



Advantage of the second-order formalism in double space T-dualization of type II superstring

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Abstract In this article we present bosonic T-dualization in double space of the type II superstring theory in the pure spinor formulation. We use the action with constant background fields obtained from the general case under some physically and mathematically justified assumptions. Unlike Nikolić and Sazdović (EPJ C 77:197, 2017), where we used the first-order theory, in this article fermionic momenta are integrated out. Full T-dualization in double space is represented as a permutation of the initial x^μ and T-dual coordinates y_μ . Requiring that a T-dual transformation law of the T-dual double coordinate ${}^*Z^M = (y_\mu, x^\mu)$ to be of the same form as for initial one $Z^M = (x^\mu, y_\mu)$, we obtain the form of the T-dual background fields in terms of the initial ones. The advantage of using the action with integrated fermionic momenta is that it gives all T-dual background fields in terms of the initial ones. In the case of the first-order theory Nikolić and Sazdović (2017) a T-dual R-R field strength was obtained out of the double space formalism under additional assumptions.

1 Introduction

T-duality is a feature which cannot be met in the point-particle theory and represents a novelty brought about by string theory [2–9]. The basic mathematical framework, in which T-dualization is performed, is the Buscher procedure [5,6]. The starting point of the procedure is the existence of global isometries along some directions. In the next step we localize that symmetry introducing world-sheet covariant derivatives (instead of ordinary ones) and gauge fields. In order to make the gauge field an unphysical degree of freedom, a term with

Lagrange multipliers is added to the action. The final phase of the procedure is using gauge freedom to fix the initial coordinates. Variation of the gauge fixed action with respect to the Lagrange multipliers produces the initial action, while variation with respect to the gauge fields gives a T-dual action. Combining these equations of motion the relations connecting initial and T-dual coordinates are obtained. These relations are known in the literature as T-dual transformation laws.

Why is T-duality so important? The answer is concerned with M-theory. Five consistent superstring theories are connected by a web of T- and S-dualities. It is a well-known fact in the case of type II superstring theories that T-dualization along one spatial dimension transforms type IIA(B) to type IIB(A) theory, while T-dualization along the time-like direction produces type II* theory, of which the R-R field strength is the initial one multiplied by the imaginary unit [1,10]. Using double space enables one to unify all three theories. This could be a way toward better understanding M-theory.

The basic presumption for implementing the Buscher T-dualization procedure is the existence of global isometry along some directions. Effectively, it means that we can find the coordinate basis in which background fields do not depend on those directions [5–9,11,12].

Except the standard Buscher procedure, there is a generalized Buscher procedure dealing with T-dualization along directions on which the background fields depend. In the generalized Buscher procedure, to be compared with the standard one, an additional ingredient is present and that is an invariant coordinate, $x_{inv}^\mu = \int d\xi^\alpha D_\alpha x^\mu$, where D_α is a world-sheet covariant derivative. So far the generalized procedure was applied in two cases: the bosonic string moving in the weakly curved background [13–15] and the case where the metric is quadratic in the coordinates and the Kalb–Ramond field is a linear function of the coordinates [16]. In the first case isometry is not obvious but actually exists, while in the second case isometry is absent.

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The Buscher T-dualization procedure was used in Refs. [17–24] in the context of closed string noncommutativity. In these articles there was considered the coordinate dependent background—a constant metric and a Kalb–Ramond field with only one nonzero component, $B_{xy} = Hz$, where the field strength H is infinitesimal.

The Buscher T-dualization procedure can be considered as a definition of T-dualization. But there is a illuminating way of representing T-duality representation using double space and a permutation group. The name “double” comes from the way it is constructed. The double space coordinate Z^M consists of initial coordinates x^μ and their T-dual ones, y_μ , $Z^M = (x^\mu, y_\mu)$ ($\mu = 0, 1, 2, \dots, D - 1$). The formalism emerged about 20 years ago and it was addressed in Refs. [25–29]. In recent years the interest for this formalism was revived [30–37]. In these recent articles T-duality is related with $O(d, d)$ transformations. On the other hand, in Refs. [1, 25, 38–40], T-dualization along some subset of directions is represented as a permutation of that subset of initial coordinates and the corresponding T-dual ones. T-duality becomes a symmetry transformation in double space.

