Eur. Phys. J. C (2019) 79:855
https://doi.org/10.1140/epjc/s10052-019-7356-5

THE EUROPEAN PHYSICAL JOURNAL C

Regular Article - Theoretical Physics

Poisson–Lie identities and dualities of Bianchi cosmologies

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Received: 24 July 2019 / Accepted: 29 September 2019 / Published online: 17 October 2019

Abstract We investigate a special class of Poisson–Lie T-plurality transformations of Bianchi cosmologies invariant with respect to non-semisimple Bianchi groups. For six-dimensional semi-Abelian Manin triples $\mathfrak{b} \bowtie \mathfrak{a}$ containing Bianchi algebras $\mathfrak{b}$ we identify general forms of Poisson–Lie identities and dualities. We show that these can be decomposed into simple factors, namely automorphisms of Manin triples, B-shifts, β-shifts, and "full" or "factorized" dualities. Further, we study effects of these transformations and utilize the decompositions to obtain new backgrounds which, supported by corresponding dilatons, satisfy Generalized Supergravity Equations.

Contents

1 Introduction

Duality transformations play crucial role in the study of various aspects of string theory and related fields. They connect theories in different coupling regimes or, in the case of T-duality, backgrounds with distinct curvature properties. Both Abelian T-duality [1] and its non-Abelian generalization [2,3] rely on the presence of symmetries of sigma model backgrounds. Dual sigma model related to the original one by T-duality is obtained by gauging of the symmetry and introduction of Lagrange multipliers. However, the symmetries are not preserved in the non-Abelian case, meaning we may not be able to return to the original model by dualization. Despite this serious issue we see renewed interest in (non-) Abelian T-duality (NATD). The procedure was extended to RR fields in [4,5] and is used frequently to find new supergravity solutions, see e.g. [6,7] and references therein. It also applies in the study of integrable models [8–10]. In this paper we are going to present new solutions of the Generalized

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Supergravity Equations obtained by action of Poisson–Lie identities and Poisson–Lie dualities introduced in [11] on the Bianchi cosmologies [12]. This extends list of results obtained in [13].

Poisson–Lie T-duality [14] introduces Drinfel'd double as the underlying algebraic structure of T-duality and replaces symmetry of the sigma model background by the so-called Poisson–Lie symmetry [15]. This allows us to treat both models equally and solves the above mentioned problem. We shall use this formalism through the whole paper. In the case of (non-) Abelian T-duality the Lie group $\mathcal{D}$ of Drinfel'd double splits into Lie subgroups $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ of equal dimension, where the former represents symmetries of the original background while the latter is Abelian. In this paper we consider only these semi-Abelian Drinfel'd doubles as we focus on dualization of particular backgrounds and the presence of symmetries remains crucial in such case.[1] Poisson–Lie duality exchanges roles of $\mathcal{G}$ and $\widetilde{\mathcal{G}}$, and we understand it as a change of decomposition $(\mathcal{G}|\widetilde{\mathcal{G}})$ of $\mathcal{D}$ to $(\widetilde{\mathcal{G}}|\mathcal{G})$ and vice versa. Beside $(\mathcal{G}|\widetilde{\mathcal{G}})$ and $(\widetilde{\mathcal{G}}|\mathcal{G})$ there might be other decompositions $(\widehat{\mathcal{G}}|\bar{\mathcal{G}})$, $(\bar{\mathcal{G}}|\widehat{\mathcal{G}})$ of a Drinfel'd double $\mathcal{D}$ that can be used to construct mutually related sigma models. The corresponding transformation between sigma models was denoted Poisson–Lie T-plurality [17]. Decompositions of low-dimensional Drinfel'd doubles were classified in papers [18–20] in terms of Manin triples $(\mathfrak{d}, \mathfrak{g}, \tilde{\mathfrak{g}})$ that represent decompositions of Lie algebra $\mathfrak{d}$ of the Drinfel'd double $\mathcal{D}$ into subalgebras $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ corresponding to subgroups $\mathcal{G}$ and $\widetilde{\mathcal{G}}$.

In our recent paper [11] we noted that besides (non-) Abelian T-duality there exist other transformations that either preserve or exchange the algebras $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ of the Manin triple $(\mathfrak{d}, \mathfrak{g}, \tilde{\mathfrak{g}})$. We shall call them *Poisson–Lie identities* and *Poisson–Lie dualities*. Similar transformations were studied in [21] to get insight into the structure of the so-called NATD group of T-duality transformations. Beside others, this group contains automorphisms of the algebras forming Manin triples, B-shifts, β-shifts,[2] "factorized" dualities and their compositions. These, however, have to be understood as special cases of Poisson–Lie T-plurality. We continue the investigation of the NATD group probing its structure for low-dimensional Drinfel'd doubles, where general forms of Poisson–Lie identities and dualities can be identified. We show that all transformations are finite compositions of the special elements of NATD group that were mentioned earlier. It turns out that the effect of automorphisms and B-shifts on resulting backgrounds can be often eliminated by a change of coordinates, hence, we identify what parameters of the transformations are relevant.

[1] See [16] for discussion on this topic.

[2] β-shifts are also referred to as TsT transformations, see [22,23].

Long-lasting problem appearing in discussion of non-Abelian T-duality is that dualization with respect to non-semisimple group $\mathcal{G}$ leads to mixed gauge and gravitational anomaly, see [24], proportional to the trace of structure constants of $\mathfrak{g}$. Authors of paper [13] have found non-Abelian T-duals of Bianchi cosmologies [12] and have shown that instead of standard beta function equations the dual backgrounds satisfy the so-called Generalised Supergravity Equations containing Killing vector $\mathcal{J}$ whose components are given by the trace of structure constants. For Bianchi V cosmology this was observed already in [25]. Therefore, it is natural to ask if backgrounds and dilatons obtained from Bianchi cosmologies by Poisson–Lie identities and dualities satisfy Generalised Supergravity Equations as well and what Killing vectors have to be used. We show that components of $\mathcal{J}$ are determined by trace of structure constants, however, one must carefully inspect what groups (or subgroups of) $\mathcal{G}$ truly participate in the duality transformation. For Yang–Baxter deformations of $AdS_5 \times S^5$ the problem of finding $\mathcal{J}$ was addressed in [26,27] and physical meaning of $\mathcal{J}$ was found in [28]. Discussion on the Poisson–Lie duality and plurality of the Generalized Supergravity Equations can be found e.g. in [29] and [30].

We start with a short description of Poisson–Lie T-plurality in Sect. 2, where necessary formulas are summarized and general forms of transformed backgrounds are presented. In Sects. 3–7 we investigate various transformations of Bianchi cosmologies focusing on groups that are not semisimple. Since calculations with general transformations often result in rather complicated backgrounds that cannot be displayed, detailed description is given only for special elements of the NATD group. Summary of transformed backgrounds can be found in the Appendix.

2 Basics of Poisson–Lie T-plurality

In the first two subsections we recapitulate Poisson–Lie T-plurality with spectators [14,17,31]. We follow the summary given in [11].

2.1 Sigma models

Let $\mathcal{M}$ be $(n+d)$-dimensional (pseudo-)Riemannian target manifold and consider sigma model on $\mathcal{M}$ given by Lagrangian

$$\mathcal{L} = \partial_- \phi^\mu \mathcal{F}_{\mu\nu}(\phi) \partial_+ \phi^\nu, \qquad \phi^\mu = \phi^\mu(\sigma_+, \sigma_-),$$
$$\mu = 1, \ldots, n+d$$

where tensor field $\mathcal{F} = \mathcal{G} + \mathcal{B}$ on $\mathcal{M}$ defines metric and torsion potential (Kalb–Ramond field) of the target manifold. Assume that there is a d-dimensional Lie group $\mathcal{G}$ with free

action on $\mathscr{M}$ that leaves the tensor invariant. The action of $\mathscr{G}$ is transitive on its orbits, hence we may locally consider $\mathscr{M} \approx (\mathscr{M}/\mathscr{G}) \times \mathscr{G} = \mathscr{N} \times \mathscr{G}$, and introduce adapted coordinates

$$\{s_\alpha, x_a\}, \quad \alpha = 1, \ldots, n = \dim \mathscr{N}, \quad a = 1, \ldots, d = \dim \mathscr{G}$$

where s_α label the orbits of $\mathscr{G}$ and are treated as spectators, and x_a are group coordinates.[3] Dualizable sigma model on $\mathscr{N} \times \mathscr{G}$ is given by tensor field $\mathcal{F}$ defined by $(n+d) \times (n+d)$ matrix $E(s)$ as

$$\mathcal{F}(s, x) = \mathcal{E}(x) \cdot E(s) \cdot \mathcal{E}^T(x), \qquad \mathcal{E}(x) = \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & e(x) \end{pmatrix} \tag{1}$$

where $e(x)$ is $d \times d$ matrix of components of right-invariant Maurer–Cartan form $(dg)g^{-1}$ on $\mathscr{G}$.

Using non-Abelian T-duality one can find dual sigma model on $\mathscr{N} \times \mathscr{A}$, where $\mathscr{A}$ is Abelian subgroup of semi-Abelian Drinfel'd double $\mathscr{D} = (\mathscr{G}|\mathscr{A})$. The necessary formulas will be given in the following subsection as a special case of Poisson–Lie T-plurality. In this paper the groups $\mathscr{G}$ will be non-semisimple Bianchi groups. Bianchi cosmologies are defined on four-dimensional manifolds, hence $d = 3$, $n = 1$, and we denote the spectator as $t := s_1$. Elements of the group $\mathscr{G}$ shall be parametrized as $g = e^{x_1 T_1} e^{x_2 T_2} e^{x_3 T_3}$ where $e^{x_2 T_2} e^{x_3 T_3}$ and $e^{x_3 T_3}$ are normal subgroups of $\mathscr{G}$. Similarly, elements of $\mathscr{A}$ are parametrized as $\tilde{g} = e^{\tilde{x}_1 \widetilde{T}^1} e^{\tilde{x}_2 \widetilde{T}^2} e^{\tilde{x}_3 \widetilde{T}^3}$.

2.2 Poisson–Lie T-plurality with spectators

For certain Drinfel'd doubles several decompositions may exist. Suppose that $\mathscr{D} = (\mathscr{G}|\widetilde{\mathscr{G}})$ splits into another pair of subgroups $\widehat{\mathscr{G}}$ and $\bar{\mathscr{G}}$. Then we can apply the full framework of Poisson–Lie T-plurality [14,17] and find sigma model on $\mathscr{N} \times \widehat{\mathscr{G}}$.

The $2d$-dimensional Lie algebra $\mathfrak{d}$ of the Drinfel'd double $\mathscr{D}$ is equipped with an ad-invariant non-degenerate symmetric bilinear form $\langle ., . \rangle$. Let $\mathfrak{d} = \mathfrak{g} \bowtie \tilde{\mathfrak{g}}$ and $\mathfrak{d} = \hat{\mathfrak{g}} \bowtie \bar{\mathfrak{g}}$ be two decompositions (Manin triples $(\mathfrak{d}, \mathfrak{g}, \tilde{\mathfrak{g}})$ and $(\mathfrak{d}, \hat{\mathfrak{g}}, \bar{\mathfrak{g}})$) of $\mathfrak{d}$ into double cross sum of subalgebras [34] that are maximally isotropic with respect to $\langle ., . \rangle$. The pairs of mutually dual bases $T_a \in \mathfrak{g}$, $\widetilde{T}^a \in \tilde{\mathfrak{g}}$ and $\widehat{T}_a \in \hat{\mathfrak{g}}$, $\bar{T}^a \in \hat{\mathfrak{g}}$, $a = 1, \ldots, d$, satisfying

$$\begin{aligned} &\langle T_a, T_b \rangle = 0, \qquad \langle \widetilde{T}^a, \widetilde{T}^b \rangle = 0, \qquad \langle T_a, \widetilde{T}^b \rangle = \delta_a^b, \\ &\langle \widehat{T}_a, \widehat{T}_b \rangle = 0, \qquad \langle \bar{T}^a, \bar{T}^b \rangle = 0, \qquad \langle \widehat{T}_a, \bar{T}^b \rangle = \delta_a^b \end{aligned} \tag{2}$$

then must be related by transformation

$$\begin{pmatrix} \widehat{T} \\ \bar{T} \end{pmatrix} = C \cdot \begin{pmatrix} T \\ \widetilde{T} \end{pmatrix} \tag{3}$$

where C is an invertible $2d \times 2d$ matrix. Due to ad-invariance of the bilinear form $\langle ., . \rangle$ the algebraic structure of $\mathfrak{d}$ is given both by

$$\begin{aligned} &[T_i, T_j] = f_{ij}^k T_k, \qquad [\widetilde{T}^i, \widetilde{T}^j] = \tilde{f}_k^{ij} \widetilde{T}^k, \\ &[T_i, \widetilde{T}^j] = f_{ki}^j \widetilde{T}^k + \tilde{f}_i^{jk} T_k \end{aligned} \tag{4}$$

and

$$\begin{aligned} &[\widehat{T}_i, \widehat{T}_j] = \hat{f}_{ij}^k \widehat{T}_k, \qquad [\bar{T}^i, \bar{T}^j] = \bar{f}_k^{ij} \bar{T}^k, \\ &[\widehat{T}_i, \bar{T}^j] = \hat{f}_{ki}^j \bar{T}^k + \bar{f}_i^{jk} \widehat{T}_k. \end{aligned} \tag{5}$$

Given the structure constants F_{ij}^k of $\mathfrak{d} = \mathfrak{g} \bowtie \tilde{\mathfrak{g}}$ and $\widehat{F}_{ij}^k$ of $\mathfrak{d} = \hat{\mathfrak{g}} \bowtie \bar{\mathfrak{g}}$, the matrix C has to satisfy equation

$$C_a^p C_b^q F_{pq}^r = \widehat{F}_{ab}^c C_c^r. \tag{6}$$

To preserve the bilinear form $\langle ., . \rangle$ and thus (2), C also has to satisfy

$$C_a^p C_b^q (D_0)_{pq} = (D_0)_{ab} \tag{7}$$

where $(D_0)_{ab}$ are components of matrix D_0 that can be written in block form as

$$D_0 = \begin{pmatrix} \mathbf{0}_d & \mathbf{1}_d \\ \mathbf{1}_d & \mathbf{0}_d \end{pmatrix}. \tag{8}$$

In other words, C is an element of $O(d, d)$ but, unlike the case of Abelian T-duality, not every element of $O(d, d)$ is allowed in (3).

For the following formulas it will be convenient to introduce $d \times d$ matrices P, Q, R, S as

$$\begin{pmatrix} T \\ \widetilde{T} \end{pmatrix} = C^{-1} \cdot \begin{pmatrix} \widehat{T} \\ \bar{T} \end{pmatrix} = \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \cdot \begin{pmatrix} \widehat{T} \\ \bar{T} \end{pmatrix} \tag{9}$$

and extend these to $(n + d) \times (n + d)$ matrices

$$\mathcal{P} = \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & P \end{pmatrix}, \qquad \mathcal{Q} = \begin{pmatrix} \mathbf{0}_n & 0 \\ 0 & Q \end{pmatrix}, \qquad \mathcal{R} = \begin{pmatrix} \mathbf{0}_n & 0 \\ 0 & R \end{pmatrix}, \qquad \mathcal{S} = \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & S \end{pmatrix}$$

to accommodate the spectator fields. It is also advantageous to introduce block form of $E(s)$ as

$$E(s) = \begin{pmatrix} E_{\alpha\beta}(s) & E_{\alpha b}(s) \\ E_{a\beta}(s) & E_{ab}(s) \end{pmatrix}, \qquad \alpha, \beta = 1, \ldots, n, \quad a, b = 1, \ldots, d.$$

The sigma model on $\mathscr{N} \times \widehat{\mathscr{G}}$ related to (1) via Poisson–Lie T-plurality is given by tensor

[3] Detailed discussion of the process of finding adapted coordinates can be found e.g. in [31–33].

$$\widehat{\mathcal{F}}(s,\hat{x}) = \widehat{\mathcal{E}}(\hat{x}) \cdot \widehat{E}(s,\hat{x}) \cdot \widehat{\mathcal{E}}^T(\hat{x}), \qquad \widehat{\mathcal{E}}(\hat{x}) = \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & \widehat{e}(\hat{x}) \end{pmatrix} \tag{10}$$

where $\widehat{e}(\hat{x})$ is $d \times d$ matrix of components of right-invariant Maurer–Cartan form $(d\hat{g})\hat{g}^{-1}$ on $\widehat{\mathscr{G}}$,

$$\begin{aligned} &\widehat{E}(s,\hat{x}) = (\mathbf{1}_{n+d} + \widehat{E}(s) \cdot \widehat{\Pi}(\hat{x}))^{-1} \cdot \widehat{E}(s) \\ &= (\widehat{E}^{-1}(s) + \widehat{\Pi}(\hat{x}))^{-1}, \\ &\widehat{\Pi}(\hat{x}) = \begin{pmatrix} \mathbf{0}_n & 0 \\ 0 & \widehat{b}(\hat{x}) \cdot \widehat{a}^{-1}(\hat{x}) \end{pmatrix}, \end{aligned} \tag{11}$$

and matrices $\widehat{b}(\hat{x})$ and $\widehat{a}(\hat{x})$ are submatrices of the adjoint representation

$$ad_{\hat{g}^{-1}}(\widehat{T}) = \widehat{b}(\hat{x}) \cdot \bar{T} + \widehat{a}^{-1}(\hat{x}) \cdot \widehat{T}.$$

The matrix $\widehat{E}(s)$ is obtained by formula

$$\widehat{E}(s) = (\mathcal{P} + E(s) \cdot \mathcal{R})^{-1} \cdot (\mathcal{Q} + E(s) \cdot \mathcal{S}) \tag{12}$$

so it is necessary[4] that

$$\det\,(\mathcal{P} + E(s) \cdot \mathcal{R}) \neq 0 \neq \det\,(\mathcal{Q} + E(s) \cdot \mathcal{S}).$$

Formulas (10)–(12) reduce to those for full Poisson–Lie duality if we choose $P = S = \mathbf{0}_d$ and $Q = R = \mathbf{1}_d$. Furthermore, for a semi-Abelian Drinfel'd double the well-known Buscher rules for (non-) Abelian T-duality are restored. If there are no spectators the plurality is called atomic.

