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Spectrum of the *uudcc̄* hidden charm pentaquark with an SU(4) flavor-spin hyperfine interaction

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Abstract We study a few of the lowest states of the pentaquark *uudcc̄*, of positive and negative parity, in a constituent quark model with an SU(4) flavor-spin hyperfine interaction. For positive parity we introduce space wave functions of appropriate permutation symmetry with one unit of orbital angular momentum in the internal motion of the four-quark subsystem or an orbital excitation between the antiquark and the four quark subsystem which remains in the ground state. We show that the lowest positive parity states $1/2^+$, $3/2^+$ are provided by the first alternative and are located below the $1/2^-$ and the $1/2^+$ states with all quarks in the ground state. We compare our results with the LHCb three narrow pentaquark structures reported in 2019.

1 Introduction

A considerable wave of interest has been raised by the 2015 observation of the LHCb collaboration of two resonances $P_c^+(4380)$ ($\Gamma = 205 \pm 18 \pm 86$ MeV) and $P_c^+(4450)$ ($\Gamma = 39 \pm 5 \pm 19$ MeV) in the $\Lambda_b^0 \to J/\psi K^- p$ decay, interpreted as hidden charm pentaquarks of structure *uudcc̄* [1]. The preferred quantum numbers, which gave the best solution, were $3/2^-$, $5/2^+$. However, the quantum numbers $3/2^+$, $5/2^-$ and $5/2^-$, $3/2^+$ were also acceptable. Various possible interpretations of the 2015 LHCb resonances as kinematical effects, molecular states or compact pentaquarks were reviewed, see for example Refs. [2–4].

Recently the LHCb Collaboration have updated their analysis of the $\Lambda_b^0 \to J/\psi K^- p$ decay [5,6] and observed three narrow peaks with masses and widths as given in Table 1, where the entirely new $P_c^+(4312)$ has a 7.3σ statistical significance and the other two resonances replace the previous $P_c^+(4450)$ now resolved at 5.4σ significance. The broad $P_c^+(4380)$ resonance observed in 2015 awaits confirmation

and a proper identification of the spin-parity is necessary in all cases.

The mass of $P_c^+(4312)$ lies a few MeV below the $\Sigma_c^+ \overline{D}^0$ threshold (4318 MeV) and the masses of $P_c^+(4440)$ and $P_c^+(4457)$ lie a few MeV below the $\Sigma_c^+ \overline{D}^{*0}$ threshold (4460 MeV). Proximity of the $\Sigma_c^+ \overline{D}^0$ and $\Sigma_c^+ \overline{D}^{*0}$ thresholds to these narrow peaks made their interpretation as molecular S-wave of $\Sigma_c^+ \overline{D}^0$ or $\Sigma_c^+ \overline{D}^{*0}$ quite natural [7–12]. In such an interpretation, they all acquire a negative parity. In Ref. [13] it is shown that the usual molecular scenarios cannot explain the production rate of P_c states in Λ_b decays. Moreover, the paper points out the need to couple the $\Sigma_c^+ \overline{D}^{*0}$ and the $\Lambda_c(2595)\overline{D}$ channels due to the very close proximity of their thresholds. As a result, the second and the third resonances of Table 1 acquire opposite parities, namely $J^P(4440) = \frac{3}{2}^-$ and $J^P(4457) = \frac{1}{2}^+$ and respectively.

The 2019 LHCb pentaquarks have also been analysed in the compact diquark model [14]. This analysis has been extended in Ref. [15], and presently includes a spin–spin, a spin–orbit and a tensor interaction, obtained from the integration in the color space of the one gluon exchange (OGE) interaction between quarks [16]. The basis contains only the color $\bar{3}$ configuration, therefore is smaller than the basis used in the present paper. Also there is a belief that the spin 0 diquarks are more tightly bound than the spin 1 diquarks. The lowest positive and negative parity states were calculated and it was found that the lowest state has negative parity.

Here we explore the spectrum of pentaquarks within a quark model [17–19], which has a flavor dependent hyperfine interaction. Contrary to the OGE model result, the lightest pentaquark has positive parity, as shown below. As a general feature, the chromomagnetic (color-spin) and the flavor-spin interactions predict contradictory results regarding the parity of open charm exotic systems, as the tetraquark $uu\bar{c}\bar{c}$, the pentaquarks $uudd\bar{c}$, $uuds\bar{c}$ and the hexaquark $uuddsc$ [20].

Table 1 Masses and decay widths of the 2019 LHCb resonances [5,6]

Resonance	Mass (MeV)	Width (MeV)
$P_c^+(4312)$	$4311.9 \pm 0.7^{+6.8}_{-0.6}$	$9.8 \pm 2.7^{+3.7}_{-4.5}$
$P_c^+(4440)$	$4440.3 \pm 1.3^{+4.1}_{-4.7}$	$20.6 \pm 4.9^{+8.7}_{-10.1}$
$P_c^+(4457)$	$4457.3 \pm 0.6^{+4.1}_{-1.7}$	$6.4 \pm 2.0^{+5.7}_{-1.9}$

In Refs. [17,19] the hyperfine splitting in hadrons is obtained from the short-range part of the Goldstone boson exchange interaction between quarks. The main merit of the model is that it reproduces the correct ordering of positive and negative parity states in the baryons spectrum in contrast to the OGE model. On the other hand it does not apply to the hyperfine splitting in mesons because it does not explicitly contain a quark-antiquark interaction.

Here we extend the model of Refs. [17,19] from SU(3) to SU(4) in order to incorporate the charm quark. The extension is made in the spirit of the phenomenological approach of Ref. [21] where, in addition to Goldstone bosons of the hidden approximate chiral symmetry of QCD, the flavor exchange interaction is augmented by additional exchange of D mesons between u, d and c quarks and of D_s mesons between s and c quarks. The model provided a satisfactory description of the heavy flavor baryons known at that time. With the SU(4) extension we give a more fundamental ground to the work of Ref. [17] and test the corresponding approximations. In the following we shall use the name of flavor-spin (FS) model for the SU(4) extension. The extended flavor exchange allows a complete antisymmetrization of quarks even for quarks of widely different mass, since the Young diagrams (or partitions) serve to label the irreducible representations both of the permutation and of the SU(4) group [22].

Presently we study the pentaquarks of structure $uudc\bar{c}$ in order to see whether or not the FS model can accommodate the new resonances observed at LHCb [6]. The approach is similar to that previously used in Ref. [23] for the positive parity $uudd\bar{c}$ and $uuds\bar{c}$ pentaquarks. That work preceded the H1 experiment [24] which has found evidence, so far not confirmed by other experiments, for a narrow baryon resonance in the $D^{*\pm}p$ invariant mass, interpreted as a $uudd\bar{c}$ pentaquark. The mass prediction made in Ref. [23] is not far from the observation of the H1 experiment.

The parity of the pentaquark is given by P$= (-)^{\ell+1}$. For the lowest positive parity states (see below) the subsystem of four quarks is defined to carry an angular momentum $\ell = 1$ in their internal motion. This implies that this subsystem must be in a state of orbital symmetry $[31]_O$. Although the kinetic energy of such a state is higher than that of the totally symmetric $[4]_O$ state, a crude estimate based on the simplified interaction (1), suggests that the $[31]_O$ symmetry leads

to a stable pentaquark of positive parity and the lowest state of symmetry $[4]_O$ is unbound.