In Ref. [1] we demonstrated the equivalence of the Buscher approach and the double space one for type II superstring theory. But there is one detail which has to be emphasized. In the mentioned article we used the pure spinor type II superstring action with constant background fields in the form of the first-order theory, i.e., the fermionic momenta, π_α and $\bar{\pi}_\alpha$, are not integrated out. In that case the process of T-dualization is mathematically simple, but the price to pay is that the T-dual R-R field strength $P^{\alpha\beta}$ could not be obtained within the double space formalism. The reason is that the R-R field strength is coupled only with the fermionic degrees of freedom which are not dualized. To reproduce the Buscher form of the T-dual R-R field strength we made some additional assumptions.

In this article we will integrate out the momenta and obtain the theory in terms of the derivatives of the bosonic coordinates, x^μ , and the fermionic coordinates, θ^α and $\bar{\theta}^\alpha$. After the fermionic momenta are integrated out, the R-R field strength is coupled with $\partial_\pm x^\mu$. It turns out that Buscher T-dualization with such an action is slightly more complicated, but, as expected, gives the same result as in the case of the first-order theory. The mathematical framework for double space T-dualization is the same as in [1]. We rewrite the T-dual transformation laws in terms of the double space coordinates Z^M , introducing the generalized metric $\check{\mathcal{H}}_{MN}$, the generalized current $\check{J}_{\pm M}$ and the permutation matrix \mathcal{T}^M_N , which swaps the initial coordinates x^μ and T-dual ones y_μ . Requiring that the T-dual double space coordinates, $\star Z^M = \mathcal{T}^M_N Z^N$, satisfy the transformation law of the same form as the initial coordinates, Z^M , we obtain the expressions for the T-dual generalized metric, $\star\check{\mathcal{H}}_{MN} = (\mathcal{T}\check{\mathcal{H}}\mathcal{T})_{MN}$, and T-dual current, $\star\check{J}_{\pm M} = (\mathcal{T}\check{J}_\pm)_M$.

There is an advantage when we perform T-duality within the double space formalism. The main benefit of the using the action with integrated fermionic momenta is that we get all T-dual background fields.

2 Buscher T-dualization of type II superstring theory with integrated fermionic momenta

In this section we will introduce the type II superstring action in a pure spinor formulation [41–48] in the approximation of constant background fields and up to the quadratic terms. Then we will integrate out the fermionic momenta and apply the standard Buscher procedure. This leads to more complicated calculations, but in double space an advantage occurs.

2.1 Type II superstring in the pure spinor formulation

The general form of the action is borrowed from [49] and it is of the form

$$S = \int_\Sigma d^2\xi (X^T)^M A_{MN} \bar{X}^N + S_\lambda + S_{\bar{\lambda}}, \tag{2.1}$$

where the vectors X^M and \bar{X}^N are the left and right chiral supersymmetric variables

$$X^M = \begin{pmatrix} \partial_+ \theta^\alpha \\ \Pi_+^\mu \\ d_\alpha \\ \frac{1}{2} N_+^{\mu\nu} \end{pmatrix}, \quad \bar{X}^M = \begin{pmatrix} \partial_- \bar{\theta}^\alpha \\ \Pi_-^\mu \\ \bar{d}_\alpha \\ \frac{1}{2} \bar{N}_-^{\mu\nu} \end{pmatrix}, \tag{2.2}$$

of which the components are defined as

$$\begin{aligned} \Pi_+^\mu &= \partial_+ x^\mu + \frac{1}{2} \theta^\alpha (\Gamma^\mu)_{\alpha\beta} \partial_+ \theta^\beta, \\ \Pi_-^\mu &= \partial_- x^\mu + \frac{1}{2} \bar{\theta}^\alpha (\Gamma^\mu)_{\alpha\beta} \partial_- \bar{\theta}^\beta, \end{aligned} \tag{2.3}$$

$$\begin{aligned} d_\alpha &= \pi_\alpha - \frac{1}{2} (\Gamma_\mu \theta)_\alpha \left[\partial_+ x^\mu + \frac{1}{4} (\theta \Gamma_\mu \partial_+ \theta) \right], \\ \bar{d}_\alpha &= \bar{\pi}_\alpha - \frac{1}{2} (\Gamma_\mu \bar{\theta})_\alpha \left[\partial_- x^\mu + \frac{1}{4} (\bar{\theta} \Gamma_\mu \partial_- \bar{\theta}) \right], \end{aligned} \tag{2.4}$$

$$N_+^{\mu\nu} = \frac{1}{2} w_\alpha (\Gamma^{[\mu\nu]})^\alpha_\beta \lambda^\beta, \quad \bar{N}_-^{\mu\nu} = \frac{1}{2} \bar{w}_\alpha (\Gamma^{[\mu\nu]})^\alpha_\beta \bar{\lambda}^\beta. \tag{2.5}$$