2.3 Poisson–Lie identities and Poisson–Lie dualities

Let us now restrict our considerations to mappings (3) that preserve the Manin triple, i.e. $\hat{\mathfrak{g}} = \mathfrak{g}$, $\bar{\mathfrak{g}} = \tilde{\mathfrak{g}}$, and that satisfy (6) and (7). They are the Poisson–Lie identities. The Maurer–Cartan form $(dg)g^{-1}$ remains unchanged but $E(s)$ transforms as in (12). Moreover, for non-Abelian T-duality the algebra $\tilde{\mathfrak{g}}$ is Abelian, i.e. $\tilde{\mathfrak{g}} = \mathfrak{a}$, thus $\widehat{b}$ and $\widehat{\Pi}$ vanish, and we may write[5]

$$\widehat{\mathcal{F}}(s,x) = \mathcal{E}(x) \cdot \widehat{E}(s) \cdot \mathcal{E}^T(x).$$

Let us note that both backgrounds $\mathcal{F}(s,x)$ and $\widehat{\mathcal{F}}(s,x)$ are invariant with respect to the group $\mathscr{G}$.

For special transformations mentioned in the introduction we can further specify the resulting backgrounds. Namely, matrices

$$I_A = \begin{pmatrix} A & 0 \\ 0 & A^{-T} \end{pmatrix} \tag{13}$$

are always among the transformations (3) preserving the Manin triple $\mathfrak{d} = \mathfrak{g} \bowtie \mathfrak{a}$ if A is an automorphism[6] of $\mathfrak{g}$. Transformed $\widehat{E}(s)$ then reads

$$\widehat{E}(s) = \mathcal{A} \cdot E(s) \cdot \mathcal{A}^T, \qquad \mathcal{A} = \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & A \end{pmatrix}.$$

Transformations (3) of the form

$$I_B = \begin{pmatrix} \mathbf{1}_d & B \\ 0 & \mathbf{1}_d \end{pmatrix}, \qquad B^T = -B \tag{14}$$

are called B-shifts, since the background $\widehat{\mathcal{F}}(s,x)$ obtained by this transformation is given by

$$\widehat{E}(s) = \left(E(s) - \bar{B}\right), \qquad \bar{B} = \begin{pmatrix} \mathbf{0}_n & 0 \\ 0 & B \end{pmatrix}.$$

Tensor $\widehat{\mathcal{F}}$ differs from the original one by an antisymmetric term $\mathcal{B}' = -\mathcal{E}(x) \cdot \bar{B} \cdot \mathcal{E}(x)^T$ that, however, for solvable Bianchi algebras does not produce supplementary torsion. Therefore, for all investigated Bianchi cosmologies $\widehat{\mathcal{F}}$ is gauge equivalent to the initial tensor $\mathcal{F}$.

β-shifts are generated by transformation matrices

$$I_\beta = \begin{pmatrix} \mathbf{1}_d & 0 \\ \beta & \mathbf{1}_d \end{pmatrix}, \qquad \beta^T = -\beta$$

and the transformed $\widehat{\mathcal{F}}(s,x)$ is given by

$$\widehat{E}(s) = \left(\mathbf{1} - E(s) \cdot \bar{\beta}\right)^{-1} \cdot E(s), \qquad \bar{\beta} = \begin{pmatrix} \mathbf{0}_n & 0 \\ 0 & \beta \end{pmatrix}.$$

For invertible $E(s)$, we may write $\widehat{E}(s) = (E(s)^{-1} - \bar{\beta})^{-1}$.

Beside these transformations we may also encounter mappings I_F that switch some of the basis vectors $T_i \leftrightarrow \widetilde{T}_i$ while preserving structure coefficients of the Manin triple. These "factorized" dualities can be interpreted as dualization with respect to subgroups of $\mathscr{G}$. In general, these cannot be written concisely in block form and we do not discuss them here. We shall see many examples in the following sections.

Let us further investigate Poisson–Lie dualities, i.e. mappings (3) that change Manin triple $\mathfrak{d} = \mathfrak{g} \bowtie \mathfrak{a}$ to $\mathfrak{d} = \mathfrak{a} \bowtie \mathfrak{g}$. Equation (3) implies that

$$\begin{pmatrix} \widetilde{T} \\ T \end{pmatrix} = D_0 \cdot \begin{pmatrix} T \\ \widetilde{T} \end{pmatrix} = D_0 \cdot I \cdot \begin{pmatrix} T \\ \widetilde{T} \end{pmatrix}.$$

Therefore, Poisson–Lie dualities are composed of Poisson–Lie identities I and "full" T-duality D_0 that exchanges all generators of $\mathfrak{g}$ and $\mathfrak{a}$ as $T_i \leftrightarrow \widetilde{T}_i$ for $i = 1, \ldots, d$. In this way we can define dual B-shifts, dual β-shifts and dual automorphisms.

[4] Invertibility of $\widehat{E}(s)$ is not required in the first expression in (11) and only $\det\,(\mathcal{P} + E(s) \cdot \mathcal{R}) \neq 0$ is required. However, for regular $\widehat{E}(s)$ the formulas simplify.

[5] Since $\widehat{\mathscr{G}} = \mathscr{G}$ we omit the hat over group coordinates and write simply x.

[6] Our approach differs from [21], where the authors consider vector space isomorphisms of Lie algebra of the Drinfel'd double rather than automorphisms of a chosen Manin triple.

Backgrounds on $\widetilde{\mathscr{G}} = \mathscr{A}$ obtained from (1) by Poisson–Lie dualities have the form

$$\widetilde{\mathcal{F}}(s,\tilde{x}) = \left(\widehat{E}^{-1}(s) + \widetilde{\Pi}(\tilde{x})\right)^{-1}, \qquad \widetilde{\Pi}(\tilde{x}) = \begin{pmatrix} \mathbf{0}_n & 0 \\ 0 & \widetilde{b}(\tilde{x}) \end{pmatrix}$$

because $\widetilde{e}(\tilde{x}) = \widetilde{a}(\tilde{x}) = \mathbf{1}_d$. For solvable groups $\mathscr{G}$ we have $\widetilde{b}_{ab}(\tilde{x}) = f^c_{ab}\tilde{x}_c$.

It is possible to be more specific when we restrict to specific elements of the NATD group. In the presence of spectators, however, the formulas for Poisson–Lie dualities are quite complicated. Fortunately, we do not need their full form since $E_{a\beta}(s)$ and $E_{\alpha b}(s)$ in $E(s)$ vanish for the backgrounds discussed in the rest of the paper. Hence $\widehat{\mathcal{F}}_{\alpha\beta}(s) = \mathcal{F}_{\alpha\beta}(s)$ and plurality affects only $\mathcal{F}_{ab}(s,x)$. The transformations we are interested in, therefore, concern only the $\widehat{\mathcal{F}}_{ab}(s,\hat{x})$ block of the resulting background tensor $\widehat{\mathcal{F}}$.

For the dual B-shift

$$D_B = D_0 \cdot I_B = \begin{pmatrix} 0 & \mathbf{1}_d \\ \mathbf{1}_d & B \end{pmatrix}, \quad B^T = -B$$

and vanishing $E_{a\beta}(s)$, $E_{\alpha b}(s)$ the matrix $\left(\widehat{E}^{-1}(s) + \widetilde{\Pi}(\tilde{x})\right)$ has the form

$$\begin{pmatrix} E^{-1}_{\alpha\beta}(s) & 0 \\ 0 & E_{ab}(s) - B_{ab} + \widetilde{b}_{ab}(\tilde{x}) \end{pmatrix}.$$

For solvable groups $(\widetilde{b}(\tilde{x}) - B)_{ab} = f^c_{ab}\tilde{x}_c - B_{ab}$. As we shall see, for some groups this enables us to get rid of some parameters of B_{ab} by coordinate transformations.

General formulas for dual β-shift

$$D_\beta = D_0 \cdot I_\beta = \begin{pmatrix} \beta & \mathbf{1}_d \\ \mathbf{1}_d & 0 \end{pmatrix}, \quad \beta^T = -\beta$$

in the presence of spectators are complicated and not particularly illuminating. For vanishing $E_{a\beta}(s)$, $E_{\alpha b}(s)$ one gets

$$\widehat{E}(s) = \begin{pmatrix} E_{\alpha\beta}(s) & 0 \\ 0 & E^{-1}_{ab}(s) - \beta \end{pmatrix}.$$

Let us focus on the role of automorphisms I_A and their duals now. As conjectured in [21], it turns out that all Poisson–Lie identities and Poisson–Lie dualities are generated by automorphisms of Manin triples, B-shifts, β-shifts and factorised dualities. Moreover, in most of the examples of Bianchi cosmologies discussed later we find that the general C matrix in (3) splits as

$$C = I_{A_2} \cdot C' \cdot I_{A_1}$$

where C' is either I_B, I_β, I_F or their duals. Transformed backgrounds then can be written as

$$\widehat{\mathcal{F}}(s,\hat{x}) = \widehat{\mathcal{E}}(\hat{x}) \cdot \mathcal{A}_2 \cdot \widehat{E}'(s,\hat{x}) \cdot \mathcal{A}_2^T \cdot \widehat{\mathcal{E}}^T(\hat{x}) \tag{15}$$

where

$$\widehat{E}'(s,\hat{x}) = \left(\left(\mathcal{Q}' + \mathcal{A}_1 E(s)\mathcal{A}_1^T \mathcal{S}'\right)^{-1}\left(\mathcal{P}' + \mathcal{A}_1 E(s)\mathcal{A}_1^T \mathcal{R}'\right) + \mathcal{A}_2^T \widehat{\Pi}(\hat{x})\mathcal{A}_2\right)^{-1}$$

and $\mathcal{P}', \mathcal{Q}', \mathcal{R}', \mathcal{S}'$ are found from C'. As expected, these expressions can be interpreted as application of Poisson–Lie T-plurality on a background given by matrix $E'(s) = \mathcal{A}_1 E(s)\mathcal{A}_1^T$. Important is that $\mathcal{A}_2$ can be eliminated from (15) by suitable transformation of group coordinates. Using the transformation properties of the covariant tensor $\widehat{\mathcal{F}}$ we may try to integrate the Jacobi matrix

$$J^\mu_\lambda = \frac{\partial \hat{x}'^\mu}{\partial \hat{x}^\lambda} = \widehat{e}(\hat{x})^k_\lambda (A_2)^p_k (\widehat{e}^{-1}(\hat{x}))^\mu_p \tag{16}$$

to find coordinates $\hat{x}'$ such that (15) simplifies to

$$\widehat{\mathcal{F}}(s,\hat{x}) = \widehat{\mathcal{E}}(\hat{x}) \cdot \widehat{E}'(s,\hat{x}) \cdot \widehat{\mathcal{E}}^T(\hat{x}).$$

These transformations can be always found for Poisson–Lie dualities on semi-Abelian Drinfel'd double where $\tilde{\mathfrak{g}} = \mathfrak{a}$ and $\widehat{e}(\hat{x}) = \mathbf{1}$. In this case $J = A$, the transformation is linear and can be even combined with the coordinate shifts mentioned earlier for dual B-shifts.

2.4 Generalized supergravity equations and transformation of dilaton

One of the goals of this paper is to verify whether backgrounds obtained from Poisson–Lie identities and dualities satisfy Generalized Supergravity Equations of Motion. We adopt convention used in [13] so the equations read[7]

$$0 = R_{\mu\nu} - \frac{1}{4}H_{\mu\rho\sigma}H_\nu{}^{\rho\sigma} + \nabla_\mu X_\nu + \nabla_\nu X_\mu, \tag{17}$$

$$0 = -\frac{1}{2}\nabla^\rho H_{\rho\mu\nu} + X^\rho H_{\rho\mu\nu} + \nabla_\mu X_\nu - \nabla_\nu X_\mu, \tag{18}$$

$$0 = R - \frac{1}{12}H_{\rho\sigma\tau}H^{\rho\sigma\tau} + 4\nabla_\mu X^\mu - 4X_\mu X^\mu \tag{19}$$

where ∇ is covariant derivative and

$$X_\mu = \partial_\mu \Phi + \mathcal{J}^\nu \mathcal{F}_{\nu\mu}.$$

For vanishing vector $\mathcal{J}$ the usual one-loop beta function equations, i.e. conformal invariance conditions, are recovered.

Under (non-) Abelian T-duality dilaton transforms as

$$\widetilde{\Phi} = \Phi + \frac{1}{2}\ln\det M \tag{20}$$

where matrix M is given by "group block" E_{ab} of $E(s)$ and submatrices of adjoint representation as

$$M = (E_{ab}(s) + \widetilde{b}(\tilde{x}) \cdot \widetilde{a}^{-1}(\tilde{x}))^{-1} = (\widetilde{E}^{-1}_{ab}(s) + \widetilde{b}(\tilde{x}) \cdot \widetilde{a}^{-1}(\tilde{x}))^{-1}.$$

[7] We consider purely bosonic backgrounds.

The formula (20) can be utilized not only for "full" duality given by D_0, but also for factorized dualities. However, for successful application of this rule it is necessary to identify the dualized directions, meaning we have to consider only subgroups of $\mathscr{G}$ and corresponding submatrices $E_{ab}, \widetilde{a}, \widetilde{b}$.

For general Poisson–Lie T-plurality the dilaton transformation rule was given in [17] and further studied in [37]. In the current notation we can write it as

$$\begin{aligned}\widehat{\Phi}(s,\hat{x}) &= \Phi(s,x) - \frac{1}{2}\ln\left|\det\left(\left(N+\widehat{\Pi}(\hat{x})M\right)\widehat{a}(\hat{x})\right)\right| \\ &+ \frac{1}{2}\ln\left|\det\left(\left(\mathbf{1}+\Pi(x)E(s)\right)a(x)\right)\right|, \\ M &= \mathcal{S}^T\cdot E(s) - \mathcal{Q}^T, \qquad N = \mathcal{P}^T - \mathcal{R}^T E(s). \end{aligned} \tag{21}$$

From the two possible decompositions of elements of Drinfel'd double

$$\begin{aligned} &l = g(x)\tilde{h}(\tilde{x}) = \widehat{g}(\hat{x})\bar{h}(\bar{x}), \\ &l\in\mathscr{D}, \quad g\in\mathscr{G},\ \tilde{h}\in\widetilde{\mathscr{G}},\ \hat{g}\in\widehat{\mathscr{G}},\ \bar{h}\in\bar{\mathscr{G}} \end{aligned}$$

we can in principle express coordinates x in terms of $\hat{x}$ and $\bar{x}$. The expression is thus nonlocal in the sense that $\widehat{\Phi}$ may depend also on coordinates $\bar{x}$ of $\bar{\mathscr{G}}$. For Poisson–Lie identities we do not encounter this problem so it is plausible to use (21) to calculate dilatons corresponding to B-shifts and β-shifts. For semi-Abelian Drinfel'd double we find that dilaton does not change under B-shifts I_B, while under I_β it transforms as

$$\widehat{\Phi}(s,x) = \Phi(s,x) - \frac{1}{2}\ln\left|\det\mathbf{1} - \beta\cdot E_{ab}(s)\right|. \tag{22}$$

For duals $D_B = D_0\cdot I_B$ and $D_\beta = D_0\cdot I_\beta$ we get the correct dilaton by formula (20) applied on the dilaton and background obtained earlier from identities I_B and I_β.

