Rotating and non-rotating AdS black holes in $f(\mathcal{T})$ gravity non-linear electrodynamics

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Received: 23 August 2019 / Accepted: 19 October 2019 / Published online: 11 November 2019
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Abstract We derive new exact charged d -dimensional black hole solutions for quadratic teleparallel equivalent gravity, $f(\mathcal{T}) = a_0 + a_1\mathcal{T} + a_2\mathcal{T}^2$, where $\mathcal{T}$ is the torsion scalar, in the case of non-linear electrodynamics. We give a specific form of electromagnetic function and find out the form of the unknown functions that characterize the vielbeins in presence of the electromagnetic field. It is possible to show that the black holes behave asymptotically as AdS solutions and contain, in addition to the monopole and quadrupole terms, other higher order terms whose source is the non-linear electrodynamics field. We calculate the electromagnetic Maxwell field and show that our d -dimensional black hole solutions coincide with the previous obtained one (Awad et al. in J High Energy Phys 13:1706.01773, 2017). The structure of the solutions show that there is a central singularity that is much mild in comparison with the respective one in general relativity. Finally, the thermodynamical properties of the solutions are investigated by calculating the entropy, the Hawking temperature, the heat capacity, and other physical quantities. The most important result of thermodynamics is that the entropy is not proportional to the area of the black hole. This inattention points out that we must have a constrain on the quadrupole term to get a positive entropy otherwise we get a negative value.

1 Introduction

Understanding of the gravitational interaction at large scales is considered a main issue of theoretical physics and cosmology [2]. For example, Einstein’s General Relativity (GR) is not able to explain the accelerated expansion epoch of our universe [3–7]. This issue can be solved in the framework of GR if some cosmic flow, having exotic properties, is assumed, the so-called dark energy, or a cosmological constant is involved into the field equation [8]. Moreover, the rotation curves of spiral galaxy are out the domain of validity of GR unless one assumes the existence of cold and pressureless dark matter [9].

Despite of this state of art, GR has achieved brilliant successes in many aspects like solar system dynamics, gravitational wave detection, relativistic stellar structure up to cosmology [10]. Einstein’s GR has a question mark when it is confronted with large scales or when quantization is taken into account. Therefore, there is the necessity for a self-consistent theory that is capable of describing the gravitational interaction ranging from quantum to cosmic scales and coinciding with GR in the limits where it is successful.

According to this philosophy, there are several proposals to extend or modify GR in view of obtaining a self-consistent theory at any scale. The so called $f(R)$ gravity is one of these proposal. Here R is the Ricci scalar and, for $f(R) = R$, its Lagrangian corresponds to the Hilbert–Einstein Lagrangian and GR is recovered [11]. In some sense, $f(R)$ gravity is the minimal extension of GR. Another proposal is $f(\mathcal{T})$ gravity in which $\mathcal{T}$ represents torsion scalar. Also in this case, for $f(\mathcal{T}) = \mathcal{T}$, the theory reduces to the so called Teleparallel Equivalent General Relativity (TEGR) which is

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constructed by the Weitzenböck geometry and it is endowed with a nonsymmetric connection, characterized by no curvature and non-vanishing torsion [12–15]. The TEGR torsion tensor plays the dynamical role of curvature and the vielbein plays the role of metric tensor, that is the gravitational potentials. Einstein used TEGR theory to construct a unification between gravitational and electromagnetism fields [15–18]. Although, TEGR is constructed using a different geometry from that of GR based on Riemann geometry, that is the Weitzenböck geometry, TEGR and GR are completely equivalent from a dynamical point of view. However, assuming generic functions $f(R)$ and $f(T)$, they are inequivalent [19–21]. Therefore, those theories are interesting and can be considered to solve the problems of dark energy and dark matter [22–36]. In this study, we are considering $f(T)$ gravitational theories.

There are many applications of $f(T)$ in solar system as well as in cosmological frame. For example, exact solutions, black hole solutions and stellar models are discussed in [37–42, 42–47, 47, 47–54]. Spherically symmetric solutions with constant torsion scalar have been derived in [55, 56]. In the solar system, it is possible to obtain a weak field solution [57] for the form $f(T) = T + \alpha T^2$ for which the authors constrain the dimensional parameter α [58]. $f(T)$ has extra degrees of freedom which are related to the non-invariance of the theory under local Lorentz transformations. Recently, an invariant $f(T)$ gravitational theory under local Lorentz transformation has been derived [59]. In the frames of cosmology and spherically symmetric geometry, it is shown that the diagonal ansatz is not a suitable vielbein to be used [60]. A non diagonal spherically symmetric vielbein has been applied to the field equations of $f(T)$ gravity and a weak field solution has been obtained [61]. Recently, new charged black holes for quadratic and cubic form of $f(T)$ have been derived using flat horizons spacetimes [1, 62]. It is the purpose of the present paper to study the effect of the non-linear electrodynamics in $f(T)$ gravity on a cylindrical spacetime. This approach could have interesting physical applications both for gravitational and electromagnetic fields.

The fact that electromagnetism is described by massless particles with integer spin is an hypothesis due to the experimental results. The photon mass, in fact, has never been experimentally observed and the entire theory is constructed to give massless interaction particles. In particular, Special Relativity takes into account the speed of light as the maximal achievable velocity. According to this principle, standard massive particles cannot reach the speed of light, neglecting, implicitly, the mass of photon. Nevertheless, the photon is the only massless particle in the Standard Model and a massive photon would not be in contrast with Special Relativity, according to a more general interpretation. Assuming that the maximal speed does not necessarily coincide with the speed

of light, Special Relativity keeps holding but electromagnetism changes radically. First of all, the D'Alembert wave equation should be modified including massive modes. Furthermore electromagnetism would lose the invariance under conformal transformations. This is a property strictly related to massless fields. According to experimental data, a possible photon mass would be very small and in most cases negligible, of the order of $10^{-50} \text{ kg} < m_\gamma < 10^{-60} \text{ kg}$ [63–68]. In general, massive photons come out as soon as the linearity in Maxwell invariant $\mathcal{F} = \frac{1}{4} \mathcal{F}_{\alpha\beta} \mathcal{F}^{\alpha\beta}$ is relaxed. In other words, higher orders in $\mathcal{F}$ can imply massive photons. Such terms could be present in the extreme conditions of a black hole and could be interesting as candidates for dark matter. Here, we want to investigate their possible relevance in black hole solutions derived from non-linear TEGR.

