



Dynamics and cosmological evolution in Λ -varying cosmology

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Abstract We study the dynamical properties of a large body of varying vacuum cosmologies for which dark matter interacts with vacuum. In particular, performing the critical point analysis we investigate the existence and the stability of cosmological solutions which describe de-Sitter, radiation and matter dominated eras. We find several cases of varying vacuum models that admit stable critical points, hence they can be used in describing the cosmic history.

1 Introduction

The detailed analysis of the recent cosmological observations [1–6] indicates that in large scales our universe is spatially flat and it consists of $\sim 4\%$ baryonic matter, $\sim 26\%$ dark matter and $\sim 70\%$ of dark energy (DE). Dark energy is an “exotic” fluid source with a negative equation of state which attributes the cosmological acceleration. The origin and nature of the DE is a complete mystery still, though some of its properties are widely accepted, namely the fact that it has a negative pressure. Obviously this has been a starting point that has given birth to numerous alternative cosmological scenarios, which mainly generalize the nominal Einstein–Hilbert action of General Relativity either by the addition of extra fields [7–14], or a non-standard gravity theory that increases the number of degrees of freedom [15–21]. These are two different approaches in the dark energy problem which are still under debate in the scientific community.

The introduction of a cosmological constant term, is one of the simplest ways to modify the Einstein–Hilbert action. In the concordance Λ CDM model, the cosmological constant coexists with the component of cold dark matter (CDM)

and baryonic matter. Although this model does describe the observed universe quite accurately, it suffers from two basic problems, namely the expected value of the vacuum energy density and the coincidence problem [22–25]. An interesting approach for solving those problems is to allow Λ to vary with cosmic time, see [26–31] and references therein. These models [32–49] are based on a dynamical Λ term that evolves as a power series of the Hubble rate [50–54]. It was found that in the latter models the spacetime can be the physical result of a non-singular initial de Sitter vacuum stage, that also provides a graceful transition out of the inflation and into the radiation era. It has been found that these running vacuum scenarios accommodate the radiation and matter dominated era as well as the late time cosmic acceleration [27, 29, 55].

In this context, matter is allowed to interact with dark energy [31, 56–66]. Although, this interaction is not imposed by a fundamental principle, it has its roots in the particle physics theory, where any two matter fields can interact with each other. Such an interaction has been found to be a very efficient way to explain the cosmic coincidence problem and at the same time approach the mismatched value of the Hubble constant H_0 from the global Λ CDM based Planck and local measurements. Thus, in the present work we shall consider several interacting cosmological models of Λ varying cosmologies. The structure of the manuscript is as follows.

In Sect. 2, we briefly introduce the concept of the running Λ varying cosmologies and the interacting models that we shall study. Section 3, includes the main analysis of our work where we study the dynamical behaviour of our models and present the main results of this work. More specifically we study the critical points and their stability. Each critical point describes a specific exact solution for the field equation which correspond to the cosmic history. By studying the stability of the solutions of the critical points we are able to reconstruct the cosmic history and infer about the cosmological viability

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of these models. Finally, in Sect. 4, we summarize our results and we draw our conclusions.

2 Λ -varying cosmology

We consider a universe with a perfect fluid with energy density ρ , and pressure $p = w\rho$; such that the energy-momentum tensor is given by $T_{\mu\nu} = -p g_{\mu\nu} + (\rho + p)U_\mu U_\nu$. In addition we consider the Λ -varying cosmological term, $T_{\mu\nu}^{(\Lambda)} = \rho_\Lambda(t) g_{\mu\nu}$, $\rho_\Lambda = \Lambda(t)/(8\pi G)$ where the effective energy momentum tensor is written as $\tilde{T}_{\mu\nu} \equiv T_{\mu\nu} + g_{\mu\nu}\rho_\Lambda$.

In General Relativity ρ_Λ is considered to be constant; however in varying vacuum cosmology, Λ is considered to be a function of the cosmic time, or of any collection of homogeneous and isotropic dynamical variables, i.e. $\Lambda = \Lambda(\chi(t))$.

The Einstein field equations are written as,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G \tilde{T}_{\mu\nu}. \quad (1)$$

where on the lhs part is the Einstein tensor and on the rhs the effective energy momentum tensor. For spatially flat FLRW spacetime with line element

$$ds^2 = -dt^2 + a^2(t) \left(dx^2 + dy^2 + dz^2\right), \quad (2)$$

the Friedmann equations are

$$3H^2(t) = \Lambda(t) + \rho(t), \quad (3)$$

$$-2\dot{H}(t) - 3H^2(t) = -\Lambda(t) + p(t), \quad (4)$$

where we have set $8\pi G \equiv c \equiv 1$ and $H(t) = \frac{\dot{a}(t)}{a(t)}$ is the Hubble function.

In this work we shall consider a universe with radiation, dark and baryonic pressureless matter as well as the varying Λ term, hence the Friedmann equations (3), (4) take the following form

$$3H^2 = (\rho_m + \rho_r + \rho_\Lambda), \quad (5)$$

$$2\dot{H} + 3H^2 = -\left(\frac{1}{3}\rho_r - \rho_\Lambda\right), \quad (6)$$

where we have used $\rho_m = \rho_{DM} + \rho_b$. Assuming that baryons and radiation are self-conserved, namely the corresponding densities evolve in the nominal way, $\rho_r = \rho_{r0}a^{-4}$ for the radiation density and $\rho_b = \rho_{b0}a^{-3}$ for the baryon density. In this way we only consider interaction between the Dark Matter and the varying vacuum sectors. Thus the Bianchi identity gives:

$$\dot{\rho}_{DM} + 3H\rho_{DM} = -\dot{\rho}_\Lambda = Q, \quad (7)$$

where Q is the interaction term between the Dark Matter and the varying vacuum component, which we will study in this work in order to define their dynamical behavior. Here we investigate the generic evolution of the solution which

is described by the field Eqs. (5), (6) and (7) for specific functional forms of the interaction term Q . Specifically, we shall consider five different cases:

The first case that we study is the *running vacuum* model (RVM) (see [26, 27, 29, 31]). Theoretical motivations for this model arise from quantum field dynamics (QFT) in curved space-time, by associating Renormalization Group's running scale μ (in our context the dynamical parameter $\chi(t)$) with a characteristic energy threshold for cosmological scales. Thus, $\chi(t)$ is chosen to be the Hubble rate H , for reviews see [67–69].