3 Bianchi V cosmology

As a warm up we shall study the well-known Bianchi V cosmology. Let us consider six-dimensional semi-Abelian Drinfel'd double[8] $\mathscr{D} = (\mathscr{B}_V|\mathscr{A})$ whose Lie algebra $\mathfrak{d} = \mathfrak{b}_V \bowtie \mathfrak{a}$ is spanned by basis $(T_1, T_2, T_3, \widetilde{T}^1, \widetilde{T}^2, \widetilde{T}^3)$. The non-trivial commutation relations of the generators of $\mathfrak{b}_V$ are

$$[T_1, T_2] = T_2, \qquad [T_1, T_3] = T_3. \tag{23}$$

The group $\mathscr{B}_V$ is not semisimple and trace of its structure constants does not vanish.

[8] By $\mathscr{B}_V$, resp. $\mathfrak{b}_V$, we denote the Bianchi V group, resp. its Lie algebra. $\mathscr{A}$ and $\mathfrak{a}$ denote three dimensional Abelian group and its Lie algebra respectively. Similar notation will be used in the following sections.

The sigma model background[9] is given by metric ($\mathcal{B} = 0$ and $\mathcal{F} = \mathcal{G}$)

$$\mathcal{F}(t,x_1) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & t^2 & 0 & 0 \\ 0 & 0 & e^{2x_1}t^2 & 0 \\ 0 & 0 & 0 & e^{2x_1}t^2 \end{pmatrix}. \tag{24}$$

Left-invariant vector fields that satisfy (23) and generate symmetries of this background are

$$V_1 = \partial_{x_1} - x_2\,\partial_{x_2} - x_3\,\partial_{x_3}, \quad V_2 = \partial_{x_2}, \quad V_3 = \partial_{x_3}.$$

In fact, the background is flat and torsionless so the standard beta function equations are satisfied if we choose zero dilaton $\Phi = 0$. This background was studied already in [38], where it was first noticed that duals with respect to non-semisimple groups are not conformal. The related gravitational-gauge anomaly was later investigated in [39].

3.1 Poisson–Lie identities and dualities

Mappings C that preserve the algebraic structure of Manin triple $(\mathfrak{d}, \mathfrak{b}_V, \mathfrak{a})$ and generate Poisson–Lie identities are given by matrices

$$I_1 = \begin{pmatrix} 1 & c_{12} & c_{13} & -c_{12}c_{15} - c_{13}c_{16} & c_{15} & c_{16} \\ 0 & c_{22} & c_{23} & -c_{15}c_{22} - c_{16}c_{23} & 0 & 0 \\ 0 & c_{32} & c_{33} & -c_{15}c_{32} - c_{16}c_{33} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{c_{13}c_{32}-c_{12}c_{33}}{c_{22}c_{33}-c_{23}c_{32}} & \frac{c_{33}}{c_{22}c_{33}-c_{23}c_{32}} & \frac{c_{32}}{c_{23}c_{32}-c_{22}c_{33}} \\ 0 & 0 & 0 & \frac{c_{13}c_{22}-c_{12}c_{23}}{c_{23}c_{32}-c_{22}c_{33}} & \frac{c_{23}}{c_{23}c_{32}-c_{22}c_{33}} & \frac{c_{22}}{c_{22}c_{33}-c_{23}c_{32}} \end{pmatrix},$$

$$I_2 = \begin{pmatrix} -1 & c_{12} & c_{13} & c_{12}c_{15} + c_{13}c_{16} & c_{15} & c_{16} \\ 0 & 0 & 0 & c_{12}c_{25} + c_{13}c_{26} & c_{25} & c_{26} \\ 0 & 0 & 0 & c_{12}c_{35} + c_{13}c_{36} & c_{35} & c_{36} \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & \frac{c_{36}}{c_{25}c_{36}-c_{26}c_{35}} & \frac{c_{35}}{c_{26}c_{35}-c_{25}c_{36}} & \frac{c_{16}c_{35}-c_{15}c_{36}}{c_{26}c_{35}-c_{25}c_{36}} & 0 & 0 \\ 0 & \frac{c_{26}}{c_{26}c_{35}-c_{25}c_{36}} & \frac{c_{25}}{c_{25}c_{36}-c_{26}c_{35}} & \frac{c_{16}c_{25}-c_{15}c_{26}}{c_{25}c_{36}-c_{26}c_{35}} & 0 & 0 \end{pmatrix}.$$

One can see that for $c_{15} = c_{16} = 0$ the matrix I_1 simplifies to the block form (13) given by automorphisms of algebra $\mathfrak{b}_V$ that in general read

$$A = \begin{pmatrix} 1 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix}. \tag{25}$$

For $c_{12} = c_{13} = c_{23} = c_{32} = 0$ and $c_{22} = c_{33} = 1$ matrix I_1 reduces to B-shift (14) of the form

$$I_B = \begin{pmatrix} 1 & 0 & 0 & 0 & c_{15} & c_{16} \\ 0 & 1 & 0 & -c_{15} & 0 & 0 \\ 0 & 0 & 1 & -c_{16} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{26}$$

[9] $E(s)$ is restored from $\mathcal{F}(s,x)$ by setting group coordinates to zero.

On the other hand, for $c_{12} = c_{13} = c_{26} = c_{35} = 0$, $c_{25} = c_{36} = 1$ matrix I_2 equals to

$$I_F = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}. \tag{27}$$

Matrix I_F switches basis vectors T_2, T_3 and $\widetilde{T}^2$, $\widetilde{T}^3$. We identify its action as factorized duality with respect to Abelian subgroup generated by T_2, T_3. The change of sign of T_1 is necessary for being an automorphism of $\mathfrak{b}_V \bowtie \mathfrak{a}$.

To study models generated by I_1 and I_2 we decompose these matrices into product of special elements of NATD group. Namely, we note that I_1 can be written as

$$I_1 = I_A \cdot I_B$$

where I_A is given by (25) and I_B is the B-shift (26). Similarly, I_2 can be decomposed as

$$I_2 = I_{A_2} \cdot I_F \cdot I_{A_1}$$

for automorphisms A_1 and A_2 of the form (25). This decomposition is not unique. To identify relevant parameters of I_2 we choose the simplest possible I_{A_1} while including the rest of the parameters in I_{A_2} as follows:

$$A_1 = \begin{pmatrix} 1 & -c_{12} & -c_{13} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 1 & c_{15} & c_{16} \\ 0 & c_{25} & c_{26} \\ 0 & c_{35} & c_{36} \end{pmatrix}. \tag{28}$$

Matrices generating Poisson–Lie dualities can be obtained from those above by left-multiplication by matrix (8) representing canonical or "full" duality. This way we get dual automorphisms generated by

$$D_A = D_0 \cdot I_A = \begin{pmatrix} \mathbf{0}_d & (A^T)^{-1} \\ A & \mathbf{0}_d \end{pmatrix},$$

dual B-shifts generated by

$$D_B = D_0 \cdot I_B = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & c_{15} & c_{16} \\ 0 & 1 & 0 & -c_{15} & 0 & 0 \\ 0 & 0 & 1 & -c_{16} & 0 & 0 \end{pmatrix},$$

and factorized duality

$$D_F = D_0 \cdot I_F = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{29}$$

that can be interpreted as Buscher duality with respect to T_1 accompanied by a change of sign in the dual coordinate.

3.2 Transformed backgrounds

3.2.1 B-shifts

Let us now apply Poisson–Lie identities on the sigma model (24). Plugging I_1 into formulas (9)–(12) we get rather complicated background tensor. Nevertheless, I_1 decomposes as $I_1 = I_A \cdot I_B$ and we can get rid of the parameters that come from I_A by a change of coordinates found by integrating the Jacobi matrix (16). Indeed, after coordinate transformation

$$y_1 = x_1, \quad y_2 = -c_{12}e^{-x_1} + c_{22}x_2 + c_{32}x_3,$$
$$y_3 = -c_{13}e^{-x_1} + c_{23}x_2 + c_{33}x_3$$

we find that the symmetric part of $\widehat{\mathcal{F}}$ equals to the original metric (24). The antisymmetric part

$$\widehat{\mathcal{B}}(y_1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -e^{y_1}c_{15} & -e^{y_1}c_{16} \\ 0 & e^{y_1}c_{15} & 0 & 0 \\ 0 & e^{y_1}c_{16} & 0 & 0 \end{pmatrix}$$

generated by the B-shift represents a torsionless B-field. Up to coordinate transformations we would get the same background using I_B instead of the full I_1 so, from the point of view of Poisson–Lie identity, we consider these matrices equivalent. In other words, Poisson–Lie identity with respect to I_B and I_1 is just a gauge transformation of the original background, there is no change in the dilaton field, and $\widehat{\Phi} = \Phi$ satisfies beta function equations.

Background calculated using $D_1 = D_0 \cdot I_1$ is too extensive to be displayed. Nevertheless, a change of coordinates (16) simplifies it to the form that one would obtain using D_B. Subsequent coordinate shift

$$\tilde{x}_1 = c_{12}(\tilde{y}_2 - c_{15}) + c_{13}(\tilde{y}_3 - c_{16}) + \tilde{y}_1,$$
$$\tilde{x}_2 = c_{22}(\tilde{y}_2 - c_{15}) + c_{23}(\tilde{y}_3 - c_{16}),$$
$$\tilde{x}_3 = c_{32}(\tilde{y}_2 - c_{15}) + c_{33}(\tilde{y}_3 - c_{16})$$

that agrees with the discussion in Sect. 2.3 eliminates the parameters of D_1 completely, producing tensor

$$\widetilde{\mathcal{F}}(t,\tilde{y}_2,\tilde{y}_3) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{t^2}{t^4+\tilde{y}_2^2+\tilde{y}_3^2} & \frac{\tilde{y}_2}{t^4+\tilde{y}_2^2+\tilde{y}_3^2} & \frac{\tilde{y}_3}{t^4+\tilde{y}_2^2+\tilde{y}_3^2} \\ 0 & -\frac{\tilde{y}_2}{t^4+\tilde{y}_2^2+\tilde{y}_3^2} & \frac{t^4+\tilde{y}_3^2}{t^2\left(t^4+\tilde{y}_2^2+\tilde{y}_3^2\right)} & -\frac{\tilde{y}_2\tilde{y}_3}{t^2\left(t^4+\tilde{y}_2^2+\tilde{y}_3^2\right)} \\ 0 & -\frac{\tilde{y}_3}{t^4+\tilde{y}_2^2+\tilde{y}_3^2} & -\frac{\tilde{y}_2\tilde{y}_3}{t^2\left(t^4+\tilde{y}_2^2+\tilde{y}_3^2\right)} & \frac{t^4+\tilde{y}_2^2}{t^2\left(t^4+\tilde{y}_2^2+\tilde{y}_3^2\right)} \end{pmatrix}. \tag{30}$$

The same background can be obtained via full duality using D_0, and, as discussed in [38,39], it is not conformal. The standard beta function equations cannot be satisfied by any dilaton $\widetilde{\Phi}$. On the other hand, dilaton

$$\widetilde{\Phi}(t,\tilde{y}_2,\tilde{y}_3) = -\frac{1}{2}\ln\left(t^2\left(\tilde{y}_2^2+\tilde{y}_3^2+t^4\right)\right)$$

together with background (30) satisfy Generalized Supergravity Equations (17)–(19) if we choose $\mathcal{J} = (0,2,0,0)$. Components of the Killing vector $\mathcal{J}$ are given by trace of structure constants[10] of $\mathfrak{b}_V$ as $\mathcal{J}^a = f^i_{ai}$. The dilaton agrees with the formula (20), and we conclude that up to a coordinate transformation the background (30) found using D_B or D_1 is equivalent to non-Abelian T-dual investigated in [13].

3.2.2 *Factorized dualities*

Using $I_2 = I_{A_2}\cdot I_F\cdot I_{A_1}$ in formulas (9)–(12) we get background that can be brought to the form

$$\widehat{\mathcal{F}}(t,y_1) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & t^2 & -e^{y_1}c_{12} & -e^{y_1}c_{13} \\ 0 & e^{y_1}c_{12} & \frac{e^{2y_1}}{t^2} & 0 \\ 0 & e^{y_1}c_{13} & 0 & \frac{e^{2y_1}}{t^2} \end{pmatrix}$$

by coordinate transformation

$$y_1 = x_1,\quad y_2 = -c_{15}e^{-x_1} + c_{25}x_2 + c_{35}x_3,$$
$$y_3 = -c_{16}e^{-x_1} + c_{26}x_2 + c_{36}x_3$$

whose Jacobi matrix (16) is determined by A_2 in (28). The background differs from $\widehat{\mathcal{F}}$ calculated using I_F since I_{A_1} changes $E(s)$ before the factorized duality is applied. However, the only difference is in the antisymmetric part $\widehat{\mathcal{B}}$. For I_2 there is a torsionless B-field, while for I_F the B-field vanishes completely. The metric has vanishing scalar curvature but is not flat. Further coordinate transformation

$$t = \sqrt{-2u\,v+2u+z_3^2+z_4^2}, \qquad y_2 = u\,z_3,$$
$$y_1 = \frac{1}{2}\ln\left(\frac{-2u\,v+2u+z_3^2+z_4^2}{u^2}\right), \qquad y_3 = u\,z_4$$

brings it to the Brinkmann form of plane parallel wave [40] with

$$ds^2 = 2\frac{z_3^2+z_4^2}{u^2}du^2 + 2du\,dv + dz_3^2 + dz_4^2.$$

Corresponding dilaton follows from the formula (20) if the factorized duality (27) is interpreted as Buscher duality[11] with respect to two-dimensional Abelian subgroup generated by left-invariant fields $V_2 = \partial_{x_2}$, $V_3 = \partial_{x_3}$. Metric (24) is written in coordinates adapted to the action of this subgroup and for the duality given by I_F we can write

$$\widehat{\Phi}(t,x_1) = \frac{1}{2}\ln\det M = \frac{1}{2}\ln\det\begin{pmatrix} \frac{e^{2x_1}}{t^2} & 0 \\ 0 & \frac{e^{2x_1}}{t^2} \end{pmatrix}$$
$$= -\ln t^2 + 2y_1 = -2\ln u.$$

Dual dilaton for background given by I_2 is derived from the altered $E'(s) = \mathcal{A}_1 E(s)\mathcal{A}_1^T$ and differs from the previous expression by a constant. We again conclude that backgrounds found using I_F and I_2 differ only by a coordinate and gauge transformation and can be considered equivalent. They satisfy beta function equations, or Generalized Supergravity Equations (17)–(19) where $\mathcal{J}$ is zero vector.

Background obtained by Poisson–Lie transformation using matrix D_F has the form

$$\widetilde{\mathcal{F}}(t,\tilde{y}_2,\tilde{y}_3) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{1}{t^2\left(\tilde{y}_2^2+\tilde{y}_3^2+1\right)} & \frac{\tilde{y}_2}{\tilde{y}_2^2+\tilde{y}_3^2+1} & \frac{\tilde{y}_3}{\tilde{y}_2^2+\tilde{y}_3^2+1} \\ 0 & -\frac{\tilde{y}_2}{\tilde{y}_2^2+\tilde{y}_3^2+1} & \frac{t^2\left(\tilde{y}_3^2+1\right)}{\tilde{y}_2^2+\tilde{y}_3^2+1} & -\frac{t^2\tilde{y}_2\tilde{y}_3}{\tilde{y}_2^2+\tilde{y}_3^2+1} \\ 0 & -\frac{\tilde{y}_3}{\tilde{y}_2^2+\tilde{y}_3^2+1} & -\frac{t^2\tilde{y}_2\tilde{y}_3}{\tilde{y}_2^2+\tilde{y}_3^2+1} & \frac{t^2\left(\tilde{y}_2^2+1\right)}{\tilde{y}_2^2+\tilde{y}_3^2+1} \end{pmatrix}. \tag{31}$$

The same background is obtained using $D_2 = D_0\cdot I_2 = D_0\cdot I_{A_2}\cdot I_F\cdot I_{A_1}$ after change of coordinates

$$\tilde{x}_1 = c_{15}(\tilde{y}_2 - c_{12}) + c_{16}(\tilde{y}_3 - c_{13}) + \tilde{y}_1,$$
$$\tilde{x}_2 = c_{25}(\tilde{y}_2 - c_{12}) + c_{26}(\tilde{y}_3 - c_{13}),$$
$$\tilde{x}_3 = c_{35}(\tilde{y}_2 - c_{12}) + c_{36}(\tilde{y}_3 - c_{13}).$$

Thus, we are able to eliminate all parameters appearing in D_2. The background is torsionless and together with dilaton

$$\widetilde{\Phi}(t,\tilde{y}_2,\tilde{y}_3) = -\frac{1}{2}\ln\left(t^2\left(\tilde{y}_2^2+\tilde{y}_3^2+1\right)\right) \tag{32}$$

satisfies beta function equations, i.e. the Killing vector in the Generalized Supergravity Equations is zero. Explanation is that we can interpret the factorized duality (29) as Buscher duality of (24), this time with one-dimensional

[10] Compared to [13], in the present nomenclature the matrices $\mathcal{F}$ representing background tensors are transposed. This results in change of sign of $\mathcal{J}$.