In the exact SU(4) limit, like in the exact SU(3) limit, the flavor-spin interaction introduced in the next section takes the following form

$$V_\chi = -C_\chi \sum_{i<j} \lambda_i^F \cdot \lambda_j^F \, \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j \tag{1}$$

with λ_i^F the Gell-Mann matrices, $\boldsymbol{\sigma}_i$ the Pauli matrices and C_χ an equal strength constant for all pairs.

Let us consider two totally antisymmetric states with the orbital symmetry $[31]_O$, allowed by Pauli principle. They are written in the flavor-spin (FS) coupling scheme, as given in Table 2. They are consistent with the isospin and J^P combinations of Table 2 of Ref. [25].

The expectation value of (1), as shown in Table 2 (see the next section for the derivation of the last column), is $-27\,C_\chi$ for $|1\rangle$ and $-21\,C_\chi$ for $|2\rangle$.

First we consider the lowest state, i.e. $|1\rangle$. The quark-antiquark interaction is neglected in the description of mesons as well, so that the meson Hamiltonian contains a kinetic and a confinement term only. If unstable against strong decays, the pentaquark would split into a baryon q^3 and a meson $Q\overline{Q}$, where Q stands for a heavy flavor. Then, in discussing the stability we introduce the quantity

$$\Delta E = E(qqqQ\overline{Q}) - E(q^3) - E(Q\overline{Q}) \tag{2}$$

where E defines the energy (mass) of a system with the content defined in the bracket. In our schematic estimate, we suppose that the confinement energy roughly cancels out in ΔE. Then, the kinetic energy contribution to ΔE is $\Delta KE = 5/4\,\hbar\omega$ in a harmonic oscillator model [26].

Table 2 shows that the FS contribution to the state $|1\rangle$ is $-27\,C_\chi$. For the nucleon or Λ_c it is $-27/2\,C_\chi$ (see next section). Then we have $\Delta V_\chi = -27/2\,C_\chi$. With $\hbar\omega \approx 157$ MeV and $C_\chi \approx 20$ MeV [21] this gives

$$\Delta E = \frac{5}{4}\hbar\omega - 27/2\,C_\chi \approx -74\,\text{MeV}, \tag{3}$$

i.e. a considerable binding. This is to be contrasted with the negative parity pentaquarks similar to those containing light quarks, studied in Ref. [27]. For the lowest negative parity state $|3\rangle$ of $uudc\bar{c}$ one gets

$$\Delta E = 3/4\,\hbar\omega - 3/2\,C_\chi \approx 88\,\text{MeV}, \tag{4}$$

i.e. instability. Thus the FS interaction overcomes the excess of kinetic contribution in $[31]_O$ and generates a lower expectation value for $[31]_O$ than for $[4]_O$.

The paper is organized as follows. In Sect. 2 we introduce the model Hamiltonian and generalize the two-body matrix elements of the FS interaction from SU(3) to SU(4). Section 3

Table 2 The simplified interaction - V_χ/C_χ, Eq. (1), integrated in the flavor-spin (FS) space for four quark subsystems in three different states defined in column 2 in the FS coupling. The notation is: O for orbital, C for color, F for flavor, S for spin

State	Definition	L	S	$\langle \sum_{i<j} \lambda_i^F . \lambda_j^F \, \boldsymbol{\sigma}_i . \boldsymbol{\sigma}_j \rangle$
$|1\rangle$	$\left([31]_O[211]_C[1^4]_{OC} \, ; \, [22]_F[22]_S[4]_{FS}\right)$	1	0	27
$|2\rangle$	$\left([31]_O[211]_C[1^4]_{OC} \, ; \, [31]_F[31]_S[4]_{FS}\right)$	1	1	21
$|3\rangle$	$\left([4]_O[211]_C[211]]_{OC} \, ; \, [211]_F[22]_S[31]_{FS}\right)$	0	0	15

describes the orbital part of the four quark subsystem constructed to be translationally invariant both for positive and negative parity states. Section 4 summarizes a few analytic details. Section 5 exhibits the numerical results for masses and distances between quarks/antiquarks. A qualitative discussion of the decay widths is given in Sect. 6. A comparison with previous studies of hidden charm pentaquark based on various versions of the FS model is made in Sect. 7. In the last section we draw some conclusions. Appendix A gives the matrices λ^F of SU(4). Appendix B is a reminder of useful group theory formulae for SU(n). Appendix C exhibits the variational solution for the baryon masses relevant for the present study. In Appendix D we construct the explicit form of flavor states of content $uudc$ in the Young Yamanouchi basis for specific irreducible representations $[f]_F$.

2 The Hamiltonian

The nonrelativistic FS Hamiltonian has the general form [17]

$$H = \sum_i m_i + \sum_i \frac{\boldsymbol{p}_i^2}{2m_i} - \frac{(\sum_i \boldsymbol{p}_i)^2}{2 \sum_i m_i} + \sum_{i<j} V_{\text{conf}}(r_{ij})$$
$$+ \sum_{i<j} V_\chi(r_{ij}), \tag{5}$$

with m_i and $\boldsymbol{p}_i$ denoting the quark masses and momenta respectively and r_{ij} the distance between the interacting quarks i and j. The Hamiltonian contains the internal kinetic energy and the linear confining interaction

$$V_{\text{conf}}(r_{ij}) = -\frac{3}{8}\lambda_i^c \cdot \lambda_j^c \, C \, r_{ij} . \tag{6}$$

The hyperfine part $V_\chi(r_{ij})$ has a flavor-spin structure which presently is extended to SU(4), similarly to Refs. [21,28], except that in Ref. [21], the term $V_{\eta'}\lambda_i^0 \cdot \lambda_j^0$ of Eq. (7) has been ignored. One has

$$V_\chi(r_{ij}) = \left\{ \sum_{F=1}^{3} V_\pi(r_{ij})\lambda_i^F\lambda_j^F + \sum_{F=4}^{7} V_K(r_{ij})\lambda_i^F\lambda_j^F \right.$$
$$+ V_\eta(r_{ij})\lambda_i^8\lambda_j^8 + V_{\eta'}(r_{ij})\lambda_i^0\lambda_j^0$$

$$+ \sum_{F=9}^{12} V_D(r_{ij})\lambda_i^F\lambda_j^F + \sum_{F=13}^{14} V_{D_s}(r_{ij})\lambda_i^F\lambda_j^F$$
$$\left. + V_{\eta_c}(r_{ij})\lambda_i^{15}\lambda_j^{15} \right\} \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j, \tag{7}$$

with λ_i^F ($F = 1, 2, \ldots, 15$) defined in Appendix A and $\lambda_i^0 = \sqrt{2/3}\,\mathbf{1}$, where $\mathbf{1}$ is the 4×4 unit matrix. Note that the norm of λ_i^0 is not that satisfying the general property $\text{tr}(\lambda^a\lambda^b) = 2\delta^{ab}$ if $\mathbf{1}$ is the 4×4 unit matrix [23], but we maintain the norm from the SU(3) version of the flavor-spin interaction in order to use the same coupling constant for η and η', which gave a good spectrum for strange baryons. At this stage it is useful to mention the study of Ref. [29] where it has been shown that η' has a significant charm component.