The supermatrix A_{MN} is of the form

$$A_{MN} = \begin{pmatrix} A_{\alpha\beta} & A_{\alpha\nu} & E_\alpha^\beta & \Omega_{\alpha,\mu\nu} \\ A_{\mu\beta} & A_{\mu\nu} & \bar{E}_\mu^\beta & \Omega_{\mu,\nu\rho} \\ E_\beta^\alpha & E_\nu^\alpha & P^{\alpha\beta} & C^\alpha_{\mu\nu} \\ \Omega_{\mu\nu,\beta} & \Omega_{\mu\nu,\rho} & \bar{C}_{\mu\nu}^\beta & S_{\mu\nu,\rho\sigma} \end{pmatrix}. \tag{2.6}$$

The world sheet Σ is parameterized by $\xi^m = (\xi^0 = \tau, \xi^1 = \sigma)$ and $\partial_{\pm} = \partial_{\tau} \pm \partial_{\sigma}$. Superspace is spanned by the bosonic coordinates x^{μ} ($\mu = 0, 1, 2, \dots, 9$) and the fermionic ones θ^{α} and $\bar{\theta}^{\alpha}$ ($\alpha = 1, 2, \dots, 16$). The variables π_{α} and $\bar{\pi}_{\alpha}$ are canonically conjugate momenta to θ^{α} and $\bar{\theta}^{\alpha}$, respectively. The actions for the pure spinors, S_{λ} and $S_{\bar{\lambda}}$, are the free field actions

$$S_{\lambda} = \int d^2\xi w_{\alpha} \partial_{-} \lambda^{\alpha}, \quad S_{\bar{\lambda}} = \int d^2\xi \bar{w}_{\alpha} \partial_{+} \bar{\lambda}^{\alpha}, \quad (2.7)$$

where λ^{α} and $\bar{\lambda}^{\alpha}$ are pure spinors and w_{α} and \bar{w}_{α} are their canonically conjugate momenta, respectively. The pure spinors satisfy the so-called pure spinor constraints

$$\lambda^{\alpha} (\Gamma^{\mu})_{\alpha\beta} \lambda^{\beta} = \bar{\lambda}^{\alpha} (\Gamma^{\mu})_{\alpha\beta} \bar{\lambda}^{\beta} = 0. \quad (2.8)$$

This action (2.1) for type II superstring in the pure spinor formulation is general and it is constructed as an expansion in powers of θ^{α} and $\bar{\theta}^{\alpha}$ (for details see [49]).

Our plan is to implement full T-dualization, which means that we T-dualize along all bosonic directions x^{μ} . Consequently, we will assume that the background fields do not depend on them. On the other hand, because of the way how the action is constructed, for practical reasons (mathematical simplification), we will consider just the first components in the expansion in powers of θ^{α} and $\bar{\theta}^{\alpha}$. Effectively, this means that the nonzero background fields are constant. The background fields from the first and last columns and rows in the matrix A_{MN} are zero (a detailed explanation could be found in [1, 49]). The fields surviving these approximations are known in the literature as physical superfields because their first components are supergravity fields.

Finally, all our assumptions produce

$$\Pi_{\pm}^{\mu} \rightarrow \partial_{\pm} x^{\mu}, \quad d_{\alpha} \rightarrow \pi_{\alpha}, \quad \bar{d}_{\alpha} \rightarrow \bar{\pi}_{\alpha}, \quad (2.9)$$

where the physical superfields take the form

$$A_{\mu\nu} = \kappa \left(\frac{1}{2} g_{\mu\nu} + B_{\mu\nu} \right) + \frac{1}{4\pi} \eta_{\mu\nu} \Phi, \quad E_{\nu}^{\alpha} = -\Psi_{\nu}^{\alpha}, \\ \bar{E}_{\mu}^{\alpha} = \bar{\Psi}_{\mu}^{\alpha}, \quad P^{\alpha\beta} = \frac{1}{2\kappa} P^{\alpha\beta}. \quad (2.10)$$