The layout of the paper is the following: In Sect. 2 we summarize $f(T)$ gravity and derive the field equations in presence of non-linear electrodynamics. In Sect. 3 we derive charged static AdS solutions, analyzing the structure of singularities. In Sect. 4 we obtain charged rotating AdS solution in non-linear electrodynamics in the context of $f(T)$ gravity. Section 5 is devoted to thermodynamics considering entropy, Hawking temperature, heat capacity, Gibbs free energy. The most interesting feature of these calculations is the fact that the entropy is not proportional to the area of the black hole in addition to the possibility of negative values of entropy. Section 6 is devoted to discussion and conclusions.

2 Basic concepts of $f(T)$ gravity

In Riemannian geometry, the metric of the spacetime has the form

$$ds^2 := g_{\mu\nu} dx^\mu dx^\nu, \quad (1)$$

where $g_{\mu\nu}$ is a second order symmetric tensor. Using the vielbein one can write Eq. (1) as

$$ds^2 := g_{\mu\nu} dx^\mu dx^\nu = \eta_{ij} \vartheta^i \vartheta^j, \quad \text{where } \vartheta^i = h^i_\mu dx^\mu, \quad (2)$$

with η_{ij} being the Minkowskian metric that is defined as: $\eta_{ij} = \text{diag}(-1, +1, +1, \dots, +1)$ and h^i_μ is the covariant vielbein that satisfies the orthogonality conditions

$$h^i_\mu h^j_\nu = \delta^i_j, \quad h^i_\mu h^i_\nu = \delta_\mu^\nu. \quad (3)$$

To construct a spacetime with vanishing curvature and a non-vanishing torsion one has to define the Weitzenböck connection which is

$$\Gamma^\lambda_{\nu\mu} = h^i_\lambda \partial_\mu h^i_\nu = -h^i_\nu \partial_\mu h^i_\lambda, \quad \text{where } \partial_\mu = \frac{\partial}{\partial x^\mu}. \quad (4)$$

Using Eq. (4), the torsion and contorsion tensors are

$$T^\alpha_{\mu\nu} := \Gamma^\alpha_{\nu\mu} - \Gamma^\alpha_{\mu\nu},$$

$$\gamma^{\mu\nu}_\alpha := -\frac{1}{2}(T^{\mu\nu}_\alpha - T^{\nu\mu}_\alpha - T^\alpha_{\mu\nu}). \quad (5)$$

From Eq. (5), one can define the superpotential tensor

$$S^{\lambda\mu}_\nu = \frac{1}{2}(\gamma^{\mu\nu}_\alpha + \delta^\mu_\alpha T^{\beta\nu}_\beta - \delta^\nu_\alpha T^{\beta\mu}_\beta). \quad (6)$$

From Eqs. (5) and (6) the torsion scalar is provided as

$$\mathcal{T} := T^\alpha_{\mu\nu} S^{\mu\nu}_\alpha. \quad (7)$$

The Lagrangian of TEGR theory is constructed from the torsion scalar given by Eq. (7).

Let us now consider $f(\mathcal{T})$ gravity minimally coupled with non-linear electrodynamics. Thence, the action of this theory is given by

$$\mathcal{L} := \frac{1}{2\kappa} \int |h| f(\mathcal{T}) d^d x + \int |h| \mathcal{L}(\mathcal{F}) d^d x, \quad (8)$$

with $|h| = \sqrt{-g} = \det(h^a_\mu)$ being the determinant of the metric and κ a dimensional constant with the form $\kappa = 2(d-3)\Omega_{d-1}G_d$, with G_d being the Newtonian gravitational constant in d -dimensions and Ω_{d-1} a $(d-1)$ -dimensional unitary volume with the form

$$\Omega_{d-1} = \frac{2\pi^{(d-1)/2}}{\Gamma((d-1)/2)}, \quad (9)$$

where Γ is the Γ -function (when $d = 4$, it is $2(d-3)\Omega_{d-1} = 8\pi$). The electromagnetic Lagrangian $\mathcal{L}(\mathcal{F})$ is gauge-invariant and depends on the invariant $\mathcal{F}$ defined as $\mathcal{F} = \frac{1}{4}\mathcal{F}_{\alpha\beta}\mathcal{F}^{\alpha\beta}$ [69]. The antisymmetric Faraday tensor is defined as

$$\mathcal{F}_{\alpha\beta} = \mathcal{E}_{\alpha,\beta} - \mathcal{E}_{\beta,\alpha}, \quad (10)$$

where $\mathcal{E}_\mu$ is its gauge potential 1-form. In the Maxwell theory, the Lagrangian $\mathcal{L}(\mathcal{F})$ is $\mathcal{L}(\mathcal{F}) = 4\mathcal{F}$. Here, we consider a more general choice of the electromagnetic Lagrangian. From Action (8), the non-linear electrodynamics is described by nonlinear terms in $\mathcal{F}_{\alpha\beta}$ and its invariants. However, we can provide a dual representation in terms of an auxiliary field $\mathcal{P}_{\alpha\beta}$. This method is proved to be highly benefit to derive exact solutions in GR, specifically for the electric case [70, 71]. The dual form can be obtained adopting the Legendre transformation below:

$$\mathfrak{N} = 2\mathcal{F}\mathcal{L}_{\mathcal{F}} - \mathcal{L}, \quad \text{where} \quad \mathcal{L}_{\mathcal{F}} = \frac{\partial \mathcal{L}}{\partial \mathcal{F}}, \quad (11)$$

where $\mathfrak{N}$ is an arbitrary function depending on the invariant $\mathcal{P}$, defined as $\mathcal{P} = \frac{1}{4}\mathcal{P}_{\alpha\beta}\mathcal{P}^{\alpha\beta}$. From Eq. (11), non-linear

electrodynamics can be recast in terms of $\mathcal{P}$ according to the formulas

$$\mathcal{P}_{\mu\nu} = \mathcal{L}_{\mathcal{F}}\mathcal{F}_{\mu\nu}, \quad \mathcal{F}_{\mu\nu} = \mathfrak{N}_{\mathcal{P}}\mathcal{P}_{\mu\nu}, \quad \mathcal{L} = 2\mathcal{P}\mathfrak{N}_{\mathcal{P}} - \mathfrak{N}, \quad (12)$$

where the standard Maxwell theory is obtained for $\mathcal{L}_{\mathcal{F}} = 1$. As it is clear from the above equations, $\mathfrak{N}$ is a function of $\mathcal{P}$, where [70, 71]

$$\mathfrak{N}_{\mathcal{P}} = \frac{\partial \mathfrak{N}}{\partial \mathcal{P}}.$$