Returning to our definition of $\Lambda(t) = \Lambda(\chi(t))$, we may express the running vacuum as a power series of the Hubble function:

$$\Lambda(t) = \Lambda(H(t)) = c_0 + \sum_k \alpha_k H^k(t).$$

It has been shown in previous works, that only even powers of H can be theoretically motivated, as the odd powers of the Hubble function are incompatible with the general covariance of the effective action [70, 71]. For that reason we shall exclude odd powers of H from the series. Furthermore, high powers of H can be very useful when treating the evolution of the early universe, but they are negligible in the matter and dark energy eras respectively [29]. In this study we are restricting our analysis to the simplified model [50, 72–75]:

$$\Lambda(H) = c_0 + nH^2, \quad (8)$$

where n is a dimensionless parameter, linked to the strength of the interaction. For consistency, the condition $\rho_\Lambda(H_0) = \rho_{\Lambda 0} = \frac{\Omega_\Lambda}{\rho_{crit}}$ fixes the value of c_0 at $c_0 = H_0^2(\Omega_\Lambda - n)$ [76]. In the case of the RVM, the interaction term is taken by solving the continuity equation (7) for the specific form of $\rho_\Lambda = \frac{3}{8\pi G}\Lambda(H) = \rho_{\Lambda 0} + \frac{3}{8\pi G}nH$, and is given by:

$$Q_A = nH(3\rho_{DM} + 3\rho_b + 4\rho_\Lambda). \quad (9)$$

In the second vacuum scenario used in this study the corresponding interaction term is taken ad hoc to be proportional to the density of dark matter [77]. In particular, the interaction term is given by $Q_B = 3nH\rho_{DM}$ where, as before, the dimensionless parameter n is an indicator of the interaction strength. Then we examine a third vacuum scenario which is presented in [31, 78] where the interaction term is written as $Q_C = 3nH\rho_\Lambda$. Motivated by interesting results on the above models, we also considered two additional scenarios.

The fourth model of our study is $Q_D = \frac{3n}{H}\rho_b\rho_{DM}$ where the interaction is dependent also on the baryonic density as well as dark matter, while for the last model of our study we assume $Q_E = 3nH\rho_{tot}$, in which the total density affects the interaction term.

To this end, from the observational viewpoint the values of n are found to be quite small, pointing a small (but not zero) deviation from the usual Λ CDM model. Indeed, the

concordance model is recovered in any case when n is set to 0. For the first three vacuum models, n is treated as a free parameter along with other cosmological parameters and it is found to be of the order $\sim 10^{-3}$ or less, see for example [76, 79], where these authors found $n = 0.00013 \pm 0.00018$, $n = 0.00014 \pm 0.00103$). Interactions Q_A , Q_B , Q_C and Q_E can be seen as linear interaction terms while Q_D is a nonlinear function.

3 Dynamical analysis

In this section, we study the cosmological evolution of the aforementioned cosmological scenarios by using methods of dynamical systems [82, 83]. Specifically we study the critical points of the field equations in order to identify the cosmological eras that are provided by the theory. The stability of those cosmological eras are determined by calculating the eigenvalues of the linearized system at the critical point. The way we approach this analysis is described as follows.

We define proper dimensionless variables to rewrite the field equations so that our analysis can be universal. Then we proceed by producing the first-order ordinary differential equations from our dimensionless variables. The critical points of the system are those sets of variables for which every differential equation of our system is equal to zero. These sets of variables represent different epochs of the cosmos that we further study in order to consider them as potential candidates that actually describe the observed universe. The eigenvalues of those points are important tools towards characterizing the stability of the critical points [84].

If a critical point is stable/attractor then the corresponding eigenvalues will need to have negative real parts. Thus, the eigenvalues can be used in order to understand the behavior of the dynamical system around the critical point [85]. Our approach is as follows. We consider a dynamical system of any number of equations:

$$\dot{x}^A = f^A(x^B),$$

then a critical point of the system, namely $P = P(x^B)$ satisfies $f^A(P) = 0$. The linearized system around P is written as

$$\delta \dot{x}^A = J_B^A \delta x^B, \quad J_B^A = \frac{\partial f^A(P)}{\partial x^B}.$$

where J_B^A is the respective Jacobian matrix. We calculate the eigenvalues and eigenvectors and write the general solution on the respective points as their expression. Since the linearized solutions are expressed in terms of the eigenvalues λ_i as functions of $e^{\lambda_i t}$, when all those terms have negative real parts the solution on the critical point is apparently stable.

3.1 Dimensional system

In order to study the generic evolution of the cosmological models of our consideration we prefer to work in the H -normalization where define the dimensionless variables [82, 83]

$$\Omega_{DM} = \frac{\rho_{DM}}{3H^2}, \quad \Omega_r = \frac{\rho_r}{3H^2}, \quad \Omega_b = \frac{\rho_b}{3H^2}, \quad \Omega_\Lambda = \frac{\rho_\Lambda}{3H^2}.$$

Consequently, the constraint equation (5) becomes

$$\Omega_{DM} + \Omega_r + \Omega_b + \Omega_\Lambda = 1, \tag{10}$$

while the rest of the field equations can be written as the following four-dimensional first-order ordinary differential equations

$$\frac{d\Omega_{DM}}{d \ln a} = -\Omega_{DM} \left(3 + 2 \frac{\dot{H}}{H^2} \right) - \frac{Q}{3H^3}, \tag{11}$$

$$\frac{d\Omega_r}{d \ln a} = -2\Omega_r \left(2 + \frac{\dot{H}}{H^2} \right), \tag{12}$$

$$\frac{d\Omega_b}{d \ln a} = -2\Omega_b \left(\frac{3}{2} + \frac{\dot{H}}{H^2} \right), \tag{13}$$

$$\frac{d\Omega_\Lambda}{d \ln a} = -2\Omega_\Lambda \frac{\dot{H}}{H^2} - \frac{Q}{3H^3}, \tag{14}$$

in which

$$\frac{\dot{H}}{H^2} = \frac{1}{2}(3\Omega_\Lambda - \Omega_r - 3). \tag{15}$$

and as new independent variable we consider the number of e-fold $N = \ln a$.