Here $g_{\mu\nu}$ is a symmetric and $B_{\mu\nu}$ is an antisymmetric tensor. Consequently, the full action S is

$$S = \kappa \int_{\Sigma} d^2\xi \left[\partial_{+} x^{\mu} \Pi_{+\mu\nu} \partial_{-} x^{\nu} + \frac{1}{4\pi\kappa} \Phi R^{(2)} \right] \\ + \int_{\Sigma} d^2\xi \left[-\pi_{\alpha} \partial_{-} (\theta^{\alpha} + \Psi_{\mu}^{\alpha} x^{\mu}) \right. \\ \left. + \partial_{+} (\bar{\theta}^{\alpha} + \bar{\Psi}_{\mu}^{\alpha} x^{\mu}) \bar{\pi}_{\alpha} + \frac{1}{2\kappa} \pi_{\alpha} P^{\alpha\beta} \bar{\pi}_{\beta} \right], \quad (2.11)$$

where $G_{\mu\nu} = \eta_{\mu\nu} + g_{\mu\nu}$ is the metric tensor and

$$\Pi_{\pm\mu\nu} = B_{\mu\nu} \pm \frac{1}{2} G_{\mu\nu}. \quad (2.12)$$

We will neglect the Tseytlin term in the further analysis because, for a constant dilaton field Φ , it is proportional to the Euler characteristic. Consequently, on some given manifold that term is constant. The actions S_{λ} and $S_{\bar{\lambda}}$ are decoupled from the rest and the action, in its final form, is ghost independent.

2.2 Full bosonic T-dualization using Buscher rules

Let us now integrate out the fermionic momenta from the action (2.11) and obtain the theory expressed in terms of the supercoordinates $(x^{\mu}, \theta^{\alpha}, \bar{\theta}^{\alpha})$ and their world-sheet derivatives. For the equations of motion for fermionic momenta π_{α} and $\bar{\pi}_{\alpha}$,

$$\pi_{\alpha} = -2\kappa \partial_{+} (\bar{\theta}^{\beta} + \bar{\Psi}_{\mu}^{\beta} x^{\mu}) (P^{-1})_{\beta\alpha}, \\ \bar{\pi}_{\alpha} = 2\kappa (P^{-1})_{\alpha\beta} \partial_{-} (\theta^{\beta} + \Psi_{\mu}^{\beta} x^{\mu}), \quad (2.13)$$

the action gets the form

$$S = \kappa \int_{\Sigma} d^2\xi \partial_{+} x^{\mu} \left[\Pi_{+\mu\nu} + 2\bar{\Psi}_{\mu}^{\alpha} (P^{-1})_{\alpha\beta} \Psi_{\nu}^{\beta} \right] \partial_{-} x^{\nu} \\ + 2\kappa \int_{\Sigma} d^2\xi \left[\partial_{+} \bar{\theta}^{\alpha} (P^{-1})_{\alpha\beta} \partial_{-} \theta^{\beta} \right. \\ \left. + \partial_{+} \bar{\theta}^{\alpha} (P^{-1})_{\alpha\mu} \partial_{-} x^{\mu} + \partial_{+} x^{\mu} (\bar{\Psi} P^{-1})_{\mu\alpha} \partial_{-} \theta^{\alpha} \right]. \quad (2.14)$$

We will perform bosonic T-dualization of the action (2.14) along all directions x^{μ} using the Buscher T-dualization rules. In order to gauge global symmetry $\delta x^{\mu} = \lambda^{\mu}$, we introduce covariant derivatives, $D_{\pm} x^{\mu} = \partial_{\pm} x^{\mu} + v_{\pm}^{\mu}$, instead of the ordinary ones, $\partial_{\pm} x^{\mu}$, where v_{\pm}^{μ} are gauge fields. Fixing the gauge ($x^{\mu} = const.$) means effectively that ordinary derivatives $\partial_{\pm} x^{\mu}$ are replaced with gauge fields v_{\pm}^{μ} in the initial action (2.14), while, in order to make v_{\pm}^{μ} unphysical degrees of freedom, we add to the action

$$S_{add} = \frac{\kappa}{2} \int d^2\xi (v_{+}^{\mu} \partial_{-} y_{\mu} - v_{-}^{\mu} \partial_{+} y_{\mu}). \quad (2.15)$$

