



New five-dimensional Bianchi type magnetically charged hairy topological black hole solutions in string theory

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Abstract We construct black hole solutions to the leading order of string effective action in five dimensions with the source given by dilaton and magnetically charged anti-symmetric gauge B -field. Presence of the considered B -field leads to the unusual asymptotic behavior of the solutions which are neither asymptotically flat nor asymptotically (A)dS. We consider the three-dimensional space part to correspond to the Bianchi classes and so the horizons of these topological black hole solutions are modeled by seven homogeneous Thurston's geometries of E^3 , S^3 , H^3 , $H^2 \times E^1$, \widetilde{SL}_2R , nilgeometry, and solvegeometry. Calculating the quasi-local mass, temperature, entropy, dilaton charge, and magnetic potential, we show that the first law of black hole thermodynamics is satisfied by these quantities and the dilaton charge is not independent of mass and magnetic charge. Furthermore, for Bianchi type V , the T -dual black hole solution is obtained which carries no charge associated with B -field and the entropy turns to be invariant under the T -duality.

1 Introduction

The low energy string effective action contains a rich variety of black hole solutions, which being characterized by charges of dilaton, Yang–Mills fields and the antisymmetric tensor gauge field (B -field), possess qualitatively different properties from those of general relativity [1–4]. In this context, the rotating and charged dilaton black hole solutions in string theories have been obtained, for instance, with constant dilaton for the dilaton-graviton system in [5] and with including a coupling of dilaton to the Maxwell field within a generalized class of theories that arise in low energy string effective action in [6]. Furthermore, taking into account a non-trivial B -field, or more generally the p -form fields, the

charged and rotating string black hole solutions have been found for example in [6–12]. In this category, the well known Kerr-Sen rotating charged black hole solution in low energy effective field theory of heterotic string theory is the first constructed solution by transforming the Kerr solution using the twisting method [8].

The equations of motion of low energy string frame effective action are equivalent to the one-loop β -function equations which are the conformal invariance condition of the corresponding σ -model. Being differential equations, like Einstein field equations, they characterize the local properties of spacetime and not the global structure, namely the topology of spacetime. The restriction on the topology of black holes was first noticed in Hawking's theorem where the event horizon cross section of a stationary asymptotically flat 4-dimensional black hole, which satisfies the dominant energy condition, was determined to be S^2 [13, 14]. A generalization of this result to the higher dimensional black holes was then provided in [15], where the event horizon of a stationary black hole in arbitrary dimension is required to admit a positive scalar curvature metric. The Emparan and Reall black ring solution with $S^2 \times S^1$ horizon topology is a relevant example in this category [11]. Nevertheless, the black hole solutions with non-trivial horizon topology, namely the topological black holes, are allowed to be considered if some of the assumptions of these theorems are relaxed [16–19]. For example, in the asymptotically anti-de Sitter (AdS) spacetimes, where the dominant energy condition and asymptotic flatness are both violated, the black hole solutions with compact Riemann surfaces of any genus g as horizon have been studied in [20–24]. Also, the rotating generalization of these solutions in AdS₄ with $g > 0$ has been presented in [25, 26], whose horizons are not compactifiable and their described spacetimes are rotating black branes [27].

Usually, the horizon geometries of topological black hole solutions are considered to be spherical, hyperbolic or flat. For the five-dimensional black hole solutions the situation

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can be more extensive. The geometries of three-dimensional homogeneous manifolds has been classified by Thurston into eight types [28], where in addition to three isotropic constant curvature cases of spherical S^3 , Hyperbolic H^3 , and Euclidean E^3 , there are five anisotropic alternative three-dimensional geometries of $S^2 \times R$, $H^2 \times R$, \widetilde{SL}_2R , solvgeometry, and nilgeometry. Admitting homogeneous metrics, these model geometries have close correspondence with the homogeneous geometries of Bianchi types and Kantowski-Sachs [29,30]. The Bianchi types are defined based on the simply-transitive three-dimensional Lie groups classification. Since the isometries of the Riemannian manifold form a Lie group, the Bianchi classification is widely used for studying the spatially homogeneous spacetimes in relativity and cosmology [31–35]. The spacetimes in these models are assumed to possess a symmetry, namely the spatial homogeneity [36]. Families of five-dimensional black hole solutions of gravity theories where the horizons are modeled by some of the Thurston 3-geometries have been presented in [37–41]. Also, extremal black branes whose geometries are classified by nine Bianchi types have been studied in [42].

Recently, the black hole solutions in string theory have attracted an increasing interest [43–49]. Nevertheless, the effect of the magnetically charged B -field has not been discussed so much in previous works [6–11]. In this work, we are interested in five-dimensional black hole solutions in string theory including a dilaton field and a 3-form field strength tensor of B -field associated with a magnetic charge. The inclusion of B -field in this class will affect the asymptotic properties of the black hole spacetime where the asymptotic flatness condition can be violated, which brings up the possibility of considering the unusual geometries. The three-dimensional horizons will be assumed to be homogeneous spaces corresponding to Bianchi types with closed spatial sections.

Generally, presence of a non-trivial dilaton, which introduces a dilaton charge, leads to dependence of mass on the asymptotic value of dilaton at infinity, ϕ_∞ . Dilaton charge has a critical role in the physics of black holes. The no-hair theorems, for example in Brans-Dicke theory with minimally coupled dilaton to gravity [50], indicate that the dilaton is not allowed to carry a charge. However, in some cases, the scalar field can have a non-trivial profile introducing a dilaton charge which is not an extra independent characteristic of the black hole. This is usually referred as the scalar hair of second kind [51]. For this kind of solutions, a cosmological scenario has been provided in [52], where ϕ_∞ does vary and so the scalar charge appears in the first law of thermodynamics. The problem with this modified first law of thermodynamics is that the dilaton charge, not being protected by a gauge symmetry, is not a conserved charge [53,54]. This charge is non-localized, exists entirely outside the hori-

zon and usually corresponds to a secondary hair [52,53]. Also, in the AdS spacetimes, scalar charges are not compatible with AdS/CFT since no such charge exist in dual CFT. In fact, it is proved in complete generality that for asymptotically AdS black holes the first law is satisfied without charge changes [55]. Noting that, even though the energy does depend on ϕ_∞ in general, see for instance the holographic stress–energy tensor expressions (from which the energy can be computed) in [56,57]. An intriguing physical discussion about the variation of ϕ_∞ is presented in [58], where using the solution phase space method and integrability of charge variations, it is shown that for black hole solutions of Einstein–Maxwell–(Axion)–Dilaton theories the ϕ_∞ is a redundant (non-physical) parameter, whose variation is not included in the first law of thermodynamics. Then consolidated by the idea that some redefinitions based on shift symmetry can eliminate the dependence of the mass and charges on ϕ_∞ [53,54], the applicability of the result in any other theory is asserted, declaring that the ϕ_∞ is generally a redundant parameter. Alternatively, inspired by the counterterm subtraction method which has been developed to define the conserved charges of AdS spacetimes in [59–62], it is shown in [63] that a well defined variational principal can be obtained by including a boundary term (counterterm) for dilaton field in Brown–York formalism which leads to a total energy with the contribution of scalar field where the first law of thermodynamics for this total energy does not contain the variation term of ϕ_∞ . Motivated by this, we will consider an appropriate boundary term for the dilaton field and investigate its consequence on the first law of black hole thermodynamics.

The paper is organized as follows: in Sect. 2, a review on low energy D -dimensional string effective action and its equations of motion is presented. In Sect. 3, the string equations of motion are solved to obtain new five-dimensional black hole solutions whose horizons are homogeneous spaces corresponding to the Bianchi types. Then, the thermodynamic behavior and the extremal condition of the solutions are investigated in Sect. 4. Furthermore, for Bianchi type V case the T -dual solutions and their thermodynamic properties are investigated in Sect. 5. Finally, some concluding remarks are presented in Sect. 6.

2 Low energy string effective action and one-loop β -functions

For a σ -model, the one-loop β -function equations of the background fields of metric $g_{\mu\nu}$, dilaton ϕ and antisymmetric tensor gauge B -field are given by [64,65]

$$R_{\mu\nu} - \frac{1}{4}H_{\mu\nu}^2 - \nabla_\mu \nabla_\nu \phi = 0, \quad (1)$$

$$\nabla^\mu (e^\phi H_{\mu\nu\rho}) = 0, \tag{2}$$

$$R - \frac{1}{12} H^2 + 2\nabla_\mu \nabla^\mu \phi + (\partial_\mu \phi)^2 - \Lambda = 0, \tag{3}$$

where $H_{\mu\nu}^2 = H_{\mu\rho\sigma} H_\nu^{\rho\sigma}$ and H is the field strength of B -field defined by $H_{\mu\nu\rho} = 3\partial_{[\mu} B_{\nu\rho]}$. These equations can be also obtained by variation of the following string frame effective action with respect to metric, B -field and dilaton

$$S = \frac{-1}{2\lambda_s^{D-2}} \int d^D x \sqrt{g} e^\phi \left(R - \frac{1}{12} H^2 + (\nabla\phi)^2 + \Lambda \right), \tag{4}$$

where λ_s is the string length and the Λ is related to the central charge deficit of theory which is given in non-critical D -dimensional bosonic theory as $\Lambda = \frac{2(26-D)}{3\alpha'}$ [66]. We will set $\Lambda = 0$ in our analysis. This may be a good approximation if the curvature and/or kinetic energy are large compared to the Λ , i.e. $\Lambda \ll R, \nabla\phi, H^2$ [67]. Alternatively, the Einstein frame can be introduced whose metric $\tilde{g}_{\mu\nu}$ is related to the string frame metric $g_{\mu\nu}$ in D -dimensional spacetime by

$$\tilde{g}_{\mu\nu} = e^{\frac{2}{D-2}\phi} g_{\mu\nu}. \tag{5}$$

Performing this conformal transformation, the Einstein frame effective action for bosonic string can be obtained as [64,65]

$$S = -\frac{1}{2\kappa_D^2} \int d^D x \sqrt{\tilde{g}} \left(\tilde{R} - \frac{(\tilde{\nabla}\phi)^2}{D-2} - \frac{1}{12} e^{\frac{4\phi}{D-2}} H^2 \right), \tag{6}$$

in which $\tilde{\nabla}$ indicates the covariant derivative with respect to \tilde{g} , and $\kappa_D^2 = 8\pi G_D = \lambda_s^{D-2} e^{-\phi} = \lambda_p^{D-2}$, where λ_p is the Planck length and G_D is the D -dimensional gravitational Newton constant. Principally, understanding the gravitational phenomena is more convenient in Einstein frame in which the dilaton coupling to Ricci scalar in (4) has been eliminated. In this frame, the one-loop β -functions (1)-(3) can be rewritten in the following form of Einstein field equations that can be also obtained by the variation of the action (6) with respect to $\tilde{g}_{\mu\nu}$ [65]

$$\tilde{R}_{\mu\nu} - \frac{1}{2} \tilde{R} \tilde{g}_{\mu\nu} = \kappa_D^2 T_{\mu\nu}^{(\text{eff})}, \tag{7}$$

where the effective energy-momentum tensor is defined as follows

$$T_{\mu\nu}^{(\text{eff})} = T_{\mu\nu}^{(\phi)} + T_{\mu\nu}^{(B)}, \tag{8}$$

and

$$\kappa_D^2 T_{\mu\nu}^{(\phi)} = \frac{1}{D-2} \left(\tilde{\nabla}_\mu \phi \tilde{\nabla}_\nu \phi - \frac{1}{2} \tilde{g}_{\mu\nu} (\tilde{\nabla}\phi)^2 \right), \tag{9}$$

$$\kappa_D^2 T_{\mu\nu}^{(B)} = \frac{e^{\frac{4\phi}{D-2}}}{4} \left(H_{\mu\kappa\lambda} H_\nu^{\kappa\lambda} - \frac{1}{6} H^2 \tilde{g}_{\mu\nu} \right), \tag{10}$$

Also, the equations of motion of dilaton and B -field are given by [65]

$$\tilde{\nabla}^2 \phi - \frac{1}{12} e^{\frac{4\phi}{D-2}} H^2 = 0, \tag{11}$$

$$\tilde{\nabla}_\mu \left(e^{\frac{4\phi}{D-2}} H^\mu_{\nu\rho} \right) = 0. \tag{12}$$

3 Black hole solutions of equations of motion of low energy string theory in five dimensions

In this paper we focus on $D = 5$ case, looking for black hole solutions of low energy string theory effective equations of motion on five-dimensional spacetimes where the r and t constant hypersurfaces are given by homogeneous space corresponding to a Bianchi types. In this regard, we start with the string frame metric ansatz

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -F dt^2 + \frac{1}{F U^2} dr^2 + g_i^2 \sigma^i \sigma^i, \tag{13}$$

in which the metric components are function of radial coordinate, $F = F(r)$, $U = U(r)$, and $g_i = g_i(r)$, where $i = 1, 2, 3$ and σ^i are left invariant basis 1-forms. The Bianchi type classification and their left invariant 1-forms are presented in Appendix A. Respecting the homogeneity of spacetime, the dilaton field is taken to be a function of r only. The contribution of field strength tensor H for Bianchi classes with diagonal metrics can be classified based on the orientation of its Hodge dual with respect to the three-dimensional hypersurface of homogeneity sections. Here, the field strength tensor of B -field, satisfying $dH = 0$, is chosen to be in the following class [32]

$$H = \frac{1}{3!} b \sigma^1 \wedge \sigma^2 \wedge \sigma^3, \tag{14}$$

where b is a constant. Then, using (192)–(194), the (t, t) , (r, r) and (i, i) components of β -function of metric (1) and β -function of dilaton (3) reduce to following coupled set of differential equations

$$-\frac{1}{2} \frac{F''}{F} - \frac{1}{2} \ln F' \left(\sum \ln g'_i + \phi' + \ln U' \right) = 0, \tag{15}$$

$$-\phi'' - \sum \frac{g_i''}{g_i} - \frac{1}{2} \frac{F''}{F} - \frac{1}{2} \ln U' \ln F' - \frac{1}{2} (2 \ln U' + \ln F') \left(\sum \ln g'_i + \phi' \right) = 0, \tag{16}$$

$$-\frac{g_i''}{g_i} - \ln g'_i \left(\sum_{j \neq i} \ln g'_j + \phi' + \ln U' + \ln F' \right) - \frac{1}{2} b^2 (g_1 g_2 g_3 U)^{-2} F^{-1} - Y_i = 0, \tag{17}$$

$$\begin{aligned}
 &2\phi'' + \sum \left(2 \frac{g_i''}{g_i} + \frac{F_i''}{F_i} + 2 \ln g_i'(\phi' + \ln F' + \ln U') + Y_i \right) \\
 &+ 2\phi' \left(\ln U' + \ln F' + \frac{1}{2}\phi' \right) + \ln U' \ln F' \\
 &+ 2 \sum_{i < j} \ln g_i' \ln g_j' + \frac{1}{2} b^2 (g_1 g_2 g_3 U)^{-2} F^{-1} = 0, \tag{18}
 \end{aligned}$$

where the prime stands for derivative with respect to r and the Y_i terms are Bianchi type dependent terms which will be given in the following subsections. Furthermore, the equation (1) imposes the the following constraint equations in Bianchi types of class B which possess non-diagonal components of Ricci tensor

$$R_{\mu\nu} = 0, \quad (\mu \neq \nu). \tag{19}$$

Now, adding (15), (16), and summed over i of (17) to (18) leads to the following equation

$$\begin{aligned}
 &\phi'' + \phi' \left(\sum \ln g_i' + \ln(FUe^\phi)' - b^2(g_1 g_2 g_3 U) \right)^{-2} \\
 &F^{-1} = 0. \tag{20}
 \end{aligned}$$

