THE EUROPEAN
PHYSICAL JOURNAL C

Universal character, phase model and topological strings on ℂ³

Na Wang[1], **Chuanzhong Li**[2,a]

[1] Department of Mathematics and Statistics, Henan University, Kaifeng 475001, China
[2] School of Mathematics and Statistics, Ningbo University, Ningbo 315211, China

Abstract In this paper, we consider two different subjects: the algebra of universal characters $S_{[\lambda,\mu]}(\mathbf{x}, \mathbf{y})$ (a generalization of Schur functions) and the phase model of strongly correlated bosons. We find that the two-site generalized phase model can be realized in the algebra of universal characters, and the entries in the monodromy matrix of the phase model can be represented by the vertex operators $\Gamma_i^{\pm}(z)(i = 1, 2)$ which generate universal characters. Meanwhile, we find that these vertex operators can also be used to obtain the A-model topological string partition function on $\mathbb{C}^3$.

1 Introduction

Symmetric functions were used to determine irreducible characters of highest weight representations of the classical groups [1]. The universal character, as a generalization of Schur function, describes the character of an irreducible rational representation of $GL(n)$ [2], which upgrades that Schur function is the character of an irreducible polynomial representation of $GL(n)$. Symmetric functions also appear in mathematical physics, especially in integrable models. The group in the Kyoto school uses Schur functions in a remarkable way to understand the KP and KdV hierarchies [3]. Tsuda defined the UC hierarchy which is a generalization of KP hierarchy and obtained that the tau functions of UC hierarchy can be realized in terms of the universal characters. He also proved that the UC hierarchy has relations with Painlevé equations [4] by similar reductions. In this paper, we consider two different subjects: the algebra of universal characters and the phase model of strongly correlated bosons.

One purpose of this paper is to give the representation of the two-site generalized phase model on the algebra of universal characters $S_{[\lambda,\mu]}(\mathbf{x}, \mathbf{y})$. The phase model, which is the so-called crystal limit of the quantum group [5], is an integrable model and can be solved in the formulism of the

quantum inverse scattering method [6]. Our results will show that the limit of the quantum inverse scattering method has an interpretation in terms of the algebra of universal characters. The crucial elements in our discussion are vertex operators $\Gamma_i^{\pm}(z)$, $(i = 1, 2)$. The Fermions can be defined from these vertex operators. In the special case $\mu = \emptyset$, the universal character $S_{[\lambda,\mu]}(\mathbf{x}, \mathbf{y})$ will be reduced to the Schur function $S_\lambda(\mathbf{x})$, and the correspondence between vertex operators and fermions in this special case is a part of the well-known Boson-Fermion correspondence.

The relation between the algebra of Schur functions and the phase model is known from [7,8]. There is the following isometry between states in the phase model and Schur functions

$$\bigotimes_{i=0}^{M} |n_i\rangle_i \mapsto S_\lambda(\mathbf{x}), \quad \lambda = 1^{n_1} 2^{n_2} \dots.$$

The actions of the entries in monodromy matrix $T(u)$ on Schur function are obtained from the truncated expansions of the vertex operators

$$\tilde{\Gamma}^+(z) = e^{\xi(\mathbf{x}, z)}, \quad \tilde{\Gamma}^-(z) = e^{\xi(\tilde{\partial}_{\mathbf{x}}, z^{-1})}$$

where $\xi(\mathbf{x}, z) = \sum_{n=1}^{\infty} x_n z^n$ and

$$\tilde{\partial}_{\mathbf{x}} = \left(\partial_{x_1}, \frac{1}{2}\partial_{x_2}, \frac{1}{3}\partial_{x_3}, \dots\right), \quad \partial_{x_i} = \frac{\partial}{\partial x_i}.$$

In this paper, we generalize vertex operators to the type

$$\Gamma_1^-(z) = e^{\xi(\mathbf{x}-\tilde{\partial}_{\mathbf{y}}, z)}, \quad \Gamma_1^+(z) = e^{\xi(\tilde{\partial}_{\mathbf{x}}, z^{-1})},$$
$$\Gamma_2^-(z) = e^{\xi(\mathbf{y}-\tilde{\partial}_{\mathbf{x}}, z)}, \quad \Gamma_2^+(z) = e^{\xi(\tilde{\partial}_{\mathbf{y}}, z^{-1})},$$

these vertex operators can generate universal characters and fermions can be defined from them. We define the map from the state in the two-site generalized phase model to the universal character by

$$\bigotimes_{i=0}^{M_1} |n_i\rangle_i^{(1)} \bigotimes_{i=0}^{M_2} |m_i\rangle_i^{(2)} \mapsto S_{[\lambda,\mu]}(\mathbf{x}, \mathbf{y})$$

[a] e-mail: lichuanzhong@nbu.edu.cn

with

$$\lambda = 1^{n_1} 2^{n_2} \ldots, \quad \mu = 1^{m_1} 2^{m_2} \ldots.$$

The actions of creation operators $B_1(u)$ and $B_2(u)$, which are entries in the monodromy matrix $T(u)$ of the phase model, on universal characters are determined by the truncated expansions of $\Gamma_1^-(z)$ and $\Gamma_2^-(z)$ respectively, and the annihilation operators $C_1(u)$ and $C_2(u)$ (multiplied by a coefficient) are adjoint to the operators $B_1(u)$ and $B_2(u)$, and determined by the truncated expansions of $\Gamma_1^+(z)$ and $\Gamma_2^+(z)$ respectively.

Another purpose of this paper is to discuss the relations between the vertex operators $\Gamma_i^-(z)$, $\Gamma_i^+(z)$ $(i = 1, 2)$ and the MacMahon functions. It is known that the A-model topological string partition function $Z_{\mathbb{C}^3}^{top}$ on $\mathbb{C}^3$ can be written as a Fermionic correlator involving the vertex operators $\tilde{\Gamma}^+(z)$ and $\tilde{\Gamma}^-(z)$ with a particular specialization of the values of $z = q^{\pm 1/2}, q^{\pm 3/2}, q^{\pm 5/2} \cdots$. In this paper, we will give that $Z_{\mathbb{C}^3}^{top}$ can also be obtained from the vertex operators $\Gamma_i^-(z)$ and $\Gamma_i^+(z)$ $(i = 1, 2)$ with the same specialization of the values of z.

The paper is organized as follows. In Sect. 2, we recall the definition of universal character and its vertex operator realization, then we give the actions of the vertex operators on $S_{[\lambda,\mu]}(\mathbf{x}, \mathbf{y})$ which is helpful for our discussion. In Sect. 3, we recall the phase model. In Sect. 4, we define the representation of the two-site generalized phase model on the algebra of universal characters, and we find that the actions of the entries in monodromy matrix on universal characters are obtained from the truncated expansions of the vertex operators discussed in Sect. 2. In Sect. 5, we give that the MacMahon function can be obtained from these vertex operators.

