Analytical studies on the hoop conjecture in charged curved spacetimes

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Abstract Recently, with numerical methods, Hod clarified the validity of Thorne hoop conjecture for spatially regular static charged fluid spheres, which were considered as counterexamples against the hoop conjecture. In this work, we provide an analytical proof on Thorne hoop conjecture in the spatially regular static charged fluid sphere spacetimes.

1 Introduction

One famous conjecture in general relativity is the Thorne hoop conjecture, which states that horizons appear when and only when a mass $\mathcal{M}$ gets compacted into a region whose circumference C in every direction is $C \leq 4\pi\mathcal{M}$ [1,2]. This upper bound can be saturated in the case of Schwarzschild black hole with the horizon radius $r_0 = 2\mathcal{M}$. If generically true, such conjecture would signify that black holes form if matter/energy is enclosed in a small enough region. At present, there are a lot of works addressing the hoop conjecture, see [3–29] and references therein.

Intriguingly, a few counterexamples against hoop conjecture were also presented [30,31]. In particular, Ref. [30] constructed the horizonless charged fluid sphere configurations with uniform charge densities. For $\frac{M}{r_0} = 0.65$ and $\frac{Q^2}{r_0^2} = 0.39$, the horizonless charged sphere satisfies a relation $\frac{C(r_0)}{4\pi M} \simeq 0.769 < 1$, where M is the total mass of the spacetime, Q is the sphere charge and r_0 is the sphere radius. According to the relation $\frac{C(r_0)}{4\pi M} < 1$ in the horizonless spacetime, the author claimed that Thorne hoop conjecture can be violated in horizonless charged fluid sphere spacetimes [30].

However, as stated by Hod, it is physically more appropriate to interpret the mass term $\mathcal{M}$ in Thorne hoop conjecture as the gravitational mass $M(r_0)$ contained within the radius r_0 and not as the total mass M of the entire curved spacetime [32]. In fact, there is electric energy outside the charged

sphere. For the same parameters $\frac{M}{r_0} = 0.65$ and $\frac{Q^2}{r_0^2} = 0.39$ as [30], Hod reexamined the validity of hoop conjecture for charged fluid spheres and numerically obtained the relation $\frac{C(r_0)}{4\pi M(r_0)} \simeq 1.099 > 1$, which is in fact in accordance with the hoop conjecture in charged spacetime [32]. Along this line, it is still meaningful to analytically examine Thorne hoop conjecture for spatial regular charged fluid spheres with generic parameters.

The rest of the paper is organized as follows. We shall introduce the gravity model of spatial regular static charged fluid spheres. We provide an analytical proof on Thorne hoop conjecture for horizonless charged fluid spheres with generic parameters. Finally, we will briefly summarize our results.

2 Validity of the hoop conjecture for charged fluid spheres

It was widely believed that Thorne hoop conjecture is a fundamental property of classical general relativity. A lot of works indeed support the hoop conjecture [3–18]. However, counterexamples against hoop conjecture were also presented in [30,31]. In particular, Ref. [30] constructed a gravity model of horizonless static charged fluid spheres. And the charged fluid sphere reads [33–36]

$$ds^2 = e^v dt^2 - e^\lambda dr^2 - r^2 (d\theta^2 + \sin^2\theta d\phi^2). \quad (1)$$

The interior metric solution with uniform charge density was obtained in [30] as

$$e^{-\lambda} = [1 - (1 - f_0)v^2]^2, \quad (2)$$

$$e^v = \frac{1}{4f_0} \left[(1 - \varepsilon - f_0)v^2 + \left(1 - 3\varepsilon + f_0 + \frac{2Q^2}{r_0^2} \right) \right]^2 e^{\lambda/2}. \quad (3)$$

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We define $v = \frac{r}{r_0} \leq 1$, $\varepsilon = \frac{M}{r_0}$ and $f_0 = \sqrt{1 - \frac{2M}{r_0} + \frac{Q^2}{r_0^2}}$, where M is the total mass of the spacetime, Q is the sphere charge and r_0 is the sphere radius.

In the exterior region $r \geq r_0$, the background is the Reissner–Nordström solution given by [37–42]

$$e^\nu = e^{-\lambda} = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}. \quad (4)$$

At the radius r_0 , interior metric (2–3) and exterior metric (4) coincide with each other.

Thorne hoop conjecture states that horizons appear when and only when a mass $\mathcal{M}$ gets compacted into a region whose circumference C in every direction is $C \leq 4\pi\mathcal{M}$ [1, 2]. The author in [30] claimed that Thorne hoop conjecture was violated by a relation

$$\frac{C(r_0)}{4\pi M} \simeq 0.769 < 1, \quad (5)$$

with $\frac{M}{r_0} = 0.65$ and $\frac{Q^2}{r_0^2} = 0.39$. However, in the charged background, as stated in [32], it is physically more appropriate to interpret the mass term $\mathcal{M}$ in Thorne hoop conjecture as the gravitational mass $M(r_0)$ contained within the radius r_0 and not as the total mass M of the entire curved spacetime. With the same parameters $\frac{M}{r_0} = 0.65$ and $\frac{Q^2}{r_0^2} = 0.39$, Hod reexamined the model and numerically obtained the relation for horizonless spheres as

$$\frac{C(r_0)}{4\pi M(r_0)} \simeq 1.099 > 1, \quad (6)$$

which is in accordance with the hoop conjecture in charged spacetime [32].