In the SU(4) version the interaction (7) contains $\gamma = \pi, K, \eta, D, D_s, \eta_c$ and η' meson-exchange terms. Every $V_\gamma(r_{ij})$ is a sum of two distinct contributions: a Yukawa-type potential containing the mass of the exchanged meson and a short-range contribution of opposite sign, the role of which is crucial in baryon spectroscopy [30]. For a given meson γ the meson exchange potential is

$$V_\gamma(r) = \frac{g_\gamma^2}{4\pi} \frac{1}{12m_im_j} \left\{ \theta(r - r_0)\mu_\gamma^2 \frac{e^{-\mu_\gamma r}}{r} \right.$$
$$\left. - \frac{4}{\sqrt{\pi}}\alpha^3 \exp(-\alpha^2(r - r_0)^2) \right\}. \tag{8}$$

In the present calculations we use the parameters of Ref. [30] to which we add the μ_D mass and the coupling constant $\frac{g_{Dq}^2}{4\pi}$. These are

$$\frac{g_{\pi q}^2}{4\pi} = \frac{g_{\eta q}^2}{4\pi} = \frac{g_{Dq}^2}{4\pi} = 0.67, \quad \frac{g_{\eta'q}^2}{4\pi} = 1.206,$$
$$r_0 = 0.43 \,\text{fm}, \quad \alpha = 2.91 \,\text{fm}^{-1}, \quad C = 0.474 \,\text{fm}^{-2},$$
$$\mu_\pi = 139 \,\text{MeV}, \quad \mu_\eta = 547 \,\text{MeV}, \quad \mu_{\eta'} = 958 \,\text{MeV},$$
$$\mu_D = 1867 \,\text{MeV}.$$

The meson masses correspond to the experimental values from the Particle Data Group [31]. There is no K- and D_s-exchange in the $uudc\bar{c}$ pentaquark and, as discussed in the following, we ignore η_c-exchange. Therefore the K, D_s and η_c masses are not needed in these calculations.

For the quark masses we take the values determined variationally in Ref. [28]

$$m_{u,d} = 340 \text{ MeV}, \quad m_c = 1350 \text{ MeV}. \tag{9}$$

They were adjusted to reproduce the average mass $\overline{M} = (M + 3M^*)/4 = 2008$ MeV of the D mesons. With these values one obtains an average mass of 2985 MeV for J/ψ and η_c.

After integrating in the flavor space, the two-body matrix elements which generalize Eq. (3.3) of Ref. [17] to SU(4), one obtains a result containing new contributions due to charm. This is

$$
V_{ij} = \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j
\begin{cases}
V_\pi + \frac{1}{3} V_\eta^{uu} + \frac{1}{6} V_{\eta_c}^{uu}, & [2]_F, I = 1 \\[6pt]
2V_K - \frac{2}{3} V_\eta^{us}, \quad 2V_D^{uc} - \frac{1}{2} V_{\eta_c}^{uc} & [2]_F, I = \frac{1}{2} \\[6pt]
2V_{D_s}^{sc} - \frac{1}{2} V_{\eta_c}^{sc} & [2]_F, I = 0 \\[6pt]
\frac{4}{3} V_\eta^{ss} + \frac{3}{2} V_{\eta_c}^{cc} & [2]_F, I = 0 \\[6pt]
-2V_{D_s}^{sc} - \frac{1}{2} V_{\eta_c}^{sc} & [11]_F, I = 0 \\[6pt]
-2V_K - \frac{2}{3} V_\eta^{us}, \quad -2V_D^{uc} - \frac{1}{2} V_{\eta_c}^{uc} & [11]_F, I = \frac{1}{2} \\[6pt]
-3V_\pi + \frac{1}{3} V_\eta^{uu} + \frac{1}{6} V_{\eta_c}^{uu}, & [11]_F, I = 0
\end{cases}
\tag{10}
$$

In Eq. (10) the pair of quarks ij is either in a symmetric $[2]_F$ or in an antisymmetric $[11]_F$ flavor state and the isospin I is defined by the quark content. The upper index of V exhibits the flavor of the two quarks interchanging a meson specified by the lower index. In order to keep close to the notations of Ref. [17] the upper index of π and K is not indicated. Obviously, in every sum/difference of Eq. (10) the upper index is the same for all terms.

Note that the term $\frac{1}{2} V_{\eta_c}^{uc}$ was missing in Ref. [28]. In practice it can be neglected but theoretically it is important because it recovers the exact SU(4) limit exhibited in Table 2.

Using the flavor wave functions given in Appendix D and the expressions (10) one can calculate the matrix elements of the flavor-spin interaction (7) for four quark states. They are presented in Table 3. To recover the SU(4) limit of Table 2 for the $uudc$ configuration, one has to take $V_\pi = V_\eta^{uu} = V_{\eta_c}^{uu} = V_D^{uc} = V_{\eta_c}^{uc} = -C_\chi$ and $V_{\eta'}^{uu} = V_{\eta'}^{uc} = 0$.

In the exact SU(4) limit the states $[2]_F$ and $[11]_F$ of Eq. (10) containing u, d, c quarks become $-3/2\,C_\chi$ and $5/2\,C_\chi$ respectively. Then, making use of the spin matrix elements $\langle \sigma_i \cdot \sigma_j \rangle = 1$ and $\langle \sigma_i \cdot \sigma_j \rangle = -3$ for $S = 1$ and $S = 0$ respectively, one finds that the contribution of the FS interaction to the baryon masses are $V_\chi(N) = V_\chi(\Lambda_c) = -27/2\,C_\chi$ and $V_\chi(\Delta) = -9/2\,C_\chi$, which are close to the SU(3) limits $-14\,C_\chi$ and $-4\,C_\chi$ respectively. As a matter of fact one can

also see that the SU(4) limit given in $-C_\chi$ units, as listed in Table 2, namely 27, 21 and 15 are quite close to the SU(3) limit (three flavors) of 28, 64/3 and 16 respectively, which can easily be obtained using Appendix B.

The exact SU(3) or SU(4) limits are useful in indicating the sequence of positive and negative parity states in the spectrum, which is maintained at a broken unitary symmetry.

3 Orbital space

This section follows closely the derivation of the orbital part of the four-quark wave function of symmetry $[31]_O$ as presented in Ref. [23].