The variation of Lagrangian (8) with respect to the vielbeins leads to

$$\zeta^\nu_\mu = S_\mu^{\rho\nu} \partial_\rho \mathcal{T} f_{\mathcal{T}} \mathcal{T} + \left[h^{-1} h^i_\mu \partial_\rho (h h_i^\alpha S_\alpha^{\rho\nu}) - T^\alpha_{\lambda\mu} S_\alpha^{\nu\lambda} \right] f_{\mathcal{T}} - \frac{f}{4} \delta^\nu_\mu + \frac{1}{2} \kappa \mathfrak{T}^{nlem\nu}_\mu \equiv 0, \quad (13)$$

and the Maxwell equations for non-linear electrodynamics become [70]

$$\partial_\nu (\sqrt{-g} \mathcal{P}^{\mu\nu}) = 0. \quad (14)$$

The stress-energy tensor of non-linear electrodynamic is

$$\mathfrak{T}^{nlem\nu}_\mu := 2(\mathfrak{N}_{\mathcal{P}} \mathcal{P}_{\mu\alpha} \mathcal{P}^{\nu\alpha} - \delta^\nu_\mu [2\mathcal{P}\mathfrak{N}_{\mathcal{P}} - \mathfrak{N}]). \quad (15)$$

It is worth saying that Eq. (15) has a non-vanishing trace unlike the stress-energy tensor coincides with the Maxwell one. It is worth noticing that the electric field of linear electrodynamics is obtained as

$$\mathcal{E} = \mathcal{F}_{tr} = \mathfrak{N}_{\mathcal{P}} \mathcal{P}_{tr}. \quad (16)$$

In this context, black hole solutions can be found.

3 Anti-de-Sitter black hole solutions in non-linear electrodynamics

Let us search now for charged AdS black hole solutions in non-linear electrodynamics assuming, in general, d -dimensions in the framework of $f(\mathcal{T})$ gravity. Using the following vierbein diagonal ansatz in d -dimensions $(t, r, \eta_1, \eta_2, \dots, \eta_n, \xi_1, \xi_2, \dots, \xi_l)$, with $l = 1, 2, \dots, d - n - 2$, in which $0 \leq r < \infty$, $-\infty < t < \infty$, $0 \leq \eta_n < 2\pi$ and $-\infty < \xi_k < \infty$, we assume the vielbein [49, 62]:

$$(h^i_\mu) = \left(\sqrt{A(r)}, \frac{1}{\sqrt{A(r)g(r)}}, r, r, r, \dots \right), \quad (17)$$

which corresponds to the metric

$$ds^2 = -A(r)dt^2 + \frac{1}{A(r)g(r)}dr^2 + r^2 \left(\sum_{i=1}^n d\eta_i^2 + \sum_{l=1}^{d-n-2} d\xi_l^2 \right), \quad (18)$$

where $A(r)$ and $g(r)$ are functions depending only on the radial coordinate r . Substituting the vielbein (17) into the torsion scalar in (7), we get

$$\mathcal{T} = (d-2)\frac{A'g}{r} + (d-2)(d-3)\frac{Ag}{r^2}, \quad (19)$$

where $A'(r) \equiv \frac{dA(r)}{dr}$ and $g'(r) \equiv \frac{dg(r)}{dr}$. Finally, since the $f(\mathcal{T})$ power law gravity seems the model with the best agreement with observational data [72–74], we will focus on the choice

$$f(\mathcal{T}) = a_0 + a_1\mathcal{T} + a_2\mathcal{T}^2, \quad (20)$$

where a_0, a_1 and a_2 are the model parameters.

3.1 Asymptotically static AdS black holes

Inserting the vielbein (17) into field Eqs. (13) and (14), we obtain the following non-vanishing components:

$$\begin{aligned} \zeta^r_r &= 2\mathcal{T}f_{\mathcal{T}} + 2a_0 - f - 4\aleph = 0, \\ \zeta^{\eta_1}_{\eta_1} &= \zeta^{\eta_2}_{\eta_2} = \dots = \zeta^{\eta_n}_{\eta_n} = \zeta^{\xi_1}_{\xi_1} \\ &= \zeta^{\xi_2}_{\xi_2} = \dots = \zeta^{\xi_{d-n-2}}_{\xi_{d-n-2}} \\ &= \frac{f_{\mathcal{T}}\mathcal{T}[r^2\mathcal{T} + (d-3)A]\mathcal{T}'}{r} \\ &\quad + \frac{f_{\mathcal{T}}}{2r^2} \left\{ 2r^2gA'' + rg'(2(d-3)A + rA') \right. \\ &\quad \left. + g(2(d-3)^2A + (3d-8)rA') \right\} \\ &\quad - f + 2a_0 - 4\aleph \\ &\quad + \frac{8q'g\aleph'}{q'g' + 2gq''} = 0, \\ \zeta^t_t &= \frac{2(d-2)Agf_{\mathcal{T}}\mathcal{T}'}{r} \\ &\quad + \frac{(d-2)f_{\mathcal{T}}}{r^2} \left\{ 2[(d-3)Ag + rgA'] + rAg' \right\} \\ &\quad - f - 4\aleph + 2a_0 = 0, \end{aligned} \quad (21)$$

where $q(r) \equiv q$ is the gauge 1-form of the non-linear electrodynamics that is defined as $\mathcal{P}_{Tr} = q'$ and $q' = \frac{dq}{dr}$, $q'' = \frac{d^2q}{dr^2}$. In the case of $f(\mathcal{T})$ with the form given by (20), the above equations reduce to

$$\zeta^r_r = a_1\mathcal{T} + 3a_2\mathcal{T}^2 + a_0 - 4\aleph = 0, \quad (22)$$