2 Universal characters and vertex operators

In this section, we review the definition of universal character and its vertex operator realization as in papers [2,4], then we review the definition of the UC hierarchy as in paper [4]. We also give a new proposition in this section which is helpful for the discussion in Sect. 4. Let $\mathbf{x} = (x_1, x_2, \ldots)$ and $\mathbf{y} = (y_1, y_2, \ldots)$. The operators $h_n(\mathbf{x})$ are determined by the generating function:

$$\sum_{n=0}^{\infty} h_n(\mathbf{x}) z^n = e^{\xi(\mathbf{x},z)}, \quad \xi(\mathbf{x}, z) = \sum_{n=1}^{\infty} x_n z^n \tag{1}$$

and set $h_n(\mathbf{x}) = 0$ for $n < 0$. The operators $h_n(\mathbf{x})$ can be explicitly written as

$$h_n(\mathbf{x}) = \sum_{k_1 + 2k_2 + \cdots nk_n = n} \frac{x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}}{k_1! k_2! \cdots k_n!}.$$

Note that $h_n(\mathbf{x})$ is the complete homogeneous symmetric function of $t_1, t_2, \ldots$ if we replace $i x_i$ with the power sum $p_i = \sum_{j=1}^{\infty} t_j^i$.

For a pair of Young diagrams $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_{l'})$, the universal character $S_{[\lambda,\mu]} = S_{[\lambda,\mu]}(\mathbf{x}, \mathbf{y})$ is a polynomial of variables $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{C}[\mathbf{x}, \mathbf{y}]$ defined by the twisted Jacobi–Trudi formula [2]:

$$S_{[\lambda,\mu]}(\mathbf{x}, \mathbf{y})$$
$$= \det \begin{pmatrix} h_{\mu_{l'-i+1}+i-j}(\mathbf{y}), & 1 \le i \le l' \\ h_{\lambda_{i-l'}-i+j}(\mathbf{x}), & l'+1 \le i \le l+l' \end{pmatrix}_{1 \le i,j \le l+l'}. \tag{2}$$

Define the degree of each variables x_n, y_n, $n = 1, 2, \ldots$ by

$$\deg x_n = n, \quad \deg y_n = -n$$

then $S_{[\lambda,\mu]}(\mathbf{x}, \mathbf{y})$ is a homogeneous polynomial of degree $|\lambda| - |\mu|$, where $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_l$ is called the weight of λ. Note that $S_\lambda(\mathbf{x})$ is a special case of the universal character: $S_\lambda(\mathbf{x}) = \det(h_{\lambda_i - i + j}(\mathbf{x})) = S_{[\lambda,\emptyset]}(\mathbf{x}, \mathbf{y})$.

Introduce the following vertex operators

$$\Gamma_1^-(z) = e^{\xi(\mathbf{x} - \tilde{\partial}_\mathbf{y}, z)}, \quad \Gamma_1^+(z) = e^{\xi(\tilde{\partial}_\mathbf{x}, z^{-1})}, \tag{3}$$

$$\Gamma_2^-(z) = e^{\xi(\mathbf{y} - \tilde{\partial}_\mathbf{x}, z)}, \quad \Gamma_2^+(z) = e^{\xi(\tilde{\partial}_\mathbf{y}, z^{-1})}. \tag{4}$$

Define

$$X^\pm(z) = \sum_{n \in \mathbb{Z}} X_n^\pm z^n = e^{\pm \xi(\mathbf{x} - \tilde{\partial}_\mathbf{y}, z)} e^{\mp \xi(\tilde{\partial}_\mathbf{x}, z^{-1})}, \tag{5}$$

$$Y^\pm(z^{-1}) = \sum_{n \in \mathbb{Z}} Y_n^\pm z^{-n} = e^{\pm \xi(\mathbf{y} - \tilde{\partial}_\mathbf{x}, z^{-1})} e^{\mp \xi(\tilde{\partial}_\mathbf{y}, z)}. \tag{6}$$

The operators $X_i^\pm$ satisfy the following Fermionic relations:

$$X_i^\pm X_j^\pm + X_{j-1}^\pm X_{i+1}^\pm = 0,$$
$$X_i^+ X_j^- + X_{j+1}^- X_{i-1}^+ = \delta_{i+j,0}.$$

The same relations hold also for $Y_i^\pm$, and $X_i^\pm$ and $Y_i^\pm$ are commutative. The operators X_i^+ and Y_i^+ are raising operators for the universal characters such that

$$S_{[\lambda,\mu]}(\mathbf{x}, \mathbf{y}) = X_{\lambda_1}^+ \cdots X_{\lambda_l}^+ Y_{\mu_1}^+ \cdots X_{\mu_{l'}}^+ \cdot 1 \tag{7}$$

where the Young diagrams $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_{l'})$.

For an unknown function $\tau = \tau(\mathbf{x}, \mathbf{y})$, the bilinear relations [4]

$$\sum_{i+j=-1} X_i^- \tau \otimes X_j^+ \tau = \sum_{i+j=-1} Y_i^- \tau \otimes Y_j^+ \tau = 0 \tag{8}$$

is called the UC hierarchy.

It turns out that $\tau(\mathbf{x}, \mathbf{y})$ equals

$$\tau(\mathbf{x}, \mathbf{y}) = \tau_1(\mathbf{x} - \tilde{\partial}_\mathbf{y}) \tau_2(\mathbf{y} - \tilde{\partial}_\mathbf{x}) \cdot 1$$

where $\tau_1(\mathbf{x})$ and $\tau_2(\mathbf{x})$ are tau functions of KP hierarchy. In special case, the universal character $S_{[\lambda,\mu]}(\mathbf{x}, \mathbf{y})$ is also the solution of UC hierarchy, and

$$S_{[\lambda,\mu]}(\mathbf{x}, \mathbf{y}) = S_\lambda(\mathbf{x} - \tilde{\partial}_\mathbf{y}) S_\mu(\mathbf{y} - \tilde{\partial}_\mathbf{x}) \cdot 1. \qquad (9)$$

From this, we get the conclusion which is helpful for the following discussion.