The gauged fixed action is of the form

$$S_{fix} = S + S_{add} \\ = \kappa \int d^2\xi \left[v_{+}^{\mu} \Pi_{+\mu\nu} v_{-}^{\nu} + 2(\partial_{+} \bar{\theta}^{\alpha} + \bar{\Psi}_{\mu}^{\alpha} v_{+}^{\mu}) (P^{-1})_{\alpha\beta} \right. \\ \left. (\partial_{-} \theta^{\beta} + \Psi_{\nu}^{\beta} v_{-}^{\nu}) + \frac{1}{2} (v_{+}^{\mu} \partial_{-} y_{\mu} - v_{-}^{\mu} \partial_{+} y_{\mu}) \right]. \quad (2.16)$$

Varying the gauge fixed action (2.16) with respect to the Lagrange multipliers y_{μ} , we find that the field strength for gauge fields v_{\pm}^{μ} is equal to zero,

$$\partial_+ v_-^\mu - \partial_- v_+^\mu = 0 \Rightarrow v_\pm^\mu = \partial_\pm x^\mu. \tag{2.17}$$

In this way we restore the initial theory from the gauge fixed action. Let us note that we omitted the dilaton term because this is a classical analysis while the dilaton is treated within the quantum formalism.

Varying the gauge fixed action with respect to the gauge fields v_+^μ and v_-^μ , we get the equations, respectively,

$$\Pi_{+\mu\nu} v_-^\nu + 2\check{\Psi}^\alpha{}_\mu (P^{-1})_{\alpha\beta} (\partial_- \theta^\beta + \Psi^\beta{}_\nu v_-^\nu) + \frac{1}{2} \partial_- y_\mu = 0, \tag{2.18}$$

$$v_+^\nu \Pi_{+\nu\mu} + 2(\partial_+ \bar{\theta}^\alpha + \check{\Psi}^\alpha{}_\nu v_+^\nu) (P^{-1})_{\alpha\beta} \Psi^\beta{}_\mu - \frac{1}{2} \partial_+ y_\mu = 0. \tag{2.19}$$

Here we introduce the notation

$$\check{\Pi}_{+\mu\nu} \equiv \Pi_{+\mu\nu} + 2\check{\Psi}^\alpha{}_\mu (P^{-1})_{\alpha\beta} \Psi^\beta{}_\nu = \check{B}_{\mu\nu} + \frac{1}{2} \check{G}_{\mu\nu}, \tag{2.20}$$

where $\check{B}_{\mu\nu}$ and $\check{G}_{\mu\nu}$ are the antisymmetric and symmetric parts of $\check{\Pi}_{+\mu\nu}$, respectively. These expressions are in fact the Kalb–Ramond field $B_{\mu\nu}$ and metric $G_{\mu\nu}$ improved by some expressions consisting of the NS-R and R-R background fields. The above expressions for the gauge fields can be rewritten in the form

$$\check{\Pi}_{+\mu\nu} v_-^\nu + 2\check{\Psi}^\alpha{}_\mu (P^{-1})_{\alpha\beta} \partial_- \theta^\beta + \frac{1}{2} \partial_- y_\mu = 0, \tag{2.21}$$

$$v_+^\nu \check{\Pi}_{+\nu\mu} + 2\partial_+ \bar{\theta}^\alpha (P^{-1})_{\alpha\beta} \Psi^\beta{}_\mu - \frac{1}{2} \partial_+ y_\mu = 0. \tag{2.22}$$

Using the relations

$$\check{\Theta}_-^{\mu\nu} \check{\Pi}_{+\nu\rho} = \frac{1}{2\kappa} \delta^\mu{}_\rho, \quad \check{\Theta}_-^{\mu\nu} = -\frac{2}{\kappa} \left(\check{G}_E^{-1} \check{\Pi}_- \check{G}_E^{-1} \right)^{\mu\nu}, \tag{2.23}$$

where

$$\check{\Theta}_-^{\mu\nu} = \Theta_-^{\mu\nu} - 4\kappa \Theta_-^{\mu\rho} \check{\Psi}^\alpha{}_\rho (\check{P}^{-1})_{\alpha\beta} \Psi^\beta{}_\lambda \Theta_-^{\lambda\nu} = \check{\Theta}^{\mu\nu} + \frac{1}{\kappa} \left(\check{G}_E^{-1} \right)^{\mu\nu}, \tag{2.24}$$