$$\begin{aligned} \zeta^{\eta_1}_{\eta_1} &= \zeta^{\eta_2}_{\eta_2} = \dots = \zeta^{\eta_n}_{\eta_n} = \zeta^{\xi_1}_{\xi_1} \\ &= \zeta^{\xi_2}_{\xi_2} = \dots = \zeta^{\xi_{d-n-2}}_{\xi_{d-n-2}} \\ &= \frac{2a_2[r^2\mathcal{T} + (d-3)A]\mathcal{T}'}{r} \\ &\quad + \frac{(a_1 + 2a_2\mathcal{T})}{2r^2} \left\{ 2r^2gA'' + rg'(2(d-3)A + rA') \right. \\ &\quad \left. + g(2(d-3)^2A + (3d-8)rA') \right\} \\ &\quad - a_1\mathcal{T} - a_2\mathcal{T}^2 + a_0 - 4\aleph + \frac{8q'g\aleph'}{q'g' + 2gq''} = 0, \end{aligned} \quad (23)$$

$$\begin{aligned} \zeta^t_t &= \frac{4a_2(d-2)Ag\mathcal{T}'}{r^2} \\ &\quad + \frac{(a_1 + 2a_2\mathcal{T})(d-2)}{r} \\ &\quad \left\{ 2[(d-3)Ag + rgA'] + rAg' \right\} \\ &\quad - \mathcal{T} - a_2\mathcal{T}^2 - 4\aleph + a_0 = 0, \end{aligned} \quad (24)$$

where $\mathcal{T}' \equiv d\mathcal{T}(r)/dr$ is calculated through (19).

We have to note that Eq. (22) is a second-order algebraic equation and it gives $\mathcal{T} = \mathcal{T}_0 = \text{const.}$ On the other hand, Eq. (19) for $\mathcal{T}$ gives the solution

$$A(r) = \Lambda_{eff}r^2 - \frac{m}{r^{d-3}}, \quad (25)$$

where m is a constant related to the mass, and the function $g(r)$ is obtained by (25) into (23). Equation (24) gives

$$\begin{aligned} g(r) &= q(r) = c, \\ \aleph(r) &= \frac{a_1\mathcal{T}_0 + 3a_2\mathcal{T}_0^2 + a_0}{4}. \end{aligned} \quad (26)$$

In the above expressions the constant Λ_{eff} , is given by

$$\Lambda_{eff} = \frac{\mathcal{T}_0}{3c}. \quad (27)$$

It is straightforward to see that this is an effective cosmological constant given by the torsion scalar. It is important to stress that such an effective cosmological constant is a specific feature coming from the non-linearity of $f(\mathcal{T})$ gravity model used here. Specifically, the structure of the adopted $f(\mathcal{T})$ model yields an effective cosmological constant which gives an AdS/dS solution depending on its sign. This property has been discussed in the framework of $f(\mathcal{T})$ gravity in [57, 75]. In the present study, we show explicitly how it appears for $d \geq 4$.

The horizons of solution (25) is given by

$$m = \Lambda_{eff}r^{d-1}, \quad (28)$$

which, in 4-dimensions, gives $m = \Lambda_{eff}r^3$.

Let us continue our analysis for the general case where the torsion scalar has non-trivial values. For a non-constant torsion scalar, we get the following solution:

$$\begin{aligned} A(r) &= \frac{1}{r^{d-3}} \left(\int \left[r^{4(d-2)} q'^3 c_2 - \frac{a_1 r^{3(d-2)} q'^2}{2a_2(d-2)c_1} \right] dr + c_3 \right), \\ \aleph(r) &= \frac{1}{16a_2} [a_1^2 - 4a_0a_2 - 8(d-2)r^{d-2}q'a_1a_2c_1c_2 \\ &\quad + 12(d-2)^2r^{2(d-2)}q'^2a_2^2c_1^2c_2^2], \\ g(r) &= \frac{c_1}{q^2r^{2(d-2)}}, \quad q(r) = q(r), \end{aligned} \quad (29)$$

where c_i , $i = 1 \dots 3$ are integration constants. We can assume a given form for the arbitrary function $\aleph(r)$ and calculate the other functions from it. So let us fix the arbitrary function to have the form

$$\aleph(r) = -\frac{P \operatorname{sech}^2\left(\frac{q_1}{(d-3)mr^{d-3}}\right)}{r^{2(d-2)}}. \quad (30)$$

It is worth noticing that, in case $d = 4$, Eq. (30) is identical to that given in [70]. Using Eq. (30) in (29) we get

$$\begin{aligned} A(r) &= -\frac{1}{r^{d-3}} \left\{ \int \frac{1}{216(d-2)^3r^{2(d-2)}a_2^3c_1^3c_2^2(1 + e^{\frac{2q_1}{(d-3)mr^{d-3}}})^3} \left[2a_1r^{d-2}[e^{\frac{2q_1}{(d-3)mr^{d-3}}} + 1] \right. \right. \\ &\quad \left. \left. - \sqrt{\aleph r^{2(d-2)}e^{\frac{4q_1}{(d-3)mr^{d-3}}} + 2[\aleph r^{2(d-2)} - 96Pa_2]e^{\frac{2q_1}{(d-3)mr^{d-3}}} + \aleph r^{2(d-2)}}} \right]^2 \right. \\ &\quad \left. + \sqrt{\aleph r^{2(d-2)}e^{\frac{4q_1}{(d-3)mr^{d-3}}} + 2[\aleph r^{2(d-2)} - 96Pa_2]e^{\frac{2q_1}{(d-3)mr^{d-3}}} + \aleph r^{2(d-2)}}} \right] dr + c_3 \Big\}, \\ g(r) &= \frac{36(d-2)^2a_2^2r^{2(d-2)}c_1^3c_2^2(1 + e^{\frac{2q_1}{(d-3)mr^{d-3}}})^2}{[2a_1r^{(d-2)}[e^{\frac{2q_1}{(d-3)mr^{d-3}}} + 1] - \sqrt{\aleph r^{2(d-2)}e^{\frac{4q_1}{(d-3)mr^{d-3}}} + 2[\aleph r^{2(d-2)} - 96Pa_2]e^{\frac{2q_1}{(d-3)mr^{d-3}}} + \aleph r^{2(d-2)}}}]^2}, \\ q(r) &= \int \frac{2a_1r^{d-2}[e^{\frac{2q_1}{(d-3)mr^{d-3}}} + 1] - \sqrt{\aleph r^{2(d-2)}e^{\frac{4q_1}{(d-3)mr^{d-3}}} + 2[\aleph r^{2(d-2)} - 96Pa_2]e^{\frac{2q_1}{(d-3)mr^{d-3}}} + \aleph r^{2(d-2)}}}}{6(d-2)r^{2(d-2)}a_2c_1c_2(1 + e^{\frac{2q_1}{(d-3)mr^{d-3}}})} dr + c_3, \end{aligned} \quad (31)$$

where $\aleph = (a_1^2 + 12a_0a_2)$.