By using the constraint equation (10) we are able to reduce the latter dynamical system into the following three-dimensional system

$$\frac{d\Omega_r}{d \ln a} = -\Omega_r(-1 - 3\Omega_\Lambda + \Omega_r), \tag{16}$$

$$\frac{d\Omega_b}{d \ln a} = -\Omega_b(3\Omega_\Lambda - \Omega_r), \tag{17}$$

$$\frac{d\Omega_\Lambda}{d \ln a} = -\Omega_\Lambda(3\Omega_\Lambda - \Omega_r - 3) - \frac{Q}{3H^3}. \tag{18}$$

The latter equation depends on the functional form of Q , which is necessary to be defined in order to continue with our analysis.

3.2 Case A: Q_A

For the first model of our consideration in which $Q_A = nH(3\rho_{DM} + 3\rho_b + 4\rho_r)$, Eq. (18) becomes

$$\frac{d\Omega_\Lambda}{d \ln a} = -\Omega_\Lambda(3\Omega_\Lambda - \Omega_r - 3) - n(3 - 3\Omega_\Lambda + \Omega_r). \tag{19}$$

Hence, by assuming the rhs of Eqs. (16), (17), (19) to be zero we determine the critical points of the dynamical system.

Table 1 Critical points and physical quantities for Case A

Point	$\{\Omega_{DM}, \Omega_b, \Omega_\Lambda, \Omega_r\}$	Existence	w	Acceleration	Eigenvalues	Stability
A_1	$\{0, 0, 1, 0\}$	Always	-1	Yes	$\{-4, -3, -3(1 - n)\}$	Stable for $n < 1$
B_1	$\{-4n, 0, n, 1 + 3n\}$	$-\frac{1}{4} \leq n \leq 0$	$\frac{1}{3}$	No	$\{4, 1, 1 + 3n\}$	Unstable
C_1	$\{1 - n, 0, n, 0\}$	$0 \leq n \leq 1$	$-n$	Yes for $n > \frac{1}{3}$	$\{-1 - 3n, -3(n - 1), -3n\}$	Unstable

Every point P has coordinates $P = \{\Omega_{DM}, \Omega_b, \Omega_\Lambda, \Omega_r\}$ and describes a specific cosmological solution. For every point we determine the physical cosmological variables as also the equation of the state parameter. In order to determine the stability of each critical point the eigenvalues of the linearized system around the critical point P are derived. Therefore, the dynamical system (16), (17), (19) admits the three critical points with coordinates $A_1 = \{0, 0, 1, 0\}$, $B_1 = \{-4n, 0, n, 1 + 3n\}$ and $C_1 = \{1 - n, 0, n, 0\}$

Point A_1 describes a de Sitter universe with equation of state parameter $w = -1$, where only the cosmological constant term contributes in the evolution of the universe. The eigenvalues of the linearized system are found to be $\{-4, -3, -3(1 - n)\}$ from where we infer that point A_1 is an attractor for $n < 1$. This is in agreement with the expected values of n and thus this point is of physical interest.

Point B_1 is physical accepted when $-\frac{1}{4} \leq n \leq 0$. In this area these points correspond to a universe where radiation, dark matter and the cosmological constant coexist and dynamically it behaves like a radiation dominated universe ($w = \frac{1}{3}$) which is the case for $n \rightarrow 0$. The eigenvalues of the linearized system at the point B_1 are derived to be $\{4, 1, 1 + 3n\}$ from where we conclude that the point is a source (unstable point).

Point C_1 describes a universe where only the cosmological constant and the dark matter fluids contribute to the total cosmic fluid. Indeed it describes the Λ -CDM universe where now the parameter n is the energy density of the cosmological constant, i.e. $\Omega_\Lambda = n$. The point is physical accepted when $0 \leq n \leq 1$, while for $n = 1$ it is reduced to point A_1 . The eigenvalues of the linearized system are determined to be $\{-1 - 3n, 3(1 - n), -3n\}$ from where we infer that the solution of the critical point is always unstable. The critical point analysis of the above system yields three critical points that are shown in Table 1. In Figs. 1 and 2 the phase space diagram of the dynamical system Q_A is presented for $n < 1$ ($n = -0, 1$) from where we can see that the unique attractor is the de Sitter point A_1 .

3.3 Case B: $Q = 3nH\rho_{DM}$

In this case our system of study are Eqs. (16), (17) and

$$\frac{d\Omega_\Lambda}{d \ln a} = -\Omega_\Lambda(3\Omega_\Lambda - \Omega_r - 3) - 3n(1 - \Omega_b - \Omega_\Lambda - \Omega_r) \quad (20)$$

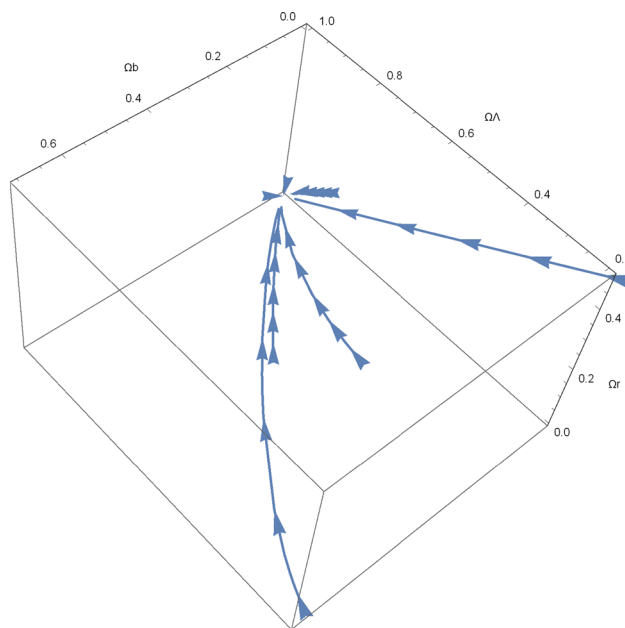


Fig. 1 Phase space diagram for the dynamical system (16), (17), (19). We consider (a) $\Omega_b = 0.2996, \Omega_r = 0.0004, \Omega_\Lambda = 0.7$, (b) $\Omega_b = 0, \Omega_r = 0.1, \Omega_\Lambda = 0.9$, (c) $\Omega_b = 0.3, \Omega_r = 0.2, \Omega_\Lambda = 0.5$, (d) $\Omega_b = 0, \Omega_r = 0.5, \Omega_\Lambda = 0.2$, (e) $\Omega_b = 0.7, \Omega_r = 0.1, \Omega_\Lambda = 0.2$, for $n < 1$. The unique attractor is the de Sitter point A_1