[11] Followed by a change of sign in the spectator coordinate x_1.

Abelian subgroup generated by left-invariant field $V_1 = \partial_{x_1} - x_2\,\partial_{x_2} - x_3\,\partial_{x_3}$. In adapted coordinates $\{s_1, s_2, s_3, y_1\}$

$$t = s_1, \quad x_1 = y_1, \quad x_2 = s_2 e^{-y_1}, \quad x_3 = s_3 e^{-y_1},$$

where $V_1 = \partial_{y_1}$, the tensor (24) is manifestly invariant with respect to shifts in y_1 since

$$\mathcal{F}(s_1, s_2, s_3) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & s_1^2 & 0 & -s_1^2 s_2 \\ 0 & 0 & s_1^2 & -s_1^2 s_3 \\ 0 & -s_1^2 s_2 & -s_1^2 s_3 & s_1^2\left(s_2^2 + s_3^2 + 1\right) \end{pmatrix}.$$

Buscher duality with respect to y_1 then restores the tensor (31) and dilaton (32) agrees with formula (20).

To sum up, in this section we have shown that backgrounds emerging from general Poisson–Lie identities or dualities differ from those obtained from special elements of NATD group only by a coordinate or gauge transformation. From now on we shall display results for these special elements and only comment on the general cases.

4 Bianchi *III* cosmology

Several results for Bianchi *III* cosmology are similar to those for Bianchi *V*. The algebra $\mathfrak{d} = \mathfrak{b}_{III} \bowtie \mathfrak{a}$ of six-dimensional semi-Abelian Drinfel'd double $(\mathscr{B}_{III}|\mathscr{A})$ is spanned by basis $(T_1, T_2, T_3, \widetilde{T}^1, \widetilde{T}^2, \widetilde{T}^3)$. Non-trivial commutation relations of the generators of $\mathfrak{b}_{III}$ are

$$[T_1, T_3] = -T_3, \tag{33}$$

while $\mathfrak{a}$ is Abelian. The trace of structure constants does not vanish and group $\mathscr{B}_{III}$ is not semisimple. The background given by metric

$$\mathcal{F}(t, x_1) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & t^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & t^2 e^{-2x_1} \end{pmatrix} \tag{34}$$

is flat, torsionless, and invariant with respect to symmetries generated by left-invariant vector fields

$$V_1 = \partial_{x_1} + x_3\,\partial_{x_3}, \qquad V_2 = \partial_{x_2}, \qquad V_3 = \partial_{x_3}$$

satisfying (33). As the background is flat and torsionless the dilaton Φ can be chosen zero. Authors of [41] mention this background in their analysis and note that its non-Abelian dual does not satisfy the standard beta function equations.

4.1 Poisson–Lie identities and dualities

Table 1 summarizes all eight types of solutions of equations (6) and (7) with structure constants $F = \widehat{F}$. These give rise to Poisson–Lie identities and dualities of $(\mathscr{B}_{III}|\mathscr{A})$. Nevertheless, all the identities are composed of automorphisms (13) with

$$A = \begin{pmatrix} 1 & a_{12} & a_{13} \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}, \tag{35}$$

B-shifts of the form (26), and factorized dualities[12]

$$I_{F_1} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \tag{36}$$

and

$$I_{F_2} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{37}$$

Matrices generating Poisson–Lie dualities can be again obtained from those above by left-multiplication by the matrix (8) representing full duality.

4.2 Transformed backgrounds

4.2.1 B-shifts

Using I_B (26) in the formulas (9)–(12) we find that the background $\widehat{\mathcal{F}}$ has the same metric as the original model (34). In addition to that, a torsionless B-field

$$\widehat{\mathcal{B}}(x_1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -c_{15} & -c_{16} e^{-x_1} \\ 0 & c_{15} & 0 & 0 \\ 0 & c_{16} e^{-x_1} & 0 & 0 \end{pmatrix} \tag{38}$$

appears. This agrees with the interpretation of action of I_B as gauge transformation. There is no change in the dilaton and $\widehat{\Phi} = \Phi$. With the full solutions I_4 and I_8 we get the same background as for I_B. Indeed, both these matrices decompose as

$$I_4 = I_A \cdot I_B, \qquad I_8 = I_A \cdot I_B$$

with I_A given by (35). A linear change of coordinates (16) thus restores the metric (34) and torsionless B-field (38).[13]

Dual background calculated using matrix $D_B = D_0 \cdot I_B$ produces tensor

[12] I_A and I_B appear as special cases of I_4 and I_8, factorized dualities I_{F_1}, I_{F_2} and their composition appear in $I_1, I_2, I_3, I_5, I_6, I_7$ and their duals.

[13] The parameter c_{15} has to be replaced by $-\frac{c_{14}+c_{13}c_{16}}{c_{12}}$ for I_4.

Table 1 PLT-identities of Drinfel'd double $(\mathscr{B}_{III}|\mathscr{A})$

$(\mathscr{B}_{III}\|\mathscr{A})$	C matrix
I_1	$\begin{pmatrix} -1 & c_{12} & c_{13} & c_{14} & \frac{c_{14}-c_{13}c_{16}}{c_{12}} & c_{16} \\ 0 & 0 & 0 & \frac{c_{12}}{c_{52}} & \frac{1}{c_{52}} & 0 \\ 0 & 0 & 0 & c_{13}c_{36} & 0 & c_{36} \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & c_{52} & 0 & \frac{(c_{14}-c_{13}c_{16})c_{52}}{c_{12}} & 0 & 0 \\ 0 & 0 & \frac{1}{c_{36}} & \frac{c_{16}}{c_{36}} & 0 & 0 \end{pmatrix}$
I_2	$\begin{pmatrix} -1 & c_{12} & c_{13} & c_{14} & \frac{c_{14}-c_{13}c_{16}}{c_{12}} & c_{16} \\ 0 & c_{22} & 0 & \frac{(c_{14}-c_{13}c_{16})c_{22}}{c_{12}} & 0 & 0 \\ 0 & 0 & 0 & c_{13}c_{36} & 0 & c_{36} \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & \frac{c_{12}}{c_{22}} & \frac{1}{c_{22}} & 0 \\ 0 & 0 & \frac{1}{c_{36}} & \frac{c_{16}}{c_{36}} & 0 & 0 \end{pmatrix}$
I_3	$\begin{pmatrix} 1 & c_{12} & c_{13} & c_{14} & -\frac{c_{14}+c_{13}c_{16}}{c_{12}} & c_{16} \\ 0 & 0 & 0 & -\frac{c_{12}}{c_{52}} & \frac{1}{c_{52}} & 0 \\ 0 & 0 & c_{33} & -c_{16}c_{33} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & c_{52} & 0 & \frac{(c_{14}+c_{13}c_{16})c_{52}}{c_{12}} & 0 & 0 \\ 0 & 0 & 0 & -\frac{c_{13}}{c_{33}} & 0 & \frac{1}{c_{33}} \end{pmatrix}$
I_4	$\begin{pmatrix} 1 & c_{12} & c_{13} & c_{14} & -\frac{c_{14}+c_{13}c_{16}}{c_{12}} & c_{16} \\ 0 & c_{22} & 0 & \frac{(c_{14}+c_{13}c_{16})c_{22}}{c_{12}} & 0 & 0 \\ 0 & 0 & c_{33} & -c_{16}c_{33} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\frac{c_{12}}{c_{22}} & \frac{1}{c_{22}} & 0 \\ 0 & 0 & 0 & -\frac{c_{13}}{c_{33}} & 0 & \frac{1}{c_{33}} \end{pmatrix}$
I_5	$\begin{pmatrix} -1 & 0 & c_{13} & c_{13}c_{16} & c_{15} & c_{16} \\ 0 & 0 & 0 & 0 & \frac{1}{c_{52}} & 0 \\ 0 & 0 & 0 & c_{13}c_{36} & 0 & c_{36} \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & c_{52} & 0 & c_{15}c_{52} & 0 & 0 \\ 0 & 0 & \frac{1}{c_{36}} & \frac{c_{16}}{c_{36}} & 0 & 0 \end{pmatrix}$
I_6	$\begin{pmatrix} -1 & 0 & c_{13} & c_{13}c_{16} & c_{15} & c_{16} \\ 0 & c_{22} & 0 & c_{15}c_{22} & 0 & 0 \\ 0 & 0 & 0 & c_{13}c_{36} & 0 & c_{36} \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{c_{22}} & 0 \\ 0 & 0 & \frac{1}{c_{36}} & \frac{c_{16}}{c_{36}} & 0 & 0 \end{pmatrix}$
I_7	$\begin{pmatrix} 1 & 0 & c_{13} & -c_{13}c_{16} & c_{15} & c_{16} \\ 0 & 0 & 0 & 0 & \frac{1}{c_{52}} & 0 \\ 0 & 0 & c_{33} & -c_{16}c_{33} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & c_{52} & 0 & -c_{15}c_{52} & 0 & 0 \\ 0 & 0 & 0 & -\frac{c_{13}}{c_{33}} & 0 & \frac{1}{c_{33}} \end{pmatrix}$
I_8	$\begin{pmatrix} 1 & 0 & c_{13} & -c_{13}c_{16} & c_{15} & c_{16} \\ 0 & c_{22} & 0 & -c_{15}c_{22} & 0 & 0 \\ 0 & 0 & c_{33} & -c_{16}c_{33} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{c_{22}} & 0 \\ 0 & 0 & 0 & -\frac{c_{13}}{c_{33}} & 0 & \frac{1}{c_{33}} \end{pmatrix}$

$$\widetilde{\mathcal{F}}(t,\tilde{x}_3) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{t^2}{t^4+c_{15}^2t^2+(c_{16}-\tilde{x}_3)^2} & \frac{t^2c_{15}}{t^4+c_{15}^2t^2+(c_{16}-\tilde{x}_3)^2} & \frac{c_{16}-\tilde{x}_3}{t^4+c_{15}^2t^2+(c_{16}-\tilde{x}_3)^2} \\ 0 & -\frac{t^2c_{15}}{t^4+c_{15}^2t^2+(c_{16}-\tilde{x}_3)^2} & \frac{t^4+(c_{16}-\tilde{x}_3)^2}{t^4+c_{15}^2t^2+(c_{16}-\tilde{x}_3)^2} & \frac{c_{15}(\tilde{x}_3-c_{16})}{t^4+c_{15}^2t^2+(c_{16}-\tilde{x}_3)^2} \\ 0 & \frac{\tilde{x}_3-c_{16}}{t^4+c_{15}^2t^2+(c_{16}-\tilde{x}_3)^2} & \frac{c_{15}(\tilde{x}_3-c_{16})}{t^4+c_{15}^2t^2+(c_{16}-\tilde{x}_3)^2} & \frac{t^2+c_{15}^2}{t^4+c_{15}^2t^2+(c_{16}-\tilde{x}_3)^2} \end{pmatrix} \tag{39}$$

whose curvature and torsion do not vanish. We can get rid of the parameter c_{16} by shift in $\tilde{x}_3$, but c_{15} remains. As earlier, backgrounds calculated using $D_4 = D_0 \cdot I_4 = D_0 \cdot I_A \cdot I_B$ or $D_8 = D_0 \cdot I_8 = D_0 \cdot I_A \cdot I_B$ differ from $\widetilde{\mathcal{F}}$ only by a transformation of coordinates. For nonzero c_{15} the tensor $\widetilde{\mathcal{F}}$ is not the same as non-Abelian dual of (34) that can be found using D_0. Nevertheless, if we understand the duality with respect to $D_B = D_0 \cdot I_B$ as full duality applied to background changed by I_B, the correct dilaton can be found from (20) as

$$\widetilde{\Phi}(t,\tilde{x}_3) = -\frac{1}{2}\ln\left(t^4 + c_{15}^2t^2 + (c_{16}-\tilde{x}_3)^2\right). \tag{40}$$

Such $\widetilde{\Phi}$ satisfies the Generalized Supergravity Equations for Killing vector $\mathcal{J} = (0,-1,0,0)$ whose components are given by trace of structure constants of $\mathfrak{b}_{III}$ as suggested in [13].

4.2.2 Factorized dualities

Poisson–Lie identities (36) and (37) can be interpreted as Buscher dualities with respect to one-dimensional Abelian subgroups generated by left-invariant fields $V_3 = \partial_{x_3}$ resp. $V_2 = \partial_{x_2}$.

Dualization with respect to V_2 does not change $\mathcal{F}$ at all due to the form of the metric (34). The background is invariant with respect to I_{F_2}. Its dual given by $D_{F_2} = D_0 \cdot I_{F_2}$ needs to be understood as dual with respect to non-Abelian group generated by V_1, V_3 that is not semisimple. The background and dilaton are the same as for the full duality D_0. We can read them from (39), (40) setting $c_{15} = c_{16} = 0$. The same results, up to a coordinate or gauge transformation, are obtained for the full solutions I_3, I_7, see Table 1, and their duals D_3, D_7 since

$$I_3 = I_{A_2} \cdot I_{F_2} \cdot I_B \cdot I_{A_1}, \qquad I_7 = I_{A_2} \cdot I_{F_2} \cdot I_B.$$

Dualization with respect to V_3, i.e. Poisson–Lie identity I_{F_1}, produces metric

$$\widehat{\mathcal{F}}(t,x_1) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & t^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{e^{-2x_1}}{t^2} \end{pmatrix} \tag{41}$$

whose scalar curvature vanishes. In coordinates

$$t = \sqrt{z_3^2 - 2u(v-1)}, \qquad x_2 = z_4,$$
$$x_1 = -\frac{1}{2}\ln\left(\frac{z_3^2-2u(v-1)}{u^2}\right), \qquad x_3 = uz_3$$

it acquires the Brinkmann form of a plane parallel wave with

$$ds^2 = 2\frac{z_3^2}{u^2}du^2 + 2du\,dv + dz_3^2 + dz_4^2.$$

As expected, dilaton calculated via formula (20)

$$\widehat{\Phi}(t,x_1) = \frac{1}{2}\ln\det M = \frac{1}{2}\ln\det\begin{pmatrix} 1 & 0 \\ 0 & \frac{e^{-2x_1}}{t^2} \end{pmatrix}$$
$$= -\frac{1}{2}\ln\, t^2 - x_1$$

satisfies beta function equations, or Generalized Supergravity Equations with $\mathcal{J} = 0$, since we have dualized with respect to Abelian subgroup of $\mathscr{B}_{III}$. Poisson–Lie identities I_1, I_2, I_5 and I_6 decompose as

$$I_1 = I_{A_2} \cdot I_{F_1} \cdot I_{F_2} \cdot I_{A_1}, \qquad I_2 = I_{A_2} \cdot I_{F_1} \cdot I_B \cdot I_{A_1},$$
$$I_5 = I_{A_2} \cdot I_{F_1} \cdot I_{F_2} \cdot I_{A_1}, \qquad I_6 = I_{A_2} \cdot I_{F_1} \cdot I_B \cdot I_{A_1}.$$

Resulting backgrounds differ from (41) only by a change of coordinates and torsionless B-field of the form (38) and can be found in Table 3 in the Appendix.

Dual background produced by $D_{F_1} = D_0 \cdot I_{F_1}$ reads

$$\widetilde{\mathcal{F}}(t,\tilde{x}_3) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{1}{t^2(\tilde{x}_3^2+1)} & 0 & -\frac{\tilde{x}_3}{\tilde{x}_3^2+1} \\ 0 & 0 & 1 & 0 \\ 0 & \frac{\tilde{x}_3}{\tilde{x}_3^2+1} & 0 & \frac{t^2}{\tilde{x}_3^2+1} \end{pmatrix}.$$

Together with the dilaton

$$\widetilde{\Phi}(t,\tilde{x}_3) = -\frac{1}{2}\ln\left(t^2\left(\tilde{x}_3^2+1\right)\right)$$

found from (20) this background satisfies beta function equations. Factorized duality given by D_{F_1} can be once again interpreted as Buscher duality with respect to symmetry generated by V_1, V_2. The same result is obtained for $D_5 = D_0 \cdot I_5$. For D_1, D_2, D_6 the tensor $\widetilde{\mathcal{F}}$ and dilaton $\widetilde{\Phi}$ contain a parameter that cannot be eliminated by coordinate or gauge transformation. Interested reader may find its full form in Table 3 in the Appendix.