Now if we assume the constraint $\aleph = 0$, i.e., $a_0 = -\frac{a_1^2}{12a_2}$, in Eq. (22) we get the same solution given by Eq. (29) except the arbitrary function $\aleph(r)$ which takes the form

$$\aleph(r) = \frac{(a_1 - 3(d-2)r^{(d-2)}q'a_2c_1c_2)^2}{12a_2}. \quad (32)$$

Considering the function (30) in (32), we get

$$\begin{aligned} A(r) &= \Lambda_{eff}r^2 - \frac{4(d-3)^2mq^2P}{(d-2)a_1^2c_1r^{d-2}q_1^2[e^{\frac{2q_1}{(d-3)mr^{d-3}}} + 1]^2} \\ &\quad \left(2\sqrt{3P|a_2|}e^{\frac{q_1}{(d-3)mr^{d-3}}}[q_1\{e^{\frac{2q_1}{(d-3)mr^{d-3}}} - 1\}] \right. \\ &\quad \left. + [1 + e^{\frac{2q_1}{(d-3)mr^{d-3}}}] (d-2)mr^{d-3} \right] \end{aligned}$$

$$\begin{aligned} &+ 3a_1q_1r^{d-2}[e^{\frac{2q_1}{(d-3)mr^{d-3}}} + 1]) + \frac{q^2}{(d-2)a_1^2c_1} \\ &\quad \int \frac{8(d-3)^2P\sqrt{3P|a_2|}(q_1^2 - (d-2)m^2r^{2(d-3)})e^{\frac{q_1}{(d-3)mr^{d-3}}}}{q_1^2r^{2(d-2)}[e^{\frac{2q_1}{(d-3)mr^{d-3}}} + 1]} dr \\ &\quad + \frac{c_3}{r^{d-3}}, \\ &\approx \Lambda_{eff}r^2 - \frac{M}{r^{d-3}} + \frac{Q^2}{r^{2(d-3)}} + \frac{Q_1^4}{r^{3(d-8)}} + \frac{Q_2^4}{r^{4(d-3)}} + \dots, \\ g(r) &= \frac{a_1^2c_1r^{2(d-2)}[e^{\frac{2q_1}{(d-3)mr^{d-3}}} + 1]^2}{q^2(d-3)^2[a_1r^{d-2}[e^{\frac{2q_1}{(d-3)mr^{d-3}}} + 1] - 4\sqrt{3P|a_2|}e^{\frac{q_1}{(d-3)mr^{d-3}}}]^2} \\ &\approx \frac{c_1}{(d-3)^2q^2} \left[1 + \frac{4(d-3)^2\sqrt{3P|a_2|}}{a_1r^{d-2}} \right. \\ &\quad \left. - \frac{2q_1^2\sqrt{3P|a_2|}}{a_1(d-3)^2m^2r^{3d-8}} - \frac{36Pa_2}{a_1^2r^{2(d-2)}} + \dots \right], \\ q(r) &= \int \frac{a_1r^{d-2}[e^{\frac{2q_1}{(d-3)mr^{d-3}}} + 1] - 4e^{\frac{q_1}{(d-3)mr^{d-3}}}\sqrt{3P|a_2|}}{3(d-2)r^{2(d-2)}a_2c_1c_2(1 + e^{\frac{2q_1}{(d-3)mr^{d-3}}})} dr + c_3 \\ &\approx c_3 + \frac{q}{r^{d-3}} - \frac{2(d-3)\sqrt{3|a_2|P}q}{a_1(2d-5)r^{2d-5}} \\ &\quad + \frac{\sqrt{3|a_2|P}qq_1^2}{(d-3)(4d-11)a_1m^2r^{4d-11}} + \dots, \end{aligned} \quad (33)$$

where $\Lambda = -\frac{a_1^3}{54(d-1)(d-2)^3a_2^3c_1^3c_2^2}$, $q = -\frac{a_1}{3(d-2)(d-3)a_2c_1c_2}$, $M = c_3 - \frac{6(d-3)^2Pq^2}{q_1a_1c_1(d-2)}$, $Q^2 = \frac{6(d-3)Pq^2}{a_1c_1(d-2)}$, $Q_1^4 = -\frac{8(d-3)^2Pq^2\sqrt{3P|a_2|}}{c_1(d-2)(2d-5)}$ and $Q_2^4 = -\frac{2Pq_1^2q^2}{m^2a_1c_1(d-2)(d-3)}$. Using Eq. (16) we get the linear electrodynamics in the following form

$$\begin{aligned} \mathcal{E}(r) &= \frac{\phi}{r^{d-3}} + \frac{6\phi^2(d-2)(d-3)^2\sqrt{3|a_2|P}}{a_1r^{(2d-5)}(2d-5)} \\ &\quad + \frac{q_1^2\phi^2(2d-5)}{m^2r^{3(d-3)}(d-3)} + \dots, \end{aligned} \quad (34)$$

where $\phi = -\frac{1}{3(d-2)(d-3)}$. To get this result, we have put $2a_2c_1^2c_2 = a_1P$. It is interesting to note that Eq. (34) coin-

cides with that given in [1] for $q_1 = 0$. The parameter q_1 is responsible for deviations from linear electrodynamics as Eq. (30) indicates. It is straightforward to show that, from (30) for $q_1 = 0$, we return to Maxwell electrodynamics and, for $q_1 \neq 0$, we have non-linear electrodynamics. Explicitly, the effect of parameter q_1 appears in Eq. (34) showing that the gauge potential is different from the one presented in [1].