Proposition 2.1 *The vertex operators in* (3) *and* (4) *act on the universal characters* $S_{[\lambda,\mu]}(\mathbf{x}, \mathbf{y})$ *in the following way,*

$$\Gamma_1^-(z) S_{[\lambda,\mu]}(\mathbf{x}, \mathbf{y}) = \sum_{n=0}^\infty z^n S_{[\lambda\cdot(n),\mu]}(\mathbf{x}, \mathbf{y})$$
$$= \sum_{\nu \succ \lambda} z^{|\nu|-|\lambda|} S_{[\nu,\mu]}(\mathbf{x}, \mathbf{y}), \qquad (10)$$

$$\Gamma_2^-(z) S_{[\lambda,\mu]}(\mathbf{x}, \mathbf{y}) = \sum_{n=0}^\infty z^n S_{[\lambda,\mu\cdot(n)]}(\mathbf{x}, \mathbf{y})$$
$$= \sum_{\nu \succ \mu} z^{|\nu|-|\mu|} S_{[\lambda,\nu]}(\mathbf{x}, \mathbf{y}), \qquad (11)$$

where the multiplication of two Young diagram λ *and* (n) *satisfies the Pieri formula* [9,10], *and* $\lambda \succ \mu$ *means that the Young diagrams* λ *and* μ *are interlaced, in the sense of* $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots$. *The operators* $\Gamma_1^+(z)$ *and* $\Gamma_2^+(z)$ *are adjoint to* $\Gamma_1^-(z)$ *and* $\Gamma_2^-(z)$ *respectively, that is,*

$$\Gamma_1^+(z) S_{[\lambda,\mu]}(\mathbf{x}, \mathbf{y}) = \sum_{n=0}^\infty z^n S_{[\lambda/(n),\mu]}(\mathbf{x}, \mathbf{y})$$
$$= \sum_{\lambda \succ \nu} z^{|\lambda|-|\nu|} S_{[\nu,\mu]}(\mathbf{x}, \mathbf{y}), \qquad (12)$$

$$\Gamma_2^+(z) S_{[\lambda,\mu]}(\mathbf{x}, \mathbf{y}) = \sum_{n=0}^\infty z^n S_{[\lambda,\mu/(n)]}(\mathbf{x}, \mathbf{y})$$
$$= \sum_{\mu \succ \nu} z^{|\mu|-|\nu|} S_{[\lambda,\nu]}(\mathbf{x}, \mathbf{y}), \qquad (13)$$

where λ/μ *denotes a skew diagram.*

3 Phase model

In this section, we review the definition of phase model as in papers [7,8] and no new result is given. We begin with a bosonic system based on the following algebra [7,8]

$$[N, \phi] = -\phi, \quad [N, \phi^\dagger] = \phi^\dagger, \quad [\phi, \phi^\dagger] = \pi, \qquad (14)$$

where $\pi = |0\rangle\langle 0|$ is the vacuum projection. The operator ϕ is one-sided isometry

$$\phi\phi^\dagger = 1, \qquad \phi^\dagger\phi = 1 - \pi.$$

This algebra can be represented in the Fock space $\mathcal{F}$ consisting of n-particle states $|n\rangle$, the operators ϕ, $\phi^\dagger$ and N acting as the phase operators and the number of particles operator, respectively,

$$\phi^\dagger|n\rangle = |n+1\rangle, \quad \phi|n\rangle = |n-1\rangle,$$
$$\phi|0\rangle = 0, \quad N|n\rangle = n|n\rangle,$$

where $|0\rangle$ is the vacuum state, the special case $n = 0$ of the n particle state.

Let the tensor product

$$\mathcal{F} = \bigotimes_{i=0}^M \mathcal{F}_i, \qquad (15)$$

be $M + 1$ copies of the Fock space. Denote by ϕ_i, $\phi_i^\dagger$, N_i the operators that act as ϕ, $\phi^\dagger$, N in (14), respectively, in the ith space and identically in the other spaces.

The phase model is a model of a periodic chain with the hamiltonian [11–13]

$$H = -\frac{1}{2} \sum_{i=0}^M \left(\phi_i^\dagger \phi_{i+1} + \phi_i \phi_{i+1}^\dagger - 2N_i \right). \qquad (16)$$

Define the operator of the total number of particles by

$$\hat{N} = \sum_{i=0}^M N_i.$$

Then the N-particle vectors in this space are of the form

$$\bigotimes_{i=0}^M |n_i\rangle_i, \quad \text{where } |n_i\rangle_i = (\phi_i^\dagger)^{n_i}|0\rangle_i, \quad N = \sum_{i=0}^M n_i, \qquad (17)$$

the numbers n_i are called the occupation numbers of the state (17).

From [6], we know that the phase model is integrable. Introduce the L-matrix

$$L_i(u) = \begin{pmatrix} u^{-1} & \phi_i^\dagger \\ \phi_i & u \end{pmatrix}, \quad i = 0, \dots, M,$$

where u is a scalar parameter, here we treat u as uI with I being the identity operator in $\mathcal{F}$. For every $i = 0, \dots, M$, the L-matrix satisfies the bilinear equation

$$R(u, v)\left(L_i(u) \otimes L_i(v)\right) = \left(L_i(v) \otimes L_i(u)\right)R(u, v), \qquad (18)$$

where R-matrix $R(u, v)$ is a 4×4 matrix given by

$$R(u, v) = \begin{pmatrix} f(v, u) & 0 & 0 & 0 \\ 0 & g(v, u) & 1 & 0 \\ 0 & 0 & g(v, u) & 0 \\ 0 & 0 & 0 & f(v, u) \end{pmatrix}, \qquad (19)$$

with

$$f(v, u) = \frac{u^2}{u^2 - v^2}, \quad g(v, u) = \frac{uv}{u^2 - v^2}.$$

Define the monodromy matrix by

$$T(u) = L_M(u) L_{M-1}(u) \cdots L_0(u),$$

which gives the solution of the phase model. It also satisfies the bilinear equation

$$R(u, v)\big(T(u) \otimes T(v)\big) = \big(T(v) \otimes T(u)\big) R(u, v). \qquad (20)$$

Let

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}, \qquad (21)$$

we have

$$\hat{N} B(u) = B(u)(\hat{N} + 1), \qquad \hat{N} C(u) = C(u)(\hat{N} - 1). \qquad (22)$$

Therefore, we call $B(u)$ the creation operator and $C(u)$ the annihilation operator. The operators $A(u)$ and $D(u)$ do not change the total number of particles.

Denote by $|0\rangle_j$ the vacuum vector in $\mathcal{F}_j$ and by $|0\rangle = \otimes_{i=0}^{M} |0\rangle_i$. Let

$$|\Psi(u_1, \ldots, u_N)\rangle = \prod_{i=1}^{N} B(u_j)|0\rangle, \qquad (23)$$

which is a N particle state.

According to [7,8], there is the following isometry between the states (17) and the Schur functions

$$\bigotimes_{i=0}^{M} |n_i\rangle_i \mapsto S_\lambda(\mathbf{x}), \quad \lambda = 1^{n_1} 2^{n_2} \ldots, \qquad (24)$$

and the operator $B(u)$ acts on Schur functions as the operator of multiplication by $u^M H_M(u^2)$, where $H_M(t) = \sum_{k=0}^{M} t^k h_k$ is the truncated generating function of the complete homogeneous symmetric functions h_k. Then the state $|\Psi(u_1, \ldots, u_N)\rangle$ has the following expansion

$$|\Psi(u_1, \ldots, u_N)\rangle = \sum_\lambda S_\lambda(u_1^2, \ldots, u_N^2) \bigotimes_{i=0}^{M} |n_i\rangle_i,$$
$$\lambda = 1^{n_1} 2^{n_2} \ldots. \qquad (25)$$

In the following, we will generalize this work [7,8] to realize the two-site generalized phase model in the algebra of universal characters, and the entries in the monodromy matrix can be obtained from the vertex operators which generate the universal characters.

