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No hair theorem for massless scalar fields outside asymptotically flat horizonless reflecting compact stars

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Abstract In a recent paper, Hod started a study on no scalar hair theorem for asymptotically flat spherically symmetric neutral horizonless reflecting compact stars. In fact, Hod's approach only rules out massive scalar fields. In the present paper, for massless scalar fields outside neutral horizonless reflecting compact stars, we provide a rigorous mathematical proof on no hair theorem. We show that asymptotically flat spherically symmetric neutral horizonless reflecting compact stars cannot support exterior massless scalar field hairs.

1 Introduction

Recently, the first ever image of a black hole has been captured by a network of eight radio telescopes around the world [1]. These discoveries open up hope to directly test various black hole theories from astronomical aspects. One remarkable property of classical black holes is the famous no hair theorem [2–9]. If generically true, such no hair theorem would signify that asymptotically flat black holes cannot support scalars, massive vectors and Abelian Higgs hairs in exterior regions, for recent references see [10–25] and reviews see [26,27]. It was believed that no hair behaviors are due to the existence of one-way absorbing horizons.

However, it was recently found that no hair behavior also appears in the background of horizonless reflecting compact stars. In the asymptotically flat gravity, Hod firstly proved no static scalar hair theorem for neutral horizonless reflecting compact stars [28]. In the asymptotically dS gravity, it was found that neutral horizonless reflecting compact stars cannot support the existence of massive scalar, vector and tensor field hairs [29]. When considering a charged background, large reflecting shells can exclude static scalar field hairs [30–32]. Similarly, static scalar field hairs cannot exist outside charged reflecting compact stars of large size [33–37]. With field-curvature couplings, such no hair theorem could also hold in the horizonless gravity [38–40]. Moreover, we proved no hair theorem for horizonless compact stars with other boundary conditions [41–43].

As is well known, scalar field mass usually plays an important role in the scalar hair formation. For massless scalar fields $\psi(r)$, no scalar hair theorem was investigated in the background of horizonless compact stars [28], where the relation $\psi(r_{peak})\psi''(r_{peak}) < 0$ at the extremum point $r = r_{peak}$ is essential in Hod's present proof. However, the general characteristic relation at the extremum point should be $\psi(r_{peak})\psi''(r_{peak}) \leqslant 0$ and in fact, $\psi''(r_{peak}) = 0$ holds for some solutions. So Hod's approach only ruled out massless scalar fields with $\psi''(r_{peak}) \neq 0$ and non-trivial solutions with $\psi''(r_{peak}) = 0$ cannot be excluded. Then it is of some importance to search for a mathematical proof on no hair theorem for massless scalar field hairs.

In the following, we consider static massless scalar fields in the background of asymptotically flat spherical neutral horizonless reflecting compact stars. We provide a rigorous mathematical proof on no hair theorem for massless scalar fields. We summarize main results in the last section.

2 No massless scalar hair for horizonless reflecting compact stars

We consider massless scalar fields outside asymptotically flat horizonless compact stars. We define the radial coordinate $r = r_s$ as the star radius. And the curved spherically symmetric spacetime is described by line element [38,39]

$$ds^2 = -e^{\nu}dt^2 + e^{\lambda}dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2), \tag{1}$$

where $\nu = \nu(r)$ and $\lambda = \lambda(r)$. Asymptotic flatness of the spacetime requires infinity behaviors $\nu(r \to \infty) \backsim O(r^{-1})$ and $\lambda(r \to \infty) \backsim O(r^{-1})$.

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The Lagrange density with massless scalar fields minimally coupled to gravity is given by

$$\mathcal{L} = R - (\partial_\alpha \psi)^2. \tag{2}$$

Here $\psi(r)$ is the scalar field and R is the scalar Ricci curvature of the spacetime.