Also, adding twice of (16) to (18) yields the initial value equation as follows

$$\begin{aligned}
 &\phi'(\phi' + \ln F') + \sum \ln g_i'(2\phi' + \ln F') + 2 \sum_{j \neq i} \ln g_i' \ln g_j' \\
 &+ \sum Y_i + \frac{1}{2} b^2 (g_1 g_2 g_3 U)^{-2} F^{-1} = 0. \tag{21}
 \end{aligned}$$

In order to have black hole solutions with regular horizon r_H , we impose the following boundary conditions

$$F_H = 0, \quad F'_H \neq 0, \tag{22}$$

with finite g_{iH} , ϕ_H and U_H . Here and in what follows, the subscript H indicates the quantities evaluated at the horizon. Also, from (20) we get

$$\phi'_H F'_H = \frac{b^2}{(g_{1H} g_{2H} g_{3H} U_H)^2}. \tag{23}$$

In the asymptotic region $r \rightarrow \infty$, we are interested in non-logarithmic branch with $F(r) = c + \mathcal{O}(r^{-1})$ and

$$\phi \sim \phi_\infty + \frac{\mathcal{D}}{r} + \frac{\mathcal{C}}{r^2}, \tag{24}$$

where c, \mathcal{D} and \mathcal{C} are finite constants. The relevant asymptotic behavior for metric functions are

$$U \sim r^2 \left(u_\infty + \frac{u_1}{r} \right), \tag{25}$$

$$g_i \sim g_{i\infty} + \frac{g_{i1}}{r}, \tag{26}$$

where $u_\infty, u_1, g_{i\infty}$ and g_{i1} are finite constants.

To solve the set of equations of (15), (16), and (20) subject to the initial value equation (21), we choose

$$U(r) = r^2 (g_1 g_2 g_3)^{-1} e^{-\phi}. \tag{27}$$

In this case, the equations (15), and (20) read

$$F'' + 2r^{-1} F' = 0, \tag{28}$$

$$\phi'' + \phi' \ln(r^2 F)' - \frac{b^2}{r^4 F} e^{2\phi} = 0. \tag{29}$$

Also, using (18) in (17) we get

$$\ln(g_i^2 e^\phi)'' + \ln(g_i^2 e^\phi)' \ln(r^2 F)' + 2 Y_i = 0. \tag{30}$$

The solutions of (28) and (29) give the $F(r)$ and $\phi(r)$, independent of the type of three-dimensional Bianchi part, as follows

$$F = c_2 + \frac{c_1}{r}, \tag{31}$$

$$\phi = -\ln \left(c_3^2 F^{\frac{1}{2}-n} - b^2 F^{\frac{1}{2}+n} \right) + \ln(2nc_1 c_3), \tag{32}$$

where the c_1, c_2, c_3 , and n are real integrating constants.

For dilaton ϕ (32) and ϕ' , which is given by

$$\phi' = -\frac{c_1 \left(n + \frac{1}{2} \right) b^2 F^{2n} + c_3 \left(n - \frac{1}{2} \right)}{r^2 F \left(c_3^2 - b^2 F^{2n} \right)}, \tag{33}$$

to be regular at the horizon and consistent with the boundary condition (23), the restriction of

$$n = \frac{1}{2}, \tag{34}$$

is required.

To have black hole interpretation in the solutions, the c_1 constant is required to be negative with positive c_2 . Also, to have well defined dilaton field given by (32) in the black hole solutions we assume that

$$c_3^2 - b^2 > 0, \quad c_3 < 0, \quad c_1 < 0. \tag{35}$$

In some cases of topological black hole solutions whose horizons are constant curvature spaces, the requirement of solution to be asymptotically AdS spacetime, i.e. Einstein space with negative cosmological constant, relates the integrating constant of type c_2 to the curvature constant of horizon [24]. Here, considering (5), (7), (13) and (14) the components of Ricci tensor in Einstein frame are given by

$$2\tilde{R}_i^i = -\tilde{R}_t^t = \frac{b^2 e^{\frac{4}{3}\phi}}{3\tilde{\gamma}}, \quad \tilde{R}_r^r = \frac{-b^2 e^{-\frac{4}{3}\phi} + r^4 \phi'^2 F e^{-\frac{2}{3}\phi}}{3\tilde{\gamma}}, \tag{36}$$

where $\tilde{\gamma}$ is the determinant of the Einstein frame metric of three-dimensional homogeneous space in (13), defined by

$$\tilde{\gamma} = (\tilde{g}_1 \tilde{g}_2 \tilde{g}_3)^2 = (g_1 g_2 g_3)^2 e^{2\phi}. \tag{37}$$

Noting the (31), (32) and (36) it can be checked that our solutions do not admit asymptotically Einstein spaces, with

$\tilde{R}_{\mu\nu} = k \tilde{g}_{\mu\nu}$. Hence, having no explicit condition to fix the c_2 parameter, we set $c_2 = 1$ for simplicity. The asymptotic value of dilaton (32) is then given by

$$\phi_\infty = \ln \left(\frac{c_1 c_3}{c_3^2 - b^2} \right). \tag{38}$$

So far, we have not considered any special kind of geometry for three-dimensional homogeneous space. The effective energy–momentum tensor (8) satisfies all of the energy conditions but the Ricci scalar in Einstein frame

$$\tilde{R} = \frac{-b^2 e^{-\frac{4}{3}\phi} + 2r^4 \phi'^2 F e^{-\frac{2}{3}\phi}}{6\tilde{\gamma}}, \tag{39}$$

has the following asymptotic form in $r \rightarrow \infty$ limit

$$\tilde{R} = \frac{3 b^2 e^{\frac{4}{3}\phi_\infty}}{2 c_3^2 \tilde{\gamma}_\infty} (2b^2 - c_3^2), \tag{40}$$

where $\tilde{\gamma}_\infty = \lim_{r \rightarrow \infty} \tilde{\gamma}$. Evidently, the presence of the field strength tensor of type (14) has affected the asymptotic behavior of these solutions. In such a way that, as long as $c_3 \neq -\sqrt{2}b$ and the $\tilde{\gamma}_\infty$ is finite, the solutions are not either asymptotically flat or (considering the (36)) asymptotically (A)dS.¹ Therefore, having violated asymptotically flatness condition, the considered spacetimes are not fundamentally forbidden to have negatively curved horizon geometries. In other words, besides the flat Bianchi type *I* and positively curved Bianchi type *IX*, we can consider the other Bianchi types with negative three-dimensional curvatures.

Determining the Bianchi type dependent terms Y_i in (30) using relations of (192)–(194), we are going to find the solutions of (30) in all Bianchi types in the following subsections to establish the form of metric of the string and Einstein frames.

3.1 Bianchi type *I*

This Bianchi type has $Y_i = 0$ and solutions of (30) give the components of homogeneous part of string frame metric (13) as

$$g_i^2 = e^{-\phi} F^{2p_i}, \tag{41}$$

where p_i are integrating constants. Substituting them into the initial value equation (21) gives the following condition on constants

$$-2 \sum p_i - 4 \sum_{i < j} p_i p_j = 0. \tag{42}$$

¹ As we will see in Sects. 4.1–4.7, all of the Bianchi type solutions have finite $\tilde{\gamma}_\infty$.

Also, the conformal transformation (5) gives the Einstein frame metric by

$$ds^2 = -W^{-\frac{2}{3}} F dt^2 + W^{\frac{1}{3}} \left(F^{-(1-\sum p_i)} \frac{dr^2}{r^4} + \sum F^{+2p_i} (\sigma^i)^2 \right), \tag{43}$$

where the $W(r)$ function (here and hereafter) is given by

$$W = \frac{c_3^2 - b^2 F}{c_3^2 - b^2} e^{-\phi_\infty}. \tag{44}$$

If the conditions (35) hold, the $W(r)$ is positive everywhere, finite at $r_H = -c_1$, and blows up at $r = 0$.

3.2 Bianchi type *II*

In this Bianchi type the Y_i terms of (30) are given by

$$-Y_1 = Y_2 = Y_3 = \frac{1}{2} g_1^2 \left(g_2^2 g_3^2 F U^2 \right)^{-1}. \tag{45}$$

The solutions of (30) give the string frame metric (13) components as

$$g_1^2 = e^{-\phi} V^{-1} F^{\frac{1}{2}(p_1 - \frac{1}{2})}, \tag{46}$$

$$g_2^2 = l_2^2 e^{-2\phi} g_1^{-2} F^{2p_2}, \tag{47}$$

$$g_3^2 = l_3^2 e^{-2\phi} g_1^{-2} F^{2p_3}, \tag{48}$$

where

$$V = \frac{2p_1 c_1 q_1}{q_1^2 - F^{2p_1}}, \tag{49}$$

and p_1, p_2, q_1, l_2 and l_3 are real constants. Substituting these solutions into initial value equation (21) gives the following constraint on integrating constants

$$-p_1^2 - 2(p_2 + p_3) - 4p_2 p_3 - \frac{1}{4} = 0. \tag{50}$$

To preserve the signature of metric at spatial infinity and near horizon, the q_1 and p_1 are required to have opposite signs with $q_1^2 - 1 > 0$. Also, using (5), the Einstein frame metric is given by

$$ds^2 = -W^{-\frac{2}{3}} F dt^2 + W^{\frac{1}{3}} V^{-1} F^{-\frac{1}{2} - p_1} \times \left(l_2^2 l_3^2 F^{2(p_2 + p_3)} \frac{dr^2}{r^4} + V^2 F^{2p_1} (\sigma^1)^2 + l_2^2 F^{1+2p_2} (\sigma^2)^2 + l_3^2 F^{1+2p_3} (\sigma^3)^2 \right). \tag{51}$$

3.3 Bianchi type *III*

Belonging to class *B*, this Bianchi type gives the following constraint equation by (r, x^3) component of (19)

$$\ln \left(g_3 g_1^{-1} \right)' = 0, \tag{52}$$

which essentially requires $SO(2)$ isometry with $g_1 = g_3$. Also, the Bianchi type dependent terms in this model are given by

$$Y_1 = Y_3 = \left(g_3^2 F U^2\right)^{-1}, \quad Y_2 = 0. \tag{53}$$

Then, the solutions of (30) are

$$g_1^2 = g_3^2 = e^{-2\phi} g_2^{-2} V^2 F^{2p_1-1}, \tag{54}$$

$$g_2^2 = l_2^2 e^{-\phi} F^{2p_2}, \tag{55}$$

in which we have defined

$$V = \frac{-2p_1 c_1 q_1}{q_1^2 + F^{2p_1}}, \tag{56}$$

and p_1, p_2, q_1 and l_1 are real constants. Substituting these solutions into the initial value equation (21) gives

$$-4\left(p_1^2 - p_2^2\right) + 2p_2 + 1 = 0. \tag{57}$$

Respecting the signature of metric at infinity and near horizon, the q_1 and p_1 constants are required to have the same signs. The Einstein frame metric is then given by

$$ds^2 = -W^{-\frac{2}{3}} F dt^2 + W^{\frac{1}{3}} V^2 F^{2p_2} \times \left(l_2^2 V^2 F^{4(p_1-p_2)-3} \frac{dr^2}{r^4} + V^{-2} l_2^2 (\sigma^2)^2 + F^{-1+2p_1-4p_2} \left((\sigma^1)^2 + (\sigma^3)^2 \right) \right). \tag{58}$$

3.4 Bianchi type IV

In this Bianchi type of class B , the constraint equation (19) leads to

$$g_3^2 g_1^{-2} = 0, \quad \ln \left(g_1^2 (g_2 g_3)^{-1} \right)' = 0, \tag{59}$$

and so this Bianchi type does not provide any non-singular solution.