One needs four internal Jacobi coordinates for pentaquarks. The convenient choice is

$$
\begin{aligned}
\boldsymbol{x} &= \boldsymbol{r}_1 - \boldsymbol{r}_2, \\
\boldsymbol{y} &= (\boldsymbol{r}_1 + \boldsymbol{r}_2 - 2\boldsymbol{r}_3)/\sqrt{3}, \\
\boldsymbol{z} &= (\boldsymbol{r}_1 + \boldsymbol{r}_2 + \boldsymbol{r}_3 - 3\boldsymbol{r}_4)/\sqrt{6}, \\
\boldsymbol{t} &= (\boldsymbol{r}_1 + \boldsymbol{r}_2 + \boldsymbol{r}_3 + \boldsymbol{r}_4 - 4\boldsymbol{r}_5)/\sqrt{10},
\end{aligned}
\tag{11}
$$

where 1,2,3 and 4 are the quarks and 5 the antiquark so that t gives the distance between the antiquark and the center of mass coordinate of the four-quark subsystem. Then one has to express the orbital wave functions of the four-quark subsystem in terms of the internal coordinates $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ for the specific permutation symmetry $[31]_O$. For the lowest positive parity states we have considered an $s^3 p$ structure for $[31]_O$ and extended Moshinski's method [32], from three to four particles, to construct translationally invariant states of definite permutation symmetry. In this way we have obtained three independent states denoted below by ψ_i, defined as Young–Yamanouchi basis vectors [22] of the irreducible representation $[31]_O$ in terms of shell model states $\langle \mathbf{r}\,|n\ell m \rangle$ where $n = 0, \ell = 1$, and, for simplicity, we took m = 0 everywhere. They read

$$\psi_1 = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1} \end{array} = \langle \mathbf{x}\,|000\rangle \langle \mathbf{y}\,|000\rangle \langle \mathbf{z}\,|010\rangle \tag{12}$$

$$\psi_2 = \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1} \end{array} = \langle \mathbf{x}\,|000\rangle \langle \mathbf{y}\,|010\rangle \langle \mathbf{z}\,|000\rangle \tag{13}$$

$$\psi_3 = \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1} \end{array} = \langle \mathbf{x}\,|010\rangle \langle \mathbf{y}\,|000\rangle \langle \mathbf{z}\,|000\rangle \tag{14}$$

One can see that the $\ell = 1$ orbital excitation is located in the relative motion between quarks defined by the internal coordinates $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$. The states ψ_1 and ψ_2 are the generalization, from three to four quarks, of $\psi_{\ell 0}^\lambda$ and ψ_3 the generalization, from three to four quarks, of $\psi_{\ell 0}^\rho$ radial wave functions of mixed symmetry [21] used in baryon spectroscopy. Thus, a more appealing notation could be $\psi_{\ell 0}^{\lambda_1}$, $\psi_{\ell 0}^{\lambda_2}$ and $\psi_{\ell 0}^\rho$ instead of ψ_i with $i = 1, 2, 3$.

In this picture there is no excitation in the relative motion between the cluster of four quarks and the antiquark defined by the coordinate **t**. Then the pentaquark orbital wave functions ψ_i^5 are obtained by multiplying each ψ_i from above by the wave function $\langle \mathbf{t} | 000 \rangle$ which describes the relative motion between the four-quark subsystem and the antiquark $\bar{c}$. Assuming an exponential behavior we introduce two variational parameters, a for the internal motion of the four-quark subsystem and b for the relative motion between the subsystem $qqqc$ and $\bar{c}$. We explicitly have

$$\psi_1^5 = N \exp\left[-\frac{a}{2} \left(x^2 + y^2 + z^2 \right) - \frac{b}{2} t^2 \right] z \, Y_{10} \left(\hat{z} \right) \tag{15}$$

$$\psi_2^5 = N \exp\left[-\frac{a}{2} \left(x^2 + y^2 + z^2 \right) - \frac{b}{2} t^2 \right] y \, Y_{10} \left(\hat{y} \right) \tag{16}$$

$$\psi_3^5 = N \exp\left[-\frac{a}{2} \left(x^2 + y^2 + z^2 \right) - \frac{b}{2} t^2 \right] x \, Y_{10} \left(\hat{x} \right) \tag{17}$$

where

$$N = \frac{2^{3/2} a^{11/4} b^{3/4}}{3^{1/2} \pi^{5/2}}. \tag{18}$$

We also need the orbital wave function of the lowest negative parity state described by the s^4 configuration of symmetry $[4]_O$ which is

$$\phi_0 = N_0 \exp\left[-\frac{a}{2} \left(x^2 + y^2 + z^2 \right) - \frac{b}{2} t^2 \right], \tag{19}$$

with

$$N_0 = \left(\frac{a}{\pi} \right)^{9/4} \left(\frac{b}{\pi} \right)^{3/4}. \tag{20}$$

4 Analytic details

The kinetic energy part T of the Hamiltonian (5) can be calculated analytically. For the positive parity states of Table 2 its expectation value becomes

$$\langle T \rangle = \frac{1}{3} \left[\left\langle \psi_1^5 | T | \psi_1^5 \right\rangle + \left\langle \psi_2^5 | T | \psi_2^5 \right\rangle + \left\langle \psi_3^5 | T | \psi_3^5 \right\rangle \right]$$
$$= \hbar^2 \left(\frac{11}{2\mu_1} a + \frac{3}{2\mu_2} b \right), \tag{21}$$

with

$$\frac{4}{\mu_1} = \frac{3}{m_q} + \frac{1}{m_Q}, \tag{22}$$

and

$$\frac{5}{\mu_2} = \frac{1}{\mu_1} + \frac{4}{m_Q}, \tag{23}$$

where $q = u, d$ and $Q = c$. Here, we have $m_q = 340$ MeV and $m_c = 1350$ MeV, as defined by Eq. (9). Taking $m_u = m_d = m_Q = m$ and setting a = b, one can recover the identical particle limit $\langle T \rangle = \frac{7}{2} \hbar \omega$ with $\hbar \omega = 2 a \hbar^2 / m$, see [26]. Note that one can obtain the desired units of all quantities involved by mutiplying/dividing them by $\hbar c \simeq 197.33$ MeV fm at the corresponding power.

By integrating in the color space, the expectation value of the confinement interaction (6) becomes

$$\langle V_{conf} \rangle = \frac{C}{2} \left(6 \, \langle r_{12} \rangle + 4 \, \langle r_{45} \rangle \right) \tag{24}$$

where the coefficients 6 and 4 account for the number of quark-quark and quark-antiquark pairs, respectively, and

$$\langle r_{ij} \rangle = \frac{1}{3} \left[\left\langle \psi_1^5 | r_{ij} | \psi_1^5 \right\rangle + \left\langle \psi_2^5 | r_{ij} | \psi_2^5 \right\rangle + \left\langle \psi_3^5 | r_{ij} | \psi_3^5 \right\rangle \right], \tag{25}$$

where $i, j = 1, 2, 3, 4, 5$ ($i \neq j$). An analytic evaluation gives

$$\langle r_{12} \rangle = \frac{20}{9} \sqrt{\frac{1}{\pi a}}, \tag{26}$$

as well as for r_{ij} with $i, j = 1, 2, 3, 4$ and

$$\langle r_{45} \rangle = \frac{1}{3\sqrt{2\pi}} \left[2\sqrt{\frac{3}{a} + \frac{5}{b}} + \sqrt{5} b \left(\frac{1}{2a} + \frac{1}{b} \right) \right], \tag{27}$$

and likewise for r_{j5} where $j = 1, 2, 3$.