5 Bianchi VI_κ cosmology

Semi-Abelian Drinfel'd double $\mathscr{D} = (\mathscr{B}_{VI_\kappa}|\mathscr{A})$ has Lie algebra $\mathfrak{d} = \mathfrak{b}_{VI_\kappa} \bowtie \mathfrak{a}$ spanned by basis $(T_1, T_2, T_3, \widetilde{T}^1, \widetilde{T}^2, \widetilde{T}^3)$ and the nontrivial comutation relations of $\mathfrak{b}_{VI_\kappa}$ are[14]

$$[T_1, T_2] = \kappa\, T_2, \qquad [T_1, T_3] = T_3, \qquad \kappa \neq -1. \tag{42}$$

Trace of structure constants does not vanish and group $\mathscr{B}_{VI_\kappa}$ is not semisimple. In the parametrization used in [13] Bianchi VI_κ cosmology is given by metric

$$\mathcal{F}(t, x_1) = \begin{pmatrix} -e^{-4\Phi(t)} a_1(t)^2 a_2(t)^2 a_3(t)^2 & 0 & 0 & 0 \\ 0 & a_1(t)^2 & 0 & 0 \\ 0 & 0 & e^{2\kappa x_1} a_2(t)^2 & 0 \\ 0 & 0 & 0 & e^{2x_1} a_3(t)^2 \end{pmatrix} \tag{43}$$

where the functions $a_i(t)$ are

$$\begin{aligned} a_1(t) &= e^{\Phi(t)} \left(\frac{p_1}{\kappa+1}\right)^{\frac{\kappa^2+1}{(\kappa+1)^2}} e^{\frac{(\kappa-1)p_2 t}{2(\kappa+1)}} \sinh^{-\frac{\kappa^2+1}{(\kappa+1)^2}}(p_1 t), \\ a_2(t) &= e^{\Phi(t)} \left(\frac{p_1}{\kappa+1}\right)^{\frac{\kappa}{\kappa+1}} e^{\frac{p_2 t}{2}} \sinh^{-\frac{\kappa}{\kappa+1}}(p_1 t), \\ a_3(t) &= e^{\Phi(t)} \left(\frac{p_1}{\kappa+1}\right)^{\frac{1}{\kappa+1}} e^{\frac{-p_2 t}{2}} \sinh^{-\frac{1}{\kappa+1}}(p_1 t). \end{aligned} \tag{44}$$

The background is invariant with respect to symmetry generated by left-invariant vector fields

$$V_1 = \partial_{x_1} - \kappa\, x_2\, \partial_{x_2} - x_3 \partial_{x_3}, \qquad V_2 = \partial_{x_2}, \qquad V_3 = \partial_{x_3}$$

satisfying (42). For dilaton $\Phi(t) = c_1 t$ the beta function equations reduce to condition

$$c_1^2 = \frac{\left(\kappa^2 + \kappa + 1\right) p_1^2}{(\kappa+1)^2} - \frac{p_2^2}{4}.$$

The background is torsionless and for $c_1 = 0$ also Ricci flat.

5.1 Poisson–Lie identities and dualities

Poisson–Lie identities of Drinfel'd double $(\mathscr{B}_{VI_\kappa}|\mathscr{A})$ are given by matrices

$$I_1 = \begin{pmatrix} 1 & c_{12} & c_{13} & -c_{12}c_{15} - c_{13}c_{16} & c_{15} & c_{16} \\ 0 & c_{22} & 0 & -c_{15}c_{22} & 0 & 0 \\ 0 & 0 & c_{33} & -c_{16}c_{33} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\frac{c_{12}}{c_{22}} & \frac{1}{c_{22}} & 0 \\ 0 & 0 & 0 & -\frac{c_{13}}{c_{33}} & 0 & \frac{1}{c_{33}} \end{pmatrix}$$

$$I_2 = \begin{pmatrix} -1 & c_{12} & c_{13} & c_{12}c_{15} + c_{13}c_{16} & c_{15} & c_{16} \\ 0 & 0 & 0 & c_{12}c_{25} & c_{25} & 0 \\ 0 & 0 & 0 & c_{13}c_{36} & 0 & c_{36} \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & \frac{1}{c_{25}} & 0 & \frac{c_{15}}{c_{25}} & 0 & 0 \\ 0 & 0 & \frac{1}{c_{36}} & \frac{c_{16}}{c_{36}} & 0 & 0 \end{pmatrix}.$$

The algebra $\mathfrak{b}_{VI_\kappa}$ admits automorphisms (35) and matrices I_A of the form (13) are among the special cases of I_1. Clearly, I_1 is a product $I_1 = I_A \cdot I_B$ of automorphisms and B-shifts (26). Matrix I_2 can be written as $I_2 = I_{A_2} \cdot I_F \cdot I_{A_1}$ where I_F is the factorized duality (27) and I_{A_1}, I_{A_2} are given by automorphisms (28). Poisson–Lie dualities are obtained by multiplication by D_0.

5.2 Transformed backgrounds

5.2.1 B-shifts

Using I_1 directly in formulas (9)–(12) we get rather complicated background tensor. However, since I_1 splits as $I_1 = I_A \cdot I_B$, the dependence of $\widehat{\mathcal{F}}$ on the parameters appearing in I_A can be eliminated by transformation (16). The background obtained using I_1 is equivalent to that obtained by B-shift (26) and reads

$$\widehat{\mathcal{F}}(t, x_1) = \begin{pmatrix} -e^{-4\Phi(t)} a_1(t)^2 a_2(t)^2 a_3(t)^2 & 0 & 0 & 0 \\ 0 & a_1(t)^2 & -e^{\kappa x_1} c_{15} & -e^{x_1} c_{16} \\ 0 & e^{\kappa x_1} c_{15} & e^{2\kappa x_1} a_2(t)^2 & 0 \\ 0 & e^{x_1} c_{16} & 0 & e^{2x_1} a_3(t)^2 \end{pmatrix}.$$

Beside the original metric (43) we have obtained a torsionless B-field. Together with the original dilaton $\Phi(t_1) = c_1 t$ the background satisfies beta function equations.

Dual background $\widetilde{\mathcal{F}}$ calculated using $D_1 = D_0 \cdot I_1 = D_0 \cdot I_A \cdot I_B$ is again too complicated to display. Nevertheless, linear transformation of coordinates (16) followed by shift in $\tilde{y}_2, \tilde{y}_3$ simplifies the background to

[14] Note that for $\kappa = 0$, or $\kappa = 1$, these are comutation relations of $\mathfrak{b}_{III}$, or $\mathfrak{b}_V$, respectively. The case $\kappa = -1$ will be treated separately in Sect. 6.

$$\widetilde{\mathcal{F}}(t,\tilde{y}_2,\tilde{y}_3)=\begin{pmatrix} -e^{-4\Phi(t)}a_1(t)^2a_2(t)^2a_3(t)^2 & 0 & 0 & 0\\ 0 & \frac{a_2(t)^2a_3(t)^2}{\Delta} & \frac{\kappa a_3(t)^2\tilde{y}_2}{\Delta} & \frac{a_2(t)^2\tilde{y}_3}{\Delta}\\ 0 & -\frac{\kappa a_3(t)^2\tilde{y}_2}{\Delta} & \frac{a_1(t)^2a_3(t)^2+\tilde{y}_3^2}{\Delta} & -\frac{\kappa\tilde{y}_2\tilde{y}_3}{\Delta}\\ 0 & -\frac{a_2(t)^2\tilde{y}_3}{\Delta} & -\frac{\kappa\tilde{y}_2\tilde{y}_3}{\Delta} & \frac{a_1(t)^2a_2(t)^2+\kappa^2\tilde{y}_2^2}{\Delta} \end{pmatrix},$$

where

$$\Delta=a_1(t)^2a_2(t)^2a_3(t)^2+\kappa^2\tilde{y}_2^2a_3(t)^2+a_2(t)^2\tilde{y}_3^2.$$

These results are the same as results obtained by full duality D_0. Dual dilaton

$$\widetilde{\Phi}(t,\tilde{y}_2,\tilde{y}_3)=c_1t-\frac{1}{2}\ln\Delta$$

found from formula (20) satisfies the generalized supergravity equations (17)–(19) where components of Killing vector $\mathcal{J}=(0,\kappa+1,0,0)$ correspond to trace of structure constants of $\mathfrak{b}_{VI_\kappa}$. Dualization with respect to D_1 can be treated as canonical duality in spite of the fact that it contains also B-shifts and automorphisms.

5.2.2 *Factorized dualities*

Poisson–Lie identity I_F in (27) can be interpreted as Buscher duality with respect to two-dimensional Abelian subgroup generated by left-invariant fields $V_2=\partial_{x_2}$, $V_3=\partial_{x_3}$. Resulting curved background

$$\widehat{\mathcal{F}}(t,x_1)=\begin{pmatrix} -e^{-4\Phi(t)}a_1(t)^2a_2(t)^2a_3(t)^2 & 0 & 0 & 0\\ 0 & a_1(t)^2 & 0 & 0\\ 0 & 0 & \frac{e^{2\kappa x_1}}{a_2(t)^2} & 0\\ 0 & 0 & 0 & \frac{e^{2x_1}}{a_3(t)^2} \end{pmatrix} \tag{45}$$

and dilaton

$$\widehat{\Phi}(t,x_1)=c_1t+\frac{1}{2}\ln\left(\frac{e^{2(\kappa+1)x_1}}{a_2(t)^2a_3(t)^2}\right) \tag{46}$$

calculated by formula (20) satisfy beta function equations with vanishing Killing vector $\mathcal{J}$ since we have dualized with respect to Abelian group.

Poisson–Lie identity $I_2=I_{A_2}\cdot I_F\cdot I_{A_1}$ gives a background whose metric can be brought to the form (45) by coordinate transformation (16). There is also a torsionless B-field depending on constants coming from I_{A_1} that transforms $E(s)$ to $E'(s)=\mathcal{A}_1E(s)\mathcal{A}_1^T$. Dilaton found by (20) from $E'(s)$ differs from (46) only by a constant shift and we may conclude that results of duality with respect to I_2 deviate from those obtained by I_F only by coordinate and gauge transformation.

After a suitable coordinate transformation we find that both matrices $D_F=D_0\cdot I_F$ and $D_2=D_0\cdot I_{A_2}\cdot I_F\cdot I_{A_1}$ produce background

$$\widetilde{\mathcal{F}}(t,\tilde{y}_2,\tilde{y}_3)=\begin{pmatrix} -e^{-4\Phi(t)}a_1^2a_2^2a_3^2 & 0 & 0 & 0\\ 0 & \frac{1}{\Delta} & \frac{\kappa a_2^2\tilde{y}_2}{\Delta} & \frac{a_3^2\tilde{y}_3}{\Delta}\\ 0 & -\frac{\kappa a_2^2\tilde{y}_2}{\Delta} & \frac{a_2^2\left(a_1^2+a_3^2\tilde{y}_3^2\right)}{\Delta} & -\frac{\kappa a_2^2a_3^2\tilde{y}_2\tilde{y}_3}{\Delta}\\ 0 & -\frac{a_3^2\tilde{y}_3}{\Delta} & -\frac{\kappa a_2^2a_3^2\tilde{y}_2\tilde{y}_3}{\Delta} & \frac{a_3^2\left(a_1^2+\kappa^2a_2^2\tilde{y}_2^2\right)}{\Delta} \end{pmatrix}$$

where

$$\Delta=a_1(t)^2+\kappa^2a_2(t)^2\tilde{y}_2^2+a_3(t)^2\tilde{y}_3^2.$$

This background is the same as the one that would be obtained by performing Buscher duality with respect to symmetry generated by V_1. Dilaton

$$\widetilde{\Phi}(t,\tilde{y}_2,\tilde{y}_3)=c_1t-\frac{1}{2}\ln\Delta$$

satisfies ordinary beta function equations.

Let us note that results of this section hold also for $\kappa=0,1$, i.e. for Bianchi III and Bianchi V. Dualities with respect to these groups were treated in Sects. 3 and 4 with different initial backgrounds.

6 Bianchi VI_{-1} cosmology

For Bianchi VI_{-1} cosmology we shall consider Manin triple $(\mathfrak{d},\mathfrak{b}_{VI_{-1}},\mathfrak{a})$ whose algebraic structure is given by comutation relations (42) with $\kappa=-1$. Structure coefficients of Lie algebra $\mathfrak{b}_{VI_{-1}}$ are traceless and the group $\mathscr{B}_{VI_{-1}}$ is not semisimple. Metric has the form (43) with functions

$$a_1(t)=\sqrt{p_1}\exp\left(\left(\frac{e^{2p_2t}+p_1t}{2}\right)+\Phi(t)\right),$$
$$a_2(t)=a_3(t)=\sqrt{p_2}e^{\frac{p_2t}{2}+\Phi(t)} \tag{47}$$

and dilaton is again $\Phi(t)=c_1t$. The beta function equations are satisfied if

$$c_1^2=\frac{1}{4}(2p_1p_2+p_2^2).$$

6.1 Poisson–Lie identities and dualities

Poisson–Lie identities of Drinfel'd double $(\mathscr{B}_{VI_{-1}}|\mathscr{A})$ are given by matrices

$$I_1 = \begin{pmatrix} -1 & c_{12} & c_{13} & c_{12}c_{15}+c_{13}c_{16} & c_{15} & c_{16} \\ 0 & 0 & \frac{c_{32}}{c_{32}c_{56}-c_{36}c_{52}} & \frac{c_{16}c_{32}-c_{12}c_{36}}{c_{32}c_{56}-c_{36}c_{52}} & \frac{c_{36}}{c_{36}c_{52}-c_{32}c_{56}} & 0 \\ 0 & c_{32} & 0 & c_{15}c_{32}+c_{13}c_{36} & 0 & c_{36} \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & c_{52} & 0 & c_{15}c_{52}+c_{13}c_{56} & 0 & c_{56} \\ 0 & 0 & \frac{c_{52}}{c_{36}c_{52}-c_{32}c_{56}} & \frac{c_{16}c_{52}-c_{12}c_{56}}{c_{36}c_{52}-c_{32}c_{56}} & \frac{c_{56}}{c_{32}c_{56}-c_{36}c_{52}} & 0 \end{pmatrix}$$

and

$$I_2 = \begin{pmatrix} 1 & c_{12} & c_{13} & -c_{12}c_{15}-c_{13}c_{16} & c_{15} & c_{16} \\ 0 & \frac{c_{33}}{c_{33}c_{55}-c_{35}c_{53}} & 0 & \frac{c_{15}c_{33}-c_{13}c_{35}}{c_{35}c_{53}-c_{33}c_{55}} & 0 & \frac{c_{35}}{c_{35}c_{53}-c_{33}c_{55}} \\ 0 & 0 & c_{33} & -c_{16}c_{33}-c_{12}c_{35} & c_{35} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & c_{53} & -c_{16}c_{53}-c_{12}c_{55} & c_{55} & 0 \\ 0 & \frac{c_{53}}{c_{35}c_{53}-c_{33}c_{55}} & 0 & \frac{c_{15}c_{53}-c_{13}c_{55}}{c_{33}c_{55}-c_{35}c_{53}} & 0 & \frac{c_{55}}{c_{33}c_{55}-c_{35}c_{53}} \end{pmatrix}.$$

As special cases we find two types of automorphisms I_A and $I_{A'}$ given by

$$A = \begin{pmatrix} 1 & a_{12} & a_{13} \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}, \qquad A' = \begin{pmatrix} -1 & a_{12} & a_{13} \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{pmatrix},$$

B-shifts generated by matrix

$$I_B = \begin{pmatrix} 1 & 0 & 0 & 0 & b_{12} & b_{13} \\ 0 & 1 & 0 & -b_{12} & 0 & b_{23} \\ 0 & 0 & 1 & -b_{13} & -b_{23} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{48}$$

β-shifts

$$I_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \beta_{23} & 0 & 1 & 0 \\ 0 & -\beta_{23} & 0 & 0 & 0 & 1 \end{pmatrix},$$

and factorized dualities (27). To analyze results following from application of Poisson–Lie identities I_1 and I_2 it is helpful to find their decomposition into products of special elements of NATD group. Depending on the values of parameters the matrices can be written as

$$I_1 = \begin{cases} I_{A'} \cdot I_B \cdot I_\beta & \text{for } c_{56} \neq 0 \\ I_{A'} \cdot I_\beta \cdot I_B & \text{for } c_{56} = 0, c_{32} \neq 0 \\ I_A \cdot I_B \cdot I_F & \text{for } c_{56} = c_{32} = 0 \end{cases}$$

and

$$I_2 = \begin{cases} I_A \cdot I_B \cdot I_\beta & \text{for } c_{55} \neq 0 \\ I_A \cdot I_\beta \cdot I_B & \text{for } c_{55} = 0, c_{33} \neq 0 \\ I_{A'} \cdot I_B \cdot I_F & \text{for } c_{55} = c_{33} = 0 \end{cases}$$

for some I_A, $I_{A'}$, I_B, I_β and I_F. The parameters rising from I_A and $I_{A'}$ can be again eliminated by coordinate transformation (16). It is thus sufficient to discuss backgrounds obtained from I_B, I_β, I_F and their products. Multiplying these matrices by D_0 we get Poisson–Lie dualities.