Additionally, Ξ is defined as

$$\Xi := \sqrt{1 - \sum_{j=1}^{\omega} \frac{n_j^2}{\lambda^2}}. \quad (37)$$

Adopting the transformations (35) to the (17), we obtain

$$(h^i{}_{\mu}) = \begin{pmatrix} \Xi\sqrt{A(r)} & 0 & -n_1\sqrt{A(r)} & -n_2\sqrt{A(r)} & \cdots & -n_{\omega}\sqrt{A(r)} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{\sqrt{A(r)g(r)}} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \frac{n_1 r}{\lambda^2} & 0 & -\Xi r & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \frac{n_2 r}{\lambda^2} & 0 & 0 & -\Xi r & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{n_{\omega} r}{\lambda^2} & 0 & 0 & 0 & \cdots & -\Xi r & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & r & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & r & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & r \end{pmatrix}, \quad (38)$$

If we calculate the invariants of the black hole solution (33), we get the same asymptotic behavior presented in [1, 62]. These invariants show that there is a singularity at $r = 0$. Approaching to $r = 0$, these invariants assume the form $(K, R_{\mu\nu}R^{\mu\nu}) \sim \sqrt[3]{r^{-4(d-2)}}$, and $(R, T) \sim \sqrt[3]{r^{-2(d-2)}}$, differently to the black holes of Maxwell electrodynamics in either GR or TEGR theories which have the forms $(K, R_{\mu\nu}R^{\mu\nu}) \sim r^{-2d}$ and $(R, T) \sim r^{-d}$, respectively. The above results indicate in a clear way that the singularity of the non-linear charged black hole is milder than the one emerging in GR and TEGR for the charged case.

Finally, if we calculate the energy of solution (33) we get the same formula presented in [1, 62] up to the leading order, i.e. $E = \frac{(d-2)M}{4(d-3)G_d}$. In other words, this feature assures the consistency of the solution.

4 Rotating black holes in Maxwell- $f(T)$ gravity

Let us derive now rotating black hole solutions satisfying the field equations of the above (20) $f(T)$ gravity. We start assuming the above static solution as a constraint. Taking into account the following transformations:

$$\bar{\eta}_i = -\Xi \eta_i + \frac{n_i}{l^2} t, \quad \bar{t} = \Xi t - \sum_{i=1}^{\omega} n_i \eta_i, \quad (35)$$

with n_i are rotation parameters (their number is $\omega = \lfloor (d-1)/2 \rfloor$ where $\lfloor \dots \rfloor$ marks the integer part), and where we can define a parameter λ connected to the Λ_{eff} of the static solution through

$$\lambda = -\frac{(d-2)(d-1)}{2\Lambda_{eff}}. \quad (36)$$

where $A(r)$ and $g(r)$ are given in (33). Hence, for the electromagnetic potential (31) we get

$$\bar{q}(r) = -q(r) \left[\sum_{j=1}^{\omega} n_j d\bar{\eta}_j - \Xi d\bar{t} \right]. \quad (39)$$

We have to note here that transformation (35) does not alter local spacetime properties, however it changes global properties (see [76]). This feature comes out from the fact that it mixes compact and noncompact coordinates. As a consequence, vielbeins (17) and (38) can be locally transformed into each other but this property does not hold globally [76, 77].

According to the vielbein (38), the metric can be written as

$$\begin{aligned} ds^2 = & -A(r) \left[\Xi d\bar{t} - \sum_{i=1}^{\omega} n_i d\bar{\phi} \right]^2 + \frac{dr^2}{A(r)g(r)} \\ & + \frac{r^2}{\lambda^4} \sum_{i=1}^{\omega} \left[n_i d\bar{t} - \Xi \lambda^2 d\bar{\phi}_i \right]^2 \\ & + r^2 d\xi_k^2 + \frac{r^2}{\lambda^2} \sum_{i < j}^{\omega} (n_i d\bar{\phi}_j - n_j d\bar{\phi}_i)^2, \end{aligned} \quad (40)$$

where $0 \leq r < \infty$, $-\infty < t < \infty$, $0 \leq \eta_i < 2\pi$, $i = 1, 2, \dots, \omega$ and $-\infty < \xi_k < \infty$. Here $d\xi_k^2$ is the Euclidean metric on $(d - \omega - 2)$ dimensions and $k = 1, 2, \dots, d - 3$. It is worth mentioning that the static configuration (18) is recovered as a particular case of the above general metric as soon as the rotation parameters n_j are going to zero. Furthermore, it is worth stressing that the line-element (40) is derived

when the Minkowski metric (2) is written in cylindrical coordinates, that is

$$(\eta_{ij}) = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 + \frac{n_\omega^2}{\lambda^2 \Xi^2} & -\frac{n_\omega n_1}{l^2 \Xi^2} & -\frac{n_\omega n_2}{\lambda^2 \Xi^2} & \dots & -\frac{n_\omega n_{\omega-1}}{l^2 \Xi^2} & 0 & 0 & \dots & 0 \\ 0 & 0 & -\frac{n_\omega n_1}{\lambda^2 \Xi^2} & 1 + \frac{n_{\omega-1}^2}{l^2 \Xi^2} & -\frac{n_{\omega-1} n_1}{\lambda^2 \Xi^2} & \dots & -\frac{n_{\omega-1} n_{\omega-2}}{\lambda^2 \Xi^2} & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & -\frac{n_\omega a_{\omega-1}}{\lambda^2 \Xi^2} & -\frac{n_{\omega-1} n_{\omega-2}}{\lambda^2 \Xi^2} & -\frac{n_{\omega-2} n_{\omega-3}}{\lambda^2 \Xi^2} & \dots & 1 + \frac{n_1^2}{\lambda^2 \Xi^2} & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 \end{pmatrix}. \quad (41)$$

Here, the interesting feature is that the torsion components are vanishing.

5 Thermodynamical stability and phase transitions

Black hole thermodynamics is a fundamental subject in physics, because it investigates the relation between gravitational and quantum regimes and it is strictly related to the thorny problem of quantum gravity. In general, there are two main approaches to deal with black hole thermodynamics: the first has been proposed by Gibbons and Hawking [78, 79], studies the thermal properties of Schwarzschild solution by applying the Euclidean continuation. The second method identifies the gravitational surface and defines the temperature of black holes [80–82].

In this study, we are going to apply the second approach to understand the thermodynamics of the AdS black hole, derived in Eq. (33), and then to study its stability by calculating the heat capacity and the Gibbs free energy. The black hole (33) is characterized by the mass, M , the charges (monopole, Q , dipole and higher order, Q_1 and Q_2) and also by a cosmological constant Λ_{eff} .

To calculate the horizons of solution (33), we have to put the function $A(r) = 0$. The plot of Fig. 1a shows the two roots of $A(r)$ which determines, respectively, the event horizon r_b and the cosmological horizon r_c of the solution (33) when the dimensional parameter a_2 has a negative value. However, when a_2 has a positive value, we have only one horizon as Fig. 1a shows. In 4-dimensions, solutions with two horizons can be obtained for Schwarzschild–de Sitter and Kerr–Schild black holes [83–85], for Reissner–Nordström black holes [86], for minimal model of regular black holes [87], and for spherically symmetric Bardeen black holes of non-commutative geometry [88–91].