One also needs to derive the expressions of $\langle r_{12} \rangle$ and $\langle r_{45} \rangle$ in the case where the four quarks are in the s^4 configuration. These are

$$\langle r_{12} \rangle = \sqrt{\frac{4}{\pi a}}, \tag{28}$$

and

$$\langle r_{45} \rangle = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{3}{a} + \frac{5}{b}}. \tag{29}$$

The matrix elements of the spin-flavor operators of Eq. (7) have been calculated using the fractional parentage technique described in Ref. [22] based on Clebsch–Gordan coefficients of the group S_4 [33]. In this way, each matrix element reduces

Table 3 The hyperfine interaction V_χ, Eq. (7), integrated in the flavor-spin space, for the quark subsystem $uudc$. $V_\gamma^{q_a q_b}$ are defined in Eq. (10) where the upper index $q_a q_b$ indicates the flavor of the interacting quark pair

State	V_χ
$\lvert[31]_O\,[22]_F\,[22]_S[4]_{FS}\rangle$	$15\,V_\pi - V_\eta^{uu} - 2V_{\eta'}^{uu} + 12V_D^{uc} - \frac{1}{2}V_{\eta_c}^{uu} + \frac{3}{2}V_{\eta_c}^{uc} - 2V_{\eta'}^{uc}$
$\lvert[31]_O\,[31]_F\,[31]_S[4]_{FS}\rangle$	$3\,V_\pi + V_\eta^{uu} + \frac{1}{3}V_{\eta'}^{uu} + 14V_D^{uc} + \frac{1}{2}V_{\eta_c}^{uu} + \frac{5}{2}V_{\eta_c}^{uc} - \frac{10}{3}V_{\eta'}^{uc}$
$\lvert[4]_O\,[211]_F\,[22]_S[31]_{FS}\rangle$	$7\,V_\pi - \frac{7}{9}V_\eta^{uu} - \frac{14}{9}V_{\eta'}^{uu} + + \frac{22}{3}V_D^{uc} - \frac{7}{18}V_{\eta_c}^{uu} + \frac{11}{6}V_{\eta_c}^{uc} - \frac{22}{9}V_{\eta'}^{uc}$

to a linear combination of two-body matrix elements of either symmetric or antisymmetric states for which Eq. (10) can be used to integrate in the spin-flavor space. For positive parity states with one unit of orbital excitation the resulting linear combinations contain orbital two-body matrix elements of type $\langle ss\,|V_\gamma^{q_a q_b}|\,ss\rangle$, $\langle sp\,|V_\gamma^{q_a q_b}|\,sp\rangle$ and $\langle sp\,|V_\gamma^{q_a q_b}|\,ps\rangle$ where $\gamma = \pi, D, \eta, \eta'$ and $q_a q_b$ is a pair of quarks from Eq. (10).

It is useful to note that the ratio of the orbital matrix elements of the D meson exchange interaction between the pair uc of quarks, denoted here by V_D^{uc} and the matrix elements of the π meson exchange interaction V_π, see Eq. (10), turned out to be about 0.25, in agreement with the mass ratio $m_{u,d}/m_c$ of Eq. (9).
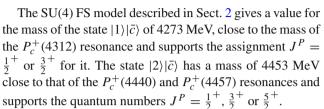
5 Results

The four quark states described in the previous sections couple to the antiquark to a total angular momentum $\mathbf{J} = \mathbf{L} + \mathbf{S} + \mathbf{s}_Q$, with $\mathbf{L}$, $\mathbf{S}$ the angular momentum and spin of the four-quark cluster and $\mathbf{s}_Q$ the spin of the heavy antiquark.

We have searched for variational solutions for the pentaquark described by the Hamiltonian defined in Sect. 2. The numerical results are presented in Table 4. The wave functions are the product of the four quarks subsystem states of flavor-spin structure defined in Table 3 and the charm antiquark wave function denoted by $|\bar{c}\rangle$. The orbital wave functions of the pentaquark are given by Eqs. (15)–(20) containing the variational parameters a and b. We have neglected the contribution of $V_{\eta_c}^{uu}$ and of $V_{\eta_c}^{uc}$, because little $u\bar{u}$ and $d\bar{d}$ is expected in η_c. We have also neglected $V_{\eta'}^{uc}$ assuming a little $c\bar{c}$ component in η', which means that we took

$$V_{\eta_c}^{uu} = V_{\eta_c}^{uc} = V_{\eta'}^{uc} = 0. \tag{30}$$

The optimal values found for the parameters a and b are indicated in each case. The eigenvalues of $|1\rangle|\bar{c}\rangle$ and $|2\rangle|\bar{c}\rangle$ states are degenerate for the allowed values of J in each case. Moreover, for the wave function $|2\rangle|\bar{c}\rangle$ the states with $J^P = \frac{1}{2}^+$ and $\frac{3}{2}^+$ have multiplicity 2. Presently there are more states than the observed ones. In the case of pentaquarks with heavy flavor, Ref. [14] argues that only part of the spectrum is reachable in Λ_b decays.

The SU(4) FS model described in Sect. 2 gives a value for the mass of the state $|1\rangle|\bar{c}\rangle$ of 4273 MeV, close to the mass of the $P_c^+(4312)$ resonance and supports the assignment $J^P = \frac{1}{2}^+$ or $\frac{3}{2}^+$ for it. The state $|2\rangle|\bar{c}\rangle$ has a mass of 4453 MeV close to the $P_c^+(4440)$ and $P_c^+(4457)$ resonances and supports the quantum numbers $J^P = \frac{1}{2}^+$, $\frac{3}{2}^+$ or $\frac{5}{2}^+$.

As discussed above, in the exact limit one obtains binding for $|1\rangle|\bar{c}\rangle$, see Eq. (3), but this is not the case for the variational solution in the broken SU(4), which is larger than the lowest thresholds, $N\eta_c(3920)$ or $NJ/\psi(4040)$. It is not surprising, because in the exact limit the matrix elements are equal for all exchanged mesons, which means that the matrix elements of η, η' and D exchanges are overestimated, thus introducing more attraction. However the SU(4) exact limit brings useful information about the relative position of various states.

The lowest negative parity state $|3\rangle|\bar{c}\rangle$ is located above the positive parity states, as expected. It has a mass of 4487 MeV close that of the $P_c^+(4440)$ or the $P_c^+(4457)$ resonances. It supports the quantum number $J^P = 1/2^-$.

It may be useful to look at the compactness of the present pentaquarks by estimating the distances $\langle r_{12}\rangle$ and $\langle r_{45}\rangle$ from Eqs. (25)–(29). They are exhibited in Table 4. In the $|1\rangle|\bar{c}\rangle$ state, which is the lowest, one can see that at equilibrium the four quarks are clustered together and the antiquark is somewhat far apart. For the other states the distances $\langle r_{12}\rangle$ and $\langle r_{45}\rangle$ are comparable.

The values of the above convenient relative distances were exhibited in order to see whether or not some clustering of four-quark formation is possible. The effect is quite small. One can use alternative relative coordinates convenient either for a molecular-type structure, or for the hadrocharmonium picture. They are both deuteron-like. Unitary transformations can relate various coordinate systems. This can be considered in the future.