3.5 Bianchi type V

The Bianchi type dependent terms are given here by

$$Y_1 = Y_2 = Y_3 = 2 \left(g_1^2 F U^2 \right)^{-1}. \tag{60}$$

Also, the following equation is given by the (r, x^1) component of (19) in this Bianchi type of class B

$$\ln \left(g_1^2 (g_2 g_3)^{-1} \right)' = 0. \tag{61}$$

It imposes the constraint of $g_1^2 = g_2 g_3$, which is satisfied by the following solutions of (30)

$$g_1^2 = V F^{p_1-\frac{1}{2}} e^{-\phi}, \tag{62}$$

$$g_2^2 = F^{2p_2} g_1^2, \tag{63}$$

$$g_3^2 = F^{-2p_2} g_1^2, \tag{64}$$

where we have defined

$$V = \frac{c_1 q_1 p_1}{F^{2p_1} + q_1^2}. \tag{65}$$

Here the p_1, p_2 , and q_1 are real constants and the q_1 and p_1 need to have opposite signs. The initial value equation (21) gives

$$-3 p_1^2 + 4 p_2^2 - \frac{3}{4} = 0. \tag{66}$$

Also, the Einstein frame metric is given by

$$ds^2 = -W^{-\frac{2}{3}} F dt^2 + W^{\frac{1}{3}} V F^{-\frac{1}{2}+p_1} \left(V^2 F^{2p_1-2} \frac{dr^2}{r^4} + (\sigma^1)^2 + F^{2p_2} (\sigma^2)^2 + F^{-2p_2} (\sigma^3)^2 \right). \tag{67}$$

3.6 Bianchi type VI_h

In this Bianchi type we have

$$(h^2 + 1)^{-1} Y_1 = (h(h + 1))^{-1} Y_2 = (h + 1)^{-1} Y_1 = \left(g_1^2 F U^2 \right)^{-1}. \tag{68}$$

Also, the (r, x^1) component of (19) is

$$-(1 + h) \ln g_1' + h \ln g_2' + \ln g_3' = 0, \tag{69}$$

which imposes the following restriction

$$\frac{g_3 g_2^h}{g_1^{h+1}} = \frac{l_2^{h-1}}{l_1^{h+1}}, \tag{70}$$

where l_1 and l_2 are real constants. In this Bianchi type, the $h = 0, 1$ cases give rise to Bianchi types III and V , respectively. Excluding these two types, only the VI_{-1} admits closed spatial section [30].² Since the black hole is usually assumed to have compact horizon [41], here we investigate the solution of the $h = -1$ case which gives the following solutions

$$g_1^2 = F^{2p_1} e^{-16q_1^{-2} p_2^2 F^{-2p_2}} e^{-\phi}, \tag{71}$$

$$g_2^2 = 4 l_2 p_2^2 c_1 q_1^{-1} F^{-p_2-\frac{1}{2}} e^{-\phi}, \tag{72}$$

$$g_3^2 = l_2^{-2} g_2^2, \tag{73}$$

where p_1, p_2 , and q_1 are real constants. Assuming the l_2 to be positive, q_1 should be negative. Substituting these solution into the initial value equation (21) gives

$$-p_2^2 + p_2(4p_1 + 1) + \frac{3}{4} = 0. \tag{74}$$

² The closed spatially homogeneous hypersurface is compact without boundary.

The Einstein frame metric is then given by

$$\begin{aligned}
 ds^2 = & -W^{-\frac{2}{3}}F dt^2 + W^{\frac{1}{3}}F^{-\frac{1}{2}} \\
 & \times \left(16 p_2^4 c_1^2 q_1^{-2} F^{-\frac{3}{2}+2 p_1-2 p_2} e^{-\frac{16 p_2^2}{q_1^2 F^{2 p_2}}} \frac{dr^2}{r^4} \right. \\
 & + 4 c_1 l_2 p_2^2 q_1^{-1} F^{-p_2} \left((\sigma^2)^2 + l_2^{-2} (\sigma^3)^2 \right) \\
 & \left. + F^{\frac{1}{2}+2 p_1} e^{-\frac{16 p_2^2}{q_1^2 F^{2 p_2}}} (\sigma^1)^2 \right). \tag{75}
 \end{aligned}$$

3.7 Bianchi type VII_h

In this case, the Bianchi type dependent terms in (30) are

$$\begin{aligned}
 Y_1 = & -\frac{g_1^4 - g_2^4}{2 F (g_1 g_2 g_3 U)^2}, \\
 Y_2 = & \frac{g_1^4 - g_2^4}{2 F (g_1 g_2 g_3 U)^2} - \frac{h^2}{F g_3^2 U^2}, \tag{76} \\
 Y_3 = & \frac{(g_1^2 - g_2^2)^2}{2 F (g_1 g_2 g_3 U)^2} - \frac{h^2}{F g_3^2 U^2}.
 \end{aligned}$$

Also, the constraint equations given by (19) are

$$h \ln(g_3 g_2^{-1})' = 0, \quad h (g_2^2 g_3^{-2}) = 0, \tag{77}$$

which restrict the solutions to be of type $h = 0$. In this case, the equations of (30) can be integrated only with $g_1 = g_2$ which, leading to $Y_i = 0$, reduces the equations to those of Bianchi type I and consequently the solutions can be recovered from there.

3.8 Bianchi type VIII

Here the Y_i terms are given by

$$\begin{aligned}
 Y_1 = & -\frac{1}{2} \left(g_1^2 - (g_2^2 + g_3^2)^2 \right) \left((g_1 g_2 g_3 U)^2 F \right)^{-1}, \\
 Y_2 = & -\frac{1}{2} \left(g_2^2 - (g_1^2 + g_3^2)^2 \right) \left((g_1 g_2 g_3 U)^2 F \right)^{-1}, \tag{78} \\
 Y_3 = & -\frac{1}{2} \left(g_3^2 - (g_1^2 - g_2^2)^2 \right) \left((g_1 g_2 g_3 U)^2 F \right)^{-1}.
 \end{aligned}$$

Then, the solutions of (30) give the metric components of string frame (13) as follows

$$g_1^2 = g_2^2 = e^{-2\phi} V_1^2 F^{2 p_1-1} g_3^{-2}, \tag{79}$$

$$g_3^2 = e^{-\phi} V_2 F^{p_2-\frac{1}{2}}, \tag{80}$$

where

$$V_1 = \frac{2 c_1 q_1 p_1}{q_1^2 + F^{2 p_1}}, \quad V_2 = \frac{-2 c_1 q_2 p_2}{q_2^2 - F^{2 p_2}}. \tag{81}$$

The p_i , q_1 and q_2 are integrating constants, where q_2 and p_2 should have the same sign and $q_2^2 - 1 > 0$. Also, as a consequence of initial value equation (21), p_i are subject to the following constraint

$$p_2^2 - 4 p_1^2 + \frac{3}{4} = 0. \tag{82}$$

Performing the conformal transformation (5) on (13) we get the Einstein frame metric in the following form

$$\begin{aligned}
 ds^2 = & -W^{-\frac{2}{3}}F dt^2 + W^{\frac{1}{3}}F^{-\frac{1}{2}-p_2} \left(V_1^4 V_2^{-1} F^{4 p_1-2} \frac{dr^2}{r^4} \right. \\
 & \left. + V_2 F^{2 p_2} (\sigma^3)^2 + V_1^2 V_2^{-1} F^{2 p_1} ((\sigma^1)^2 + (\sigma^2)^2) \right). \tag{83}
 \end{aligned}$$

3.9 Bianchi type IX

Here, the Bianchi type dependent terms Y_i are given by

$$Y_i = \frac{(g_j^2 - g_k^2)^2 - g_i^2}{2 F (g_1 g_2 g_3 U)^2}. \tag{84}$$

where (ijk) is taken to be cyclically as (123). The equations (30) can be integrated by setting $g_1 = g_3$. Then, we get

$$g_1^2 = g_3^2 = V_1^2 F^{2 p_1-1} e^{-2\phi} g_2^{-2}, \tag{85}$$

$$g_2^2 = V_2 F^{p_2-\frac{1}{2}} e^{-\phi}, \tag{86}$$

where

$$V_i = \frac{2 c_1 q_i p_i}{q_i^2 - F^{2 p_i}}, \quad i = 1, 2, \tag{87}$$

where q_i and p_i are constants and the q_2 and p_2 are especially required to have opposite signs with $q_2^2 - 1 > 0$. Substituting these solutions into the initial value equation (21) yields

$$-4 p_1^2 + p_2^2 + \frac{3}{4} = 0. \tag{88}$$

Furthermore, the Einstein frame metric reads

$$\begin{aligned}
 ds^2 = & -W^{-\frac{2}{3}}F dt^2 + W^{\frac{1}{3}}F^{-\frac{1}{2}-p_2} \left(V_1^4 V_2^{-1} F^{4 p_1-2} \frac{dr^2}{r^4} \right. \\
 & \left. + F^{2 p_2} V_2 (\sigma^2)^2 + V_1^2 V_2^{-1} F^{2 p_1} ((\sigma^1)^2 + (\sigma^3)^2) \right). \tag{89}
 \end{aligned}$$

4 Thermodynamic properties of the topological black hole solutions

Having found the solutions of low energy string effective action equations of motion in the previous section, we come to investigate the physical properties of the solutions. The black hole interpretation of the solutions with a horizon located at $r_H = -c_1$ requires the p_i constants to have some appropriate values such that the \tilde{g}_{rr} change its sign crossing the r_H . Assuming that the solutions are black hole solutions, the relevant values of p_i will be obtained demanding some special properties of the solutions. Also, the singularity properties of these topological black hole solutions and verification of the first law of thermodynamics will be investigated.

All of the obtained solutions for various Bianchi types have finite $\tilde{\gamma}_\infty$ in (40). Therefore, as we mentioned earlier, with $c_3 \neq -\sqrt{2}b$ the solutions are neither asymptotically flat nor asymptotically (A)dS. Hence, the considering of Bianchi classes with negative three-dimensional curvature as the horizons has no conflict with the area theorems [15], whose asymptotic flatness condition is violated.

For calculating the mass of these non-asymptotically flat solutions we use the Brawn–York formalism which defines the quasi-local conserved mass by [68]

$$M = \frac{1}{\kappa_5^2} \int_{^3B} d^3x \sqrt{\tilde{\sigma}} \left(K_{ab} - \tilde{h}_{ab} K \right) n^a \tilde{\xi}^b, \tag{90}$$

in which κ_5 is the five-dimensional Newton constant, 3B is the three-dimensional boundary, n^a is the time-like unit normal vector to the boundary 3B , K_{ab} is the extrinsic curvature of the 4-dimensional boundary $\partial\mathcal{M}$ with induced metric \tilde{h}_{ab} , and K is the trace of K_{ab} . We assume that the metrics have a normalized asymptotically time-like Killing vector $\tilde{\xi}^\mu = e^{-\frac{\phi_\infty}{3}} \delta_t^\mu$ such that $\tilde{\xi}^2 = \tilde{\xi}^\mu \tilde{\xi}^\nu g_{\mu\nu} \rightarrow -1$ for $r \rightarrow \infty$. The $\tilde{\sigma}$ is determinant of the metric of 3B in Einstein frame where, noting the relation between the coordinate and non-coordinate basis (191), we have $\tilde{\sigma} = \tilde{\gamma} |e_\mu^a|^2$ in which $\tilde{\gamma}$ has been defined by (37) and $|e_\mu^a|$ is determinant of vielbeine matrix whose components for each Bianchi type are presented in Appendix A. Then, considering the metric (13) along with (27) and (37), the mass expression (90) recasts the following general form

$$M = \frac{-c_1}{\kappa_5^2} \lim_{r \rightarrow \infty} F^{\frac{1-\lambda}{2}} W^{-\frac{1}{6}} \left(\frac{d}{dF} \sqrt{\tilde{\gamma}} \right) \omega_3, \tag{91}$$

where the volume element $\omega_3 = \frac{1}{3!} \int \sigma^1 \wedge \sigma^2 \wedge \sigma^3$ and the $F(r)$ and $W(r)$ functions have been given by (31) and (44), respectively. Using the obtained solutions with considering the given conditions in (35), the mass per unit volume for Bianchi type solutions reads

$$\mathcal{M} = -\frac{c_1 e^{-\frac{1}{3}\phi_\infty}}{\kappa_5^2} \left(\frac{b^2}{2(c_3^2 - b^2)} + \alpha - \frac{\lambda}{2} \right), \tag{92}$$

in which, being different for each Bianchi type, the λ and α are given for each type as

$$I \& VI I_0: \lambda = 2 \sum p_i, \alpha = 0, \tag{93}$$

$$II: \lambda = \frac{1}{2} + 2(p_2 + p_3) - p_1, \alpha = \frac{p_1}{q_1^2 - 1}, \tag{94}$$

$$III: \lambda = 2 - 4p_1 + 2p_2, \alpha = \frac{4p_1}{q_1^2 + 1}, \tag{95}$$

$$V: \lambda = 3p_1 - \frac{3}{2}, \alpha = \frac{3p_1}{q_1^2 + 1}, \tag{96}$$

$$VI_{-1}: \lambda = -\frac{3}{2} + 2p_1 - 2p_2, \alpha = \frac{-16p_2^3}{q_1^2}, \tag{97}$$

$$VIII: \lambda = 4p_1 - p_2 - \frac{3}{2}, \alpha = \frac{4p_1}{q_1^2 + 1} + \frac{p_2}{q_2^2 - 1}, \tag{98}$$

$$IX: \lambda = 4p_1 - p_2 - \frac{3}{2}, \alpha = \frac{4p_1}{q_1^2 - 1} + \frac{p_2}{q_2^2 - 1}. \tag{99}$$

It is worth mentioning that the quasi-local mass (92) coincides with the mass obtained by the Abbott–Deser approach [69] using the normalized killing vector $\tilde{\xi}^\mu$.

Also, the area of horizon is generally given by

$$A_H = \sqrt{\tilde{\gamma}_H} \omega_3, \tag{100}$$

where $\tilde{\gamma}_H$ is determinant of induced metric (37) on the horizon. A common feature of the obtained Bianchi type solutions is that their $\tilde{\gamma}$ is proportional to F^λ multiplied by a factor which is finite on the horizon. Therefore, to have non-zero area of horizon the $\lambda = 0$ is required and then the last term in (92) vanishes.

Furthermore, the surface gravity is defined by³

$$\begin{aligned} \kappa &= \sqrt{-\frac{1}{2} (\tilde{\nabla}_\mu \tilde{\xi}^\nu) (\tilde{\nabla}^\mu \tilde{\xi}^\nu)} \Big|_{r=r_H} \\ &= \frac{-e^{\frac{\phi_\infty}{3}}}{2} \sqrt{-\tilde{g}^{rr}(r_H) \tilde{g}^{tt}(r_H) \tilde{g}'_{tt}(r_H)}. \end{aligned} \tag{101}$$

The finite and non-zero surface gravity demands finite non-vanishing $\tilde{g}^{rr} \tilde{g}^{tt}$ on the horizon. Noting (27) and (37) and the fact that $\tilde{\gamma}$ is proportional to F^λ , the $\tilde{g}^{rr} \tilde{g}^{tt}$ is proportional to $F^{-\lambda}$. Hence, the finite surface gravity requires again $\lambda = 0$. Then, the following general form for surface gravity can be obtained

$$\kappa = \frac{-c_1 e^{-\frac{\phi_\infty}{3}}}{2\sqrt{\tilde{\gamma}_H}}. \tag{102}$$

³ Here, we need to choose the normalized killing vector to have the first law of black hole thermodynamics satisfied.

