we obtain the maximum efficiency $\eta_{max} = E_{3max}$ and decreases monotonously with the increase of b . Finally, in the PMP case[C], we get the maximum efficiency $\eta_{max} = S_{max}$ which decreases when the b increases. All our results can reduce to the Kerr situation [10] when $b = 0$. The Compton scating and inverse Compton scating of spinless particle in Kerr background is discussed in [6], and our results shows when the spin take into account, the maximum efficiency can be greatly improved.

In summarize, for extreme Kerr-Sen black hole, decrease the charge parameter $b = Q^2/2M$ always increase the maximum efficiency of energy extraction.

Acknowledgements This work is supported by NSFC with No. 11775082. The authors could like to thank prof. Kazumasa Okabayashi for helpful discussion.

Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors' comment: This is a theoretical study and therefore no experimental data has been listed.]

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5 Appendix

$$\begin{aligned} \mathcal{U} = & -a^2b^4 \Delta r^2 s^4 (C_{i3} - BrC_{i4})^2 \\ & + 4ab^2 C_{ia2} r^3 s^2 (BrC_{i2} + C_{i1})(C_{i3} - BrC_{i4}) \\ & + 2f_{v2} r (C_{i3} - BrC_{i4})(ab^2 s^2 (BrC_{i2} + C_{i1}) \\ & + C_{ia2} r^3 (C_{i3} - BrC_{i4})) \end{aligned}$$

$$\begin{aligned} & + 2b^2 r s^2 (BrC_{i2} + C_{i1})(ab^2 \Delta s^2 (C_{i3} - BrC_{i4}) \\ & - C_{ia2} r (BrC_{i2} + C_{i1})) - b^4 \Delta s^4 (BrC_{i2} \\ & + C_{i1})^2 + f_{v2}^2 r^2 (C_{i3} - BrC_{i4})^2 \\ & + a^2 b^4 s^4 (BrC_{i2} + C_{i1})^2 \end{aligned} \tag{5.1}$$

$$C_{ia2} = \mathcal{F}/r^2; \tag{5.2}$$

$$C_{i1} = \sqrt{\Sigma}(a^2 r (2b + r)^2 + as((b + r)(2b + r) + r)\sqrt{\Sigma} + (2b + r)\Sigma^2); \tag{5.3}$$

$$C_{i2} = \sqrt{\Sigma}(-ar(2b + r)^2 - rs\sqrt{\Sigma}); \tag{5.4}$$

$$C_{i3} = (2b + r)^2(a\sqrt{\Sigma} + s(b + r)); \tag{5.5}$$

$$C_{i4} = \sqrt{\Sigma}(2b + r)^2; \tag{5.6}$$

$$f_1(s, b) = -b^4 - 2b^3 + 2b - s^2 + 1; \tag{5.7}$$

$$f_2(s, b) = b^3 - \sqrt{1 - b^2}s + b^2 - b - 1; \tag{5.8}$$

$$k_1(E, s, b, \alpha) = \sqrt{E^2 k_{i2}(s, b, \alpha) - f_1(s, b)^2}; \tag{5.9}$$

$$\begin{aligned} k_2(s, b, \xi) = & -\xi(2(b^2 - 3b + 1)\sqrt{1 - b^2}(b + 1)s^2 \\ & - 2(b - 1)(b + 2)\sqrt{1 - b^2}(b + 1)^4 \\ & + 2(b - 1)bs^3 + 2(b - 3)(b - 1)(b + 1)^3s \\ & - (b + 1)((b^2 + 2b - 1)s^3 \\ & - 2\sqrt{1 - b^2}(b + 1)s^2 + (b - 1)(b^2 + 2b - 1)(b + 1)^3s \\ & - 2(b - 1)\sqrt{1 - b^2}(b + 1)^4); \end{aligned} \tag{5.10}$$

$$\begin{aligned} k_3(s, b) = & (b - 1)(b + 1)(-(b - 1)(3b \\ & - 7)\sqrt{1 - b^2}(b + 1)^4s + (b^5 + 3b^4 + 7b + 1)s^4 \\ & + (b - 1)^3(b + 1)^6 - (2b^3 + 2b^2 \\ & - 3b + 11)\sqrt{1 - b^2}(b + 1)s^3 \\ & + (b - 1)(2b^3 - 2b^2 - 13b + 9)(b + 1)^3s^2); \end{aligned} \tag{5.11}$$