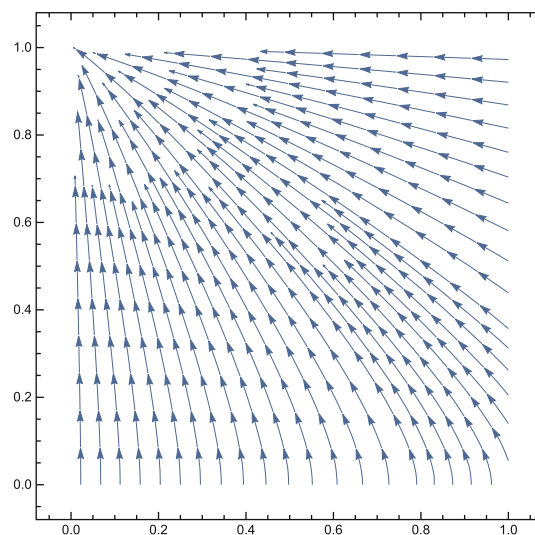


Fig. 2 Phase space diagram for the dynamical system (16), (17), (19) in the space of variables Ω_b, Ω_Λ for $n < 1$ and $\Omega_r = 10^{-4}$. The unique attractor is the de Sitter point A_1

Table 2 Critical points and physical quantities for Case B

Point	$\{\Omega_{DM}, \Omega_b, \Omega_\Lambda, \Omega_r\}$	Existence	w_{tot}	Acceleration	Eigenvalues	Stability
A_2	$\{0, 0, 1, 0\}$	Always	-1	Yes	$\{-4, -3, -3 + n\}$	Stable for $n < 3$
B_2	$\{\frac{3-n}{3}, 0, \frac{n}{3}, 0\}$	$0 \leq n \leq 3$	$-\frac{n}{3}$	Yes for $n > 1$	$\{3 - n, -n, -n - 1\}$	Stable for $n > 3$
C_2	$\{0, 1, 0, 0\}$	Always	0	No	$\{3, -1, n\}$	Unstable
D_2	$\{0, 0, 0, 1\}$	Always	$\frac{1}{3}$	No	$\{4, 1, 1 + n\}$	Unstable

thus the dynamical system (16), (17), (20) admits four critical points with coordinates $A_2 = \{0, 0, 1, 0\}$, $B_2 = \{\frac{3-n}{3}, 0, \frac{n}{3}, 0\}$ and $C_2 = \{0, 1, 0, 0\}$, $D_2 = \{0, 0, 0, 1\}$

Point A_1 describes a de Sitter universe with an equation of state parameter $w = -1$, where only the cosmological constant term contributes in the evolution of the universe. The eigenvalues of the linearized system are found to be $\{-4, -3, -3 + n\}$ and thus we can conclude that point A_2 is an attractor for $n < 3$. Taking into account the literature values of n [76], this is a valid point.

Point B_2 provides a Λ CDM scenario where the components of the fluid are $\Omega_{DM} = 1 - \frac{n}{3}$ and $\Omega_\Lambda = \frac{n}{3}$. Apparently this family of points exists only for $0 \leq n \leq 3$, but it can be an accelerating point only for $n > 1$. For $n = 3$ this point reduces to a deSitter one. In terms of stability, the eigenvalues of the linearized system are $\{3 - n, -n, -n - 1\}$, hence, this point is an attractor, i.e. stable for $n > 3$, while it is a source for $n < 3$.

Point C_2 describes a baryon dominated universe, while the solution at this point is always unstable since there is always a positive eigenvalue, namely the corresponding eigenvalues are $\{3, -1, n\}$.

Point D_2 describes a radiation dominated universe that does not accelerate, the corresponding eigenvalues are $\{4, 1, 1 + n\}$, hence the current point is a source.

The critical point analysis of the above system yields four critical points that are shown in Table 2.

In Figs. 3 and 4 the phase space diagram of the dynamical system Q_B is presented for $n < 1$ ($n = -0, 1$) from where we can see that the unique attractor is the de Sitter point A_2 .

3.4 Case C: $Q = 3nH\rho_\Lambda$

For the third model of our study, the system of equations is (16), (17) and

$$\frac{d\Omega_\Lambda}{d \ln a} = -\Omega_\Lambda (3\Omega_\Lambda - \Omega_r - 3 + n), \tag{21}$$

The dynamical system (16), (17), (21) admits four critical points, namely $A_3 = \{1 - \Omega_b, \Omega_b, 0, 0\}$, $B_3 = \{\frac{n}{3}, 0, \frac{3-n}{3}, 0\}$ and $C_3 = \{1, 0, 0, 0\}$, $D_3 = \{0, 0, 0, 1\}$.