The Bekenstein–Hawking entropy in the 4-dimension for $f(T)$ gravity can be defined as [92]

$$S(r_b) = \frac{1}{4} A f_T = \pi r_b^2 f_T, \quad (42)$$

where r_b is the event horizon in Planck units and A is the event horizon area. Using Eqs. (33) in (42), we get the entropy in d -dimension as

$$S(r_b) \approx \frac{r_b^{d-2} \Omega_{d-2}}{6} \left[a_1 - \frac{2\sqrt{3P|a_2|}}{r_b^{d-2}} + \frac{q_1^2 \sqrt{3P|a_2|}}{(d-3)^2 m^2 r_b^{3d-8}} + O\left(\frac{1}{r_b^{2(d-1)}}\right) + \dots \right], \quad (43)$$

where Ω_{d-2} is the volume of the unit $(d-2)$ -sphere. Eq. (43) shows that, if we neglect the higher order terms of $O\left(\frac{1}{r_b^{2(d-1)}}\right)$, the term $\left(a_1 - \frac{2\sqrt{3P|a_2|}}{r_b^{d-2}} + \frac{q_1^2 \sqrt{3P|a_2|}}{(d-3)^2 m^2 r_b^{3d-8}}\right)$ must be positive in order to have a positive entropy. This leads to $q_1 \geq \frac{\pm m r_b^{d-3} \sqrt{x(2x-a_1 r_b^{d-2})}}{x}$ and $x > \frac{a_1 r_b^{d-2}}{2}$ where $x = \sqrt{3P|a_2|}$, otherwise we have a negative entropy.

It is important to stress the fact that if the above condition is not satisfied, one gets a negative entropy. In this case, f_T assumes a negative value which yields $T < -\frac{a_1}{a_2}$. This means that the functional form of $f(T)$ would be different from Eq. (20) and therefore, black hole (31) is no more a solution. Using the data coming from galaxy–galaxy weak lensing, a novel test on GR has been developed [93]. Specifically, from the analysis of these data, the upper bound on the parameter a_2 , which indicates deviations from GR, can be calculated. It is $a_2 \leq 0.76_{-0.58}^{+2.45} pc^2$, compatible with observational tests [93, 94].

The thermodynamical stability is related to the heat capacity C_b . In particular with the sign of this quantity. Below, we will take into account the thermal stability of the black holes via their heat capacity [95–97]

$$C_b = \frac{dE_b}{dT_b} = \frac{\partial m}{\partial r_b} \left(\frac{\partial T}{\partial r_b} \right)^{-1}, \quad (44)$$

where E_b is the energy. If $C_b > 0$ ($C_b < 0$), the black hole is stable (unstable) from thermodynamical point of view. To

Fig. 1 Schematic plot of horizons of solution (33). The plot of **a** shows the black hole event horizon, r_b , and the cosmological horizon, r_c , while of **b** shows the horizon-mass radius relation. Here we take $d = 4$ and

$a_1 = 1$. The units are in terms of the event horizon r_b assumed equal to 1. In other words, the plots are in terms of Schwarzschild gravitational radius $r_g = 2GM/c^2$ ruled by the mass of the black hole

better understand this phenomenon, let us assume that, due to thermal fluctuations, the black hole absorbs more radiation than it emits. When this happens, its heat capacity is positive. According to this situation, the black hole mass increases. On the other hand, if the black hole emits more radiation than it absorbs, the heat capacity becomes negative. In this situation, the black hole mass decreases and it can completely evaporate. In conclusion, black holes with negative heat capacities are unstable from a thermodynamical point of view.

In order to calculate Eq. (44), we have to derive the formulae of $M_b \equiv M(r_b)$ and $T_b \equiv T(r_b)$. Firstly, we calculate the black hole mass within an even horizon r_b . We set $A(r_b) = 0$, then we obtain

$$M_{b_{Eq.(33)}} = r_b^{d-3} \left(\Lambda_{eff} r_b^2 + \frac{Q^2}{r_b^{2(d-3)}} + \frac{Q_1^4}{r_b^{(3d-8)}} + \frac{Q_2^4}{r_b^{4(d-3)}} + \dots \right). \quad (45)$$

The above equation shows that the total mass of the black hole is given by a function of the charge and the horizon radius. It is straightforward to calculate the degenerate horizon by the condition $\partial M_h / \partial r_b = 0$, which gives

$$r_{dg} = \pm \frac{(6a_2P)^{1/6} \sqrt{(3q_1^2a_1 + 4m^2\sqrt{3P|a_2|} + q_1\sqrt{a_1}\sqrt{9a_1q_1^2 + 24m^2\sqrt{3P|a_2|}})^{3/2} - 2\sqrt{6}(m^4Pa_2)^{1/3}}}{\sqrt{a_1}m^{1/3}(3a_1q_1^2 + 4m^2\sqrt{3P|a_2|} + q_1\sqrt{a_1}\sqrt{9a_1q_1^2 + 24m^2\sqrt{3P|a_2|}})^{1/6}},$$

in 4 dimensions. As seen from Eq. (45) and Fig. 2a, the horizon mass-radius relation is given by

$$M(r_b \rightarrow 0) \rightarrow \infty, \quad M(r_b \rightarrow \infty) \rightarrow \infty. \quad (46)$$

The Hawking temperature of black holes is derived by requiring no singularity at the horizon of the Euclidean sector of solutions. Furthermore, it is possible to obtain the temperature related with the outer event horizon $r = r_b$ as [98]

$$T = \frac{\kappa}{2\pi}, \quad \kappa = \frac{A'(r_b)}{2}. \quad (47)$$

where κ is the surface gravity. The Hawking temperature associated with the black hole solution (33) is

$$T_{b_{Eq.(33)}} = \frac{1}{4\pi} \left\{ (d-1)r_b\Lambda_{eff} - \frac{(d-3)Q^2}{r_b^{2d-5}} - \frac{(2d-5)Q_1^4}{r_b^{(3d-7)}} + \frac{3(d-3)Q_2^4}{r_b^{(4d-11)}} + \dots \right\}, \quad (48)$$

where T_b is calculated at the event horizon. In Fig. 2b, it is shown that the horizon temperature T_h is zero at the degenerate horizon $r_b = r_{dg}$. For $r_b < r_{dg}$, the horizon temperature

evolves below the absolute zero giving rise to an ultra-cold black hole. As pointed out in [99], there is no reason from