Also, the Hawking temperature can be derived from the Euclidean regularity methods [70] in our considered metric by⁴

$$T_H(r) = \frac{\sqrt{(\tilde{g}_{rr}^{-1})' \tilde{g}'_{tt}}|_{r=r_H}}{4\pi \sqrt{\tilde{g}_{tt}}}, \tag{104}$$

which is infinite on the horizon and has the finite non-vanishing value at infinity, given by

$$T_H = \frac{-c_1 e^{-\frac{\phi_\infty}{3}}}{2\sqrt{\tilde{\gamma}(r_H)}} = \frac{\kappa}{2\pi}. \tag{105}$$

Furthermore, the black hole entropy can be obtained using the Wald's formula [71]

$$S = -2\pi \int_H d^3x \sqrt{\tilde{\sigma}} \frac{\delta \mathcal{L}}{\delta \tilde{R}_{\mu\nu\rho\sigma}} \epsilon_{\mu\alpha} \epsilon_{\rho\nu} = -\frac{\pi}{\kappa_5^2} \epsilon_{\mu\nu} \epsilon^{\mu\nu} A_H, \tag{106}$$

which is evaluated on the horizon and $\epsilon_{\mu\nu}$ is the binormal to the horizon whose normalization is usually chosen as $\epsilon_{\mu\nu} \epsilon^{\mu\nu} = -2$.

Presence of the 3-form field strength tensor (14) introduces a charge associated to the antisymmetric B -field. Having no $H_{t\mu\nu}$ component, the Noether electric charge of B -field is zero, as well as its electric type potential, which is actually defined by the difference of the values of $B_{t\alpha}$ component of tensor gauge field at infinity and at the horizon [11]. On the other hand, a topological magnetic charge can be defined here by [72]

$$Q_m = \frac{1}{\sqrt{2\kappa_5}} \int_{3B} H = \frac{b}{\sqrt{2\kappa_5}} \omega_3, \tag{107}$$

whose conservation is associated with the Bianchi identity $dH = 0$. In the following, we will mention this charge by its density defined by $\mathcal{Q}_m = \frac{b}{\sqrt{2\kappa_5}}$. To calculate the conjugate potential of this magnetic charge, similar to the derivation of electric and magnetic potentials in Einstein–Maxwell–dilatons theory [73], we can use the electric-magnetic duality [74]. The equations of motion in Einstein frame are invariant under the transformation ($H \rightarrow \tilde{H} = *He^{-\frac{4\phi}{3}}, \phi \rightarrow -\phi$), where the $*$ stands for Hodge dual operation. Considering the field strength tensor (14), the \tilde{H} is a two-form corresponding to an electrically charged Maxwell field

$$\tilde{H} = \frac{b e^{2\phi}}{12 r^2} dt \wedge dr, \tag{108}$$

⁴ The same result can be obtained via the normalized temperature definition [18]

$$T_H(r) = \frac{\sqrt{-(\nabla_\mu \xi_\nu)(\nabla^\mu \xi^\nu)}|_{r=r_H}}{4\pi \sqrt{-\xi_\mu \xi^\mu}}, \tag{103}$$

which is independent of the normalization of the horizon generator.

whose vector potential in a gauge where the scalar potential vanishes on the horizon is

$$\tilde{A} = \left(\frac{c_3}{12 b W(r)} - \Phi_H \right) dt, \tag{109}$$

where $\Phi_H = \frac{(c_3 - b^2)e^{\phi_\infty}}{12 b c_3}$. The electric charge of this dual vector field, defined by $Q_e = \frac{1}{\sqrt{2\kappa_5}} \int_{3B} e^{\frac{4}{3}\phi_\infty} * \tilde{H}$, equals Q_m (107).⁵ Also, the electric potential is given by the time component of the dualized tensor gauge potential by

$$\tilde{\Phi}_e = \tilde{\xi} \cdot \tilde{A}|_{r=r_H} - \tilde{\xi} \cdot \tilde{A}|_{r=\infty} = \frac{1}{\sqrt{2\kappa_5}} \frac{-c_1 b e^{-\frac{\phi_\infty}{3}}}{(c_3^2 - b^2)}. \tag{110}$$

Hence, our solutions can be alternatively described by Maxwell electric Hodge dual field to the three form H (14). Now, as a matter of fact that the roles of Maxwell electric and B -field magnetic charges are exchanged under the extended Hodge dualization, the magnetic potential Φ_m in our solution can be interpreted as the electric potential in the dual frame $\tilde{\Phi}_e$, which can be rewritten in the following form using (38) and (107)

$$\Phi_m = -\frac{c_1 Q_m e^{\frac{2\phi_\infty}{3}}}{c_3}. \tag{111}$$

To check the first law of black hole thermodynamics, the relation between the solution parameters and physical ones are required. Reminding the conditions (35) where the c_1 and c_3 have been required to be negative, the inverse relations for Bianchi type I are given by

$$-c_1 = e^{\frac{\phi_\infty}{3}} \left(\frac{Q_m^2 e^{\frac{4\phi_\infty}{3}} - 4 \kappa_5^2 \mathcal{M}^2}{2 \mathcal{M}} \right), \tag{112}$$

$$-c_3 = \frac{Q_m^2 e^{\frac{2\phi_\infty}{3}}}{\mathcal{M}}, \tag{113}$$

where for the other Bianchi types we have⁶

$$-c_1 = e^{\frac{\phi_\infty}{3}} \left(\frac{\kappa_5^2 \mathcal{M}}{\alpha} - \frac{2 Q_m^2 e^{\frac{4\phi_\infty}{3}}}{\mathcal{M} + \sqrt{8 \left(\alpha - \frac{1}{2} \right) \alpha \kappa_5^{-2} Q_m^2 e^{\frac{4\phi_\infty}{3}} + \mathcal{M}^2}} \right), \tag{114}$$

$$-c_3 = \frac{\kappa_5^2 \mathcal{M} + \sqrt{8 \left(\alpha - \frac{1}{2} \right) \alpha Q_m^2 e^{\frac{4\phi_\infty}{3}} + \kappa_5^2 \mathcal{M}^2}}{2 \alpha e^{\frac{2\phi_\infty}{3}}}. \tag{115}$$

⁵ Since in D dimensions for a p -form a we have $**a = (-1)^{D-1+p(D-p)}a$, the Q_m and Q_e have the same sign.

⁶ The α (or equivalently the q_i) will be considered as non-varying constants which will be fixed by the first law of thermodynamics. In fact, the α are related to the curvature of three-dimensional space part. This is in analogy with the topological black hole solutions where the constant curvature k of horizon appears in the mass [16–19].

Before considering the first law of thermodynamics, let us have a closer look at the dilaton field (32). Having a non-trivial configuration, the dilaton introduces a dilaton charge. Based on the definition of this charge [2]

$$D = \int d^3 \Sigma^\mu \nabla_\mu \phi, \tag{116}$$

where $d \Sigma_\mu = \frac{\sqrt{7}}{3!} \epsilon_{\mu\alpha_1\alpha_2\alpha_3} \sigma^{\alpha_1} \wedge \sigma^{\alpha_2} \wedge \sigma^{\alpha_3}$ is the dual of 3-area form and the integral is evaluated at spatial infinity, the dilaton charge per volume ω_3 is given by

$$\mathcal{D} = \frac{c_1 b^2}{c_3^2 - b^2}, \tag{117}$$

which is negative holding the conditions (35). It is worth mentioning that, the same \mathcal{D} can be obtained from the asymptotic expansion of the dilaton field (24) in which the \mathcal{C} is given by

$$\mathcal{C} = \frac{c_1^2 b^4}{2(c_3^2 - b^2)^2} = \frac{\mathcal{D}^2}{2}. \tag{118}$$

In addition, the Einstein frame effective action (6) is invariant under the following scaling symmetry in five dimensions

$$\phi \rightarrow \phi + \phi_0, \quad B_{\mu\nu} \rightarrow e^{-\frac{2}{3}\phi_0} B_{\mu\nu}, \tag{119}$$

which is a constant shift in dilaton accompanied by a scaling in the gauge B -field. When a theory exhibits the shift symmetry of dilaton, the associated Noether current can be considered to identify whether or not a non-trivial configuration for dilaton is allowed [51, 75, 76]. Especially, this kind of current and its conservation have been used to establish the no-hair theorem in [75], where the J^μ which is given in terms of ϕ' , respecting the symmetries of metric, has only the J^r as non-vanishing component and the finite value of $J_\mu J^\mu$ on the horizon requires the J^r to vanish on horizon. The last step of their proof utilizes the conservation of J , which being rewritten in the form of $\partial_\mu (\sqrt{-\tilde{g}} J^\mu) = 0$ leads to $\rho^2(r) J^r = const$ where the ρ function measures the area of spheres with constant r . Then, since the ρ remains finite and non-zero even on the horizon, therefore the constant in this equation is zero and so the J^r has to vanish everywhere implying that dilaton needs to be constant with forbidding the dilaton hair. Following these steps in our considered case with a field strength tensor of (14) and a $\phi(r)$, we can investigate the related Noether current to (119) which is given by

$$J^\mu = \partial^\mu \phi - \frac{1}{2} e^{\frac{4}{3}\phi} H^{\mu\nu\rho} B_{\nu\rho}, \tag{120}$$

and is conserved only on-shell, using the equations of motion (11) and (12). This current (120) does not generally respect either the gauge invariance of B -field, i.e. $B_{\mu\nu} \rightarrow B_{\mu\nu} + \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu$, or the symmetries of the metric (isometries).

Although, the later one can be established by imposing an extra condition especially when the T -dual solutions are of interest.⁷ Also, considering (14) and noting that the dilaton is a function of r only, the presence of the second term in (120) indicates that besides J^r , there are also non-vanishing components J^i . At the horizon, this current has finite norm $J_\mu J^\mu$ and vanishing J^r . However, the presence of J^i contests the last mentioned step of proof of [75] where the conservation of J does not require J^r to vanish everywhere. Then, having a non-trivial configuration for dilaton field is not forbidden in our considered case. This does not, however, imply that the scalar charge is an independent charge carried by the obtained black holes. In some hairy black hole solutions, the regularity condition of ϕ and ϕ' has been employed to relate the dilaton charge to the mass of black hole implying that the scalar hair is of secondary type [51]. In our case, it just fixed the integrating constant n by (34). Considering the mass expression (92) along with aforementioned $\lambda = 0$ condition, the dilaton charge (117) coincides with the first term in mass (92) and we have

$$\mathcal{D} = -2 \left(\kappa_5^2 e^{\frac{\phi_\infty}{3}} \mathcal{M} + \alpha c_1 \right). \tag{121}$$

Noting that the Bianchi type I model has $\alpha = 0$ and the other Bianchi types with non-zero α have the relation (114) for c_1 , the relation (121) actually shows that the dilaton charge can be expressed in terms of \mathcal{M} , \mathcal{Q}_m and the parameter ϕ_∞ . Therefore, the dilaton charge is not an independent characteristic of the obtained black hole solutions and corresponds to a secondary hair [78]. Also, considering the (107), (111), and (117), the following relation is relevant between the dilaton charge and the magnetic charge multiplied by its potential

$$\mathcal{D} = -2\kappa_5^2 e^{\frac{2}{3}\phi_\infty} \Phi_B \mathcal{Q}_m. \tag{122}$$

The horizon radius $r_H = -c_1$ in Bianchi type I is given by (112), but in the other types it can be also rewritten in terms of mass and dilaton charge densities

$$r_H = -c_1 = \frac{1}{2\alpha} \left(2\kappa_5^2 e^{\frac{\phi_\infty}{3}} \mathcal{M} + \mathcal{D} \right). \tag{123}$$

Evidently, the solutions have been characterized by mass, magnetic charge \mathcal{Q}_m associated to B -field and the parameter

⁷ In principle, the invariance of J^μ under the symmetries of metric is not actually indicted by the field equations, but it can be desired as an extra condition. In fact, considering the homogeneous spacetimes, there exist some Killing vectors, i.e. $\mathcal{L}_\chi g = 0$, where \mathcal{L}_χ is the Lie derivative along the generator of the symmetry χ . Here, the field strength tensor H (14) and dilaton (32) have the same invariance $\mathcal{L}_\chi H = \mathcal{L}_\chi \phi = 0$. Also, the invariance of σ -model requires $\mathcal{L}_\chi B = d\omega$; however, to make the T -dual transformation valid, one can use the gauge invariance of σ -model under $B \rightarrow B + d\omega$, to choose an adopted coordinate system where all of the background fields are independent of isometry coordinate [77]. In this case, with $\mathcal{L}_\chi B = 0$ the shift symmetry current (120) respects the symmetries of metric as well.