Also, one could introduce a third variational parameter, different from a or b, associated to the coordinate z where the charm quark is involved. In that case the analytic work would be more involved. It was found that the energy levels vary slowly with the present variational parameters, so that the minima are very shallow. In view of this observation the spectrum is expected to be a little lowered by the introduction of a new parameter, but the parity sequence will remain the same. However a more sophisticated variational solution would be important for the wave function, thus for studying

Table 4 Lowest positive and negative parity $uudc\bar{c}$ pentaquarks of quantum numbers I, J^P and symmetry structure defined in Table 2. Column 1 gives the state, column 2 the isospin, column 3 the parity and total angular momentum, column 4 the optimal variational param-eters associated to the wave functions defined in Sect. 3, column 5 the calculated mass and the last two columns the distances between quarks/antiquarks

State	I	J^P	Variational parameters		Mass (GeV)	$\langle r_{12} \rangle$ fm	$\langle r_{45} \rangle$ fm
			a (fm^{-2})	b (fm^{-2})			
$\lvert 1 \rangle \, \lvert \bar{c} \rangle$	$\frac{1}{2}$	$\frac{1}{2}^+, \frac{3}{2}^+$	2.055	1.079	4.273	0.875	1.018
$\lvert 2 \rangle \, \lvert \bar{c} \rangle$	$\frac{1}{2}, \frac{3}{2}$	$\frac{1}{2}^+, \frac{3}{2}^+, \frac{5}{2}^+$	1.541	1.027	4.453	1.010	1.085
$\lvert 3 \rangle \, \lvert \bar{c} \rangle$	$\frac{1}{2}$	$\frac{1}{2}^-$	1.284	1.079	4.487	0.996	1.062

decay properties numerically. More theoretical work should be done, in wait for more precise data.

6 Strong decay widths

Assuming a coupling to open channels, which could provide estimates of widths (and level shifts), one can qualitatively discuss the strong decay widths in the context of compact pentaquarks. First, for P-waves the centrifugal barrier will reduce the widths relative to S-waves. Thus, in the present model, the lowest state, having positive parity, will have a smaller width than the negative parity ones, contrary to the arguments of Ref. [25] inspired by OGE or molecular models. Next, the narrowness of the lowest state is related to the overlap of the color-flavor-spin-orbital wave function with an $p + J/\psi$ final state. For the lowest state, $\lvert [31]_O \, [22]_F [22]_S [4]_{FS} \rangle$, the overlap with $q^3 + Q\bar{Q}$ may be small. This problem deserves a separate investigation.

7 Comparison with other studies

The basis (15)–(17) is entirely consistent with the parity considerations made in the context of the tetrahedral group T_d [34], isomorphic with the permutation group S_4. The states $\lvert 1 \rangle$ and $\lvert 2 \rangle$ of Table 2 are states of symmetry $1^-_{A_1}$ in the notation of Ref. [34]. Likewise the state $\lvert 3 \rangle$ of the same table has symmetry $0^+_{A_1}$. There is agreement between our findings and the conclusion of Ref. [34] that the parity of a pentaquark depends on the interplay between the flavor-spin interaction and the contribution of the orbital excitation to the mass. Here we found that the flavor-spin attraction compensates the excess of kinetic energy brought in by one unit of orbital excitation in the internal motion of the four-quark subsystem containing a charmed quark, as it was the case for pentaquarks containing only light flavors like $uudd$ [23], so that the lowest state has positive parity in both cases, either for light or light + heavy quarks.

The authors of Ref. [35] have studied the $qqqc\bar{c}$ pentaquark prior to the 2015 LHCb observation of pentaquarks,

in three different models, including the flavor-spin (FS) model. The orbital wave function of the four-quark subsystem has symmetry $[4]_O$ for both parities. A symmetric state with an s^3p structure describes the spurious centre of mass motion, while the state of orbital symmetry $[31]_O$ derived in Sect. 3 is translationally invariant. Although the radial wave function was not specified, one can infer that the positive parity states of Ref. [35] were obtained by including a unit of orbital angular momentum in the relative motion between the four-quark subsystem and the antiquark. In this case the orbital wave function takes the form

$$\psi_4^5 = N_4 \exp\left[-\frac{a}{2}\left(x^2 + y^2 + z^2 \right) - \frac{b}{2} t^2 \right] t \, Y_{10}\left(\hat{t} \right),$$
(31)

where

$$N_4 = \frac{8^{1/2} a^{9/4} b^{5/4}}{3^{1/2} \pi^{5/2}}.$$
(32)

Then the expectation value of the kinetic energy becomes

$$\langle T_t \rangle = \hbar^2 \left(\frac{3}{m_q} a + \frac{3}{8\mu_1} a + \frac{1}{8\mu_2} b \right),$$
(33)

with

$$\frac{1}{\mu_1} = \frac{1}{m_q} + \frac{3}{m_Q},$$
(34)

and

$$\frac{1}{\mu_2} = \frac{3}{m_q} + \frac{17}{m_Q}.$$
(35)

The expectation value of the confinement interaction is given by Eq. (24) with

$$\langle r_{12} \rangle = \sqrt{\frac{4}{\pi a}},$$
(36)

and

$$\langle r_{45} \rangle = \frac{2}{3}\sqrt{\frac{2b}{5\pi}}\left(\frac{3}{4a} + \frac{5}{b} \right).$$
(37)

The hyperfine interaction integrated in the flavor-spin space is given by the third row of Table 3 because the symmetry state corresponds to $|[4]_O [211]_F [22]_S [31]_{FS}\rangle$.
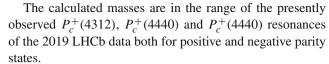
The attraction brought by the flavor-spin interaction in the four-quark state $[31]_{FS}$ is not strong enough to compensate the excess of kinetic energy due to the orbital excitation of the antiquark relative to the four quark subsystem. In our case we obtain a mass of 4573 MeV ($a = 0.040\,\text{GeV}^2$, $b = 0.037\,\text{GeV}^2$) which corresponds to $J^P = 1/2^+$ or $3/2^+$, higher than the mass 4487 GeV of the $1/2^-$ state of Table 4. This can explain why the lowest negative parity state $1/2^-$ in the calculations of Ref. [35] is lower than the lowest $1/2^+$ state.

In Ref. [36] the spectrum of the $uudc\bar{c}$ is studied in a chiral quark model where the Hamiltonian contains both the chromomagnetic and the flavor-spin interactions. For the flavor-spin part the symmetry is restricted to SU(3) with the replacement of the s quark by the c quark which, according to Eq. (10), seems to be reasonable. However, the lowest states have negative parity and the results for positive parity states are not shown, claiming that they are unbound. A possible interpretation is that the chromomagnetic dominates over the flavor-spin interaction in those calculations.

However, some support in favor of a flavor-spin interaction in exotics is given by Ref. [37]. Although restricted to tetraquarks described in a diquark basis, it has been shown that the mass difference observed in some charmed tetraquarks can be explained by an isospin dependent interaction, which is an SU(2) flavor-spin interaction. An important remark is that the contribution of the flavor-spin part relative to the one gluon exchange part must be adequate, irrespective of the basis used in the calculations.

8 Conclusions

We have calculated the lowest part of the mass spectrum of the hidden charm pentaquark $uudc\bar{c}$, within the SU(4) version of the flavor-spin model introduced in this work. This is a model which provides an internal structure for pentaquarks, contrary to the models mentioned in the introduction. An important achievement is that for positive parity states we have made a clear distinction between the case where a unit of angular momentum is located in the internal motion of the four-quark subsystem and the previously schematic case used in Ref. [35], where the angular momentum has to be located in the relative motion between the four-quark subsystem and the antiquark in order to have translationally invariant states (no center of mass motion) of specific flavor-spin symmetry. We have presented analytic expressions for the kinetic, confinement and the flavor-spin parts of the Hamiltonian for a simple variational solution. In the numerical estimates we have used model parameters from earlier studies of pentaquarks.