$$\begin{aligned} k_{i2}(s, b, \alpha) = & (b + 1)^2(\alpha^2(8\sqrt{1 - b^2}(b + 1)s \\ & - 4(b - 1)(b + 1)^3 + 4s^2) + 4\alpha((b^2 + 2b - 1)s^2 \\ & + (b - 1)(b + 3)\sqrt{1 - b^2}(b + 1)s \\ & + 2(b - 1)(b + 1)^3) + (b - 1)(b + 3) \\ & \times (b(b + 2)s^2 + (b - 1)(b + 1)^3 \\ & - 2((b + 1)\sqrt{1 - b^2}s)) \end{aligned} \tag{5.12}$$

$$\begin{aligned} h_{41}(s, b, \xi) = & (b + 1)(4(b - 1)\xi(2(b - 1)(b^2 - 2b \\ & + 2)\sqrt{1 - b^2}(b + 1)^4s \\ & + (b^4 - 2b^2 + 2b - 1)s^4 \\ & + (b^4 - 2b^2 + 8b - 5)\sqrt{1 - b^2}(b + 1)s^3 \\ & + 2(b - 2)^2(b - 1)(b + 1)^3s^2 \\ & + 2(b - 1)^2b(b + 1)^6)f_1(s, b) - (b + 1)^2(b^2 - 1) \\ & \times (b^4 + 2b^3 + 2b(\sqrt{1 - b^2}s - 1) \\ & + 2\sqrt{1 - b^2}s - s^2 - 1)f_1(s, b)^2 - 4(b - 1)\xi^2 \\ & \times ((b - 1)^3(2b - 1)(b + 1)^8 + (b - 1)^2(7b^2 \\ & - 12b + 8)\sqrt{1 - b^2}(b + 1)^6s \\ & + 2(b - 1)b^2s^6 + 2(b - 1)(6b^3 - 16b^2) \end{aligned}$$

$$\begin{aligned}
 &+16b - 13)\sqrt{1 - b^2}(b + 1)^3s^3 \\
 &-2(b - 1)^2(2b^3 - 12b^2 + 14b - 11)(b + 1)^5s^2 \\
 &-(b - 1)(2b^4 - 12b^3 \\
 &+18b^2 - 18b + 13)(b + 1)^2s^4 + (4b^4 - 4b^3 \\
 &+b^2 - 4b + 2)\sqrt{1 - b^2}s^5) \tag{5.13}
 \end{aligned}$$

$$\begin{aligned}
 h_{61}(s, b, \alpha) &= (b + 1)(2(b - 1)(b + 3)\sqrt{1 - b^2}(b + 1)^2s^3 \\
 &+2(b - 1)^2(b + 3)\sqrt{1 - b^2}(b + 1)^5s \\
 &+4\alpha(b + 1)(2\sqrt{1 - b^2}(b + 1)s^3 \\
 &+2(b - 1)\sqrt{1 - b^2}(b + 1)^4s - (b - 1)^2(b + 1)^6 \\
 &+s^4) + 2(b^3 + 3b^2 + b - 1)s^4 \\
 &+2(b - 1)(b + 1)^6s^2 + 4(b - 1)^2(b + 1)^7) \tag{5.14}
 \end{aligned}$$

$$h_{62}(s, b, \alpha) = \alpha h_{621}(s, b) + \alpha^2 h_{622}(s, b) \tag{5.15}$$

$$\begin{aligned}
 h_{621}(s, b) &= -2(b + 1)(b - 1)(- (b - 1)(b(2b + 3) \\
 &-6)\sqrt{1 - b^2}(b + 1)s^3 \\
 &-(b + 1)^3(b(b + 3) - 21) + 13)s^2 \\
 &+(b - 1)(b(b + 3) + 12)\sqrt{1 - b^2}(b + 1)^4s \\
 &+4(b - 1)(b + 1)^6 + (-b(b + 2)(2b - 1) - 1)s^4) \tag{5.16}
 \end{aligned}$$