ϕ_∞ related to the asymptotic value of dilaton. The dependence of mass on the ϕ_∞ is one of the unusual features of black hole solutions in dilatonic theories. For this type of black hole solutions a cosmological scenario has been provided in [52], where considering the ϕ_∞ as a varying parameter, the first law of black hole thermodynamics has been modified as

$$dM = \frac{1}{\kappa_5^2} \kappa dA_H + \Phi_m dQ_m + \left(\frac{\partial M}{\partial \phi_\infty} \right)_{A_H, Q_m} d\phi_\infty, \tag{124}$$

in which the last term is actually proportional to dilaton charge. As mentioned in introduction, the nature of dilaton charge, which is not a locally conserved and gauge symmetry protected charge, criticizes this modified version of first law [53,54]. Here, motivated by [63], we consider the case that besides the gravitational surface term which leads to the quasi-local mass (90), the effective action is supplemented with a dilaton field boundary term. In Hamiltonian formalism where the variation of dilaton boundary term is given by [79]

$$\delta Q_\phi[\xi] = -\frac{1}{6\kappa_5^2} \int \bar{\xi}^\perp \delta\phi \nabla_\mu \phi d^3 \Sigma^\mu, \tag{125}$$

where $\bar{\xi}^\perp = \bar{\xi} \cdot n$ with n the unit normal to space-like surface Σ and $\bar{\xi}$ is the normalized time-like killing vector. Considering the obtained metrics, the $d^3 \Sigma^r$ is proportional to r^2 and so there is neither divergent term nor contribution of \mathcal{C} in δQ_ϕ . In such a way that the only non-vanishing term in (125) at infinity is given by

$$\delta Q_\phi[\xi] = -\frac{1}{6\kappa_5^2} D e^{-\frac{\phi_\infty}{3}} \delta\phi_\infty. \tag{126}$$

In particular, the integrability of this charge needs a functional relationship between D and ϕ_∞ .⁸ In order to obtain a well defined variational principle we have to add the boundary term

$$S_\phi^{ct} = \frac{-1}{6\kappa_5^2} \int_{\partial\mathcal{M}} d^4x \sqrt{\tilde{h}} \frac{Y(\phi_\infty) e^{\frac{\phi_\infty}{3}}}{\sqrt{\tilde{\gamma}_\infty}}, \tag{127}$$

with boundary condition

$$\mathcal{D} = e^{\frac{\phi_\infty}{3}} \frac{dY(\phi_\infty)}{d\phi_\infty}. \tag{128}$$

Noting (27) and the asymptotic expansions of the metric and scalar field (24)–(26), it can be easily shown that the variation of the action which yields the following boundary term

⁸ In fact, it was first observed in AdS context in [79] that the integrability of energy in Hamiltonian formalism forces the \mathcal{D} and \mathcal{C} to be functionally related.

$$\delta(S_\phi + S_\phi^{ct}) = \frac{1}{6\kappa_5^2} \int_{\partial\mathcal{M}} d^4x \sqrt{\tilde{h}} \left[\sqrt{\tilde{g}^{rr}} \phi' \left(1 + \frac{1}{r} \frac{d\mathcal{D}}{d\phi_\infty} \right) + \frac{\mathcal{D}}{\sqrt{\tilde{\gamma}_\infty}} \right] \delta\phi_\infty, \tag{129}$$

is well defined at $r \rightarrow \infty$, i.e. $\lim_{r \rightarrow \infty} \delta(S_\phi + S_\phi^{ct}) = 0$. Now, considering (90), the total energy is given by

$$E_{Tot} = M + \frac{1}{6\kappa_5^2} \int_{3B} d^3x \sqrt{\tilde{\sigma}} h_{ab} n^a \bar{\xi}^b \frac{Y(\phi_\infty) e^{\frac{\phi_\infty}{3}}}{\sqrt{\tilde{\gamma}_\infty}} = M + \frac{1}{6\kappa_5^2} Y(\phi_\infty) \omega_3. \tag{130}$$

On the other hand, the mass (92) can be expressed in the following form using the (100), (102), and (117)⁹

$$M = \frac{1}{\kappa_5^2} \left(-\frac{1}{2} D e^{-\frac{\phi_\infty}{3}} + \alpha \kappa A_H \right). \tag{131}$$

Therefore, if the boundary term of dilaton with condition (128) is taken into account, the last term in (124) can be concealed and so the variation of the total energy in the first law of thermodynamics will be

$$dE_{Tot} = \frac{1}{\kappa_5^2} \kappa dA_H + \Phi_m dQ_m, \tag{132}$$

which, interestingly, does not contain the dilaton charge through the variation term of ϕ_∞ .

In the following subsections, we present the explicit forms of thermodynamic quantities A_H (100) and κ (102) for each Bianchi type solution to investigate the first law of black hole thermodynamics by checking the satisfiability of

$$\kappa = \kappa_5^2 \left(\frac{\partial E_{Tot}}{\partial A_H} \right)_{Q_m, \phi_\infty}, \quad \Phi_m = \left(\frac{\partial E_{Tot}}{\partial Q_m} \right)_{A_H, \phi_\infty}, \tag{133}$$

using the (112)–(115). As we have seen before, the following extra condition are required on integrating constants of solutions to ensure the finiteness of non-zero area of horizon, entropy, and surface gravity

$$\lambda = 0, \tag{134}$$

where λ for any Bianchi type as been given by (93)–(99). It will be shown that these constraints along with the initial value conditions, that have been obtained for each Bianchi type via the Eq. (21), will fix the p_i constants with compatible values with the black hole interpretation. Also, the singularity behavior of each Bianchi type solution, the classification of horizon geometries based on the correspondence between Bianchi types and Thurston geometries, and the extremal condition of the solutions will be presented.

⁹ It is not a Smarr-like formula and is given here to clarify the variation.

4.1 Bianchi types I and VII_0

The group of Bianchi type I models is isomorphic to the translation group of three-dimensional Euclidean space E^3 . This model corresponds to Thurston’s geometries of R^3 , $E^2 \times R$ and E^3 with isotropy groups of e , $SO(2)$ and $SO(3)$, respectively [30]. In addition, as we have seen in Sect. 3.7, the constraint equation (19) restricts the VII_h model to have $h = 0$ where the solutions are equivalent to those of Bianchi type I . Hence, thermodynamic properties of these two types can be investigated in parallel.

Regarding the initial condition (42) and the physical requirement of (134), in which the λ for these Bianchi types is given by (93), the allowed real values of the integrating constants are

$$p_i = 0, \tag{135}$$

which means that the generally anisotropic solutions (43) reduce to the isotropic one by imposing the physical requirement (134). Then, in the Einstein frame we have

$$ds^2 = -W^{-\frac{2}{3}}F dt^2 + W^{\frac{1}{3}}\left(\frac{dr^2}{r^4F} + \sum(\sigma^i)^2\right). \tag{136}$$

Also, the area of horizon (100) and surface gravity (102) take the following forms

$$A_H = \sqrt{c_3c_1^{-1}}\omega_3, \tag{137}$$

$$\kappa = \frac{-c_1}{2}\sqrt{c_1c_3^{-1}}e^{-\frac{\phi_\infty}{3}}. \tag{138}$$

It is straightforward to check that the first law of thermodynamics (132) is satisfied by these thermodynamic quantities. Also, the Kretschmann and Ricci scalars are proportional to the inverse of W , such that

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \propto \frac{1}{W^{\frac{14}{3}}}, \quad R \propto \frac{1}{W^{\frac{7}{3}}}. \tag{139}$$

So the metric (136) has two irremovable singularities at $r_1 = 0$ and $r_2 = \frac{c_1b^2}{c_3-b^2}$. Holding the conditions (35) the r_2 is negative. Since the metric blows up near $r = 0$, without lose of generality we will study the solutions for $r > 0$ and r_2 can be ignored. Also, it is worth mentioning that for these Bianchi type solutions the asymptotic value of γ in (40) is $\gamma_\infty = e^{-\phi_\infty}$.

As we will see in the following, all of the obtained Bianchi type solutions have the r_1 and r_2 singularities and the given discussion can be applied for all types. However, there may be other singularities in the some Bianchi types which will be discussed in any cases.

4.2 Bianchi type II

There is a correspondence between Bianchi type II and Thurston’s nilgeometry and Heisenberg group whose isotropy groups are $SO(2)$ and e , respectively [30]. The solutions in this Bianchi type have been found in Sect. 3.2. The initial condition (50) along with physical requirement of (134), in which the λ is given by (94), yields

$$p_1 = \frac{1}{2}, \quad p_2 = p_3 = 0. \tag{140}$$

With this fixed constants, the metric (51) recasts the following form

$$ds^2 = -W^{-\frac{2}{3}}F dt^2 + W^{\frac{1}{3}}V^{-1}\left(l_2^2l_3^2\frac{dr^2}{r^4F} + V^2(\sigma^1)^2 + l_2^2(\sigma^2)^2 + l_3^2(\sigma^3)^2\right), \tag{141}$$

where preserving the signature of metric at infinity and near horizon requires $q_1 < 0$ and $q_1^2 - 1 > 0$. The area of horizon (100) and surface gravity (102) are also given by

$$A_H = |l_2l_3c_1^{-1}|\sqrt{c_3q_1}\omega_3, \tag{142}$$

$$\kappa = \frac{c_1^2}{2|l_2l_3|\sqrt{c_3q_1}}e^{-\frac{\phi_\infty}{3}}, \tag{143}$$

From the first law of thermodynamics (132) point of view, if the l_1 and l_2 are taken to be non-varying constants, the first law is satisfied with $\alpha = -\frac{1}{4}$. In this case, black hole interpretation of the solution is lost because the $r_H = -c_1$ parameter becomes negative. But, a consistent example can be obtained if one sets $l_2 = l_3 = (-c_1^2q_1^{-1})^{\frac{1}{4}}$ which satisfies the first law by fixing $\alpha = \frac{1}{4}$, i.e. $q_1 = -\sqrt{3}$.

For this Bianchi type solution the asymptotic value of γ in (40) is $\gamma_\infty = -\frac{2c_1}{3}e^{-\phi_\infty}$. Also, the Ricci and Kretschmann scalars are proportional to $W^{-\frac{7}{3}}V^{-1}$ and $W^{-\frac{17}{3}}V^{-1}$, respectively. Hence, besides the $r_1 = 0$ singularity which has been mentioned in Bianchi type I , there is another initial singularity at $r_3 = \frac{c_1}{q_1^2-1}$, which is negative and can be ignored.

4.3 Bianchi type III

The Thurston geometries of $H^2 \times E^1$ (where H^2 is two-dimensional hyperbolic space) and $\widetilde{SL_2R}$ locally possess Bianchi type III symmetry with $SO(2)$ isotropy [30].¹⁰ The solutions in this type have been obtained in Sect. 3.3. The initial condition (57) along with the physical restriction (134) in which the λ for this type is given in (95), fixes the constants as

$$p_1 = \frac{1}{2}, \quad p_2 = 0. \tag{144}$$

¹⁰ The $\widetilde{SL_2R}$ can also have the Bianchi type $VIII$ symmetry.

Then, the Einstein frame metric (58) takes the following form

$$ds^2 = -W^{-\frac{2}{3}} F dt^2 + W^{\frac{1}{3}} \left(\frac{V^4}{l_2^2 r^4 F} dr^2 + l_2^2 (\sigma^2)^2 + l_2^{-2} V^2 ((\sigma^1)^2 + (\sigma^3)^2) \right), \tag{145}$$

where the V function has been defined by (56). Also, the surface gravity (102) and area of horizon (100) read

$$A_H = \frac{\sqrt{c_3 c_1^3}}{|l_2| q_1^2} \omega_3, \tag{146}$$

$$\kappa = \frac{|l_2| q_1^2}{2\sqrt{c_1 c_3}} e^{-\frac{1}{3}\phi_\infty}. \tag{147}$$

Similar to the Bianchi type II , considering the l_2 as non-varying constants, the satisfaction of first law with these thermodynamics quantities needs $\alpha = 1$. This value can not be accepted because leads to non-real c_1 and c_3 in (114) and (115). But, for example, choosing $l_2 = q_1^{-2} (-c_1)^{\frac{3}{2}}$ the first law can be satisfied consistently with $\alpha = \frac{1}{4}$, i.e. $q_1 = \sqrt{7}$.

Here the asymptotic value of γ in (40) is $\gamma_\infty = -\frac{20401c_1}{8} e^{-\phi_\infty}$. Also, investigating the behavior of Ricci and Kretschmann scalars, which are proportional to $W^{-\frac{7}{3}} V^{-5}$ and $W^{-\frac{17}{3}} V^{-11}$, respectively, shows that besides the $r_1 = 0$, there is another initial singularity at $r_3 = \frac{-c_1}{q_1^2 + 1}$, which is not naked since $r_3 < r_H$.

4.4 Bianchi type V

In isotropic case, the Bianchi type V has hyperbolic geometry H^3 with $SO(3)$ isotropy [30]. But, an anisotropic expansion is not allowed for this Bianchi type if it is required to admit a closed spatial section [30]. The solutions in this Bianchi type have been presented in Sect. 3.5, which are initially anisotropic. To have the black hole interpretation in this model, the metric (67) needs to be isotropic. Interestingly, being consistent with this demand, the solution of the set of initial value equation (66) and physical condition (134), in which the λ is given by (96), restricts the solutions to be isotropic by

$$p_1 = \frac{1}{2}, \quad p_2 = 0. \tag{148}$$

Then, the Einstein frame metric (67) recasts the following form

$$ds^2 = -W^{-\frac{2}{3}} F dt^2 + W^{\frac{1}{3}} V \left(\frac{V^2}{r^4 F} dr^2 + \sum (\sigma^i)^2 \right), \tag{149}$$

where the V function is given by (65) and the q_1 is required to be negative. Also, the (100) and (102) give

$$A_H = -c_1 \sqrt{2c_3 q_1^{-3}} \omega_3, \tag{150}$$

$$\kappa = \sqrt{2q_1^3 c_3^{-1}} e^{-\frac{1}{3}\phi_\infty}. \tag{151}$$

It can be checked that with non-dynamical q_1 , the first law of thermodynamics (132) is satisfied only with $\alpha = \frac{3}{4}$, i.e. $q_1 = -1$.

Here, the asymptotic value of γ in (40) is $\gamma_\infty = \frac{-c_1^3}{8} e^{-\phi_\infty}$. Also, the Ricci and Kretschmann scalars are proportional to $W^{-\frac{7}{3}} V^{-5}$ and $W^{-\frac{17}{3}} V^{-11}$, respectively. Then, considering the W and V functions given by (44) and (65), there are initial singularities at $r_1 = 0$ and $r_3 = \frac{-c_1}{q_1^2 + 1}$, which are hidden behind the horizon since $r_1 < r_3 < r_H$.

4.5 Bianchi type VI_{-1}

As we have mentioned in Sect. 3.6, the $h = 0, 1$ cases of VI_h are equivalent to Bianchi types III and V . Besides them, only the VI_{-1} case admits a closed spatial section and is equivalent to the Thurston geometry type of solvegeometry with Abelian isotropy [30]. Here, the integrating constants p_i are subject to the initial value condition (74). Taking into account the restrictive condition (134) along with (97), the constants are constrained to be

$$p_2 = -\frac{1}{2}, \quad p_1 = 0. \tag{152}$$

Then, the Einstein frame metric (75) recasts the following form

$$ds^2 = -W^{-\frac{2}{3}} F dt^2 + W^{\frac{1}{3}} \left(c_1^2 q_1^{-2} e^{-\frac{4F}{q_1^2}} \frac{dr^2}{r^4 F} + e^{-\frac{4F}{q_1^2}} (\sigma^1)^2 + c_1 l_2 q_1^{-1} \left((\sigma^2)^2 + l_2^{-2} (\sigma^3)^2 \right) \right), \tag{153}$$

and the thermodynamic quantities are given by

$$A_H = |q_1|^{-1} \sqrt{c_3 c_1} \omega_3, \tag{154}$$

$$\kappa = |q_1| \sqrt{c_1 c_3^{-1}} e^{-\frac{1}{3}\phi_\infty}. \tag{155}$$

Now, considering the mass of this model given by (92) and (97), it can be checked that with non-dynamical l_2 and q_1 , the first law of thermodynamics (132) can be verified by fixing $\alpha = \frac{3}{4}$, i.e. $q_1 = -2$.