4 Two-site generalized phase model and universal characters

In this section, we will give the representation of the two-site generalized phase model on the algebra of universal characters. All the results in this section are new. Now fix two positive integers M_1, M_2 and consider the tensor products

$$\mathcal{F}^{(1)} = \bigotimes_{i=0}^{M_1} \mathcal{F}_i^{(1)} \qquad \mathcal{F}^{(2)} = \bigotimes_{i=0}^{M_2} \mathcal{F}_i^{(2)}, \qquad (26)$$

which $\mathcal{F}^{(1)}$ and $\mathcal{F}^{(2)}$ are $M_1 + 1$ and $M_2 + 1$ copies of the Fock space respectively. Denote by $\phi_i^{(1)}, \phi_i^{(1)\dagger}, N_i^{(1)}$ the operators that act as $\phi, \phi^\dagger, N$ in (14), respectively, in the ith space $\mathcal{F}_i^{(1)}$ and identically in the other spaces of $\mathcal{F}^{(1)}$ and in all spaces of $\mathcal{F}^{(2)}$, and denote by $\phi_i^{(2)}, \phi_i^{(2)\dagger}, N_i^{(2)}$ the operators that act as $\phi, \phi^\dagger, N$, respectively, in the ith space $\mathcal{F}_i^{(2)}$ and identically in the other spaces of $\mathcal{F}^{(2)}$ and in all spaces of $\mathcal{F}^{(1)}$. Let

$$\mathcal{F} = \mathcal{F}^{(1)} \bigotimes \mathcal{F}^{(2)} \qquad (27)$$

so that $\mathcal{F}$ is $M_1 + M_2 + 2$ copies of the Fock space. Denote by $|0\rangle_j^{(i)}$ the vacuum vector in $\mathcal{F}_j^{(i)}$ with $i = 1, 2$.

Define the operators

$$\hat{N}_1 = \sum_{i=0}^{M_1} N_i^{(1)}, \quad \hat{N}_2 = \sum_{i=0}^{M_2} N_i^{(2)}, \quad \text{and} \quad \hat{N} = \hat{N}_1 + \hat{N}_2.$$

The (N_1, N_2)-particle vectors in space $\mathcal{F}$ are of the form

$$\bigotimes_{i=0}^{M_1} |n_i\rangle_i^{(1)} \bigotimes_{i=0}^{M_2} |m_i\rangle_i^{(2)}, \quad \text{with} \quad N_1 = \sum_{i=0}^{M_1} n_i,$$
$$N_2 = \sum_{i=0}^{M_2} m_i, \qquad (28)$$

where

$$|n_i\rangle_i^{(1)} = (\phi_i^{(1)\dagger})^{(n_i)} |0\rangle_j^{(1)}, \quad |m_i\rangle_i^{(2)} = (\phi_i^{(2)\dagger})^{(m_i)} |0\rangle_j^{(2)}.$$

Define the map $J : \mathcal{F} \to \mathbb{C}[\mathbf{x}, \mathbf{y}]$ by

$$J\left(\bigotimes_{i=0}^{M_1} |n_i\rangle_i^{(1)} \bigotimes_{i=0}^{M_2} |m_i\rangle_i^{(2)} \right) = S_{[\lambda, \mu]}(\mathbf{x}, \mathbf{y}) \qquad (29)$$

with

$$\lambda = 1^{n_1} 2^{n_2} \ldots, \quad \mu = 1^{m_1} 2^{m_2} \ldots. \qquad (30)$$

In fact, this association is not quite unique: partitions λ, μ themselves do not know about numbers n_0 and m_0 of particles. Nonetheless, if we fix the total numbers of particles N_1 and N_2, we can deduce $n_0 = N_1 - l(\lambda)$ and $m_0 = N_2 - l(\mu)$, where $l(\lambda)$ is the length of partition λ, that is, the number of

rows in λ. Note that the correspondence (7) in [8] is a special case of the map J defined above.

The L-matrices are

$$L_i^{(1)}(u) = \begin{pmatrix} u^{-1} & \phi_i^{(1)\dagger} \\ \phi_i^{(1)} & u \end{pmatrix}, \qquad i = 0, \ldots, M_1,$$

$$L_i^{(2)}(u) = \begin{pmatrix} u^{-1} & \phi_i^{(2)\dagger} \\ \phi_i^{(2)} & u \end{pmatrix}, \qquad i = 0, \ldots, M_2,$$

and the monodromy matrix is

$$T(u) = L_{M_2}^{(2)}(u) \cdots L_0^{(2)}(u) L_{M_1}^{(1)}(u) \cdots L_0^{(1)}(u).$$

Each L-matrix, as well as the monodromy matrix, satisfies the bilinear equation again

$$R(u,v)\big(L_i^{(j)}(u) \otimes L_i^{(j)}(v)\big) = \big(L_i^{(j)}(v) \otimes L_i^{(j)}(u)\big) R(u,v),$$
$$j = 1, 2,$$
$$R(u,v)\big(T(u) \otimes T(v)\big) = \big(T(v) \otimes T(u)\big) R(u,v), \quad (31)$$

with the same R-matrix in (19).

Let

$$T_i(u) = L_{M_i}^{(i)}(u) L_{M_i-1}^{(i)}(u) \cdots L_0^{(1)}(u) \tag{32}$$

$$= \begin{pmatrix} A_i(u) & B_i(u) \\ C_i(u) & D_i(u) \end{pmatrix}, \quad i = 1, 2. \tag{33}$$

The operators $B_1(u)$ and $B_2(u)$ are called the creation operators and $C_1(u)$ and $C_2(u)$ the annihilation operators, in the sense that they increase and decrease the total numbers of particles

$$\hat{N}_i B_i(u) = B_i(u)(\hat{N}_i + 1), \qquad \hat{N}_i C_i(u) = C_i(u)(\hat{N}_i - 1)$$
$$\text{for} \quad i = 1, 2. \tag{34}$$

The operators $A_i(u)$ and $D_i(u)$ ($i = 1, 2$) do not change the total number of particles.