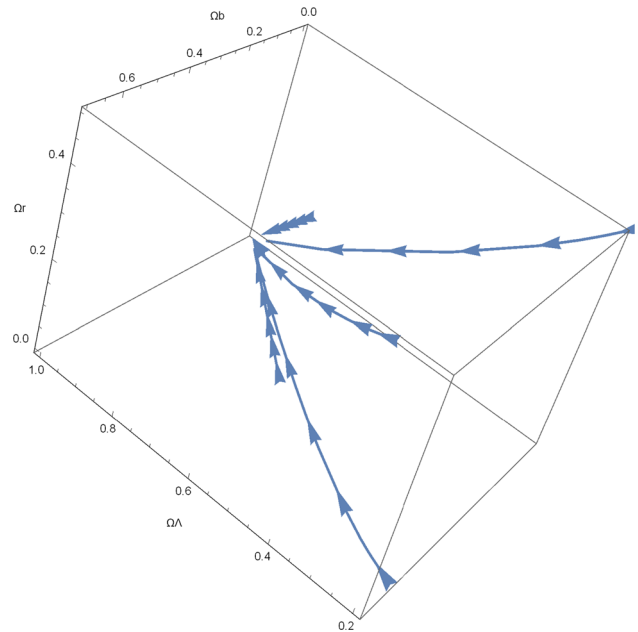


Fig. 3 Phase space diagram for the dynamical system (16), (17), (20). We consider (a) $\Omega_b = 0.2996, \Omega_r = 0.0004, \Omega_\Lambda = 0.7$, (b) $\Omega_b = 0, \Omega_r = 0.1, \Omega_\Lambda = 0.9$, (c) $\Omega_b = 0.3, \Omega_r = 0.2, \Omega_\Lambda = 0.5$, (d) $\Omega_b = 0, \Omega_r = 0.5, \Omega_\Lambda = 0.2$, (e) $\Omega_b = 0.7, \Omega_r = 0.1, \Omega_\Lambda = 0.2$, for $n < 1$. The unique attractor is the de Sitter point A_2

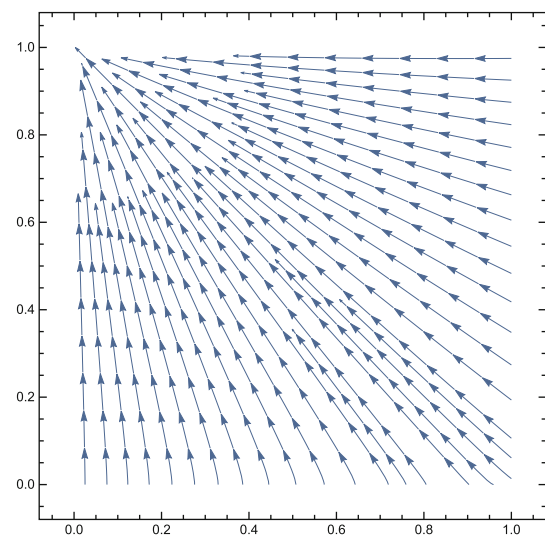


Fig. 4 Phase space diagram for the dynamical system (16), (17), (20) in the space of variables Ω_b, Ω_Λ for $n < 1$ and $\Omega_r = 10^{-4}$. The unique attractor is the de Sitter point A_2

Table 3 Critical points and physical quantities for Case C

Point	$\{\Omega_{DM}, \Omega_b, \Omega_\Lambda, \Omega_r\}$	Existence	w	Acceleration	Eigenvalues	Stability
A_3	$\{1 - \Omega_b, \Omega_b, 0, 0\}$	Always	0	No	$\{-1, 0, 3 - n\}$	Unstable
B_3	$\{\frac{n}{3}, 0, \frac{3-n}{3}, 0\}$	$0 \leq n \leq 3$	$-1 + \frac{n}{3}$	Yes for $0 \leq n < 2$	$\{n - 4, n - 3, n - 3\}$	Stable for $n < 3$
C_3	$\{1, 0, 0, 0\}$	Always	0	No	$\{-1, 0, 3 - n\}$	unstable
D_3	$\{0, 0, 0, 1\}$	Always	$\frac{1}{3}$	No	$\{1, 1, 4 - n\}$	unstable

Point A_3 describes a matter (baryons plus dark matter) dominated universe, hence it that does not accelerate ($w = 0$). The eigenvalues of the linearized system are $\{-1, 0, 3 - n\}$. For $n < 3$ the solution of A_3 is always unstable.

Point B_3 provides a Λ CDM scenario where the components of the fluid are $\Omega_{DM} = \frac{n}{3}$ and $\Omega_\Lambda = \frac{3-n}{3}$. Apparently this family of points exists only for $0 \leq n \leq 3$ and it provides cosmic acceleration ($w = \frac{n}{3} - 1$) only for $n < 2$. The eigenvalues of the critical point are found to be $\{n - 4, n - 3, n - 3\}$, hence for $n < 3$ the point is always unstable. For $n \rightarrow 0$ the solution at the point describes a stable de Sitter universe ($w = -1$) where only the cosmological constant term contributes in the evolution of the universe. Thus this is an interesting point, since cosmological data point that $n \sim 10^{-3}$. [76,79].

Point C_3 describes a dark matter dominated universe that apparently does not accelerate. The eigenvalues of the linearized system are calculated to be $\{-1, 0, 3 - n\}$. The point is a source (unstable).

Point D_3 describes a radiation dominated universe that does not accelerate. The eigenvalues of the linearized system are $\{1, 1, 4 - n\}$, from where we can infer that the solution at point D_3 is unstable.

The critical point analysis of the above system yields four critical points that are shown in Table 3.

In Figs. 5 and 6 the phase space diagram of the dynamical system Q_C is presented for $n < 1$ ($n = -0, 1$) from where we can see that the unique attractor is the point B_3 .

3.5 Case D: $Q = \frac{3n}{H} \rho_b \rho_{DM}$

For the fourth model of our consideration the dynamical system of our study consisted by the Eqs. (16), (17) and

$$\frac{d\Omega_\Lambda}{d \ln a} = -\Omega_\Lambda (3\Omega_\Lambda - \Omega_r - 3) - 3n\Omega_b(1 - \Omega_b - \Omega_\Lambda - \Omega_r), \tag{22}$$

The dynamical system (16), (17), (22) admits four critical points with coordinates $A_4 = \{0, 0, 1, 0\}$, $B_4 = \{1, 0, 0, 0\}$ and $C_4 = \{0, 1, 0, 0\}$, $D_4 = \{0, 0, 0, 1\}$.