Fig. 2 Schematic plots of the degenerate horizons of solution (33): **a** The plot of $A(r)$ shows the black hole event horizon, r_b , and the cosmological horizon, r_c , where $m > m_{\min}$. At $m = m_{\min}$ ($r_b = r_c$), the

black hole has a degenerate horizon at r_{dg} . Otherwise, $m < m_{\min}$ the black hole is naked; **b** the temperature vanishes on the horizon radius r_b . Here we take $d = 4$

thermodynamical point of view to prevent a black hole temperature to go under absolute zero. In this case, the black hole would become a naked singularity. In the range $r_b > r_{dg}$, the horizon temperature is positive. Considering also gravitational effects, we obtain that, for some high temperature $T_{\max}$, the radiation becomes unstable and the collapse starts [100]. As a consequence, the AdS solution is stable only for $T < T_{\max}$. Above $T_{\max}$, only the heavy black holes reach stable configurations [100].

Let us now calculate the heat capacity C_b horizon and substitute Eqs. (45) and (48) into Eq. (44). We have

$$C_b \approx -4\pi r_b^2 - \frac{288\pi(d-2)\alpha P}{r_b^{d-2}} + \frac{576\pi(d-2)\alpha P\sqrt{3P|\alpha|}}{r_b^{2(d-2)}} + \frac{288\pi(2d-5)\alpha q_1^2 P}{(d-3)^2 m^2 r_b^{(3d-8)}}. \quad (49)$$

In Fig. 3a, it is shown that the heat capacity is negative for $r_b < r_{dg}$ and positive for $r_b > r_{dg}$. Always considering Fig. 3a, a characteristic of the heat capacity is a second-order phase transition at r_c whereas the heat capacity shows an infinite discontinuity.

The Grand Canonical Ensemble free energy, that is the Gibbs free energy, is defined as [101]

$$G(r_b) = M(r_b) - T(r_b)S(r_b), \quad (50)$$

where $M(r_b)$, $T(r_b)$ and $S(r_b)$ are the mass, the temperature and the entropy of the black hole at the event horizon, respectively. Using Eqs. (42), (45) and (48) in Eq. (50), we get

$$G_b \approx r_b^{d-3} \left(\Lambda_{eff} r_b^2 + \frac{Q^2}{r_b^{2(d-3)}} + \frac{Q_1^4}{r_b^{3(d-8)}} + \frac{Q_2^4}{r_b^{4(d-3)}} + \dots \right) + \Omega_{d-2} \left(\frac{q^2 r_b^{d-1} (d-3)^2}{144(c-2)c_1\alpha} - \frac{q^2 r_b (d-3)^2 \sqrt{3P|\alpha|}}{72(d-2)c_1\alpha} + \frac{P}{36(d-2)^3 c_1 r_b^{d-3}} + \frac{q^2 \sqrt{3P|\alpha|}}{144m^2 (d-2) r_b^{2d-7} c_1\alpha} - \frac{5(d-3)^2 P q^2 \sqrt{3P|\alpha|}}{6(d-2) r_b^{2b-5} c_1\alpha} \right). \quad (51)$$

It is worth noticing that, as soon as the charge parameter $q_1 \rightarrow 0$, the Gibbs free energy derived from Eq. (33) is coincident with that in [102]. The Gibbs energy of our solution is represented in Fig. 3b for some values of model parameters.

6 Discussion and conclusions

In this paper, we have investigated the effect of the non-linear electrodynamics on modified TEGR theory. To this aim, we derived the charged non-linear electrodynamics field equations for $f(T)$ gravity. They reduce to the well known form

Fig. 3 Schematic plot of the heat capacity shows the locally unstable event horizon which is characterized by the negative $C_h < 0$. Also, the plot shows a second-order phase transition as C_h diverges; **b** shows free energy of black hole (33)

of Maxwell field equations assuming some constraints on the arbitrary functions. Applying these field equations to cylindrical coordinates in d -dimensions, we got a closed system of non-linear differential equations. In this framework, we obtained black hole solutions. The most interesting feature of these black hole solutions is that they behave as AdS solutions generalizing the black hole solutions derived in [1]. This generalization comes from the contribution of the parameter included in the arbitrary function (30). If this parameter set equal to zero we return to the black black hole presented in [1]. The contributions of non-linear electrodynamics clearly emerge in the above black holes discriminating the solutions with respect to the standard Maxwell field. Our black holes keep all the features of the black holes derived in [1], i.e., they show a central singularity, that is softer in comparison with the standard GR and TEGR cases. The rotating black hole solutions can be achieved by a suitable coordinate transformation.

More information on the black hole (33) is obtained by its thermodynamical properties. The most important feature in $f(T)$ gravity is that entropy is not always proportional to the horizon area [103, 104]. It is possible to show that, for constraints on the parameter q_1 characterizing the arbitrary function of the non-linear electrodynamics, one has a positive entropy. On the other hand, there are some regions of parameter q_1 where entropy is negative [103, 105–107]. Negative entropy is a familiar feature in gravitational theories: several black hole solutions have negative entropy, e.g. charged Gauss–Bonnet AdS black holes [103, 105, 106, 108]. Our results indicate that negative entropies may be explained

as a region where the parameter q_1 values have entered into an un-allowed region, or into a regime where there is a phase transition. The gravitational entropy of non-trivial solutions in $f(T)$ gravity will be the subject of future researches.

Furthermore, the heat capacity of black hole (33) has been derived and we have shown that there is a locally unstable event horizon characterized by $C_b < 0$. Furthermore there is a second-order phase transition at r_b whereas the heat capacity is characterized by an infinite discontinuity. Finally, the heat capacity of our black hole has a stable event horizon which is characterized by a positive value, i.e., $C_b > 0$ for which $r_b > r_{dg}$. Finally, we have derived the Gibbs free energy showing that the black hole solution (33) has always a positive value of this quantity for some constraints on the parameter as Fig. 3b shows. In a forthcoming study, possible astrophysical applications of these solutions will be considered.

Acknowledgements SC is supported in part by the INFN sezione di Napoli, iniziative specifiche QGSKY and MOONLIGHT2. The article is also based upon work from COST action CA15117 (CANTATA), supported by COST (European Cooperation in Science and Technology).

Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors' comment: No observational datasets have been used.]

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