6.2 Transformed backgrounds

6.2.1 B-shifts

Transformed background

$$\widehat{\mathcal{F}}(t, x_1) = \begin{pmatrix} -e^{-4\Phi(t)}a_1(t)^2a_2(t)^4 & 0 & 0 & 0 \\ 0 & a_1(t)^2 & -b_{12}e^{-x_1} & -b_{13}e^{x_1} \\ 0 & b_{12}e^{-x_1} & e^{-2x_1}a_2(t)^2 & -b_{23} \\ 0 & b_{13}e^{x_1} & b_{23} & e^{2x_1}a_2(t)^2 \end{pmatrix} \tag{49}$$

given by B-shift differs from original $\mathcal{F}$ by a torsionless B-field and together with the dilaton $\widehat{\Phi} = c_1 t$ satisfies beta function equations.

Coordinate shifts eliminate b_{12}, b_{13} in the dual obtained from $D_0 \cdot I_B$ so it reads

$$\widetilde{\mathcal{F}}(t, \tilde{y}_2, \tilde{y}_3) = \begin{pmatrix} -e^{-4\Phi(t)}a_1^2a_2^4 & 0 & 0 & 0 \\ 0 & \frac{a_2^4+b_{23}^2}{\Delta} & -\frac{\tilde{y}_2a_2^2+b_{23}\tilde{y}_3}{\Delta} & \frac{a_2^2\tilde{y}_3-b_{23}\tilde{y}_2}{\Delta} \\ 0 & \frac{a_2^2\tilde{y}_2-b_{23}\tilde{y}_3}{\Delta} & \frac{a_1^2a_2^2+\tilde{y}_3^2}{\Delta} & \frac{b_{23}a_1^2+\tilde{y}_2\tilde{y}_3}{\Delta} \\ 0 & -\frac{\tilde{y}_3a_2^2+b_{23}\tilde{y}_2}{\Delta} & \frac{\tilde{y}_2\tilde{y}_3-b_{23}a_1^2}{\Delta} & \frac{a_1^2a_2^2+\tilde{y}_2^2}{\Delta} \end{pmatrix}$$

where

$$\Delta = \left(a_2(t)^4 + b_{23}^2\right) a_1(t)^2 + a_2(t)^2 \left(\tilde{y}_2^2 + \tilde{y}_3^2\right).$$

The constant b_{23} remains. For dilaton

$$\widetilde{\Phi}(t, \tilde{y}_2, \tilde{y}_3) = c_1 t - \frac{1}{2} \ln \Delta$$

beta function equations are satisfied. Vanishing of vector $\mathcal{J}$ corresponds to the fact that structure constants of $\mathfrak{b}_{VI_{-1}}$ are traceless.

6.2.2 β-shifts

Background given by β-shift is

$$\widehat{\mathcal{F}}(t, x_1) = \begin{pmatrix} -e^{-4\Phi(t)}a_1(t)^2a_2(t)^4 & 0 & 0 & 0 \\ 0 & a_1(t)^2 & 0 & 0 \\ 0 & 0 & \frac{e^{-2x_1}a_2(t)^2}{\beta_{23}^2a_2(t)^4+1} & \frac{\beta_{23}a_2(t)^4}{\beta_{23}^2a_2(t)^4+1} \\ 0 & 0 & -\frac{\beta_{23}a_2(t)^4}{\beta_{23}^2a_2(t)^4+1} & \frac{e^{2x_1}a_2(t)^2}{\beta_{23}^2a_2(t)^4+1} \end{pmatrix} \tag{50}$$

and together with the dilaton calculated by formula (22)

$$\widehat{\Phi}(t) = c_1 t - \frac{1}{2} \ln \left(\beta_{23}^2 a_2(t)^4 + 1\right)$$

satisfy beta function equations. Although matrices I_B and I_β do not commute, backgrounds obtained from $I_B \cdot I_\beta$ and $I_\beta \cdot I_B$ are the same and differ from $\widehat{\mathcal{F}}$ in (50) only by a torsionless B-field.

The dual obtained from $D_0 \cdot I_\beta$ is

$$\widetilde{\mathcal{F}}(t,\tilde{x}_2,\tilde{x}_3) = \begin{pmatrix} -e^{-4\Phi(t)}a_1^2a_2^4 & 0 & 0 & 0 \\ 0 & \frac{a_2^2}{a_1^2a_2^2+\tilde{x}_2^2+\tilde{x}_3^2} & \frac{\beta_{23}a_2^2\tilde{x}_3-\tilde{x}_2}{a_1^2a_2^2+\tilde{x}_2^2+\tilde{x}_3^2} & \frac{\beta_{23}\tilde{x}_2a_2^2+\tilde{x}_3}{a_1^2a_2^2+\tilde{x}_2^2+\tilde{x}_3^2} \\ 0 & \frac{\beta_{23}\tilde{x}_3a_2^2+\tilde{x}_2}{a_1^2a_2^2+\tilde{x}_2^2+\tilde{x}_3^2} & \frac{a_1^2a_2^2+\left(\beta_{23}^2a_2^4+1\right)\tilde{x}_3^2}{a_2^2\left(a_1^2a_2^2+\tilde{x}_2^2+\tilde{x}_3^2\right)} & \frac{\left(\beta_{23}^2a_2^4+1\right)\tilde{x}_2\tilde{x}_3-\beta_{23}a_1^2a_2^4}{a_2^2\left(a_1^2a_2^2+\tilde{x}_2^2+\tilde{x}_3^2\right)} \\ 0 & \frac{\beta_{23}a_2^2\tilde{x}_2-\tilde{x}_3}{a_1^2a_2^2+\tilde{x}_2^2+\tilde{x}_3^2} & \frac{\beta_{23}a_1^2a_2^4+\left(\beta_{23}^2a_2^4+1\right)\tilde{x}_2\tilde{x}_3}{a_2^2\left(a_1^2a_2^2+\tilde{x}_2^2+\tilde{x}_3^2\right)} & \frac{a_1^2a_2^2+\left(\beta_{23}^2a_2^4+1\right)\tilde{x}_2^2}{a_2^2\left(a_1^2a_2^2+\tilde{x}_2^2+\tilde{x}_3^2\right)} \end{pmatrix}$$

and with dilaton

$$\widetilde{\Phi}(t,\tilde{x}_2,\tilde{x}_3) = c_1 t - \frac{1}{2}\ln\left(a_2(t)^2\left(a_1(t)^2\,a_2(t)^2+\tilde{x}_2^2+\tilde{x}_3^2\right)\right)$$

they satisfy beta function equations. Tensors $\widetilde{\mathcal{F}}$ arising from $D_0 \cdot I_B \cdot I_\beta$ and $D_0 \cdot I_\beta \cdot I_B$ are too extensive to be displayed here. Nevertheless, it is straightforward to calculate them and verify that together with corresponding dilatons they satisfy beta function equations.

$$\widetilde{\mathcal{F}}(t,\tilde{x}_2,\tilde{x}_3) = \begin{pmatrix} -e^{-4\Phi(t)}a_1^2a_2^4 & 0 & 0 & 0 \\ 0 & \frac{1}{\Delta} & -\frac{a_2^2\tilde{x}_2}{\Delta} & \frac{a_2^2\tilde{x}_3}{\Delta} \\ 0 & \frac{a_2^2\tilde{x}_2}{\Delta} & \frac{a_2^2\left(a_1^2+a_2^2\tilde{x}_3^2\right)}{\Delta} & \frac{a_2^4\tilde{x}_2\tilde{x}_3}{\Delta} \\ 0 & -\frac{a_2^2\tilde{x}_3}{\Delta} & \frac{a_2^4\tilde{x}_2\tilde{x}_3}{\Delta} & \frac{a_2^2\left(a_1^2+a_2^2\tilde{x}_2^2\right)}{\Delta} \end{pmatrix}$$

where

$$\Delta = a_1(t)^2 + a_2(t)^2\left(\tilde{x}_2^2+\tilde{x}_3^2\right).$$

This background and dilaton

$$\widetilde{\Phi}(t,\tilde{x}_2,\tilde{x}_3) = c_1 t - \frac{1}{2}\ln\Delta$$

satisfy beta function equations. Background

$$\widetilde{\mathcal{F}}(t,\tilde{y}_2,\tilde{y}_3) = \begin{pmatrix} -e^{-4\Phi(t)}a_1^2a_2^4 & 0 & 0 & 0 \\ 0 & \frac{b_{23}^2a_2^4+1}{\Delta} & -\frac{a_2^2\left(b_{23}\tilde{y}_3a_2^2+\tilde{y}_2\right)}{\Delta} & \frac{a_2^2\tilde{y}_3-b_{23}a_2^4\tilde{y}_2}{\Delta} \\ 0 & \frac{a_2^2\tilde{y}_2-b_{23}a_2^4\tilde{y}_3}{\Delta} & \frac{a_2^2\left(a_1^2+a_2^2\tilde{y}_3^2\right)}{\Delta} & \frac{a_2^4\left(b_{23}a_1^2+\tilde{y}_2\tilde{y}_3\right)}{\Delta} \\ 0 & -\frac{a_2^2\left(b_{23}\tilde{y}_2a_2^2+\tilde{y}_3\right)}{\Delta} & -\frac{a_2^4\left(b_{23}a_1^2-\tilde{y}_2\tilde{y}_3\right)}{\Delta} & \frac{a_2^2\left(a_1^2+a_2^2\tilde{y}_2^2\right)}{\Delta} \end{pmatrix}$$

6.2.3 *Factorized dualities*

Poisson–Lie identity (27), interpreted as Buscher duality with respect to symmetry generated by V_2 and V_3, produces metric (45) with $\kappa = -1$ and functions $a_2(t) = a_3(t)$ given by (47). Dilaton calculated by the formula (20)

$$\widehat{\Phi}(t) = c_1 t - \frac{1}{2}\ln a_2(t)^4 = -(c_1+p_2)\,t + const.$$

satisfies beta function equations. Background obtained from $I_B \cdot I_F$ differs from this $\widehat{\mathcal{F}}$ only by a torsionless B-field that is the same as in (49). Let us note that for $c_1 = -p_2$ the metric is Ricci flat.

Dual background produced by $D_0 \cdot I_F$ reads

$$\widetilde{\mathcal{F}}(t,\tilde{x}_2,\tilde{x}_3) = \begin{pmatrix} -e^{-4\Phi(t)}a_1^2a_2^4 & 0 & 0 & 0 \\ 0 & \frac{1}{\Delta} & -\frac{a_2^2\tilde{x}_2}{\Delta} & \frac{a_2^2\tilde{x}_3}{\Delta} \\ 0 & \frac{a_2^2\tilde{x}_2}{\Delta} & \frac{a_2^2\left(a_1^2+a_2^2\tilde{x}_3^2\right)}{\Delta} & \frac{a_2^4\tilde{x}_2\tilde{x}_3}{\Delta} \\ 0 & -\frac{a_2^2\tilde{x}_3}{\Delta} & \frac{a_2^4\tilde{x}_2\tilde{x}_3}{\Delta} & \frac{a_2^2\left(a_1^2+a_2^2\tilde{x}_2^2\right)}{\Delta} \end{pmatrix}$$

where

$$\Delta = \left(b_{23}^2a_2(t)^4+1\right)a_1(t)^2 + a_2(t)^2\left(\tilde{y}_2^2+\tilde{y}_3^2\right)$$

is obtained from $D_0 \cdot I_B \cdot I_F$ and with dilaton

$$\widetilde{\Phi}(t,\tilde{y}_2,\tilde{y}_3) = c_1 t - \frac{1}{2}\ln\Delta$$

it satisfies beta function equations.

7 Bianchi *II* cosmology

Lie algebra $\mathfrak{d} = \mathfrak{b}_{II} \bowtie \mathfrak{a}$ of the Drinfel'd double $\mathscr{D} = (\mathscr{B}_{II}|\mathscr{A})$ is spanned by basis $(T_1, T_2, T_3, \widetilde{T}^1, \widetilde{T}^2, \widetilde{T}^3)$ where nontrivial comutation relations of $\mathfrak{b}_{II}$ are

$$[T_2, T_3] = T_1. \tag{51}$$

Trace of structure constants is zero and group $\mathscr{B}_{II}$ is not semisimple.

Cosmology invariant with respect to symmetry generated by left-invariant vector fields

$$V_1 = \partial_{x_1}, \qquad V_2 = -x_3\,\partial_{x_1} + \partial_{x_2}, \qquad V_3 = \partial_{x_3}$$

satisfying (51) is given by the metric

$$\mathcal{F}(t,x_2) = \begin{pmatrix} -e^{-4\Phi(t)}a_1(t)^2a_2(t)^2a_3(t)^2 & 0 & 0 & 0 \\ 0 & a_1(t)^2 & 0 & a_1(t)^2x_2 \\ 0 & 0 & a_2(t)^2 & 0 \\ 0 & a_1(t)^2x_2 & 0 & a_1(t)^2x_2^2 + a_3(t)^2 \end{pmatrix} \tag{52}$$

where the functions $a_i(t)$ are

$$\begin{aligned} a_1(t) &= e^{\Phi(t)}\sqrt{\frac{p_1}{\cosh(p_1t)}}, \\ a_2(t) &= e^{\Phi(t)+\frac{p_2t}{2}}\sqrt{\cosh(p_1t)}, \\ a_3(t) &= e^{\Phi(t)+\frac{p_3t}{2}}\sqrt{\cosh(p_1t)} \end{aligned} \tag{53}$$

as in [13]. For dilaton $\Phi(t) = c_1\,t$ the beta function equations reduce to

$$4c_1^2 = p_3p_2 - p_1^2.$$

The background is torsionless and for $c_1 = 0$ also Ricci flat.

There are no factorized dualities satisfying (6) and (7). Poisson–Lie dualities can be obtained from identities by left-multiplication by the matrix (8) representing full duality.

7.2 Transformed backgrounds

We already know that if Poisson–Lie identity decomposes as in (54), coordinate transformations can eliminate parameters of I_A in the resulting backgrounds. Thus it is sufficient to investigate the effects of I_B, I_β and their products.

7.2.1 B-shifts

Structure coefficients of Manin triple $(\mathfrak{d}, \mathfrak{b}_{II}, \mathfrak{a})$ remain invariant under B-shift (48) that transforms the background (52) to

$$I_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \beta_{12} & \beta_{13} & 1 & 0 & 0 \\ -\beta_{12} & 0 & 0 & 0 & 1 & 0 \\ -\beta_{13} & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{55}$$

$$\widehat{\mathcal{F}}(t,x_2) = \begin{pmatrix} -e^{-4\Phi(t)}a_1^2a_2{}^2a_3{}^2 & 0 & 0 & 0 \\ 0 & a_1^2 & -b_{12} & a_1^2x_2 - b_{13} \\ 0 & b_{12} & a_2{}^2 & b_{12}x_2 - b_{23} \\ 0 & x_2a_1^2 + b_{13} & b_{23} - b_{12}x_2 & a_3{}^2 + a_1{}^2x_2^2 \end{pmatrix}.$$

7.1 Poisson–Lie identities and dualities

Unfortunately, we are not able to display general forms of matrices generating Poisson–Lie identities of Manin triple $(\mathfrak{d}, \mathfrak{b}_{II}, \mathfrak{a})$ because the expressions are too extensive. However, we were able to decompose them into products of automorphisms, B-shifts and β-shifts. To be more specific, all the solutions can be written in one of the two forms

$$I_1 = I_A \cdot I_B \cdot I_\beta, \qquad I_2 = I_A \cdot I_\beta \cdot I_B \tag{54}$$

where automorphisms have the form (13) with

$$A = \begin{pmatrix} \Lambda & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad \Lambda = \det\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix},$$

B-shifts are generated by matrix (48), and β-shifts are given by

Up to gauge transformation of the antisymmetric part it is equivalent to (52). Together with dilaton $\widehat{\Phi}(t) = c_1t$ the background satisfies beta function equations.