$$\begin{aligned}
 h_{622}(s, b) &= 4(b + 1)(-5((b - 1)(b + 1)^4\sqrt{1 - b^2})s \\
 &+(2b^3 - 2b^2 - 3b + 1)\sqrt{1 - b^2}s^3 \\
 &+(b - 1)bs^4 - (b - 2)^2(b - 1)(b + 1)^3s^2 \\
 &+(b - 1)^2(b + 2)(b + 1)^6) \tag{5.17}
 \end{aligned}$$

$$\begin{aligned}
 h_{71}(s, b) &= (b + 1)(b^2(-s) \\
 &+2b(\sqrt{1 - b^2} - s) + 2\sqrt{1 - b^2} + s) \tag{5.18}
 \end{aligned}$$

$$\begin{aligned}
 h_{72}(s, b, \alpha) &= 2\alpha(b + 1)(b\sqrt{1 - b^2} + \sqrt{1 - b^2} + s) \\
 &-h_{71}(s, b) \tag{5.19}
 \end{aligned}$$

$$\begin{aligned}
 h_{81}(s, b) &= (b + 1)^4(2\sqrt{1 - b^2}(b + 1)s \\
 &+(b - 1)(b + 1)^3 - s^2) \tag{5.20}
 \end{aligned}$$

$$\begin{aligned}
 h_{82}(s, b) &= (1 - b)(b + 1)^5(2\sqrt{1 - b^2}(b + 1)s \\
 &+(b - 1)(b + 1)^3 - s^2)f_1(s, b)^2 \tag{5.21}
 \end{aligned}$$

$$\begin{aligned}
 h_{83}(s, b) &= \sqrt{1 - b^2}s(-5(b + 2)b^2 + b \\
 &-2)f_1(s, b)f_2(s, b) - 5b(b(b^2 + b - 3) + 1) \\
 &\times (b + 1)^3f_2(s, b) + (b - 1)(b + 1)^4f_1(s, b) \\
 &-\sqrt{1 - b^2}(b(b + 2) - 1) \\
 &(b(b + 3) + 3)s^3f_2(s, b) - 2(b + 1)^2(b^2(b + 2) \\
 &-3)s^2f_2(s, b) \\
 &+(b - 1)(b + 1)^2((b - 1)(b + 1)^4f_1(s, b) \\
 &-(b(b + 10)f_1(s, b) + f_1(s, b) \\
 &+10(b - 1)b(b + 1)^3)f_2(s, b)) \tag{5.22}
 \end{aligned}$$

$$\begin{aligned}
 h_{84}(s, b) &= 4((b + 2)(b(2b(b + 2) + 9) + 6)f_2(s, b)^2s^4 \\
 &-2(4b + 3)\sqrt{1 - b^2}(b(b + 3)
 \end{aligned}$$

$$\begin{aligned}
 &+3)f_1(s, b)f_2(s, b)s^3 + 2(b - 1)(b + 1)^2 \\
 &\times (2(3b + 1)(b(b + 3) + 3)f_1(s, b) \\
 &+(b + 1)(8b(b(b + 2) + 3) - 3)f_2(s, b))f_2(s, b)s^2 \\
 &-\sqrt{1 - b^2}((b(11b + 18) \\
 &+6)f_1(s, b) + 10(b - 1)b(4b + 3)(b \\
 &+1)^3)f_1(s, b)f_2(s, b)s + (b - 1)(b + 1)^2 \\
 &\times (35(b - 1)b^2(b + 1)^4f_2(s, b)^2 \\
 &+20(b - 1)b(3b + 1)(b + 1)^3f_1(s, b)f_2(s, b) \\
 &+(2(b(13b + 10) + 1)f_2(s, b) \\
 &-(b - 1)(b + 1)^4)f_1(s, b)^2)) \tag{5.23}
 \end{aligned}$$

$$\begin{aligned}
 h_{85}(s, b, \alpha, \beta) &= f_2(s, b)((5b^5 + 10b^4 + b^2(s^2 - 10) \\
 &+b(3s^2 - 5) + 3s^2)f_2(s, b)\alpha \\
 &-(b + 1)f_1(s, b)f_2(s, b)\beta + (6b^4 + 8b^3 \\
 &-4b(\sqrt{1 - b^2}s + 2) - 3\sqrt{1 - b^2}s - 4b^2 \\
 &-2)f_1(s, b)\alpha) \tag{5.24}
 \end{aligned}$$

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