Point A_4 is a viable de Sitter point where only the cosmological constant term contributes in the evolution of the universe. This point always exists and it is always stable, since

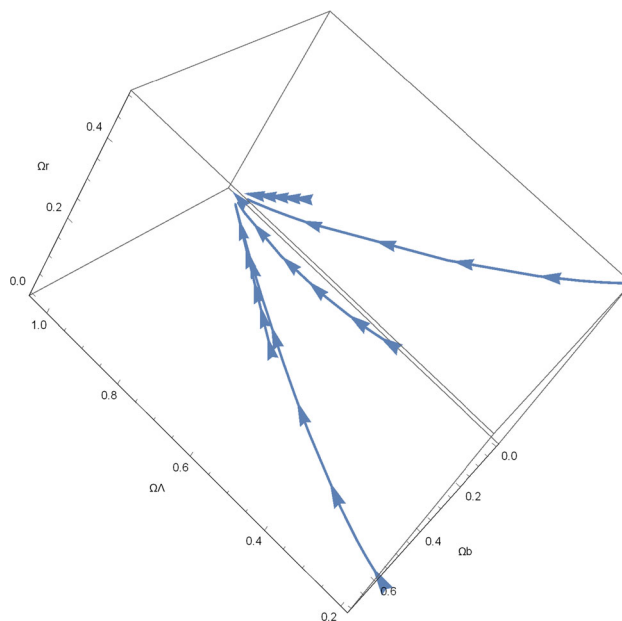


Fig. 5 Phase space diagram for the dynamical system (16), (17), (21). We consider (a) $\Omega_b = 0.2996, \Omega_r = 0.0004, \Omega_\Lambda = 0.7$, (b) $\Omega_b = 0, \Omega_r = 0.1, \Omega_\Lambda = 0.9$, (c) $\Omega_b = 0.3, \Omega_r = 0.2, \Omega_\Lambda = 0.5$, (d) $\Omega_b = 0, \Omega_r = 0.5, \Omega_\Lambda = 0.2$ (e) $\Omega_b = 0.7, \Omega_r = 0.1, \Omega_\Lambda = 0.2$, for $n < 1$. The unique attractor is the point B_3

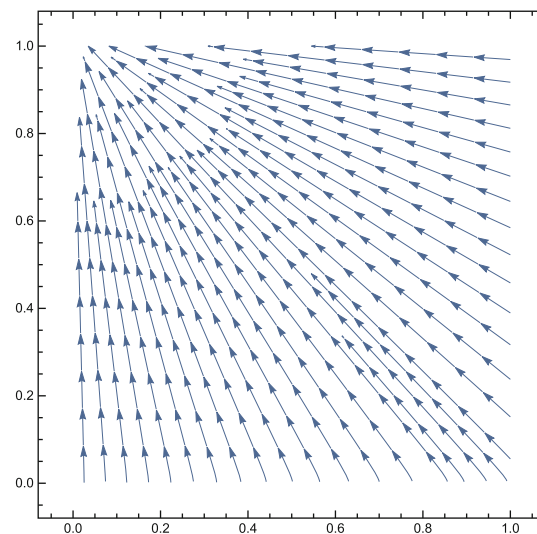


Fig. 6 Phase space diagram for the dynamical system (16), (17), (21) in the space of variables Ω_b, Ω_Λ for $n < 1$ and $\Omega_r = 10^{-4}$. The attractor is the point B_3

Table 4 Critical points and physical quantities for Case D

Point	$\{\Omega_{DM}, \Omega_b, \Omega_\Lambda, \Omega_r\}$	Existence	w	Acceleration	Eigenvalues	Stability
A_4	$\{0, 0, 1, 0\}$	Always	-1	Yes	$\{-4, -3, -3\}$	Stable
B_4	$\{1, 0, 0, 0\}$	Always	0	No	$\{3, -1, 0\}$	Unstable
C_4	$\{0, 1, 0, 0\}$	Always	0	No	$\{3, -1, 3n\}$	Unstable
D_4	$\{0, 0, 0, 1\}$	Always	$\frac{1}{3}$	No	$\{4, 1, 1\}$	Unstable

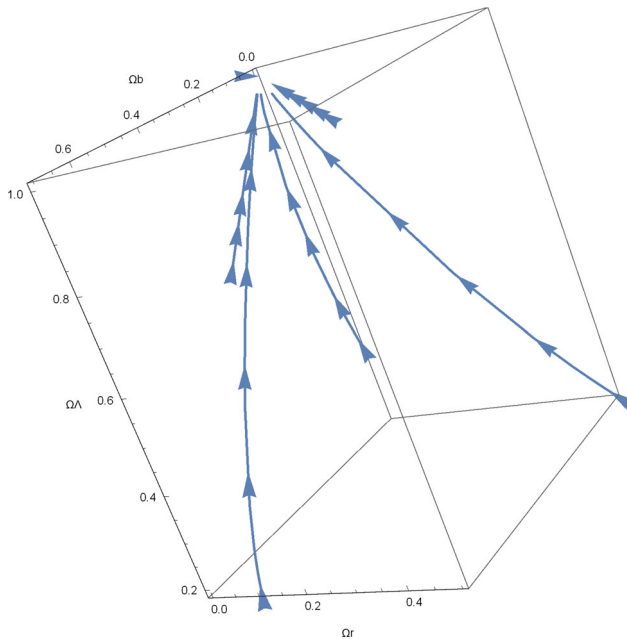


Fig. 7 Phase space diagram for the dynamical system (16), (17), (22). We consider (a) $\Omega_b = 0.2996, \Omega_r = 0.0004, \Omega_\Lambda = 0.7$, (b) $\Omega_b = 0, \Omega_r = 0.1, \Omega_\Lambda = 0.9$ (c) $\Omega_b = 0.3, \Omega_r = 0.2, \Omega_\Lambda = 0.5$, (d) $\Omega_b = 0, \Omega_r = 0.5, \Omega_\Lambda = 0.2$, (e) $\Omega_b = 0.7, \Omega_r = 0.1, \Omega_\Lambda = 0.2$, for $n < 1$. The unique attractor is the de Sitter point A_4

the eigenvalues of the linearized system at A_4 are always negative, i.e. $\{-4, -3, -3\}$.

Point B_4 describes a dark matter dominated universe that does not accelerate. The eigenvalues are derived $\{3, -1, 0\}$ from where we find that this point is a source.