From the metric (1) and the action (2), we obtain the scalar field equation

$$\psi'' + \left(\frac{2}{r} + \frac{\nu'}{2} - \frac{\lambda'}{2}\right)\psi' = 0. \tag{3}$$

Around the infinity, the scalar field equation can be approximated by

$$\psi'' + \frac{2}{r}\psi' = 0. \tag{4}$$

It yields the infinity boundary behavior

$$\psi \sim A + \frac{B}{r}, \tag{5}$$

where A and B are integral constants. The finite ADM mass condition yields that the static scalar field must asymptotically approaches zero at the infinity. So we fix $A = 0$. Then the scalar field satisfies the infinity boundary condition

$$\psi(r \to \infty) = 0. \tag{6}$$

At the star surface, we impose the reflecting boundary condition

$$\psi(r_s) = 0. \tag{7}$$

According to relations $\psi(r_s) = 0$ and $\psi(r \to \infty) = 0$, $\psi(r)$ must possesses one extremum point r_{peak}, which is between the star surface r_s and the infinity. With the symmetry $\psi \to -\psi$ of Eq. (3), it is enough for us to only consider the case of positive local maximum value in the following.

Metric solutions were assumed to be real analytic in the proof of the uniqueness of the Kerr solution [44–46]. A real function is analytic if it can be locally expressed with Taylor series. In this work, we also assume that the metric solutions are real analytic. Then a positive constant δ exists and in the range $(r_{peak}-\delta, r_{peak}+\delta)$, metric solutions can be expanded as

$$\nu(r) = \sum_{n=0}^{\infty} a_n (r - r_{peak})^n, \tag{8}$$

$$\lambda(r) = \sum_{n=0}^{\infty} b_n (r - r_{peak})^n, \tag{9}$$

where $a_n = \frac{\nu^{(n)}(r_{peak})}{n!}$ and $b_n = \frac{\lambda^{(n)}(r_{peak})}{n!}$. Solutions $\psi(r)$ of Eq. (3) are real analytic according to Cauchy–Kowalevski theorem, which states that solutions of differential equations are real analytic when coefficients are real analytic [44–47].

In the same range $(r_{peak} - \delta, r_{peak} + \delta)$, the scalar field can be expressed as

$$\psi(r) = \sum_{n=0}^{\infty} c_n (r - r_{peak})^n \tag{10}$$

with $c_n = \frac{\psi^{(n)}(r_{peak})}{n!}$, which can be obtained by putting (8) and (9) into (3) and considering terms order by order.

Now we show that a nonzero c_n $(n \geqslant 1)$ should exist for nontrivial solution $\psi(r)$. Otherwise, $\psi(r)$ is a constant in the range $(r_{peak}-\delta, r_{peak}+\delta)$. Then we can search for the largest R where $\psi(r)$ is a constant in the range $(r_{peak} - \delta, R]$. Since $\psi(r)$ is real analytic, there is a constant $\tilde{\delta} > 0$ and in the range $(R - \tilde{\delta}, R + \tilde{\delta})$, the scalar field can be expressed as

$$\psi(r) = \sum_{n=0}^{\infty} d_n (r - R)^n. \tag{11}$$

If we approach R from the left side, we find all coefficients $d_n = 0$ for $n \geqslant 1$ since $\psi(r)$ is a constant in the range $(R-\tilde{\delta}, R]$. However, if we approach R from the right side, a nonzero $d_n \neq 0$ with $n \geqslant 1$ should exist since $\psi(r)$ is not a constant in the range $[R, R+\tilde{\delta})$. This contradiction leads to the conclusion that a nonzero $c_n = \frac{\psi^{(n)}(r_{peak})}{n!}$ $(n \geqslant 1)$ exists.

By considering leading terms of (10), we obtain following conclusions.

(I) Firstly, there is $\psi'(r_{peak}) = 0$. Otherwise, $\psi(r) = \psi(r_{peak}) + \psi'(r_{peak})(r - r_{peak}) + \cdots$ and $\psi(r)$ cannot has extremum value at the point r_{peak}.