The calculated masses are in the range of the presently observed $P_c^+(4312)$, $P_c^+(4440)$ and $P_c^+(4440)$ resonances of the 2019 LHCb data both for positive and negative parity states.

One should stress that an important feature of the flavor-spin model is that it introduces an isospin dependence of the pentaquark states, necessary to discriminate between decay channels, like for tetraquarks [37].

The radial form (8) favors the p states over the s states due to the second term, which makes the FS interaction closer to the SU(4) limit. Therefore it would be interesting to find out how the pentaquark spectrum depends on the particular radial form of the FS hyperfine interaction by choosing the more realistic version of Ref. [38] with a cut-off parameter for each individual meson exchange. Based on symmetry arguments we however expect a similar conclusion for other forms of the hyperfine interaction.

In addition, one should look for more suitable coupling constants related to the presence of the charmed quark.

Presently, the most important conclusion is that the FS model predicts positive parity for the lowest state in contrast to the widely used OGE model, with or without correlated quarks/antiquarks.

Among the recent studies interpreting the 2019 LHCb data, positive parity states have been considered in Ref. [14] within a doubly-heavy triquark — light diquark model. In that model, besides the spin-spin chromomagnetic interaction, the Hamiltonian incorporates spin-orbit and orbital interactions. The fitted parametres lead to a positive contribution of the angular momentum dependent terms which make the $\ell = 1$ states to be higher than the $\ell = 0$ states. This means that the lowest state has a negative parity, contrary to our results.

The 2019 LHCb pentaquarks have also been studied as hadrocharmonium states [39]. In that model the new $P_c^+(4312)$ resonance is interpreted as a bound state of $\chi_0(1P)$ and a nucleon with $I = 1/2$ and $J^P = 1/2^+$. The other two resonances $P_c^+(4440)$ and $P_c^+(4457)$, have a negative parity. The parity sequence in the spectrum is closer to the one produced by the flavor-spin model.

The main issue of the present work is the parity sequence in the pentaquark $uudc\bar{c}$ spectrum. Thus, a crucial test is to experimentally determine the spin and parity of the 2019 LHCb resonances in order to discriminate between different interpretations, in particular between the OGE and the flavor-spin quark models, the doubly-heavy triquark — light diquark model, the molecular picture or the hadronic molecular picture.

Appendix A: The SU(4) generators

Here we reproduce the λ^F matrices which are the SU(4) generators in the fundamental representation of SU(4) [22]. Implementing them in Eq. (7) one can obtain the two-body matrix elements of Eq. (10) for each pair of quarks of a given flavor.

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{(A.1)}$$

$$\lambda^3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda^4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{(A.2)}$$

$$\lambda^5 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{(A.3)}$$

$$\lambda^7 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda^8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{(A.4)}$$

$$\lambda^9 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda^{10} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \quad \text{(A.5)}$$

$$\lambda^{11} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \lambda^{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} \quad \text{(A.6)}$$

$$\lambda^{13} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \lambda^{14} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \quad \text{(A.7)}$$

$$\lambda^{15} = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} \quad \text{(A.8)}$$

Appendix B: Exact SU(4) limit

The expectation value of the operator (1) displayed in Table 2, can be checked with the following formula [40]

$$\left\langle \sum_{i<j} \lambda_i^F \cdot \lambda_j^F \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j \right\rangle = 4C_2^{SU(2n)} - 2C_2^{SU(n)} - \frac{4}{k}C_2^{SU(2)}$$
$$-k\frac{3(n^2-1)}{n} \quad \text{(B.9)}$$

where n is the number flavors and k the number of quarks, here $n = 4$ and $k = 4$. $C_2^{SU(n)}$ is the Casimir operator eigenvalues of $SU(n)$ which can be derived from the expression [41]:

$$C_2^{SU(n)} = \frac{1}{2}[f_1'(f_1' + n - 1) + f_2'(f_2' + n - 3)$$
$$+ f_3'(f_3' + n - 5)$$
$$+ f_4'(f_4' + n - 7) + \cdots + f_{n-1}'(f_{n-1}' - n + 3)]$$
$$- \frac{1}{2n}\left(\sum_{i=1}^{n-1} f_i'\right)^2 \quad \text{(B.10)}$$

where $f_i' = f_i - f_n$, for an irreducible representation given by the partition $[f_1, f_2, \ldots, f_n]$. Equation (B.9) has been previously used for $n = 3$ and $k = 6$ in Ref. [41].

Appendix C: The baryons

The masses of baryons related to the present study were calculated variationally using a radial wave function of the form $\phi \propto exp[-\frac{a}{2}(x^2 + y^2)]$ containing the variational parameter a and the coordinates x and y defined by Eq. (11).

The results are indicated in Table 5 together with the experimental masses. One can see that condition $V_{\eta'}^{uc} \neq 0$ is disfavored because in this case the mass difference between Σ_c^* and Σ_c is negligible as compared to the experiment. This quantitatively justify the condition $V_{\eta'}^{uc} = 0$, imposed in the case of pentaquarks. (Note that the quantity $V_{\eta'}^{uc}$ is not present in the mass of Λ_c.)

The calculated charmed baryon masses with $V_{\eta'}^{uc} = 0$ are somewhat smaller than the experimental values. By increasing the charmed quark mass from $m_c = 1.35$ to $m_c = 1.45$ GeV the agreement with the experiment is much better. If the same increase is taken into account in calculating the pentaquark masses, the spectrum is shifted to larger masses but still remains at the edge of the desired experimental range. However the conclusion remains unchanged: the lowest state has positive parity.

Table 5 Variational solution for the baryon ground state masses with the flavor-spin interaction of Sect. 2. Column 1 gives the baryon, column 2 the isospin, column 3 the spin and parity column 4 the mass with $V_{\eta'}^{uc} = 0$ and $V_{\eta'}^{uc} \neq 0$, column 5 the variational parameter and the last column the experimental mass

Baryon	I	J^P	Calculated mass (GeV) $V_{\eta'}^{uc} = 0$	$V_{\eta'}^{uc} \neq 0$	a(fm^{-2})	Exp. mass (GeV)
N	$\frac{1}{2}$	$\frac{1}{2}^+$	0.960		2.594	0.940
Δ	$\frac{3}{2}$	$\frac{3}{2}^+$	1.304		2.594	1.232
Λ_c	0	$\frac{1}{2}^+$	2.180		2.055	2.283
Σ_c	1	$\frac{1}{2}^+$	2.346	2.434	2.055	2.455
Σ_c^*	1	$\frac{3}{2}^+$	2.482	2.438	2.055	2.530

In addition, we have calculated the masses of two doubly charmed baryons, namely $\Xi_{cc}^{++}(\frac{1}{2})$ and $\Xi_{cc}^{++}(\frac{3}{2})$. They are 3386 MeV and 3520 MeV respectively, when $V_{\eta'}^{uc} = 0$. From experimental point of view the situation remains controversial. Recently the LHCb collaboration announced the observation of the double charm baryon Ξ_{cc}^{++} in the $\Lambda_c^+ K^- \pi^+ \pi^+$ mass spectrum [42]. Its mass of 3621 MeV is at variance with the 3519 MeV value found earlier by the SELEX Collaboration [43]. The spin and parity are not known experimentally.