Point C_4 describes a baryon matter only dominated universe that apparently does not accelerate. The point is a source, because at least one of the eigenvalues is always positive, the eigenvalues are $\{3, -1, 3n\}$.

Point D_4 describes a radiation dominated universe that does not accelerate. The three eigenvalues are $\{4, 1, 1\}$, that is, point D_4 is a source and the solution described at point D_4 is unstable.

The critical point analysis of the above system yields four critical points that are shown in Table 4.

In Figs. 7 and 8 the phase space diagram of the dynamical system Q_D is presented for $n < 1$ ($n = -0, 1$) from where we can see that the unique attractor is the de Sitter point A_4 .

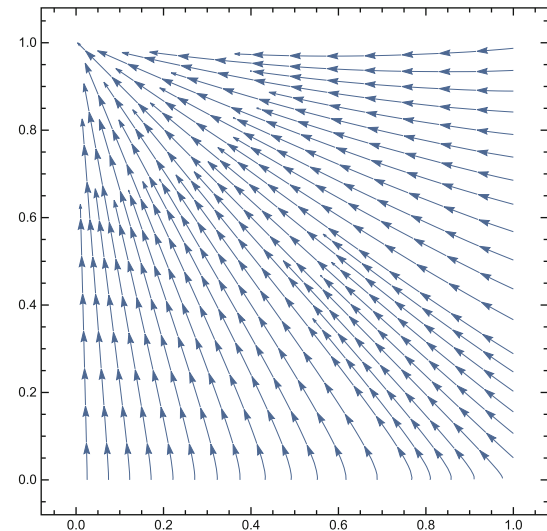


Fig. 8 Phase space diagram for the dynamical system (16), (17), (22) in the space of variables Ω_b, Ω_Λ for $n < 1$ and $\Omega_r = 10^{-4}$. The unique attractor is the de Sitter point A_4

3.6 Case E: $Q = 3nH\rho_{tot}$

For $Q = 3nH\rho_{tot}$ the dynamical system of our study consists by the Eqs. (16), (17) and

$$\frac{d\Omega_\Lambda}{d \ln a} = \frac{1}{H} \left(\frac{\dot{\rho}_\Lambda}{3H^2} - \rho_\Lambda \frac{2\dot{H}}{3H^3} \right) = -\Omega_\Lambda(3\Omega_\Lambda - \Omega_r - 3) - 3n, \tag{23}$$

The latter dynamical system admits three critical points with coordinates $A_5 = \{\frac{1}{2}(1 + \sqrt{1-4n}), 0, \frac{1}{2}(1 - \sqrt{1-4n}), 0\}$, $B_5 = \{\frac{1}{2}(1 - \sqrt{1-4n}), 0, \frac{1}{2}(1 + \sqrt{1-4n}), 0\}$ and $C_5 = \{-3n, 0, \frac{3n}{4}, \frac{9n+4}{4}\}$.

Points A_5 and B_5 describe both a Λ -CDM scenario where the dark matter and the cosmological constant contribute in the evolution of the universe. Point A_5 exists for $0 \leq n \leq \frac{1}{4}$ and can provide an accelerating universe for $\frac{2}{9} \leq n \leq \frac{1}{4}$. Moreover, point B_5 , exists for $0 \leq n \leq \frac{1}{4}$ and for the same range of values can also provide an accelerating universe. As far as the stability of these two points is concerned, the eigenvalues of the linearized system at point A_5 are $\{-\frac{3}{2}(1 - \sqrt{1-4n}), -\frac{1}{2}(5 - 3\sqrt{1-4n}), 3\sqrt{1-4n}\}_{A_5}$, while at point B_5 are $\{-\frac{1}{2}(5 + 3\sqrt{1-4n}), -\frac{3}{2}(1 + \sqrt{1-4n}), -3\sqrt{1-4n}\}$. Therefore, the

Table 5 Critical points and physical quantities for Case E

Point	$\{\Omega_{DM}, \Omega_b, \Omega_\Lambda, \Omega_r\}$	Existence	w	Acceleration
A_5	$\{\frac{1}{2}(1 + \sqrt{1 - 4n}), 0, \frac{1}{2}(1 - \sqrt{1 - 4n}), 0\}$	$ 0 \leq n \leq \frac{1}{4}$	$-\frac{1}{2}(1 - \sqrt{1 - 4n})$	$\frac{2}{9} \leq n \leq \frac{1}{4}$
B_5	$\{\frac{1}{2}(1 - \sqrt{1 - 4n}), 0, \frac{1}{2}(1 + \sqrt{1 - 4n}), 0\}$	$ 0 \leq n \leq \frac{1}{4}$	$-\frac{1}{2}(1 + \sqrt{1 - 4n})$	$n \leq \frac{1}{4}$
C_5	$\{-3n, 0, \frac{3n}{4}, \frac{9n+4}{4}\}$	$n = 0$	$\frac{1}{3}$	No

Table 6 Critical points and stability for Case E

Point	Eigenvalues	Stability
A_5	$\{-\frac{3}{2}(1 - \sqrt{1 - 4n}), -\frac{1}{2}(5 - 3\sqrt{1 - 4n}), 3\sqrt{1 - 4n}\}$	Unstable
B_5	$\{-\frac{1}{2}(5 + 3\sqrt{1 - 4n}), -\frac{3}{2}(1 + \sqrt{1 - 4n}), -3\sqrt{1 - 4n}\}$	Yes for $n < \frac{1}{4}$
C_5	$\{1, \frac{1}{2}(5 - 3\sqrt{1 - 4n}), \frac{1}{2}(5 + 3\sqrt{1 - 4n})\}$	Unstable