We call ith Fock space $\mathcal{F}_i^{(j)}$ ($j = 1, 2$) the i-energy space. Note that the correspondence (29) does not take into account the numbers n_0, m_0 of zero-energy particles, Therefore (29) gives a representation of the positive-energy space

$$\hat{\mathcal{F}} = \bigotimes_{i=1}^{M_1} \mathcal{F}_i^{(1)} \bigotimes_{i=1}^{M_2} \mathcal{F}_i^{(2)},$$

in the algebra of symmetric functions $\mathbb{C}[\mathbf{x}, \mathbf{y}]$, in fact in its subspace $\mathbb{C}_{M_1, M_2}[\mathbf{x}, \mathbf{y}]$ generated by universal characters $S_{[\lambda,\mu]}$ where the Young diagrams λ have at most M_1 columns and μ at most M_2 columns. From the definition of the twisted Jacobi–Trudi formula, we can define this subspace by supposing

$$h_{M_1+1}(\mathbf{x}) = h_{M_1+2}(\mathbf{x}) = \cdots = 0,$$
$$h_{M_2+1}(\mathbf{y}) = h_{M_2+2}(\mathbf{y}) = \cdots = 0.$$

By the definition of $\hat{N}_1$, $\hat{N}_2$, the space $\mathcal{F}$ has a decomposition into (N_1, N_2)-particle subspaces $\mathcal{F}^{N_1, N_2}$, i.e.,

$$\mathcal{F} = \bigoplus_{N_1, N_2 \geq 0} \mathcal{F}^{N_1, N_2} = \bigoplus_{N_1, N_2 \geq 0} \mathcal{F}_1^{N_1} \otimes \mathcal{F}_2^{N_2}. \tag{35}$$

Under the map (29), the subspace $\mathcal{F}^{N_1, N_2}$ corresponds to $\mathbb{C}_{M_1, M_2}^{N_1, N_2}[\mathbf{x}, \mathbf{y}]$ which is spanned by universal characters $S_{[\lambda,\mu]}$ whose diagrams λ lie in the $N_1 \times M_1$ box and μ lie in the $N_2 \times M_2$ box. That Young diagram λ lies in the $N \times M$ box means λ has at most N rows and at most M columns.

Define the projection $P : \mathcal{F} \to \hat{\mathcal{F}}$ by forgetting the zero energy states, and define the operator $\mathcal{B}_i(u) := P B_i(u) P$ for $i = 1, 2$. Then $\mathcal{B}_i(u)$ are operators acting on the space $\hat{\mathcal{F}} \cong \mathbb{C}_{M_1, M_2}[\mathbf{x}, \mathbf{y}]$. Since $B_i(u)$ are creation operators, then $\mathcal{B}_1(u)$ sends $\mathbb{C}_{M_1, M_2}^{N_1, N_2}[\mathbf{x}, \mathbf{y}]$ to $\mathbb{C}_{M_1, M_2}^{N_1+1, N_2}[\mathbf{x}, \mathbf{y}]$ and $\mathcal{B}_2(u)$ sends $\mathbb{C}_{M_1, M_2}^{N_1, N_2}[\mathbf{x}, \mathbf{y}]$ to $\mathbb{C}_{M_1, M_2}^{N_1, N_2+1}[\mathbf{x}, \mathbf{y}]$. In the following, we discuss the actions of $\mathcal{B}_i(u)$ on $\mathbb{C}_{M_1, M_2}[\mathbf{x}, \mathbf{y}]$. Define $\tilde{\mathcal{B}}_i(u)$ by $\mathcal{B}_i(u) = u^{-M_i} \tilde{\mathcal{B}}_i(u)$ for $i = 1, 2$. Then we can derive the following proposition.

Proposition 4.1 *In the space $\mathbb{C}_{M_1, M_2}[\mathbf{x}, \mathbf{y}]$,*

$$\tilde{\mathcal{B}}_1(u) = H_{M_1}(\mathbf{x} - \tilde{\partial}_{\mathbf{y}}, u^2), \tag{36}$$
$$\tilde{\mathcal{B}}_2(u) = H_{M_2}(\mathbf{y} - \tilde{\partial}_{\mathbf{x}}, u^2), \tag{37}$$

where $H_n(\mathbf{x}, t) = \sum_{k=0}^n t^k h_k(\mathbf{x})$, and $h_k(\mathbf{x})$ is defined in (1), which in fact is the complete homogeneous symmetric function.

To prove this proposition, we need the following lemma.

Lemma 4.2 *For any Schur functions S_λ and complete symmetric function h_k, we have*

$$h_k(\mathbf{y} - \tilde{\partial}_{\mathbf{x}}) S_\lambda(\mathbf{x} - \tilde{\partial}_{\mathbf{y}}) = S_\lambda(\mathbf{x} - \tilde{\partial}_{\mathbf{y}}) h_k(\mathbf{y} - \tilde{\partial}_{\mathbf{x}}). \tag{38}$$

Proof This holds since $y_n - \frac{1}{n}\partial_{x_n}$ and $x_m - \frac{1}{m}\partial_{y_m}$ are commutative. □

The proof of Proposition 4.1.

Proof We know that

$$S_{[\lambda,\mu]}(\mathbf{x}, \mathbf{y}) = S_\lambda(\mathbf{x} - \tilde{\partial}_{\mathbf{y}}) S_\mu(\mathbf{y} - \tilde{\partial}_{\mathbf{x}}) \cdot 1,$$

and in [7,8]

$$\tilde{\mathcal{B}}(u) \bigotimes_{i=0}^M |n_i\rangle_i = \sum_{k=0}^M u^{2k} h_k S_\lambda, \quad \lambda = 1^{n_1} 2^{n_2} \ldots.$$

Here the mapping sign $\jmath$ is omitted at the left of the equation and we will do the same in the following. Hence,

$$\tilde{\mathcal{B}}_1(u) \bigotimes_{i=0}^{M_1} |n_i\rangle_i^{(1)} \bigotimes \bigotimes_{i=0}^{M_2} |m_i\rangle_i^{(2)}$$

$$= \left(\tilde{\mathcal{B}}_1(u) \bigotimes_{i=0}^{M_1} |n_i\rangle_i^{(1)} \right) \bigotimes \bigotimes_{i=0}^{M_2} |m_i\rangle_i^{(2)}$$

$$= \sum_{k=0}^{M_1} u^{2k} h_k(\mathbf{x} - \tilde{\partial}_{\mathbf{y}}) S_\lambda(\mathbf{x} - \tilde{\partial}_{\mathbf{y}}) S_\mu(\mathbf{y} - \tilde{\partial}_{\mathbf{x}}) \cdot 1,$$

and

$$\tilde{\mathcal{B}}_2(u) \bigotimes_{i=0}^{M_1} |n_i\rangle_i^{(1)} \bigotimes \bigotimes_{i=0}^{M_2} |m_i\rangle_i^{(2)}$$

$$= \bigotimes_{i=0}^{M_1} |n_i\rangle_i^{(1)} \bigotimes \left(\tilde{\mathcal{B}}_2(u) \bigotimes_{i=0}^{M_2} |m_i\rangle_i^{(2)} \right)$$

$$= S_\lambda(\mathbf{x} - \tilde{\partial}_{\mathbf{y}}) \left(\sum_{k=0}^{M_2} u^{2k} h_k(\mathbf{y} - \tilde{\partial}_{\mathbf{x}}) S_\mu(\mathbf{y} - \tilde{\partial}_{\mathbf{x}}) \right) \cdot 1$$

$$= \sum_{k=0}^{M_2} u^{2k} h_k(\mathbf{y} - \tilde{\partial}_{\mathbf{x}}) S_\lambda(\mathbf{x} - \tilde{\partial}_{\mathbf{y}}) S_\mu(\mathbf{y} - \tilde{\partial}_{\mathbf{x}}) \cdot 1.$$

$\square$

Note that the truncated generating function $H_n(\mathbf{x}, t) = \sum_{k=0}^{n} t^k h_k(\mathbf{x})$ can also be regarded as the full generating function $H(\mathbf{x}, t) = \sum_{k=0}^{\infty} t^k h_k(\mathbf{x})$ under the specialization: $H_n(\mathbf{x}, t) = H(\mathbf{x}, t)|_{h_{n+1}(\mathbf{x})=h_{n+2}(\mathbf{x})=\ldots=0}$. Then we get the following corollary.