Dependence on b_{23} can be eliminated in background obtained from $D_B = D_0 \cdot I_B$ and we have

$$\widetilde{\mathcal{F}}(t,y_1) = \begin{pmatrix} -e^{-4\Phi(t)}a_1^2a_2^2a_3^2 & 0 & 0 & 0 \\ 0 & \frac{a_2^2a_3^2+\tilde{y}_1^2}{\Delta} & \frac{b_{12}a_3^2-b_{13}\tilde{y}_1}{\Delta} & \frac{b_{13}a_2^2+b_{12}\tilde{y}_1}{\Delta} \\ 0 & -\frac{b_{12}a_3^2+b_{13}\tilde{y}_1}{\Delta} & \frac{b_{13}^2+a_1^2a_3^2}{\Delta} & \frac{a_1^2\tilde{y}_1-b_{12}b_{13}}{\Delta} \\ 0 & \frac{b_{12}\tilde{y}_1-b_{13}a_2^2}{\Delta} & -\frac{\tilde{y}_1a_1^2+b_{12}b_{13}}{\Delta} & \frac{b_{12}^2+a_1^2a_2^2}{\Delta} \end{pmatrix}$$

where

$$\Delta = a_1(t)^2\left(a_2(t)^2a_3(t)^2 + \tilde{y}_1^2\right) + b_{13}^2a_2(t)^2 + b_{12}^2a_3(t)^2.$$

With dilaton

$$\widetilde{\Phi}(t,\tilde{y}_1) = c_1t - \frac{1}{2}\ln\Delta$$

given by (20) the beta function equations are satisfied. Results for the dual B-shift differ from the canonical dual obtained by D_0 not only by a shift in y_1 but also by other terms depending on b_{12}, b_{13}.

7.2.2 β-shifts

Let us now investigate the transformation of metric (52) given by β-shift (55). This Poisson–Lie identity generates

$$\widehat{\mathcal{F}}(t,x_2)=\begin{pmatrix} -e^{-4\Phi(t)}a_1^2a_2^2a_3^2 & 0 & 0 & 0\\ 0 & \frac{a_1^2}{\Delta} & \frac{a_1^2a_2^2\beta_{12}}{\Delta} & \frac{a_1^2(\beta_{13}a_3^2+x_2)}{\Delta}\\ 0 & -\frac{a_1^2a_2^2\beta_{12}}{\Delta} & \frac{a_2^2(a_1^2a_3^2\beta_{13}^2+1)}{\Delta} & -\frac{a_1^2a_2^2\beta_{12}(\beta_{13}a_3^2+x_2)}{\Delta}\\ 0 & \frac{a_1^2(x_2-a_3^2\beta_{13})}{\Delta} & \frac{a_1^2a_2^2\beta_{12}(x_2-a_3^2\beta_{13})}{\Delta} & \frac{(a_1^2a_2^2\beta_{12}^2+1)a_3^2+a_1^2x_2^2}{\Delta} \end{pmatrix}$$

where

$$\Delta=\left(a_2(t)^2\beta_{12}^2+a_3(t)^2\beta_{13}^2\right)a_1(t)^2+1.$$

Together with corresponding dilaton

$$\widehat{\Phi}(t)=c_1t-\frac{1}{2}\ln\Delta$$

the background satisfies beta function equations. Poisson–Lie identity I_B acting on this background adds a torsionless B-field, so we conclude that backgrounds obtained from $I_B\cdot I_\beta$ and I_β can be considered equivalent. Despite the fact that I_B and I_β do not commute, $\widehat{\mathcal{F}}$ obtained from $I_\beta\cdot I_B$ is exactly the same as for $I_B\cdot I_\beta$.

Dual background resulting from $D_\beta=D_0\cdot I_\beta$ is

$$\widetilde{\mathcal{F}}(t,\tilde{x}_1)=\begin{pmatrix} -e^{-4\Phi(t)}a_1^2a_2^2a_3^2 & 0 & 0 & 0\\ 0 & \frac{(a_3^2+\beta_{12}^2a_1^2\tilde{x}_1^2)a_2^2+(\beta_{13}^2a_1^2a_3^2+1)\tilde{x}_1^2}{a_1^2(a_2^2a_3^2+\tilde{x}_1^2)} & \frac{a_3^2(\beta_{13}\tilde{x}_1-\beta_{12}a_2^2)}{a_2^2a_3^2+\tilde{x}_1^2} & -\frac{a_2^2(\beta_{13}a_3^2+\beta_{12}\tilde{x}_1)}{a_2^2a_3^2+\tilde{x}_1^2}\\ 0 & \frac{a_3^2(\beta_{12}a_2^2+\beta_{13}\tilde{x}_1)}{a_2^2a_3^2+\tilde{x}_1^2} & \frac{a_3^2}{a_2^2a_3^2+\tilde{x}_1^2} & \frac{\tilde{x}_1}{a_2^2a_3^2+\tilde{x}_1^2}\\ 0 & \frac{a_2^2(\beta_{13}a_3^2-\beta_{12}\tilde{x}_1)}{a_2^2a_3^2+\tilde{x}_1^2} & -\frac{\tilde{x}_1}{a_2^2a_3^2+\tilde{x}_1^2} & \frac{a_2^2}{a_2^2a_3^2+\tilde{x}_1^2} \end{pmatrix}$$

The dilaton is

$$\widetilde{\Phi}(t,\tilde{x}_1)=c_1t-\frac{1}{2}\ln\left(a_1(t)^2\left(a_2(t)^2a_3(t)^2+\tilde{x}_1^2\right)\right)$$

and it is interesting that it does not depend on β_{12} and β_{13}. Together they satisfy beta function equations.

Dual backgrounds and dilatons found from $D_0\cdot I_B\cdot I_\beta$ and $D_0\cdot I_\beta\cdot I_B$ are too complicated to display and not particularly illuminating so we omit them here. Nevertheless, one can check that they satisfy beta function equations.

8 Conclusions

We have identified general forms of Poisson–Lie identities and Poisson–Lie dualities for six-dimensional semi-Abelian Manin triples $\mathfrak{d}=\mathfrak{b}\bowtie\mathfrak{a}$ where $\mathfrak{b}$'s are Bianchi algebras that generate isometries of Bianchi cosmologies. Subsequently we have managed decomposing both the Poisson–Lie identities and Poisson–Lie dualities into simple factors, namely automorphisms of Manin triples, B-shifts, β-shifts and "full" or "factorized" dualities. This supports the conjecture posed in [21] that NATD group is generated by these elements. Finally, we have used these decompositions to transform Bianchi cosmologies supplemented by dilaton fields [12]. For these transformations we used Poisson–Lie T-plurality and dilaton formula described in Sect. 2.

In this way we have obtained many new backgrounds and corresponding dilatons that solve the Generalized Supergravity Equations, and confirmed that the Killing vector $\mathcal{J}$ in Generalized Supergravity Equations is given by trace of structure constants [13]. One must, however, carefully evaluate what groups, more precisely what subgroups of Drinfel'd double, truly participate in the transformation since it influences resulting Killing vector. For factorized dualities these subgroups often become Abelian and the Generalized Supergravity Equations reduce to standard beta function equations. New backgrounds, dilatons and corresponding Killing vectors are summarized in the Tables in the Appendix. The backgrounds obtained by Poisson–Lie identities are again invariant with respect to Bianchi groups.

Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors' comment: Paper is self-contained, all results can be reproduced using information contained in the text.]

Funded by SCOAP[3].

Appendix

For reader's convenience we recapitulate backgrounds and dilatons yielded from Poisson–Lie identities and dualities in the following tables. We add vector $\mathcal{J}$ as well to indicate whether the backgrounds satisfy standard beta function equations (in which case $\mathcal{J} = 0$) or Generalized Supergravity Equations (17)–(19). In the first column we display which one of the special transformations was used to get the result. Automorphisms I_A are not mentioned. Nevertheless, since we want to include results obtained from general Poisson–Lie identities and dualities, some parameters appearing in the tensors may arise from automorphisms. We recommend to check details in previous sections (Tables 2, 3, 4, 5, 6).

Table 2 Results for Poisson–Lie identities and dualities of Bianchi II cosmology. Functions $a_i(t)$ are given by (53) and $\Phi(t) = c_1 t$

$\mathscr{B}_{II}$	Transformed backgrounds, dilatons and vectors $\mathcal{J}$
I_B	$\widehat{\mathcal{F}}(t,x_2) = \begin{pmatrix} -e^{-4\Phi(t)}a_1^2a_2^2a_3^2 & 0 & 0 & 0 \\ 0 & a_1^2 & -b_{12} & a_1^2x_2 - b_{13} \\ 0 & b_{12} & a_2^2 & b_{12}x_2 - b_{23} \\ 0 & x_2a_1^2 + b_{13} & b_{23} - b_{12}x_2 & a_3^2 + a_1^2x_2^2 \end{pmatrix}$ $\widehat{\Phi}(t) = c_1 t, \quad \mathcal{J} = 0$
D_B	$\widetilde{\mathcal{F}}(t,y_1) = \begin{pmatrix} -e^{-4\Phi(t)}a_1^2a_2^2a_3^2 & 0 & 0 & 0 \\ 0 & \frac{a_2^2a_3^2+\tilde{y}_1^2}{\Delta} & \frac{b_{12}a_3^2-b_{13}\tilde{y}_1}{\Delta} & \frac{b_{13}a_2^2+b_{12}\tilde{y}_1}{\Delta} \\ 0 & -\frac{b_{12}a_3^2+b_{13}\tilde{y}_1}{\Delta} & \frac{b_{13}^2+a_1^2a_3^2}{\Delta} & \frac{a_1^2\tilde{y}_1-b_{12}b_{13}}{\Delta} \\ 0 & \frac{b_{12}\tilde{y}_1-b_{13}a_2^2}{\Delta} & -\frac{\tilde{y}_1a_1^2+b_{12}b_{13}}{\Delta} & \frac{b_{12}^2+a_1^2a_2^2}{\Delta} \end{pmatrix}$ $\Delta = a_1(t)^2\left(a_2(t)^2a_3(t)^2 + \tilde{y}_1^2\right) + b_{13}^2a_2(t)^2 + b_{12}^2a_3(t)^2$ $\widetilde{\Phi}(t,\tilde{y}_1) = c_1 t - \frac{1}{2}\ln\Delta, \quad \mathcal{J} = 0$
I_β	$\widehat{\mathcal{F}}(t,x_2) = \begin{pmatrix} -e^{-4\Phi(t)}a_1^2a_2^2a_3^2 & 0 & 0 & 0 \\ 0 & \frac{a_1^2}{\Delta} & \frac{a_1^2a_2^2\beta_{12}}{\Delta} & \frac{a_1^2(\beta_{13}a_3^2+x_2)}{\Delta} \\ 0 & -\frac{a_1^2a_2^2\beta_{12}}{\Delta} & \frac{a_2^2(a_1^2a_3^2\beta_{13}^2+1)}{\Delta} & -\frac{a_1^2a_2^2\beta_{12}(\beta_{13}a_3^2+x_2)}{\Delta} \\ 0 & \frac{a_1^2(x_2-a_3^2\beta_{13})}{\Delta} & \frac{a_1^2a_2^2\beta_{12}(x_2-a_3^2\beta_{13})}{\Delta} & \frac{(a_1^2a_2^2\beta_{12}^2+1)a_3^2+a_1^2x_2^2}{\Delta} \end{pmatrix}$ $\Delta = \left(a_2(t)^2\beta_{12}^2 + a_3(t)^2\beta_{13}^2\right)a_1(t)^2 + 1$ $\widehat{\Phi}(t) = c_1 t - \frac{1}{2}\ln\Delta, \quad \mathcal{J} = 0$
D_β	$\widetilde{\mathcal{F}}(t,\tilde{x}_1) =$ $\begin{pmatrix} -e^{-4\Phi(t)}a_1^2a_2^2a_3^2 & 0 & 0 & 0 \\ 0 & \frac{(a_3^2+\beta_{12}^2a_1^2\tilde{x}_1^2)a_2^2+(\beta_{13}^2a_1^2a_3^2+1)\tilde{x}_1^2}{a_1^2(a_2^2a_3^2+\tilde{x}_1^2)} & \frac{a_3^2(\beta_{13}\tilde{x}_1-\beta_{12}a_2^2)}{a_2^2a_3^2+\tilde{x}_1^2} & -\frac{a_2^2(\beta_{13}a_3^2+\beta_{12}\tilde{x}_1)}{a_2^2a_3^2+\tilde{x}_1^2} \\ 0 & \frac{a_3^2(\beta_{12}a_2^2+\beta_{13}\tilde{x}_1)}{a_2^2a_3^2+\tilde{x}_1^2} & \frac{a_3^2}{a_2^2a_3^2+\tilde{x}_1^2} & \frac{\tilde{x}_1}{a_2^2a_3^2+\tilde{x}_1^2} \\ 0 & \frac{a_2^2(\beta_{13}a_3^2-\beta_{12}\tilde{x}_1)}{a_2^2a_3^2+\tilde{x}_1^2} & -\frac{\tilde{x}_1}{a_2^2a_3^2+\tilde{x}_1^2} & \frac{a_2^2}{a_2^2a_3^2+\tilde{x}_1^2} \end{pmatrix}$ $\widetilde{\Phi}(t,\tilde{x}_1) = c_1 t - \frac{1}{2}\ln\left(a_1(t)^2\left(a_2(t)^2a_3(t)^2 + \tilde{x}_1^2\right)\right), \quad \mathcal{J} = 0$

Table 3 Results for Poisson–Lie identities and dualities of Bianchi III cosmology and dilaton $\Phi = 0$

$\mathscr{B}_{III}$	Transformed backgrounds, dilatons and vectors $\mathcal{J}$
I_B	$\widehat{\mathcal{F}}(t,x_1)=\begin{pmatrix}-1&0&0&0\\0&t^2&-c_{15}&-c_{16}e^{-x_1}\\0&c_{15}&1&0\\0&c_{16}e^{-x_1}&0&t^2e^{-2x_1}\end{pmatrix}$, $\widetilde{\Phi}=0$, $\mathcal{J}=0$
D_B	$\widetilde{\mathcal{F}}(t,\tilde{x}_3)=\begin{pmatrix}-1&0&0&0\\0&\frac{t^2}{t^4+c_{15}^2t^2+\tilde{x}_3^2}&\frac{t^2c_{15}}{t^4+c_{15}^2t^2+\tilde{x}_3^2}&-\frac{\tilde{x}_3}{t^4+c_{15}^2t^2+\tilde{x}_3^2}\\0&-\frac{t^2c_{15}}{t^4+c_{15}^2t^2+\tilde{x}_3^2}&\frac{t^4+\tilde{x}_3^2}{t^4+c_{15}^2t^2+\tilde{x}_3^2}&\frac{c_{15}\tilde{x}_3}{t^4+c_{15}^2t^2+\tilde{x}_3^2}\\0&\frac{\tilde{x}_3}{t^4+c_{15}^2t^2+\tilde{x}_3^2}&\frac{c_{15}\tilde{x}_3}{t^4+c_{15}^2t^2+\tilde{x}_3^2}&\frac{t^2+c_{15}^2}{t^4+c_{15}^2t^2+\tilde{x}_3^2}\end{pmatrix}$, $\widetilde{\Phi}(t,\tilde{x}_3)=-\frac{1}{2}\ln\left(t^4+c_{15}^2t^2+\tilde{x}_3^2\right)$, $\mathcal{J}=(0,-1,0,0)$
I_{F_1}	$\widehat{\mathcal{F}}(t,x_1)=\begin{pmatrix}-1&0&0&0\\0&t^2&-c_{12}&-c_{13}e^{-x_1}\\0&c_{12}&1&0\\0&c_{13}e^{-x_1}&0&\frac{e^{-2x_1}}{t^2}\end{pmatrix}$, $\widehat{\Phi}(t,x_1)=-\ln t-x_1$, $\mathcal{J}=0$
D_{F_1}	$\widetilde{\mathcal{F}}(t,\tilde{x}_3)=\begin{pmatrix}-1&0&0&0\\0&\frac{1}{c_{15}^2+t^2(\tilde{x}_3^2+1)}&\frac{c_{15}}{c_{15}^2+t^2(\tilde{x}_3^2+1)}&-\frac{t^2\tilde{x}_3}{c_{15}^2+t^2(\tilde{x}_3^2+1)}\\0&-\frac{c_{15}}{c_{15}^2+t^2(\tilde{x}_3^2+1)}&\frac{t^2(\tilde{x}_3^2+1)}{c_{15}^2+t^2(\tilde{x}_3^2+1)}&\frac{c_{15}t^2\tilde{x}_3}{c_{15}^2+t^2(\tilde{x}_3^2+1)}\\0&\frac{t^2\tilde{x}_3}{c_{15}^2+t^2(\tilde{x}_3^2+1)}&\frac{c_{15}t^2\tilde{x}_3}{c_{15}^2+t^2(\tilde{x}_3^2+1)}&\frac{t^2(c_{15}^2+t^2)}{c_{15}^2+t^2(\tilde{x}_3^2+1)}\end{pmatrix}$, $\widetilde{\Phi}(t,\tilde{x}_3)=-\frac{1}{2}\ln\left(c_{15}^2+t^2\tilde{x}_3^2+t^2\right)$, $\mathcal{J}=0$