The Ricci and Kretschmann scalars in these solutions have analogs behavior with the Bianchi type I and so there is an initial singularity at the origin $r_1 = 0$. Also, the asymptotic value of γ in (40) for this type is $\gamma_\infty = \frac{c_1^2}{4} e^{-(1+\phi_\infty)}$.

4.6 Bianchi type VIII

The Thurston type geometry in this Bianchi type is \widetilde{SL}_2R and the isotropy groups are e and $SO(2)$ [30]. The solutions in this type have been presented in Sect. 3.8. Considering the initial value equation (82) and the condition (134), with using (98), the constants are fixed as follows

$$p_1 = p_2 = \frac{1}{2}. \tag{156}$$

Then, the Einstein frame metric (83) recasts the following form

$$ds^2 = -W^{-\frac{2}{3}}F dt^2 + W^{\frac{1}{3}}\left(V_1^4V_2^{-1}\frac{dr^2}{r^4F} + V_2(\sigma^3)^2 + V_1^2V_2^{-1}((\sigma^1)^2 + (\sigma^2)^2)\right), \tag{157}$$

in which the V_1 and V_2 functions are given by (81). The q_1 can have any sign but q_2 needs to be positive in such a way that $q_2^2 - 1 > 0$. Also, the thermodynamic quantities are given by

$$A_H = -c_1q_1^{-2}\sqrt{-c_3q_2}\omega_3, \tag{158}$$

$$\kappa = \frac{q_1^2}{\sqrt{-c_3q_2}}e^{-\frac{1}{3}\phi_\infty}. \tag{159}$$

By assuming q_1 and q_2 as non-varying parameters, the first law of thermodynamics (132) can be confirmed if the α parameter in the mass (92) equals to $\frac{3}{4}$, i.e. $\frac{2}{q_1^2+1} + \frac{1}{2(q_2^2-1)} = \frac{3}{4}$.

Furthermore, the Ricci and Kretschmann scalars are proportional to $W^{-\frac{7}{3}}V_1^{-6}V_2^{-1}$ and $W^{-\frac{17}{3}}V_1^{-14}V_2^{-2}$, respectively. Accordingly, besides $r_1 = 0$, there are two other initial singularities at $r_3 = \frac{-c_1}{q_1^2+1}$, and $r_4 = \frac{c_1}{q_1^2-1}$. Obviously, $r_1 < r_3 < r_H$ and r_3 is not a naked singularity. But, the requirement of $q_2^2 - 1 > 0$ implies that the r_4 is negative and can be ignored. The asymptotic value of γ in (40) for this type is $\gamma_\infty = -\frac{c_1^3q_1^4(q_2^2-1)}{q_2(q_1^2+1)}e^{-\phi_\infty}$.

4.7 Bianchi type IX

The Bianchi type IX has the spherical Thurston geometry S^3 and a group isomorphic to three-dimensional rotation group $SO(3)$ [30]. Solutions of this type are given in Sect. 3.9. The initial condition (88) along with condition (134), in which the λ is given by (99), fixes the constants as

$$p_1 = p_2 = \frac{1}{2}. \tag{160}$$

With these constants the Einstein frame metric (89) gets the form of (157), but the V_1 and V_2 functions for this Bianchi type are given by (87). Also, the q_2 needs to be negative with

$q_2^2 - 1 > 0$ but q_1 can have any sign. The area of horizon (100) and surface gravity (102) are

$$A_H = \frac{-c_1\sqrt{c_3q_2}}{q_1^2}\omega_3, \tag{161}$$

$$\kappa = \frac{q_1^2}{\sqrt{c_3q_2}}e^{-\frac{1}{3}\phi_\infty}. \tag{162}$$

Checking the first law of thermodynamics (132) with assuming the q_1 and q_2 as non-varying parameters, shows that the first law is satisfied if the α parameter in mass expression (92) for this Bianchi type equals to $\frac{3}{4}$, i.e. $\frac{2}{q_1^2-1} + \frac{1}{2(q_2^2-1)} = \frac{3}{4}$.

This Bianchi type can admit a constant curvature type horizon, i.e. with $R_{ij} = kg_{ij}$ by constant k , if one sets $\pm q_1 = q_2$. The Ricci and Kretschmann scalars are proportional to $W^{-\frac{7}{3}}V_1^{-6}V_2^{-1}$ and $W^{-\frac{17}{3}}V_1^{-14}V_2^{-2}$, respectively. Hence, there are three initial singularity at $r_1 = 0, r_3 = \frac{c_1}{q_1^2-1}$, and $r_4 = \frac{c_1}{q_2^2-1}$. Since $q_2^2 - 1 > 0$ then $r_3 < 0$ and can be ignored. However, we don't have such a condition on q_1 but in the case of $\pm q_1 = q_2$ the r_4 is negative as well. The asymptotic value of γ in (40) for this type is then $\gamma_\infty = -\frac{39\sqrt{39}c_1^3}{1000}e^{-\phi_\infty}$.

Possessing positive Ricci scalar with respect to the induced metric on the horizon, Bianchi type IX model is the only one which is allowed by the horizon theorem [15] to be asymptotically flat. This limit can be obtained by setting $c_3 = -\sqrt{2}b$, as indicated by (40). The first law of thermodynamics for this solution is verified with $\alpha = 0$. In this case, the quasi-local mass equals the factor $\frac{4}{3}$ times the Kumar energy obtained by the normalized time-like killing vector ξ .

4.8 Near-horizon, extremal and asymptotic limits of the solutions

To end this section, let us investigate some behaviors of the solutions in near-horizon, extremal, and large r limits. We have seen in Sects. 4.1–4.7 that the physical requirement (134) together with the initial value conditions on constants, obtained from (21), fixed the p_i constants, identifying the final forms of black hole metrics. Also, satisfaction of the first law of thermodynamics in Bianchi types II–IX constrained the α (or equivalently the q_i) constants to take special values. Eventually, the obtained Einstein frame metrics (136), (141), (145), (149), (153), and (157) can be written in the following general form

$$d\tilde{s}^2 = -W^{-\frac{2}{3}}F dt^2 + \frac{W^{\frac{1}{3}}X}{r^4F} dr^2 + \sum_{i=1}^3 \tilde{g}_i^2 (\sigma^i)^2, \tag{163}$$

where the $F(r)$ and $W(r)$ are given by (31) and (44) but the $X(r)$ and $\tilde{g}_i(r)$ functions are different for any Bianchi types

and can be read simply from the obtained metrics. To find the near horizon limit of these metrics we define $\xi = r - r_H$ and so as $r \rightarrow r_H$, i.e. in $\xi \rightarrow 0$ limit, (163) reduces to

$$d\tilde{s}^2 = -W_H^{-\frac{2}{3}} r_H^{-1} \xi dt^2 + \frac{W_H^{\frac{1}{3}} X_H}{r_H^3} \frac{d\xi^2}{\xi} + \sum_{i=1}^3 \tilde{g}_{iH}^2 (\sigma^i)^2, \tag{164}$$

in which the W_H , X_H , and \tilde{g}_{iH} are finite values taken by the $W(r)$, $X(r)$, and $\tilde{g}_i(r)$ functions on the horizon. Then, introducing a new coordinate x by

$$\xi = \frac{r_H^3 x^2}{4X_H W_H^{\frac{1}{3}}}, \tag{165}$$

we get

$$d\tilde{s}^2 = -\left(4W_H X_H r_H^{-2}\right)^{-1} x^2 dt^2 + dx^2 + \sum_{i=1}^3 \tilde{g}_{iH}^2 (\sigma^i)^2, \tag{166}$$

Now, a closer look at the presented solutions in 4.1–4.7 sections reveals that the factor $\left(4W_H X_H r_H^{-2}\right)^{-1}$ for each Bianchi type solution coincides the square of its surface gravity κ . Therefore, the (t, x) part of spacetime in the above metric is a flat Minkowski space written in the Rindler coordinates.

In addition, the regularity of the black hole solutions can be investigated to obtain the restriction on the amount of magnetic charge that can be carried by these black holes. From a physical point of view, the required condition is $r_H = -c_1 > 0$. For Bianchi types *I* and *VII*₀, considering (112), the black hole interpretation of the solutions will be lost unless

$$M^2 > \frac{1}{4\kappa_5^2} Q_m^2 e^{\frac{4}{3}\phi_\infty}. \tag{167}$$

Also, for the Bianchi types *II* and *III* the first law of thermodynamics has been satisfied with $\alpha = \frac{1}{4}$. In this case, the real and negative integration constants c_1 and c_3 , given by (114) and (115), require again the same condition (167) to hold. The Bianchi types *V*, *VI*₋₁, *VIII* and *IX* satisfied the first law with $\alpha = \frac{3}{4}$, where the integration constants c_1 (114) and c_3 (115) are real, but negativity of them is again guaranteed by the restriction of type (167). In other words, the condition (167) gives the extremal condition on the magnetic charge of the all obtained Bianchi type black hole solutions. For all solutions, in the extremal limit as $Q_m \rightarrow 2\kappa_5 e^{-\frac{2}{3}\phi_\infty} M$, we have $c_3 \rightarrow b$ and $r_H = -c_1 \rightarrow 0$. In fact similar to Einstein–Maxwell–dilaton solutions [1], the solutions have two horizons at $r = -c_1$ and $r = 0$. The $r = -c_1$ is the event horizon and $r = 0$ is generally singular. In extremal case these two horizons coincide at $r = 0$.

Near the horizon in (near) extremal limit the behavior of temperature and entropy depends on the type of Bianchi as follows¹¹

$$I \ \& \ VII_0: \quad T_H \rightarrow 0, S \rightarrow \infty, \tag{168}$$

$$II \ \& \ III: \quad T_H \rightarrow 0, S \rightarrow \sqrt{\sqrt{2\kappa_5 Q_m}/4\omega_3}, \tag{169}$$

$$VI_{-1}: \quad T_H \rightarrow 0, S \rightarrow 0, \tag{170}$$

$$V: \quad T_H \rightarrow \sqrt{2(\sqrt{2\kappa_5 Q_m})^{-1}}, S \rightarrow 0, \tag{171}$$

$$VIII \ \& \ IX: \quad T_H \rightarrow q_1^2 \sqrt{2|q_2|(\sqrt{2\kappa_5 Q_m})^{-1}}, S \rightarrow 0. \tag{172}$$

For Bianchi types *I* and *VII*₀ the behavior is similar to that of Rindler space times [80]. In this limits, the *II* and *III* types have finite entropy with vanishing temperature, similar to extreme near horizon Reissner–Nordstrom black hole solution. Thermodynamic analysis of black hole solutions with this behavior has been studied in [81,82]. The *VI*₋₁ type solution has finite-valued κ/A_H in this limit, considering (154) and (155). This characteristic, along with (170), reminds the Extremal Vanishing Horizon (EVH) type black holes [83,84]. However, noting (153), the \tilde{g}_i part of near horizon metric (166) in this type does not have any zero eigenvalues and hence vanishing of entropy is not accompanied by a vanishing one-cycle on the horizon. Therefore, the *VI*₋₁ solution cannot be regarded as an EVH black hole. Also, for *V*, *VIII* and *IX* type solutions the behavior of temperature and entropy is similar to that of Schwarzschild black holes in the zero-mass limit [85].

It is worth considering the attractor mechanism [86,87] here to check the consistency of the analysis. The dilaton for metric (166) is

$$\phi(x) = \frac{3}{4} \ln \left(\frac{3c_1 c_3}{2(b c_1 x + c_3)^2} \right), \tag{173}$$

and no matter what is the value of ϕ_∞ , near the horizon we have

$$e^{\phi_H} \equiv \lim_{x \rightarrow 0} e^{\phi(x)} = \left(\frac{3c_1}{2c_3} \right)^{\frac{3}{4}}, \quad \phi'_H \equiv \lim_{x \rightarrow 0} \phi(x)' = \frac{3b c_1}{2c_3}.$$

In the extremal limit, ϕ'_H vanishes, which is a fixed point and $e^{\phi_H} = \left(\frac{0}{2Q_m} \right)^{\frac{3}{4}} = 0$. Also, as discussed in Appendix B, using the equation of motion we can consider $V_{eff} = 2\kappa_5^2 Q_m^2 e^{2\phi}$, where the attractor condition $\partial_\phi V_{eff} = 0$ [87] leads again to $e^{\phi_H} = 0$.¹² Also, using the obtained expres-

¹¹ It should be noted that in the extremal limit we have $c_1 \rightarrow 0$ and $c_3 \rightarrow b$, simultaneously. It can be checked using L'Hospital's rule that the ϕ_∞ (38) is finite in this limit.

¹² In comparison with the cases of $V_{eff} = P^2 e^{\alpha_1 \phi} + Q^2 e^{-\alpha_2 \phi}$, where α_1 and α_2 have the same signs and $e^{(\alpha_1 + \alpha_2)\phi_0} = \frac{\alpha_2 Q^2}{\alpha_1 P^2}$ extremises the effective potential [54,87], here we have $Q = 0$ and P proportional to

sion for $\tilde{\gamma}_H$ in (202), noting (100) and (106), it can be easily checked that in extremal limit, this formula gives the same behavior for entropy as given in (168)–(172).

In addition, using (38), (107) and (134), the mass of black holes (92) can be rewritten as follows

$$\mathcal{M} = -\frac{Q_m^2}{c_3} e^{\frac{2}{3}\phi_\infty} + \frac{\alpha r_H}{\kappa_D^2} e^{-\frac{\phi_\infty}{3}}. \tag{174}$$

In the high energy limit $r_H \rightarrow 0$ [88], which can be alternatively regarded as extremal limit here, the mass is non zero and is govern by the magnetic charge term, where in the asymptotic limit, i.e. as $r_H \rightarrow \infty$, the dominant term is the second term which is zero for flat horizon cases.

It is worth mentioning that although the metric (163) looks singular as $r \rightarrow \infty$, its asymptotic scalar curvature given by (40) is finite in this limit, noting that the value of $\tilde{\gamma}_\infty$ for each Bianchi type solution given in previous subsections is finite and non-zero. Also, the Ricci scalar (39) on the horizon becomes $\tilde{R} = -\frac{b^2(c_1c_3^{-1})^{\frac{4}{3}}}{6\tilde{\gamma}_H}$ and $\tilde{\gamma}_H$ is finite for all Bianchi type solutions.