In the following, we provide an analytical proof on Thorne hoop conjecture for generic parameters. According to the weak energy condition (WEC), the energy density is non-negative [33, 34]. In this work, we can take a more general condition that the energy density is real (negative or nonnegative). The energy density of interior region is [30]

$$\rho_m = \frac{1 - f_0}{8\pi r_0^2} \left\{ 6 - v^2 \left[5(1 - f_0) + \frac{Q^2}{r_0^2(1 - f_0)} \right] \right\}. \quad (7)$$

It can be transformed into

$$\rho_m = -\frac{5v^2}{8\pi r_0^2}(1 - f_0)^2 + \frac{3}{4\pi r_0^2}(1 - f_0) + \frac{Q^2 v^2}{8\pi r_0^4}. \quad (8)$$

We put $1 - f_0 = a + bi$, in the general form

$$1 - f_0 = a + bi, \quad (9)$$

where a, b are real numbers.

Putting (9) into (8), we arrive at the relation

$$\rho_m = \left(-\frac{5v^2 a^2}{8\pi r_0^2} - \frac{5v^2 b^2}{8\pi r_0^2} + \frac{3a}{4\pi r_0^2} + \frac{Q^2 v^2}{8\pi r_0^4} \right) + \left(-\frac{5v^2 a}{4\pi r_0^2} + \frac{3}{4\pi r_0^2} \right) bi. \quad (10)$$

At the center $v = \frac{r}{r_0} = 0$, (10) yields

$$\rho_m(0) = \frac{3a}{4\pi r_0^2} + \frac{3b}{4\pi r_0^2} i. \quad (11)$$

Since $\rho_m(0)$ is real, there is

$$b = 0. \quad (12)$$

That is to say $f_0 = \sqrt{1 - \frac{2M}{r_0} + \frac{Q^2}{r_0^2}}$ is real according to (9), which yields

$$\frac{2M}{r_0} - \frac{Q^2}{r_0^2} \leq 1. \quad (13)$$

In the case of $\frac{2M}{r_0} - \frac{Q^2}{r_0^2} = 1$, there is $1 - \frac{2M}{r_0} + \frac{Q^2}{r_0^2} = 0$ and r_0 is horizon according to (4). Since we study horizonless sphere, the spacetime should satisfies

$$\frac{2M}{r_0} - \frac{Q^2}{r_0^2} < 1. \quad (14)$$

There is electric energy outside the charged sphere. In order to calculate $M(r_0)$, we should subtract the exterior electric energy $E(r > r_0)$ from the total energy M . Outside the sphere, one has the Maxwell field energy density $\rho(r) = \frac{Q^2}{8\pi r^4}$. The mass $E(r > r_0)$ of the Maxwell field above the radius r_0 is given by

$$E(r > r_0) = \int_{r_0}^{+\infty} 4\pi r'^2 \rho(r') dr' = \frac{Q^2}{2r_0}. \quad (15)$$

So the mass $M(r_0)$ within the radius r_0 is [32, 43, 44]

$$M(r_0) = M - E(r > r_0) = M - \frac{Q^2}{2r_0}. \quad (16)$$

With (14) and (16), we arrive at the relation

$$\frac{C(r_0)}{4\pi M(r_0)} = \frac{2\pi r_0}{4\pi(M - \frac{Q^2}{2r_0})} = \frac{1}{\frac{2M}{r_0} - \frac{Q^2}{r_0^2}} > 1. \quad (17)$$

It yields the inequality for the horizonless spacetime as

$$C(r_0) > 4\pi M(r_0) \quad (18)$$

in accordance with Thorne hoop conjecture, which states that horizons appear when and only when the mass $M(r_0)$ and circumference $C(r_0)$ satisfy the relation $C(r_0) \leq 4\pi M(r_0)$ [32]. We point out that relation (18) holds for generic parameters. Here we analytically prove Thorne hoop conjecture in spatial regular charged fluid sphere spacetimes.

3 Conclusions

We analytically examined the validity of Thorne hoop conjecture in spatial regular charged curve spacetimes. We took the natural assumption that the matter energy density is real. On this real energy density assumption, with generic parameters, we found that horizonless charged fluid sphere should satisfy the relation (18) in accordance with Thorne hoop conjecture. In summary, we provided an analytical proof on Thorne hoop conjecture in the spatially regular static charged fluid sphere spacetimes.

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