Table 4 Results for Poisson–Lie identities and dualities of Bianchi V cosmology and dilaton $\Phi = 0$

$\mathscr{B}_{V}$	Transformed backgrounds, dilatons and vectors $\mathcal{J}$
I_B	$\widehat{\mathcal{F}}(t,x_1)=\begin{pmatrix}-1&0&0&0\\0&t^2&-e^{x_1}c_{15}&-e^{x_1}c_{16}\\0&e^{x_1}c_{15}&e^{2x_1}t^2&0\\0&e^{x_1}c_{16}&0&e^{2x_1}t^2\end{pmatrix}$, $\widehat{\Phi}=0$, $\mathcal{J}=0$
D_B	$\widetilde{\mathcal{F}}(t,\tilde{x}_2,\tilde{x}_3)=\begin{pmatrix}-1&0&0&0\\0&\frac{t^2}{t^4+\tilde{x}_2^2+\tilde{x}_3^2}&\frac{\tilde{x}_2}{t^4+\tilde{x}_2^2+\tilde{x}_3^2}&\frac{\tilde{x}_3}{t^4+\tilde{x}_2^2+\tilde{x}_3^2}\\0&-\frac{\tilde{x}_2}{t^4+\tilde{x}_2^2+\tilde{x}_3^2}&\frac{t^4+\tilde{x}_3^2}{t^2(t^4+\tilde{x}_2^2+\tilde{x}_3^2)}&-\frac{\tilde{x}_2\tilde{x}_3}{t^2(t^4+\tilde{x}_2^2+\tilde{x}_3^2)}\\0&-\frac{\tilde{x}_3}{t^4+\tilde{x}_2^2+\tilde{x}_3^2}&-\frac{\tilde{x}_2\tilde{x}_3}{t^2(t^4+\tilde{x}_2^2+\tilde{x}_3^2)}&\frac{t^4+\tilde{x}_2^2}{t^2(t^4+\tilde{x}_2^2+\tilde{x}_3^2)}\end{pmatrix}$, $\widetilde{\Phi}(t,\tilde{x}_2,\tilde{x}_3)=-\frac{1}{2}\ln\left(t^2\left(x_2^2+x_3^2+t^4\right)\right)$, $\mathcal{J}=(0,2,0,0)$
I_F	$\widehat{\mathcal{F}}(t,x_1)=\begin{pmatrix}-1&0&0&0\\0&t^2&-e^{x_1}c_{12}&-e^{x_1}c_{13}\\0&e^{x_1}c_{12}&\frac{e^{2x_1}}{t^2}&0\\0&e^{x_1}c_{13}&0&\frac{e^{2x_1}}{t^2}\end{pmatrix}$, $\widehat{\Phi}(t,x_1)=-2\ln t+2x_1$, $\mathcal{J}=0$
D_F	$\widetilde{\mathcal{F}}(t,\tilde{x}_2,\tilde{x}_3)=\begin{pmatrix}-1&0&0&0\\0&\frac{1}{t^2(\tilde{x}_2^2+\tilde{x}_3^2+1)}&\frac{\tilde{x}_2}{\tilde{x}_2^2+\tilde{x}_3^2+1}&\frac{\tilde{x}_3}{\tilde{x}_2^2+\tilde{x}_3^2+1}\\0&-\frac{\tilde{x}_2}{\tilde{x}_2^2+\tilde{x}_3^2+1}&\frac{t^2(\tilde{x}_3^2+1)}{\tilde{x}_2^2+\tilde{x}_3^2+1}&-\frac{t^2\tilde{x}_2\tilde{x}_3}{\tilde{x}_2^2+\tilde{x}_3^2+1}\\0&-\frac{\tilde{x}_3}{\tilde{x}_2^2+\tilde{x}_3^2+1}&-\frac{t^2\tilde{x}_2\tilde{x}_3}{\tilde{x}_2^2+\tilde{x}_3^2+1}&\frac{t^2(\tilde{x}_2^2+1)}{\tilde{x}_2^2+\tilde{x}_3^2+1}\end{pmatrix}$, $\widetilde{\Phi}(t,\tilde{x}_2,\tilde{x}_3)=-\frac{1}{2}\ln\left(t^2\left(x_2^2+x_3^2+1\right)\right)$, $\mathcal{J}=0$

Table 5 Results for Poisson–Lie identities and dualities of Bianchi VI_κ cosmology. Functions $a_i(t)$ are given by (44), $\kappa \neq -1$ and $\Phi(t) = c_1 t$

$\mathscr{B}_{VI_\kappa}$	Transformed backgrounds, dilatons and vectors $\mathcal{J}$
I_B	$\widehat{\mathcal{F}}(t,x_1) = \begin{pmatrix} -e^{-4\Phi(t)}a_1^2a_2^2a_3^2 & 0 & 0 & 0 \\ 0 & a_1^2 & -e^{\kappa x_1}c_{15} & -e^{x_1}c_{16} \\ 0 & e^{\kappa x_1}c_{15} & e^{2\kappa x_1}a_2^2 & 0 \\ 0 & e^{x_1}c_{16} & 0 & e^{2x_1}a_3^2 \end{pmatrix}$, $\widehat{\Phi}(t) = c_1 t, \quad \mathcal{J} = 0$
D_B	$\widetilde{\mathcal{F}}(t,\tilde{x}_2,\tilde{x}_3) = \begin{pmatrix} -e^{-4\Phi(t)}a_1^2a_2^2a_3^2 & 0 & 0 & 0 \\ 0 & \frac{a_2^2a_3^2}{\Delta} & \frac{\kappa a_3^2\tilde{x}_2}{\Delta} & \frac{a_2^2\tilde{x}_3}{\Delta} \\ 0 & -\frac{\kappa a_3^2\tilde{x}_2}{\Delta} & \frac{a_1^2a_3^2+\tilde{x}_3^2}{\Delta} & -\frac{\kappa\tilde{x}_2\tilde{x}_3}{\Delta} \\ 0 & -\frac{a_2^2\tilde{x}_3}{\Delta} & -\frac{\kappa\tilde{x}_2\tilde{x}_3}{\Delta} & \frac{a_1^2a_2^2+\kappa^2\tilde{x}_2^2}{\Delta} \end{pmatrix}$, $\Delta = a_1(t)^2a_2(t)^2a_3(t)^2 + \kappa^2\tilde{x}_2^2a_3(t)^2 + a_2(t)^2\tilde{x}_3^2$, $\widetilde{\Phi}(t,\tilde{x}_2,\tilde{x}_3) = c_1 t - \frac{1}{2}\ln\Delta, \quad \mathcal{J} = (0,\kappa+1,0,0)$
I_F	$\widehat{\mathcal{F}}(t,x_1) = \begin{pmatrix} -e^{-4\Phi(t)}a_1^2a_2^2a_3^2 & 0 & 0 & 0 \\ 0 & a_1^2 & -e^{\kappa x_1}c_{12} & -e^{x_1}c_{13} \\ 0 & e^{\kappa x_1}c_{12} & \frac{e^{2\kappa x_1}}{a_2^2} & 0 \\ 0 & e^{\kappa x_1}c_{13} & 0 & \frac{e^{2x_1}}{a_3^2} \end{pmatrix}$, $\widehat{\Phi}(t,x_1) = c_1 t + (\kappa+1)x_1 - \ln(a_2(t)a_3(t)), \quad \mathcal{J} = 0$
D_F	$\widetilde{\mathcal{F}}(t,\tilde{x}_2,\tilde{x}_3) = \begin{pmatrix} -e^{-4\Phi(t)}a_1^2a_2^2a_3^2 & 0 & 0 & 0 \\ 0 & \frac{1}{\Delta} & \frac{\kappa a_2^2\tilde{x}_2}{\Delta} & \frac{a_3^2\tilde{x}_3}{\Delta} \\ 0 & -\frac{\kappa a_2^2\tilde{x}_2}{\Delta} & \frac{a_2^2(a_1^2+a_3^2\tilde{x}_3^2)}{\Delta} & -\frac{\kappa a_2^2a_3^2\tilde{x}_2\tilde{x}_3}{\Delta} \\ 0 & -\frac{a_3^2\tilde{x}_3}{\Delta} & -\frac{\kappa a_2^2a_3^2\tilde{x}_2\tilde{x}_3}{\Delta} & \frac{a_3^2(a_1^2+\kappa^2a_2^2\tilde{x}_2^2)}{\Delta} \end{pmatrix}$, $\Delta = a_1(t)^2 + \kappa^2a_2(t)^2\tilde{x}_2^2 + a_3(t)^2\tilde{x}_3^2$, $\widetilde{\Phi}(t,\tilde{x}_2,\tilde{x}_3) = c_1 t - \frac{1}{2}\ln\Delta, \quad \mathcal{J} = 0$

Table 6 Results for Poisson–Lie identities and dualities of Bianchi VI_{-1} cosmology. Functions $a_i(t)$ are given by (47) and $\Phi(t) = c_1 t$

$\mathscr{B}_{VI_{-1}}$	Transformed backgrounds, dilatons and vectors $\mathcal{J}$
I_B	$\widehat{\mathcal{F}}(t,x_1) = \begin{pmatrix} -e^{-4\Phi(t)}a_1^2a_2^4 & 0 & 0 & 0 \\ 0 & a_1^2 & -b_{12}e^{-x_1} & -b_{13}e^{x_1} \\ 0 & b_{12}e^{-x_1} & e^{-2x_1}a_2^2 & -b_{23} \\ 0 & b_{13}e^{x_1} & b_{23} & e^{2x_1}a_2^2 \end{pmatrix}$, $\widehat{\Phi} = c_1 t, \quad \mathcal{J} = 0$
D_B	$\widetilde{\mathcal{F}}(t,\tilde{y}_2,\tilde{y}_3) = \begin{pmatrix} -e^{-4\Phi(t)}a_1^2a_2^4 & 0 & 0 & 0 \\ 0 & \frac{a_2^4+b_{23}^2}{\Delta} & -\frac{\tilde{y}_2a_2^2+b_{23}\tilde{y}_3}{\Delta} & \frac{a_2^2\tilde{y}_3-b_{23}\tilde{y}_2}{\Delta} \\ 0 & \frac{a_2^2\tilde{y}_2-b_{23}\tilde{y}_3}{\Delta} & \frac{a_1^2a_2^2+\tilde{y}_3^2}{\Delta} & \frac{b_{23}a_1^2+\tilde{y}_2\tilde{y}_3}{\Delta} \\ 0 & -\frac{\tilde{y}_3a_2^2+b_{23}\tilde{y}_2}{\Delta} & \frac{\tilde{y}_2\tilde{y}_3-b_{23}a_1^2}{\Delta} & \frac{a_1^2a_2^2+\tilde{y}_2^2}{\Delta} \end{pmatrix}$, $\Delta = \left(a_2(t)^4 + b_{23}^2\right)a_1(t)^2 + a_2(t)^2\left(\tilde{y}_2^2 + \tilde{y}_3^2\right)$, $\widetilde{\Phi}(t,\tilde{y}_2,\tilde{y}_3) = c_1 t - \frac{1}{2}\ln\Delta, \quad \mathcal{J} = 0$
I_β	$\widehat{\mathcal{F}}(t,x_1) = \begin{pmatrix} -e^{-4\Phi(t)}a_1^2a_2^4 & 0 & 0 & 0 \\ 0 & a_1^2 & 0 & 0 \\ 0 & 0 & \frac{e^{-2x_1}a_2^2}{\beta_{23}^2a_2^4+1} & \frac{\beta_{23}a_2^4}{\beta_{23}^2a_2^4+1} \\ 0 & 0 & -\frac{\beta_{23}a_2^4}{\beta_{23}^2a_2^4+1} & \frac{e^{2x_1}a_2^2}{\beta_{23}^2a_2^4+1} \end{pmatrix}$, $\widehat{\Phi}(t) = c_1 t - \frac{1}{2}\ln\left(\beta_{23}^2a_2(t)^4 + 1\right), \quad \mathcal{J} = 0$

Table 6 continued

$\mathscr{B}_{VI_{-1}}$	Transformed backgrounds, dilatons and vectors $\mathcal{J}$
D_β	$\widetilde{\mathcal{F}}(t,\tilde{x}_2,\tilde{x}_3) = \begin{pmatrix} -e^{-4\Phi(t)}a_1^2a_2^4 & 0 & 0 & 0 \\ 0 & \frac{a_2^2}{a_1^2a_2^2+\tilde{x}_2^2+\tilde{x}_3^2} & \frac{\beta_{23}a_2^2\tilde{x}_3-\tilde{x}_2}{a_1^2a_2^2+\tilde{x}_2^2+\tilde{x}_3^2} & \frac{\beta_{23}\tilde{x}_2a_2^2+\tilde{x}_3}{a_1^2a_2^2+\tilde{x}_2^2+\tilde{x}_3^2} \\ 0 & \frac{\beta_{23}\tilde{x}_3a_2^2+\tilde{x}_2}{a_1^2a_2^2+\tilde{x}_2^2+\tilde{x}_3^2} & \frac{a_1^2a_2^2+(\beta_{23}^2a_2^4+1)\tilde{x}_3^2}{a_2^2(a_1^2a_2^2+\tilde{x}_2^2+\tilde{x}_3^2)} & \frac{(\beta_{23}^2a_2^4+1)\tilde{x}_2\tilde{x}_3-\beta_{23}a_1^2a_2^4}{a_2^2(a_1^2a_2^2+\tilde{x}_2^2+\tilde{x}_3^2)} \\ 0 & \frac{\beta_{23}a_2^2\tilde{x}_2-\tilde{x}_3}{a_1^2a_2^2+\tilde{x}_2^2+\tilde{x}_3^2} & \frac{\beta_{23}a_1^2a_2^4+(\beta_{23}^2a_2^4+1)\tilde{x}_2\tilde{x}_3}{a_2^2(a_1^2a_2^2+\tilde{x}_2^2+\tilde{x}_3^2)} & \frac{a_1^2a_2^2+(\beta_{23}^2a_2^4+1)\tilde{x}_2^2}{a_2^2(a_1^2a_2^2+\tilde{x}_2^2+\tilde{x}_3^2)} \end{pmatrix}$ $\widetilde{\Phi}(t,\tilde{x}_2,\tilde{x}_3) = c_1t - \frac{1}{2}\ln\left(a_2(t)^2\left(a_1(t)^2\,a_2(t)^2+\tilde{x}_2^2+\tilde{x}_3^2\right)\right), \quad \mathcal{J}=0$
I_F	$\widehat{\mathcal{F}}(t,x_1) = \begin{pmatrix} -e^{-4\Phi(t)}a_1^2a_2^4 & 0 & 0 & 0 \\ 0 & a_1^2 & 0 & 0 \\ 0 & 0 & \frac{e^{-2x_1}}{a_2^2} & 0 \\ 0 & 0 & 0 & \frac{e^{2x_1}}{a_2^2} \end{pmatrix}$ $\widehat{\Phi}(t) = c_1t - \frac{1}{2}\ln a_2(t)^4, \quad \mathcal{J}=0$
D_F	$\widetilde{\mathcal{F}}(t,\tilde{x}_2,\tilde{x}_3) = \begin{pmatrix} -e^{-4\Phi(t)}a_1^2a_2^4 & 0 & 0 & 0 \\ 0 & \frac{1}{\Delta} & -\frac{a_2^2\tilde{x}_2}{\Delta} & \frac{a_2^2\tilde{x}_3}{\Delta} \\ 0 & \frac{a_2^2\tilde{x}_2}{\Delta} & \frac{a_2^2(a_1^2+a_2^2\tilde{x}_3^2)}{\Delta} & \frac{a_2^4\tilde{x}_2\tilde{x}_3}{\Delta} \\ 0 & -\frac{a_2^2\tilde{x}_3}{\Delta} & \frac{a_2^4\tilde{x}_2\tilde{x}_3}{\Delta} & \frac{a_2^2(a_1^2+a_2^2\tilde{x}_2^2)}{\Delta} \end{pmatrix}$ $\Delta = a_1(t)^2 + a_2(t)^2\left(\tilde{x}_2^2+\tilde{x}_3^2\right)$ $\widetilde{\Phi}(t,\tilde{x}_2,\tilde{x}_3) = c_1t - \frac{1}{2}\ln\Delta, \quad \mathcal{J}=0$

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