Corollary 4.3 *In the $M_1, M_2 \to \infty$ limit, the actions of the creation operators $\tilde{\mathcal{B}}_1(u)$ and $\tilde{\mathcal{B}}_2(u)$ on the space $\hat{\mathcal{F}}$ correspond to the multiplications of $H(\mathbf{x} - \tilde{\partial}_{\mathbf{y}}, u^2)$ and $H(\mathbf{y} - \tilde{\partial}_{\mathbf{x}}, u^2)$ on the space $\mathbb{C}[\mathbf{x}, \mathbf{y}]$ respectively.*

Define

$$|\tilde{\Psi}_N(u_1, \ldots, u_N)\rangle := \prod_{j=1}^{N} B_2(u_j) B_1(u_j)|0\rangle \qquad (39)$$

then we obtain the following proposition.

Proposition 4.4 *The expansion of the (N, N)-particle vector (39) in terms of basis vector (28) is given by the formula*

$$|\tilde{\Psi}_N(u_1, \ldots, u_N)\rangle$$

$$= (u_1 \cdots u_N)^{-M_1-M_2} \sum_{\lambda,\mu} S_\lambda(u_1^2, \ldots, u_N^2) S_\mu(u_1^2, \ldots, u_N^2)$$

$$\bigotimes_{i=0}^{M_1} |n_i\rangle_i^{(1)} \bigotimes \bigotimes_{i=0}^{M_2} |m_i\rangle_i^{(2)}$$

$$= (u_1 \cdots u_N)^{-M_1-M_2} \sum_{\lambda,\mu} S_\lambda(u_1^2, \ldots, u_N^2) S_\mu(u_1^2, \ldots, u_N^2)$$

$$S_{[\lambda,\mu]}(\mathbf{x}, \mathbf{y})$$

where the sum is over Young diagrams λ with at most N rows and at most M_1 columns, Young diagrams μ with at most N rows and at most M_2 columns.

Proof The operators $B_1(u)$ and $B_2(u)$ are commutative.

$$\prod_{j=1}^{N} B_1(u_j) = (u_1 \cdots u_N)^{-M_1} \sum_\lambda S_\lambda(u_1^2, \ldots, u_N^2) S_\lambda(\mathbf{x} - \tilde{\partial}_{\mathbf{y}})$$
$$(40)$$

$$\prod_{j=1}^{N} B_2(u_j) = (u_1 \cdots u_N)^{-M_2} \sum_\mu S_\mu(u_1^2, \ldots, u_N^2) S_\mu(\mathbf{y} - \tilde{\partial}_{\mathbf{x}})$$
$$(41)$$

then we have

$$|\tilde{\Psi}_N(u_1, \ldots, u_N)\rangle$$

$$= (u_1 \cdots u_N)^{-M_1-M_2} \sum_{\lambda,\mu} S_\lambda(u_1^2, \ldots, u_N^2) S_\mu(u_1^2, \ldots, u_N^2)$$

$$S_\lambda(\mathbf{x} - \tilde{\partial}_{\mathbf{y}}) S_\mu(\mathbf{y} - \tilde{\partial}_{\mathbf{x}}) \cdot 1$$

$$= (u_1 \cdots u_N)^{-M_1-M_2} \sum_{\lambda,\mu} S_\lambda(u_1^2, \ldots, u_N^2) S_\mu(u_1^2, \ldots, u_N^2)$$

$$S_{[\lambda,\mu]}(\mathbf{x}, \mathbf{y}). \qquad (42)$$

By the restrictions on λ and μ, we obtain the conclusion. $\square$

Recall that

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}$$

$$= T_2(u) T_1(u) = \begin{pmatrix} A_2(u) & B_2(u) \\ C_2(u) & D_2(u) \end{pmatrix} \begin{pmatrix} A_1(u) & B_1(u) \\ C_1(u) & D_1(u) \end{pmatrix}$$

then

$$B(u) = A_2(u) B_1(u) + B_2(u) D_1(u).$$

The matrix entries of $T_i(u), i = 1, 2$ are related by the following formulas:

$$B_i(u) = u A_i(u) \phi_0^{(i)\dagger}, \quad C_i(u) = u^{-1} \phi_0^{(i)} A_i^\dagger(u^{-1}),$$
$$D_i(u) = \phi_0^{(i)} A_i^\dagger(u^{-1}) \phi_0^{(i)\dagger}.$$

Let $\mathcal{A}_i(u) = P A_i(u) P$, $\mathcal{C}_i(u) = P C_i(u) P$, $\mathcal{D}_i(u) = P D_i(u) P$, where P is the projection $\mathcal{F} \to \hat{\mathcal{F}}$, then we have

Lemma 4.5 *The operators $\mathcal{A}_i(u)$, $\mathcal{B}_i(u)$, $\mathcal{C}_i(u)$, $\mathcal{D}_i(u)$, $(i = 1, 2)$ are related by the following formulas*

$$\mathcal{A}_i(u) = u^{-1} \mathcal{B}_i(u), \; \mathcal{C}_i(u) = \mathcal{B}_i^\dagger(u^{-1}), \; \mathcal{D}_i(u) = u \mathcal{B}_i^\dagger(u^{-1}).$$

Since $\mathcal{B}_1(u) = u^{-M_1} H_{M_1}(\mathbf{x} - \tilde{\partial}_{\mathbf{y}}, u^2)$, we obtain the following lemma.

Lemma 4.6 *Let $\tilde{\mathcal{C}}_1(u) = u^{-M_1} \mathcal{C}_1(u)$, we have*

$$\tilde{\mathcal{C}}_1(u) = H_{M_1}^\perp(\mathbf{x} - \tilde{\partial}_{\mathbf{y}}, u^{-2}) = H_{M_1}^\perp(\mathbf{x}, u^{-2})$$

where $H_{M_1}^\perp(\mathbf{x}, t) = \sum_{k=0}^{M_1} t^k h_k^{\perp, M_1}(\mathbf{x})$, *and* $h_k^{\perp, M_1}(\mathbf{x})$ *is the adjoint to the operator of multiplication by* $h_k(\mathbf{x})$.