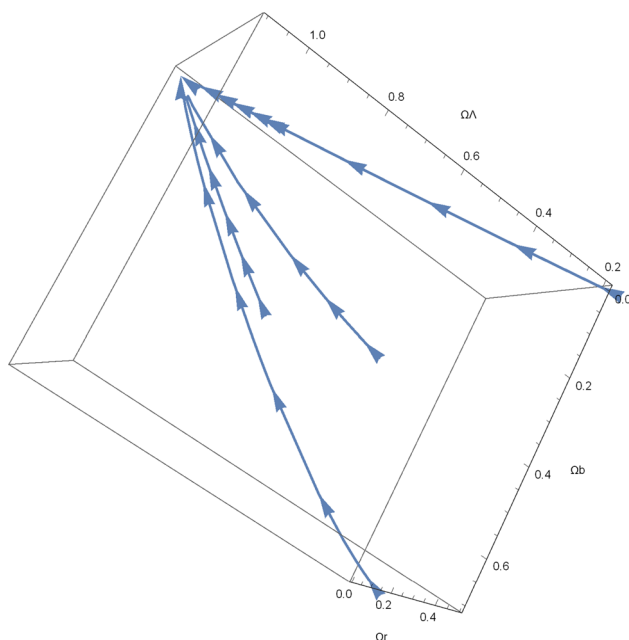


Fig. 9 Phase space diagram for the dynamical system (16), (17), (23). We consider (a) $\Omega_b = 0.2996, \Omega_r = 0.0004, \Omega_\Lambda = 0.7$, (b) $\Omega_b = 0, \Omega_r = 0.1, \Omega_\Lambda = 0.9$, (c) $\Omega_b = 0.3, \Omega_r = 0.2, \Omega_\Lambda = 0.5$, (d) $\Omega_b = 0, \Omega_r = 0.5, \Omega_\Lambda = 0.2$, (e) $\Omega_b = 0.7, \Omega_r = 0.1, \Omega_\Lambda = 0.2$, for $n < 1$. The unique attractor is point B_5

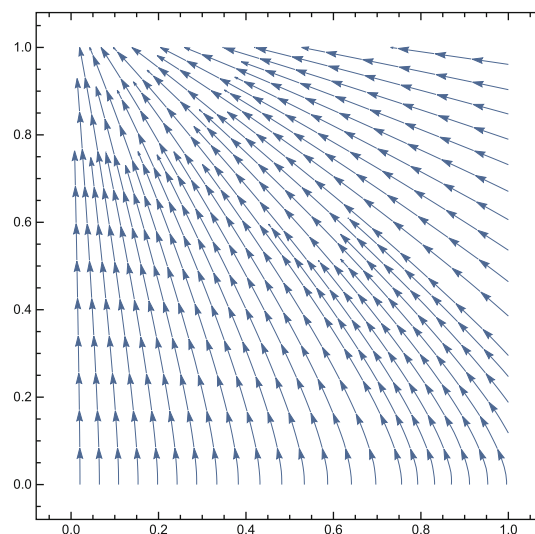


Fig. 10 Phase space diagram for the dynamical system (16), (17), (23) in the space of variables Ω_b, Ω_Λ for $n < 1$ and $\Omega_r = 10^{-4}$. The unique attractor is the point B_5

solution at point A_5 is always unstable while point B_5 is an attractor. Furthermore excluding the value $n = \frac{1}{4}$, in the same range of values it is also a stable point.

Point C_5 only exists for $n = 0$, in which case it describes a radiation dominated universe ($\Omega_r = 1$) that does not accelerate. The eigenvalues are $\{1, \frac{1}{2}(5 - 3\sqrt{1 - 4n}), \frac{1}{2}(5 + 3\sqrt{1 - 4n})\}$ which mean that the point is a source.

The above results are summarized in Tables 5 and 6.

In Figs. 9 and 10 the phase space diagram of the dynamical system Q_E is presented for $n < 1$ ($n = -0, 1$) from where we can see that the unique attractor is the point B_5 .

4 Conclusions

The current era phenomenology of the Λ -varying cosmological models has been discussed by one of the current authors and collaborators, in a number of very detailed papers. It has been found that the $\Lambda(H)$ models are not only highly consistent with the plethora of the astrophysical and cosmological data, but can also help alleviate some of the current-era tensions in data, including the σ_8 and the current value of the Hubble-parameter H_0 tensions [80, 81]. However, a complete dynamical analysis is missing from the literature. In this article we studied the dynamical behavior of several varying vacuum models. In particular, we investigated various models for which baryons and radiation are self-conserved, while interaction between the dark matter and the varying vacuum takes different forms. Bellow we summarize the main points of our analysis.

In the first case we assumed the following interaction term $Q_A = nH(3\rho_{DM} + 3\rho_b + 4\rho_r)$ from where it follows a viable de Sitter scenario (point A_1 as a future attractor for $n < 1$). In this scenario n can also have negative values and thus matter is allowed to decay into vacuum.

For our second model, namely $Q_B = 3nH\rho_{DM}$, we found two possible interesting scenarios that are described by points A_2, B_2 . Point A_2 describes again a de Sitter universe that is an attractor for $n < 3$, and point B_2 describes a Λ CDM universe that is always unstable (in the area of its existence $0 \leq n \leq 3$). This is an interesting result because this solution recovers Λ CDM with future attractor an expanding de Sitter universe.

In the third vacuum model scenario we considered $Q_C = 3nH\rho_\Lambda$, and found a unique attractor which is described by the critical point B_3 with $0 \leq n < 2$, where the exact solution of this point describes a stable and accelerating Λ CDM universe. For the fourth model $Q_D = 3n\rho_b\rho_{DM}/H$ a viable de Sitter solution is described by point A_4 which is found to be always stable. Finally, for $Q_E = 3nH\rho_{tot}$ we found two points that describe a Λ CDM universe. Specifically, point A_5 with $\frac{2}{9} \leq n \leq \frac{1}{4}$ provides an unstable Λ CDM universe, while point B_5 with $0 \leq n \leq \frac{1}{4}$ provides a stable Λ CDM model.

It is interesting to mention that in all stable critical points which produce cosmic acceleration the corresponding parameter n is found to be small, hence our theoretical results are consistent cosmological observations. Large values of n lead to a different evolution history for our universe that is not consistent with the available data. In our analysis, positive values of n mean that the vacuum decays into dark matter, whereas negative values of n imply that dark matter decays into vacuum. From our results it is clear that from the dynamical point of view the interacting varying vacuum scenarios can largely accommodate models that describe various phases of the observed behavior of the universe.

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