Furthermore, non-zero components of effective energy momentum tensor (8) are given by

$$\begin{aligned} T_i^{i(\text{eff})} = -T_r^{r(\text{eff})} &= \frac{1}{12\tilde{\gamma}} \left(3b^2 e^{\frac{4\phi}{3}} - 2c_1^2 F \left(\frac{d\phi}{dF} \right)^2 e^{-\frac{4\phi}{3}} \right), \\ -T_t^{t(\text{eff})} &= \frac{1}{12\tilde{\gamma}} \left(3b^2 e^{\frac{4\phi}{3}} + 2c_1^2 F \left(\frac{d\phi}{dF} \right)^2 e^{-\frac{4\phi}{3}} \right). \end{aligned} \tag{175}$$

In $r \rightarrow \infty$ limit

$$\begin{aligned} T_i^{i(\text{eff})} = -T_r^{r(\text{eff})} &= \frac{b^2}{12c_3^2\tilde{\gamma}_\infty} \left(3c_3^2 - 2b^2 \right) e^{\frac{4\phi_\infty}{3}}, \\ -T_t^{t(\text{eff})} &= \frac{b^2}{12c_3^2\tilde{\gamma}_\infty} \left(3c_3^2 + 2b^2 \right) e^{\frac{4\phi_\infty}{3}}, \end{aligned} \tag{176}$$

indicating that the pressures and energy density do not diverge at $r \rightarrow \infty$ limit. Also, noting (176), or equivalently the (36), the asymptotic behavior of solutions can not be regarded as asymptotically (A)dS behavior. Then, as we have mentioned before, assuming $c_3 \neq -\sqrt{2}b$ in (40), we call the solutions non-asymptotically flat, non-(A)dS topological black hole solutions. On the other hand, the $\tilde{\gamma}$ of any Bianchi type solutions diverges at its irremovable singularities, the pressures and energy density blow up at these points.

Q_m which, as mentioned before, can be interpreted as electric charge in Hodge dual theory.

5 T-dual solutions of Bianchi type V black hole

In the presence of the isometries of background fields, where the following conditions are satisfied

$$\mathcal{L}_\chi g = \mathcal{L}_\chi \phi = \mathcal{L}_\chi H = 0, \tag{177}$$

with killing vectors χ , the Buscher’s T -duality transformations are valid [89]. In this case, a convenient coordinate system can be adopted where all of the background field are independent of the isometry direction x and $\mathcal{L}_\chi = \partial_x$ [77]. Then, the T -dual transformation with respect to the isometry direction x are given by¹³

$$\bar{g}_{xx} = \frac{1}{g_{xx}}, \bar{g}_{x\mu} = \frac{B_{x\mu}}{g_{xx}}, \bar{B}_{x\mu} = \frac{g_{x\mu}}{g_{xx}}, \tag{178}$$

$$\bar{B}_{\mu\nu} = B_{\mu\nu} + \frac{(g_{x\mu}B_{\nu x} - g_{x\nu}B_{\mu x})}{g_{xx}}, \tag{179}$$

$$\bar{g}_{\mu\nu} = g_{\mu\nu} - \frac{(g_{x\mu}g_{x\nu} - B_{x\mu}B_{x\nu})}{g_{xx}}, \tag{180}$$

$$\bar{\phi} = \phi + \ln |g_{xx}|, \tag{181}$$

where $\bar{g}_{\mu\nu}$, $\bar{B}_{\mu\nu}$ and $\bar{\phi}$ are metric, antisymmetric tensor field and dilaton of the T -dual σ -model.

Here, as an example, we investigate the T -dual version of the black hole solutions of Bianchi type V. Considering the vielbeins of this Bianchi type given in the Table 1 of Appendix A, there are two isometry directions of x^2 and x^3 . In this respect, the antisymmetric B -field associated to field strength tensor (14) can be considered as¹⁴

$$B = \frac{1}{4} b e^{2x^1} dx^2 \wedge dx^3. \tag{182}$$

Now, given the dilaton (32), B -field (182), and the metric (13) whose components are given by (27), (31) and (62)–(64), we perform the T -dual transformations of (179)–(181) for example with respect to x^2 . Then, the T -dual dilaton, its asymptotic value, and dilaton charge are obtained as follows

$$\begin{aligned} \bar{\phi} &= \ln(V) + 2x^1 + \frac{1}{2} (4p_2 + 2p_1 - 1) F, \\ \bar{\phi}_\infty &= 2x^1 + \ln \frac{c_1 q_1}{2(q_1^2 + 1)}, \\ \bar{D} &= \frac{c_1}{2} (4p_2 + 2p_1 - 1) - \frac{2p_1 c_1}{q_1^2 + 1}, \end{aligned} \tag{183}$$

where the $V(r)$ function is given by (65). Furthermore, all of the components of T -dual B -field vanish and we have $\bar{B} = 0$. Investigating the properties of T -dual metric, obtained by

¹³ Here, the (181) is given based on our notation for the β -function equations of (1)–(3) in which the dilaton is minus twice of the dilaton of Refs. [65,89].

¹⁴ The (182) is the general form of B -field which leads to the field strength tensor (14) with $\mathcal{L}_\chi B = 0$, without including the constant components of B which have no relevant physical interpretation here.

(179) and (180), shows that the determinant of the three-dimensional metric in Einstein frame, $\tilde{\gamma}$, has been kept invariant. Here, the non-zero and finite area of horizon and surface gravity in (101) impose the conditions of $\lambda = 0$ and $13 p_1 + 8 p_2 - 13/2 = 0$, which along with initial condition (66) fix the integrating constants again as (148). With these fixed values, the first terms in dilaton and its charge in (183) vanish and the T -dual string frame metric takes the following form

$$ds^2 = -F dt^2 + \frac{WV^3}{F r^4} dr^2 + WV(dx^1)^2 + (WV)^{-1} \left(e^{-2x^1} (dx^2)^2 + b dx^2 dx^3 + e^{2x^1} \left((WV)^2 + b^2 \right) (dx^3)^2 \right) \tag{184}$$

where $W(r)$ function is

$$W = \frac{c_3^2 - b^2 F}{c_1 c_3}. \tag{185}$$

The Einstein frame metric can be easily obtained by performing the conformal transformation (5) using T -dual dilaton (183). It can be checked that the T -dual solutions in Einstein frame are again hairy black hole solutions where the location of the horizon is unchanged, given by $r_H = -c_1$. Here, the second term in shift symmetry current (120) vanishes but the x^1 dependence in the dilaton (183) indicates that the non-vanishing components are J^1 and J^r . This current has a finite norm on the horizon and, as a consequence of having the J^1 component, its conservation has no conflict with the presence of dilaton charge. Also, the singularity properties of the black hole have not changed and the Kretschmann and Ricci scalars of the T -dual black hole are divergent at the same non-naked singularities of $r_1 = 0$ and $r_3 = \frac{-c_1}{q_1^2 + 1}$.

The invariance of $\tilde{\gamma}$ under T -duality shows that the area of horizon and entropy remain invariant under T -duality. Also, the mass (92) (specialized for Bianchi type V with (96)), surface gravity (151) and consequently the temperature (105) will be rescaled under T -duality, but their final forms are similar to the original ones and just rewritten in terms of $\bar{\phi}_\infty$ instead of ϕ_∞ as

$$\bar{\mathcal{M}} = -\frac{c_1 e^{-\frac{1}{3}\bar{\phi}_\infty}}{2\kappa_D^2} \left(\frac{b^2}{c_3^2 - b^2} + \frac{3}{q_1^2 + 1} \right), \tag{186}$$

$$\bar{\kappa} = \frac{1}{|l_2 l_3|} \sqrt{\frac{2q_1^3}{c_3}} e^{-\frac{1}{3}\bar{\phi}_\infty}. \tag{187}$$

However, except c_1 , the other constants' role has been changed in T -dual version of mass. The $\bar{B} = 0$ indicates that no charge associated with B -field is carried by the T -dual solutions and the b is just a constant appeared in three-dimensional part of the metric (184). Also, instead of c_3 , the

q_1 is related to dilaton here and the inverse relations for c_1 and q_1 are given by

$$-c_1 = -\frac{4e^{\frac{1}{3}\bar{\phi}_\infty} \bar{\mathcal{M}}}{\beta} + \frac{12e^{\frac{5}{3}\bar{\phi}_\infty}}{\sqrt{\bar{\mathcal{M}}^2 + (-\beta^2 + 6\beta)e^{\frac{4}{3}\bar{\phi}_\infty} - \bar{\mathcal{M}}}}, \tag{188}$$

$$-q_1 = \beta^{-1} \left(-\bar{\mathcal{M}} e^{-\frac{2}{3}\bar{\phi}_\infty} + \sqrt{\bar{\mathcal{M}}^2 e^{-\frac{4}{3}\bar{\phi}_\infty} - \beta^2 + 6\beta} \right), \tag{189}$$

in which $\beta = \frac{b^2}{c_3^2 - b^2}$. Now, considering β as a non-varying constant, the first law of thermodynamics can be verified with $\beta = \frac{1}{4}$. Then, the dilaton charge can be expressed in terms of T -dual mass by

$$\bar{D} = \frac{-2e^{\frac{5}{3}\bar{\phi}_\infty}}{\bar{\mathcal{M}} - \sqrt{\bar{\mathcal{M}}^2 + 5e^{\frac{4}{3}\bar{\phi}_\infty}}}, \tag{190}$$

which indicates that the T -dual solution has also the scalar hair of second kind.

6 Conclusion

We have constructed five-dimensional topological black hole solutions at leading order of string effective action in presence of dilaton and antisymmetric B -field associated with a magnetic charge. The asymptotic behavior of the solutions at infinity have been affected by the presence of the magnetic charge such that the solutions are neither asymptotically flat nor asymptotically (A)dS. The solutions have been assumed to have horizons with Bianchi type symmetries where in the context of violated asymptotic flatness assumption of the horizon geometry theorem [15], in addition to flat Bianchi type I and positively curved Bianchi type IX , the other negatively curved Bianchi spaces were also considered. These solutions can be regarded as black hole solutions whose three-dimensional horizons are modeled on seven types of Thurston 3-geometries corresponding to Bianchi types, where there exist more possibilities for the horizon geometry than the known constant curvature types of spherical, hyperbolic or flat cases, which are given by product constant curvature type $H^2 \times R$ and twisted product types of $\widetilde{SL_2R}$, nilgeometry, and solvegeometry.

Possessing non-trivial configuration, the dilaton field introduced a dilaton charge in the solutions. We gave an argument based on the Noether current of shift symmetry, demonstrating that this current does not cover all of the assumptions of the no-hair theorem proof of [75] and so the presence of dilaton charge has no conflict with the conservation of this current. The solutions have dilaton hair which is, however, of secondary type in the sense that the dilaton charge is not

an independent charge carried by the black hole solutions and can be entirely determined by the magnetic charge and the mass of the black holes. Hence, the no-hair conjecture still holds. Also, the dilaton charge turned to be proportional to the magnetic charge multiplied by the magnetic potential through Eq. (122). It is worth mentioning that a quite similar relation has been obtained in [76] between the scalar charge and the magnetic charge, accompanied by its potential, in the case of the massless field ϕ coupled to the electromagnetic field through the second Chern character by using the Noether current of shift symmetry.

The mass and other thermodynamic quantities depend in a non-trivial way on the asymptotic value of dilaton ϕ_∞ . The problem with the drastic modification of the first law of (static) hairy black hole thermodynamics which contains the variation of ϕ_∞ has been considered. Following the idea of [63], where a boundary term for dilaton field was included in the action in the quasi-local formalism of mass, we considered the case that the effective action is also supplemented with a boundary term for dilaton. However, only the spherically symmetric solutions have been considered in [63]. We applied a similar discussion inspired by the proposed boundary term variation of dilaton in [79], where with vanishing ϕ_∞ , the \mathcal{D} and \mathcal{C} coefficients in the asymptotic expansion of the scalar field were required to be functionally related for integrability of energy. This relation which is usually imposed as a boundary condition, can be fixed uniquely where the asymptotic AdS symmetry is of interest. Particularly, the $\mathcal{C} = k\mathcal{D}^2$ with a constant k , besides preserving AdS invariance, is compatible with the conformal symmetry of the boundary [90] and makes the contribution of the scalar field in energy vanish [91]. When the conformal invariance is broken, the trace anomaly leads to an extra contribution of dilaton to the total energy with interesting interpretations [90, 92]. For instance, in dyonic black hole solution, this contribution leads to satisfactions of the first law without including an extra scalar charge dependent term [63]. In our solutions we are dealing with non-AdS cases, however the obtained dilaton solution satisfies $\mathcal{C} = \frac{\mathcal{D}^2}{2}$ if the regularity of ϕ and ϕ' on the horizon is required. The ϕ_∞ is non-zero here and δQ_ϕ gives only a term with contribution of ϕ_∞ and \mathcal{D} , which following [79], have been considered to be functionally related. Similar to [63], it turns out that, the scalar field gives a non-vanishing contribution to the total energy which leads to a well defined variational principle in such a way that even if the ϕ_∞ is assumed to vary the first law of thermodynamics does not include the scalar charge dependent term.

Thermodynamics of the solutions and satisfaction of the first law of thermodynamics have been investigated for all Bianchi type solutions that admit compact event horizons. The solutions are characterized by three parameters: the mass

M , the B -field magnetic charge Q_m , and the constant ϕ_∞ . We expressed the integrating constants of c_1 , c_3 , and b in terms of these parameters where the remaining constants have been fixed by requiring some restrictive conditions. In such a way that, the n and p_i have been fixed by certain conditions, including the initial value condition, the regularity of dilaton on the horizon, and the finite non-vanishing surface gravity and area of horizon in the non-extremal black hole cases. Also, it has been observed that the first law holds for Bianchi type I solutions and for the other types its satisfactions requires fixing the integrating constants of q_i .

The solutions have a horizon hiding the scalar curvature singularities including one at the origin for all Bianchi types and another one for some of them, where the later singularity actually appears where the g_i components of the metric vanish. At these points, the Ricci and Kretschmann scalars, the energy momentum invariants T_μ^μ and $T_{\mu\nu}T^{\mu\nu}$, energy density, and pressures of the effective matter, which has been considered to include the contributions of dilaton and B -field, blow up.

The Hodge dual of our solutions can be interpreted as electrically charged black hole solutions of Einstein–Maxwell–dilaton theory. The dual transformation keeps the Einstein frame metric fixed but changes the sign of dilaton. This implies that the properties that depend on the Einstein metric are not influenced by this transformation (for instance, the singularity behavior and thermodynamic properties). But, the dilaton charge, which was negative for magnetically charge solutions, is positive for electrically charged ones. It is worth mentioning that for electrically charged black hole solutions the action should be supplemented with a surface term which is required to make the variation of the Hamiltonian well defined [74, 93]. This surface term is zero in the magnetic cases but the variational principle for the gauge field should be considered carefully.