Define $\mathcal{B}(u) = P B(u) P$, we have

$$
\begin{aligned}
\mathcal{B}(u) &= \mathcal{B}_2(u)(u^{-1}\mathcal{B}_1(u) + u\mathcal{B}_1^\dagger(u^{-1})) \\
&= u^{-M_2} H_{M_2}(\mathbf{y} - \tilde{\partial}_\mathbf{x}, u^2)(u^{-M_1-1} H_{M_1}(\mathbf{x} - \tilde{\partial}_\mathbf{y}, u^2) \\
&\quad + u^{M_1+1} H_{M_1}^\perp(\mathbf{x} - \tilde{\partial}_\mathbf{y}, u^{-2})).
\end{aligned}
$$

We write $H_{M_1}(\mathbf{x} - \tilde{\partial}_\mathbf{y}, u^2)$ by $H_1(u^2)$ for short. From the bilinear equation (20), we have

$$
\begin{aligned}
&u_1^{M_1+1} u_2^{-M_1-1} H_1^\perp(u_1^{-2}) H_1(u_2^2) \\
&= u_1^{M_1+1} u_2^{-M_1-1} \frac{u_1^2}{u_1^2 - u_2^2} H_1(u_2^2) H_1^\perp(u_1^{-2}) \\
&\quad - u_1^{-M_1-1} u_2^{M_1+1} \frac{u_1^2}{u_1^2 - u_2^2} H_1(u_1^2) H_1^\perp(u_2^{-2}).
\end{aligned}
$$

Recall that

$$
|\Psi_N(u_1, \dots, u_N)\rangle = \prod_{j=1}^N B(u_j)|0\rangle.
$$

Then by calculation, we get

Proposition 4.7 *The operators* $u_1^{-1}\mathcal{B}_1(u_1) + u_1\mathcal{B}_1^\dagger(u_1^{-1})$ *and* $u_2^{-1}\mathcal{B}_1(u_2) + u_2\mathcal{B}_1^\dagger(u_2^{-1})$ *are commutative, which tells us that the coefficients, in the expansion* $|\Psi_N(u_1, \dots, u_N)\rangle$ *in terms of basis vector* (28), *are symmetric functions of variables* $u_1^2, \dots, u_N^2$.

Since

$$
\begin{aligned}
&\prod_{j=1}^N \mathcal{B}(u_j)|0\rangle \\
&= \prod_{j=1}^N \mathcal{B}_2(u_j)\left(u_j^{-1}\mathcal{B}_1(u_j) + u_j\mathcal{B}_1^\dagger(u_j^{-1})\right)|0\rangle \\
&= \prod_{j=1}^N \mathcal{B}_2(u_j) \prod_{j=1}^N \left(u_j^{-1}\mathcal{B}_1(u_j) + u_j\mathcal{B}_1^\dagger(u_j^{-1})\right)|0\rangle
\end{aligned}
$$

and

$$
\prod_{j=1}^N \mathcal{B}_2(u_j) = (u_1 \cdots u_N)^{-M_2} \sum_\mu S_\mu(u_1^2, \dots, u_N^2) S_\mu(\mathbf{y} - \tilde{\partial}_\mathbf{x})
$$

where μ is a Young diagram with at most N rows and at most M_2 columns. Hence, in the following, we consider the expansion of $\prod_{j=1}^N (u_j^{-1}\mathcal{B}_1(u_j) + u_j\mathcal{B}_1^\dagger(u_j^{-1}))|0\rangle$.

Proposition 4.8 *Let* $k_1, k_2 \cdots, k_i$ *be in the set* $\{1, 2, \dots, N\}$ *and satisfy* $k_1 < k_2 < \cdots < k_i$. *We denote* $u_{k_1}u_{k_2}\cdots u_{k_i}$ *by* $u_{\{k\}}$ *for short. Then we have*

$$
\prod_{j=1}^N \left(u_j^{-1}\mathcal{B}_1(u_j) + u_j\mathcal{B}_1^\dagger(u_j^{-1})\right)|0\rangle
$$

$$
= (u_1 \cdots u_N)^{M_1+1} \sum_{i=0}^N \sum_{k_1, \dots, k_i} (u_{\{k\}})^{-2M_1-2}
$$

$$
\times \prod_{j \neq k_i} \frac{u_j^2}{u_j^2 - u_{k_i}^2} H_1(u_{k_1}^2) \cdots H_1(u_{k_i}^2) \cdot 1
$$

where

$$
H_1(u_{k_1}^2) \cdots H_1(u_{k_i}^2) \cdot 1 = \sum_\lambda S_\lambda(u_{k_1}^2, \dots, u_{k_i}^2) S_\lambda(\mathbf{x} - \tilde{\partial}_\mathbf{y}) \cdot 1.
$$

Proof One can prove it by inductions. $\square$

From the discussion above, we get the expansion of $|\Psi_N(u_1, \dots, u_N)\rangle$.

Proposition 4.9 *The N-particle vector* $|\Psi_N(u_1, \dots, u_N)\rangle$ *can be written as*

$$
\begin{aligned}
|\Psi_N(u_1, \dots, u_N)\rangle &= (u_1 \cdots u_N)^{-M_2+M_1+1} \\
&\quad \times \sum_{i=0}^N \sum_{k_1, \dots, k_i} (u_{\{k\}})^{-2M_1-2} \prod_{j \neq k_i} \frac{u_j^2}{u_j^2 - u_{k_i}^2} \\
&\quad \times \sum_{\lambda, \mu} S_\lambda(u_{k_1}^2, \dots, u_{k_i}^2) \\
&\quad \times S_\mu(u_1^2, \dots, u_N^2) S_{[\lambda, \mu]}(\mathbf{x}, \mathbf{y})
\end{aligned}
$$

where λ *is a Young diagram with at most i rows and M_1 columns, and* μ *a Young diagram with at most N rows and M_2 columns.*

In a special case, we have $S_{[\lambda, \emptyset]}(\mathbf{x}, \mathbf{y}) = S_\lambda(\mathbf{x})$. Let $M_2 = \emptyset$,

$$
\begin{aligned}
|\Psi_N(u_1, \dots, u_N)\rangle &= \prod_{j=1}^N B_1(u_j)|0\rangle \\
&= (u_1 \cdots u_N)^{-M_1} \prod_{j=1}^N \tilde{\mathcal{B}}_1(u_j)|0\rangle \\
&= \sum_\lambda S_\lambda(u_1^2, \dots, u_N^2) S_\lambda(\mathbf{x})
\end{aligned}
$$

which is the same as in [7,8].

5 Vertex operators and topological strings on $\mathbb{C}^3$

The A-model topological string partition function on $\mathbb{C}^3$ is given by the MacMahon function

$$
Z_{\mathbb{C}^3}^{top} = M(q) = \prod_{n=1}^\infty \frac{1}{(1-q^n)^n} = \sum_{n=0}^\infty P(n)q^n. \tag{43}
$$

It is related to the topological vertex and $P(n)$ counts the number of plane partition whose total boxes number equals n. It is known that the generating function of plane partitions can be written as a fermionic correlator involving the standard vertex operators

$$\tilde{\Gamma}^+(z) = e^{\xi(\mathbf{x},z)}, \quad \tilde{\Gamma}^-(z) = e^{\xi(\tilde{\partial}_{\mathbf{x}},z^{-1})} \tag{44}$$

with a particular specialization of the values of $z = q^{\pm 1/2}, q^{\pm 3/2}, q^{\pm 5/2}, \ldots$.