(II) In the case of $\psi''(r_{peak}) \neq 0$, we will have $\psi''(r_{peak}) < 0$. Otherwise, ψ cannot has local maximum extremum value at the point r_{peak} since $\psi(r) = \psi(r_{peak}) + \frac{\psi''(r_{peak})}{2}(r - r_{peak})^2 + \cdots$. In this work, we only consider the case of positive local maximum value according to the symmetry $\psi \to -\psi$ of Eq. (3).

(III) In the case of $\psi''(r_{peak}) = 0$, we will have $\psi^{(3)}(r_{peak}) = 0$. Otherwise, ψ cannot have local maximum extremum value at the point r_{peak} since $\psi(r) = \psi(r_{peak}) + \frac{\psi'''(r_{peak})}{3}(r - r_{peak})^3 + \cdots$.

(IV) In the case of $\psi''(r_{peak}) = 0$ and $\psi^{(4)}(r_{peak}) \neq 0$, we will have to impose $\psi^{(4)}(r_{peak}) < 0$ to obtain a local maximum extremum value for ψ at the point r_{peak}. In this case, there is the relation $\psi(r) = \psi(r_{peak}) + \frac{\psi^{(4)}(r_{peak})}{24}(r - r_{peak})^4 + \cdots$.

Following this analysis, we can obtain an even number N, which satisfies

$$\psi' = 0,\ \psi'' = 0,\ \psi^{(3)} = 0,\ \psi^{(4)} = 0, \ldots, \psi^{(N-1)} = 0,\ \psi^{(N)} < 0 \quad \text{for} \quad r = r_{peak}. \tag{12}$$

At the extremum point $r = r_{peak}$, Eq. (3) yields relations

We define a new function $f(r)=\frac{2}{r}+\frac{\nu'}{2}-\frac{\lambda'}{2}$. Then the Eq. (3) can be expressed as

$$\psi''+f\psi'=0. \tag{13}$$

As we stated, metric solutions are assumed to be real analytic, which is the same as cases in the proof of the uniqueness of the Kerr solution [44–46]. According to Cauchy–Kowalevski theorem, as we have shown, Eq. (3) possesses a locally real analytic solution $\psi(r)$ around r_{peak}. Since metric functions and $\psi(r)$ are real analytic around r_{peak}, both $f'(r_{peak}), f''(r_{peak}), f'''(r_{peak}), f^{(4)}(r_{peak}),\ldots$ and $\psi'(r_{peak}), \psi''(r_{peak}), \psi'''(r_{peak}), \psi^{(4)}(r_{peak}),\ldots$ exist. Taking the derivative of both sides of the Eq. (13), we get the equation

$$(\psi''+f\psi')'=0, \tag{14}$$

which holds round r_{peak}. The relation (14) is equal to

$$\psi'''+f\psi''+f'\psi'=0. \tag{15}$$

From (15), we obtain the equation

$$(\psi'''+f\psi''+f'\psi')'=0. \tag{16}$$

The relation (16) can be transformed into

$$\psi^{(4)}+f\psi'''+2f'\psi''+f''\psi'=0. \tag{17}$$

Along this line, we can obtain the following relation

$$\psi^{(N)}+f\psi^{(N-1)}+(N-2)f'\psi^{(N-2)} +\cdots+(N-2)f^{(N-3)}\psi''+f^{(N-2)}\psi'=0. \tag{18}$$

At the extremum point, relations (12) are in contradiction with the relation (18). Due to this contradiction, there is no nontrivial scalar field solution of Eq. (3). We conclude that asymptotically flat spherical neutral horizonless reflecting compact stars cannot support exterior massless scalar field hairs.

3 Conclusions

We studied no hair theorem for static massless scalar fields outside the asymptotically flat spherically symmetric horizonless reflecting compact stars. We obtained the characteristic relations (12) at extremum points, which are in contradiction with the Eq. (18). That is to say there is no nontrivial scalar field solution of Eq. (3). We concluded that asymptotically flat spherically symmetric horizonless reflecting compact stars cannot support the existence of exterior massless scalar fields. In this work, we provided a rigorous mathematical proof on no hair theorem for massless scalar fields.

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