Appendix D: The flavor wave functions

Here we give the explicit form of flavor states of content *uudc* in the Young Yamanouchi basis of irreducible representations (irrep) $[f]_F$ appearing in the wave functions. We assume identical particle and use the method of Ref. [22] to derive these basis vectors. The order of particles is always 1234 in every term below and the permutation symmetry of each state is defined by its Yamanouchi symbol, which is a compact notation for a Young tableau. For a tableau with n particles it is defined by $Y = (r_n, r_{n-1}, \ldots, r_1)$ where r_i represents the row of the particle i.

For the irrep [22] there are two basis vectors

$$|[22]_F 2211\rangle = \sqrt{\frac{1}{6}}(uudc + uucd + dcuu + cduu$$
$$-\frac{1}{2}duuc - \frac{1}{2}cuud - \frac{1}{2}ucdu$$
$$-\frac{1}{2}udcu - \frac{1}{2}uduc$$
$$-\frac{1}{2}ducu - \frac{1}{2}ucud - \frac{1}{2}cudu) \quad (D.11)$$

$$|[22]_F 2121\rangle = \sqrt{\frac{1}{8}}[(ud - du)(uc - cu)$$
$$+(uc - cu)(ud - du)] \quad (D.12)$$

For the irrep [31] there are three basis vectors

$$|[31]_F 2111\rangle = \frac{1}{6}(3uudc + 3duuc + 3uduc$$
$$-cudu - cuud - cduu - ucdu - dcuu$$
$$-ucud - uucd - ducu - udcu), \quad (D.13)$$

$$|[31]_F 1211\rangle = \sqrt{\frac{1}{18}}(2uucd + 2ducu + 2udcu$$
$$-cuud - cudu - cduu$$
$$-ucud - dcuu - ucdu), \quad (D.14)$$

$$|[31]_F 1121\rangle = \sqrt{\frac{1}{6}}(ucud + dcuu + ucdu$$
$$-cuud - cduu - cudu) \quad (D.15)$$

Note that there is a conflict with the basis vectors (A.9)–(A.11) of Ref. [35] which do not form a Young Yamanouchi basis, thus, although orthogonal, they do not satisfy the correct permutation symmetry.

In the same way the three basis vectors of the irrep [211] were obtained as

$$|[211]_F 3211\rangle = \frac{1}{4}(2uudc - 2uucd - uduc - duuc + udcu$$
$$+ducu + ucud + cuud - ucdu - cudu), \quad (D.16)$$

$$|[211]_F 1321\rangle = \sqrt{\frac{1}{6}}(udcu + cudu + dcuu$$
$$-cduu - ucdu - ducu), \quad (D.17)$$

$$|[211]_F 3121\rangle = \sqrt{\frac{1}{48}}(3uduc - 3duuc + 3cuud - 3ucud$$
$$+2dcuu - 2cduu - cudu + ucdu$$
$$+ducu - udcu). \quad (D.18)$$

In the present model it is useful to rewrite the above wave functions in terms of products of symmetric $\phi_{[2]}(q_a q_b) = (q_a q_b + q_b q_a)/\sqrt{2}$ or antisymmetric $\phi_{[11]}(q_a q_b) = (q_a q_b - q_b q_a)/\sqrt{2}$ quark pair states for the pairs 12 and 34. This allows a straightforward calculation of the flavor integrated matrix elements (10) and in addition one can easily read off the isospin of the corresponding wave function.

In particular the states (D.11) and (D.12) become

$$|[22]_F 2211\rangle = \sqrt{\frac{1}{6}}[\sqrt{2}\phi_{[2]}(uu)\,\phi_{[2]}(cd)$$
$$+\sqrt{2}\phi_{[2]}(cd)\,\phi_{[2]}(uu)$$
$$-\phi_{[2]}(ud)\,\phi_{[2]}(uc)$$
$$-\phi_{[2]}(uc)\,\phi_{[2]}(ud)] \quad (D.19)$$

and

$$|[22]_F 2121\rangle = \sqrt{\frac{1}{2}}[\phi_{[11]}(ud)\,\phi_{[11]}(uc) \\ + \phi_{[11]}(uc)\,\phi_{[11]}(ud)], \qquad (D.20)$$

where (D.20) obviously has isospin $I = 1/2$ which means that the pairs 12 and 34 in (D.19) have to couple to the same isospin value as well.

For the irrep [31] one has to use the Rutherford–Young–Yamanouchi basis, which symmetrizes or antisymmetrizes the last two particles [22]. Accordingly the basis vectors (D.13) and (D.14) transform into

$$|[31]_F \overline{12}11\rangle = \sqrt{\frac{1}{6}}[\phi_{[2]}(uu)\,\phi_{[2]}(cd) + \sqrt{2}\phi_{[2]}(ud)\,\phi_{[2]}(uc) \\ - \sqrt{2}\phi_{[2]}(uc)\,\phi_{[2]}(ud) \\ - \phi_{[2]}(cd)\,\phi_{[2]}(uu)], \qquad (D.21)$$

where the pair 34 is in a symmetric state, and

$$|[31]_F \tilde{1}211\rangle = \sqrt{\frac{1}{3}}[\phi_{[2]}(uu)\,\phi_{[11]}(cd) \\ + \sqrt{2}\phi_{[2]}(ud)\,\phi_{[11]}(uc)], \qquad (D.22)$$

where the pair 34 is in an antisymmetric state.

The state (D.15) can simply be rewritten as

$$|[31]_F 1121\rangle = \sqrt{\frac{1}{3}}[\phi_{[11]}(dc)\,\phi_{[2]}(uu) \\ + \sqrt{2}\phi_{[11]}(uc)\,\phi_{[2]}(ud)], \qquad (D.23)$$

where the pair 12 is in an antisymmetric state and 34 in a symmetric state. The states (D.21)–(D.23) can have $I = 1/2$ or 3/2.

For the irrep [211] the vector (D.16) can be directly rewritten as

$$|[211]_F 3211\rangle = \frac{1}{2}(\sqrt{2}\phi_{[2]}(uu)\,\phi_{[11]}(dc) - \phi_{[2]}(ud)\,\phi_{[11]}(uc) \\ + \phi_{[2]}(uc)\,\phi_{[11]}(ud)), \qquad (D.24)$$

and the mixture of the other two in the Rutherford–Young–Yamanouchi basis reads

$$|[211]_F \overline{13}21\rangle = \frac{1}{2}[\phi_{[11]}(ud)\,\phi_{[2]}(uc) + \sqrt{2}\phi_{[11]}(dc)\,\phi_{[2]}(uu) \\ - \phi_{[11]}(uc)\,\phi_{[2]}(ud)], \qquad (D.25)$$

$$|[211]_F \tilde{1}321\rangle = \sqrt{\frac{1}{2}}[-\phi_{[11]}(ud)\,\phi_{[11]}(uc) \\ + \phi_{[11]}(uc)\,\phi_{[11]}(ud)]. \qquad (D.26)$$

The states (D.24)–(D.26) have $I = 1/2$.

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