In the following, we will give that $Z_{\mathbb{C}^3}^{top}$ can also be obtained from vertex operators $\Gamma_i^{\pm}(t)$, $i = 1, 2$. We first give the relations between vertex operators $\Gamma_i^{\pm}(t)$, $i = 1, 2$ and operators $\tilde{\mathcal{B}}_i(u)$ and $\tilde{\mathcal{C}}_i(u)$.

Proposition 5.1 *In the M_1, $M_2 \to \infty$ limit, operators $\tilde{\mathcal{B}}_i(u)$ and $\tilde{\mathcal{C}}_i(u)$ have the following vertex operator representations*

$$\tilde{\mathcal{B}}_1(u) = e^{\xi(\mathbf{x}-\tilde{\partial}_{\mathbf{y}},u^2)} = \Gamma_1^+(u^2), \tag{45}$$

$$\tilde{\mathcal{C}}_1(u) = e^{\xi(\tilde{\partial}_{\mathbf{x}},u^{-2})} = \Gamma_1^-(u^2), \tag{46}$$

$$\tilde{\mathcal{B}}_2(u) = e^{\xi(\mathbf{y}-\tilde{\partial}_{\mathbf{x}},u^2)} = \Gamma_2^+(u^2), \tag{47}$$

$$\tilde{\mathcal{C}}_2(u) = e^{\xi(\tilde{\partial}_{\mathbf{y}},u^{-2})} = \Gamma_2^-(u^2), \tag{48}$$

where the vertex operators $\Gamma_i^{\pm}(t)$, $i = 1, 2$ are defined in (3) *and* (4).

To obtain the form of $Z_{\mathbb{C}^3}^{top}$ written by a fermionic correlator involving vertex operators $\Gamma_i^{\pm}(t)$, $i = 1, 2$, we need the following lemma.

Lemma 5.2 *The following relation holds*

$$\langle 0| \prod_{m=1}^{\infty} \Gamma_2^-(q^{m-1/2})\Gamma_1^-(q^{m-1/2})|0\rangle = \prod_{n \geq 1}(1 - q^n)^n. \tag{49}$$

Proof Since

$$\Gamma_1^-(w) = e^{\xi(\mathbf{x},w)}e^{-\xi(\tilde{\partial}_{\mathbf{y}},w)},$$
$$\Gamma_2^-(z) = e^{\xi(\mathbf{y},z)}e^{-\xi(\tilde{\partial}_{\mathbf{x}},z)},$$

and

$$e^{-\xi(\tilde{\partial}_{\mathbf{x}},z)}e^{\xi(\mathbf{x},w)} = (1 - zw)e^{\xi(\mathbf{x},w)}e^{-\xi(\tilde{\partial}_{\mathbf{x}},z)},$$

then we get

$$\Gamma_2^-(z)\Gamma_1^-(w) = (1 - zw) : \Gamma_2^-(z)\Gamma_1^-(w) :$$

where the normal order is defined as usual. Using this formula step by step, we get the conclusion. □

Then we obtain the following result.

Proposition 5.3 *The A-model topological string partition function on $\mathbb{C}^3$ (the MacMahon function) equals*

$$Z_{\mathbb{C}^3}^{top} = \langle 0| \prod_{m=1}^{\infty} \Gamma_2^+(q^{-m+1/2})\Gamma_1^+(q^{-m+1/2})$$
$$\times \prod_{m=1}^{\infty} \Gamma_2^-(q^{m-1/2})\Gamma_1^-(q^{m-1/2})|0\rangle. \tag{50}$$

Proof Since

$$\Gamma_i^+(z)\Gamma_i^-(w) = \frac{1}{1 - w/z}\Gamma_i^-(w)\Gamma_i^+(z), \quad i = 1, 2,$$

$$\Gamma_i^+(z)\Gamma_j^-(w) = \Gamma_j^j(w)\Gamma_i^+(z), \quad i, j = 1, 2, \ i \neq j.$$

Then the right hand side of (50) equals

$$\frac{1}{(1 - q^n)^n}\frac{1}{(1 - q^n)^n}\langle 0| \prod_{m=1}^{\infty} \Gamma_2^-(q^{m-1/2})\Gamma_1^-(q^{m-1/2})|0\rangle.$$

By the Lemma 5.2, we get the conclusion. □

Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors' comment: This paper is devoted to connecting the mathematical object "Universal character" with a theoretical physical phase model. The study focus on the mathematical construction and computation. That is why there is no data.]

References

1. H. Weyl, *The Classical Groups; Their Invariants and Representations* (Princeton University Press, Princeton, 1946)
2. K. Koike, On the decomposition of tensor products of the representations of the classical groups: by means of the universal characters. Adv. Math. **74**, 57–86 (1989)
3. T. Miwa, M. Jimbo, E. Date, *Solitons: Differential Equations, Symmetries and Infinite Dimensional Algebras* (Cambridge University Press, Cambridge, 2000)
4. T. Tsuda, Universal characters and an extension of the KP hierarchy. Commun. Math. Phys. **248**, 501–526 (2004)
5. M. Kashivara, Crystalizing the q-analogue of universal enveloping algebras. Commun. Math. Phys. **133**, 249 (1990)
6. V. Korepin, N.M. Bogoliubov, A. Izergin, *Quantum Inverse Scattering Method and Correlation Functions* (Cambridge University Press, Cambridge, 1993)
7. P. Sułkowski, Deformed boson-fermion correspondence, Q-bosons, and topological strings on the conifold. JHEP **0810**, 104 (2008). arXiv:0808.2327 [hep-th]

8. N. Tsilevich, Quantum inverse scattering method for the q-boson model and symmetric functions. Funct. Anal. Appl. **40**(3), 207–217 (2006). arXiv:math-ph/0510073
9. I.G. Macdonald, *Symmetric Functions and Hall Polynomials. Oxford Mathematical Monographs* (Clarendon Press, Oxford, 1979)
10. W. Fulton, J. Harris, *Representation Theory, A First Course* (Springer, New York, 1991)
11. N.M. Bogoliubov, R. Bullough, J. Timonen, Critical behavior for correlated strongly coupled boson systems in 1+1 dimensions. Phys. Rev. Lett. **25**, 3933–3926 (1994)
12. N.M. Bogoliubov, A. Izergin, N. Kitanine, Correlation functions for a strongly correlated boson system. Nucl. Phys. B **516**, 501–528 (1998). arXiv:solv-int/9710002
13. N.M. Bogoliubov, Boxed Plane Partitions as an Exactly Solvable Boson Model. arXiv:cond-mat/0503748