The extremal and near horizon limit of the solutions has been also studied. We have shown that in near horizon limit 2-dimensional Rindler space can be recovered in (t, r) part of spacetime. Also, the extremal condition appeared to be the same for all Bianchi type black hole solutions so the solutions are regular only if $4\kappa_5^2 M^2 - Q_m^2 e^{\frac{4}{3}\phi_\infty} > 0$. Violation of this inequality changes the nature of the curvature singularity where the naked singularities appear for a sufficiently high magnetic charge.

Furthermore, T -dual transformation is allowed by the symmetries of the considered spacetimes. As an example, a class of T -dual solutions is obtained in Bianchi type V which has a non-trivial dilaton and vanishing B -field. This solution admits hairy black hole, where the scalar charge, being completely determined by the black hole mass, is secondary and there is no naked singularity. However, the mag-

netic charge has not been preserved and no B -field charge is carried by the T -dual black hole. Examining the properties and thermodynamics of this solution showed that the location of horizon is unchanged and the area of horizon and consequently the entropy remain invariant under T -duality. In fact, the T -duality invariance of entropy has been pointed out first in [94], where the surface gravity and temperature are invariant as well. However, unlike the considered solutions in [94], the ϕ_∞ is non-zero here and the mass, surface gravity, and temperature which depend on ϕ_∞ have been rescaled in such a way that the T -dual versions have a similar expressions to the original ones, but they have been rewritten in terms of asymptotic value of T -dual dilaton, $\bar{\phi}_\infty$.

It worth mentioning that families of five-dimensional black hole solutions of gravity theories whose three-dimensional horizons are modeled by some of the eight Thurston geometries have been obtained for instance in [37, 39–41]. Especially, the non-trivial geometries which are not constant curvature or product of constant curvature types hold attention. In this category, the solution with nilgeometry and solvegeometry horizons have been studied, but the case of \widetilde{SL}_2R Thurston geometry, having somewhat more complicated field equations, have been left open. We have found black hole solutions in two Bianchi types III and $VIII$ which correspond to \widetilde{SL}_2R geometry. The Lifshitz black hole solutions in these two Bianchi types in Einstein gravity with cosmological constant have been obtained in [38]. However, differently from their solutions which have zero entropy with non-zero temperature, we have seen that the our solutions admit non-zero entropy with some fixed values of integrating constants.

The magnetic and scalar charges, masses, and the area of horizons appeared to have a dependence on the volume of three-dimensional homogeneous part of space, ω_3 . This aspect of the solutions is similar to that of four-dimensional topological black hole solutions where some of thermodynamic quantities are proportional to the area of ω_2 [18]. The ω_2 can be completely determined by the topology of horizon in terms of the Euler characteristic of 2-manifold. But, a difficulty in the investigating of 3-manifolds has been the lack of such a good topological invariant, like the Euler characteristic of 2-manifolds. In fact, the Euler characteristic of closed 3-manifolds is zero [28]. However, in some cases the w_3 can be regarded as a topological invariant and further work is under progress in this sense.

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A Bianchi type classification

In this Appendix, the classification of real three-dimensional Bianchi type Lie algebras are presented. The relation between coordinate and non-coordinate basis is given by

$$\sigma^i = e_\alpha^i(x) dx^\alpha, \tag{191}$$

where $\{\sigma^i, i = 1, 2, 3\}$ are left invariant basis 1-forms and $e_\alpha^i(x)$ are vielbeins. For each Bianchi type the associated vielbeins, corresponding Thurston type geometries and the isotropy groups are presented in Table 1, in which for Bianchi type VII_h we have defined $X = e^{-kx^1} \cos(ax^1)$, $Y = -\frac{1}{a}e^{-kx^1} \sin(ax^1)$, $k = \frac{h}{2}$, and $a = \sqrt{1 - k^2}$.

Considering the metric (13), non-zero components of Riemann and Ricci tensors are given by

$$\begin{aligned} R^i_{jkl} &= \Gamma_{lj}^m \Gamma_{km}^i - \Gamma_{kj}^m \Gamma_{lm}^i - f_{kl}^m \Gamma_{mj}^i, \\ R^r_{jrl} &= \Gamma_{lj}^{r'} + \Gamma_{lj}^m \Gamma_{rm}^r - \Gamma_{rj}^m \Gamma_{lm}^r, \\ R^r_{trt} &= \Gamma_{tt}^{r'} + \Gamma_{tt}^r \Gamma_{rr}^r - \Gamma_{rt}^r \Gamma_{tt}^r, \\ R^r_{jkl} &= \Gamma_{lj}^m \Gamma_{km}^r - \Gamma_{kj}^m \Gamma_{lm}^r - f_{kl}^m \Gamma_{mj}^r, \\ R_{ij} &= \Gamma_{ji}^{r'} + \Gamma_{ji}^e \Gamma_{de}^d + \Gamma_{ji}^r \Gamma_{dr}^d + \Gamma_{ji}^r \Gamma_{rr}^r - \Gamma_{di}^e \Gamma_{je}^d \\ &\quad - \Gamma_{di}^r \Gamma_{jr}^d - \Gamma_{ri}^e \Gamma_{je}^r - f_{dj}^e \Gamma_{ei}^d, \\ R_{rj} &= \Gamma_{jr}^e \Gamma_{de}^d - \Gamma_{dr}^e \Gamma_{je}^d, \\ R_{rr} &= \Gamma_{rr}^{r'} - \Gamma_{jr}^i \Gamma_{rr}^j - \Gamma_{tr}^t \Gamma_{rt}^t, \end{aligned} \tag{192}$$

where the prime symbol stands for derivative with respect to r and f_{ij}^k are the structure constants of Bianchi Lie algebras. Also, the connection coefficients are given by

$$\begin{aligned} \Gamma_{ij}^k &= -g^{dk} (f_{id}^e g_{ej} + f_{jd}^e g_{ei}) + f_{ij}^k, \\ \Gamma_{ir}^i &= \ln g'_{ii}, \quad \Gamma_{tr}^t = \Gamma_{rt}^t = \frac{1}{2} \ln F' \\ \Gamma_{ij}^r &= -\frac{1}{2} F U^2 g_{ij}', \quad \Gamma_{rr}^r = \ln (F U^2)', \quad \Gamma_{tt}^r = \frac{1}{2} F U^2 F'. \end{aligned} \tag{194}$$

The Riemann and Ricci tensors in coordinate basis can be obtained by multiplying the vielbeins e_α^i , for example $R_{\alpha\beta} = e_\alpha^i e_\beta^j R_{ij}$ and $R_{\alpha 0} = e_\alpha^i R_{i0}$.

Table 1 Bianchi-type classification

Type	Class	$d\sigma^i = \frac{1}{2} f_{jk}^i \sigma^j \sigma^k$	Coordinate basis	Geometry	Isotropy
I	A	$d\sigma^1 = 0$	$\sigma^i = dx^i$	E^3	$SO(3)$
		$d\sigma^2 = 0$		$E^2 \times R$	$SO(2)$
		$d\sigma^3 = 0$		R^3	e
II	A	$d\sigma^1 = \sigma^2 \wedge \sigma^3$	$\sigma^1 = dx^2 - x^1 dx^3$	Heisenberg G Nilgeometry	e
		$d\sigma^2 = 0$	$\sigma^2 = dx^3$		$SO(2)$
		$d\sigma^3 = 0$	$\sigma^3 = dx^1$		
III	B	$d\sigma^1 = 0$	$\sigma^1 = dx^1$	$H^2 \times E^1$ $\widetilde{SL_2R}$	$SO(2)$
		$d\sigma^2 = 0$	$\sigma^2 = dx^2$		$SO(2)$
		$d\sigma^3 = \sigma^1 \wedge \sigma^3$	$\sigma^3 = e^{x^1} dx^3$		
IV	B	$d\sigma^1 = \sigma^1 \wedge \sigma^3 + \sigma^2 \wedge \sigma^3$	$\sigma^1 = e^{-x^1} dx^2 - x^1 e^{-x^1} dx^3$	-	
		$d\sigma^2 = \sigma^2 \wedge \sigma^3$	$\sigma^2 = e^{-x^1} dx^3$		
		$d\sigma^3 = 0$	$\sigma^3 = dx^1$		
V	B	$d\sigma^1 = 0$	$\sigma^1 = dx^1$	H^3	$SO(3)$
		$d\sigma^2 = \sigma^1 \wedge \sigma^2$	$\sigma^2 = e^{x^1} dx^2$		
		$d\sigma^3 = \sigma^1 \wedge \sigma^3$	$\sigma^3 = e^{x^1} dx^3$		
VI_a	B	$d\sigma^1 = 0$	$\sigma^1 = dx^1$		
		$d\sigma^2 = h\sigma^1 \wedge \sigma^2$	$\sigma^2 = e^{hx^1} dx^2$		
		$d\sigma^3 = \sigma^1 \wedge \sigma^3$	$\sigma^3 = e^{x^1} dx^3$		
VI_{-1}	B			Solvgeometry	e
VII_h		$d\sigma^1 = -\sigma^2 \wedge \sigma^3$	$\sigma^1 = (X - kY)dx^2 - Ydx^3$	$\widetilde{SL_2R}$	e
		$d\sigma^2 = \sigma^1 \wedge \sigma^3$	$\sigma^2 = Ydx^2 + (X + kY)dx^3$		
	$+ h\sigma^2 \wedge \sigma^3, d\sigma^3 = 0$	$\sigma^3 = dx^1$			
VII_0	A			E^3	$SO(3)$
VIII	A	$d\sigma^1 = \sigma^2 \wedge \sigma^3$	$\sigma^1 = dx^1 + ((x^1)^2 - 1)dx^2$	$\widetilde{SL_2R}$	$SO(2)$
			$+ (x^1 + x^2 - x^2(x^1)^2)dx^3$		
		$d\sigma^2 = -\sigma^3 \wedge \sigma^1$	$\sigma^2 = dx^1 + ((x^1)^2 + 1)dx^2$		
		$d\sigma^3 = \sigma^1 \wedge \sigma^2$	$\sigma^3 = 2x^1 dx^2 + (1 - 2x^1 x^2)dx^3$	$\widetilde{SL_2R}$	
IX	A	$d\sigma^1 = \sigma^2 \wedge \sigma^3$	$\sigma^1 = -\sin x^3 dx^1 + \sin x^1 \cos x^3 dx^2$	$SU(2) \approx S^3$	e
		$d\sigma^2 = \sigma^3 \wedge \sigma^1$	$\sigma^2 = \cos x^3 dx^1 + \sin x^1 \sin x^3 dx^2$	$SU(2) \approx S^3$	$SO(2)$
		$d\sigma^3 = \sigma^1 \wedge \sigma^2$	$\sigma^3 = \cos x^1 dx^2 + dx^3$	S^3	$SO(3)$

B Effective potential

Here, we rewrite the equations of motion (7)–(12) to derive the expression of the effective potential, V_{eff} , in attractor mechanism. We have

$$\frac{1}{\sqrt{\tilde{g}}} \partial_\mu \left(\sqrt{\tilde{g}} \partial^\mu \phi \right) = \frac{e^{\frac{4\phi}{3}}}{12} H^2, \tag{196}$$

$$\partial_\mu \left(\sqrt{\tilde{g}} e^{\frac{4\phi}{3}} H_{\nu\rho}^\mu \right) = 0. \tag{197}$$

Using (195) we have

$$\tilde{R}_{tt} = \frac{\tilde{g}_{tt}}{3\tilde{\gamma}} V_{eff}(\phi), \quad \tilde{R}_{ii} = \frac{\tilde{g}_{ii}}{6\tilde{\gamma}} V_{eff}(\phi), \tag{198}$$

where

$$V_{eff}(\phi) = 2\kappa_5^2 e^{\frac{4\phi}{3}} Q_m^2. \tag{199}$$

$$\tilde{R}_{\mu\nu} = \frac{1}{3} \tilde{\nabla}_\mu \phi \tilde{\nabla}_\nu \phi + \frac{e^{\frac{4\phi}{3}}}{4} \left(H_{\mu\kappa\lambda} H_\nu^{\kappa\lambda} - \frac{2}{9} H^2 \tilde{g}_{\mu\nu} \right), \tag{195}$$

By definition [87], the V_{eff} can be regarded as an effective potential for the scalar field, where from (196) we get

$$r^2 \partial_r (r^2 F \partial_r \phi) = \frac{1}{4} \partial_\phi V_{eff}(\phi). \tag{200}$$

Also, the rr component of (195) gives

$$2r^4 \tilde{g}_{tt} e^{-\frac{2\phi}{3}} \left(\frac{1}{2} \sum_{i < j} (\ln \tilde{g}_i^2)' (\ln \tilde{g}_j^2)' - \frac{1}{3} \phi' \right) + r^4 (\tilde{g}_{tt})' \ln(\tilde{\gamma})' e^{-\frac{2\phi}{3}} + V_{eff}(\phi) + \tilde{Y} = 0. \tag{201}$$

On the horizon \tilde{g}_{tt} vanishes and $r^2 \tilde{g}'_{tt} = -c_1 e^{\frac{2\phi_H}{3}}$. Consequently, we have

$$\tilde{\gamma}_H = \frac{-c_1^2 \frac{d\tilde{\gamma}}{dF} |_{r=r_H}}{V_{eff}(\phi_H) + \tilde{Y} |_{r=r_H}}, \tag{202}$$

in which \tilde{Y} for different Bianchi types is given as follows

$$I \& VII_0: \tilde{Y} = 0, \tag{203}$$

$$II: \tilde{Y} = V^2, \tag{204}$$

$$III: \tilde{Y} = 2l_2^2 V^2, \tag{205}$$

$$V: \tilde{Y} = 12 V^2, \tag{206}$$

$$VI_{-1}: \tilde{Y} = c_1^2, \tag{207}$$

$$VIII: \tilde{Y} = 4V_1^2 + V_2^2, \tag{208}$$

$$IX: \tilde{Y} = 4V_1^2 + V_2^2. \tag{209}$$

where with the given metrics in Sects. 4.1-4.7, the V function for each Bianchi types can be found in (49), (56), (65), (81), and (87).

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