$$\check{P}^{\alpha\beta} \equiv P^{\alpha\beta} + 4\kappa \Psi^\alpha{}_\mu \Theta_-^{\mu\nu} \check{\Psi}^\beta{}_\nu, \tag{2.25}$$

$$\Theta_-^{\mu\nu} = -\frac{2}{\kappa} \left(G_E^{-1} \Pi_- G^{-1} \right)^{\mu\nu}, \quad \Theta_-^{\mu\rho} \Pi_{+\rho\nu} = \frac{1}{2\kappa} \delta^\mu{}_\nu, \tag{2.26}$$

we get

$$v_-^\mu = -\kappa \check{\Theta}_-^{\mu\nu} \partial_- \left[y_\nu + 4\check{\Psi}^\alpha{}_\mu (P^{-1})_{\alpha\beta} \theta^\beta \right], \tag{2.27}$$

$$v_+^\mu = \kappa \partial_+ \left[y_\nu - 4\bar{\theta}^\alpha (P^{-1})_{\alpha\beta} \Psi^\beta{}_\nu \right] \check{\Theta}_-^{\nu\mu}. \tag{2.28}$$

Equation (2.23) is proved by direct calculation and using the definition of $\check{P}^{\alpha\beta}$.

Inserting the expressions (2.27) and (2.28) into the expression for the gauge fixed action (2.16) we obtain the T-dual action

$$\begin{aligned} {}^*S = \kappa \int d^2\xi \left[\frac{\kappa}{2} \partial_+ y_\mu \check{\Theta}_-^{\mu\nu} \partial_- y_\nu \right. \\ \left. + 2\kappa \partial_+ y_\mu \check{\Theta}_-^{\mu\nu} \check{\Psi}^\alpha{}_\nu (P^{-1})_{\alpha\beta} \partial_- \theta^\beta \right. \\ \left. - 2\kappa \partial_+ \bar{\theta}^\alpha (P^{-1})_{\alpha\beta} \Psi^\beta{}_\mu \check{\Theta}_-^{\mu\nu} \partial_- y_\nu \right. \\ \left. + 2\partial_+ \bar{\theta}^\alpha (P^{-1} - 4\kappa P^{-1} \Psi \check{\Theta}_- \check{\Psi} P^{-1})_{\alpha\beta} \partial_- \theta^\beta \right]. \end{aligned} \tag{2.29}$$

Introducing T-dual background fields marked by $*$, we write the T-dual action in the form of the initial action (2.14)

$$\begin{aligned} {}^*S = \kappa \int d^2\xi \left[\partial_+ y_\mu \left({}^*\Pi_+ + 2{}^*\check{\Psi}^* P^{-1} {}^*\Psi \right)^{\mu\nu} \partial_- y_\nu \right. \\ \left. + 2\partial_+ y_\mu \left({}^*\check{\Psi}^* P^{-1} \right)^\mu{}_\alpha \partial_- \theta^\alpha \right. \\ \left. + 2\partial_+ \bar{\theta}^\alpha \left({}^*P^{-1} {}^*\Psi \right)_\alpha{}^\mu \partial_- y_\mu + 2\partial_+ \bar{\theta}^\alpha \left({}^*P^{-1} \right)_{\alpha\beta} \partial_- \theta^\beta \right]. \end{aligned} \tag{2.30}$$

Comparing the last two equations, we get the T-dual background fields in terms of the initial ones,

$${}^*\Pi_+^{\mu\nu} + 2{}^*\check{\Psi}^{\mu\alpha} \left({}^*P^{-1} \right)_{\alpha\beta} {}^*\Psi^{\beta\nu} = \frac{\kappa}{2} \check{\Theta}_-^{\mu\nu}, \tag{2.31}$$

$$\begin{aligned} {}^*\check{\Psi}^{\mu\beta} \left({}^*P^{-1} \right)_{\beta\alpha} = \kappa \check{\Theta}_-^{\mu\nu} \check{\Psi}^\beta{}_\nu \left(P^{-1} \right)_{\beta\alpha}, \quad \left({}^*P^{-1} \right)_{\alpha\beta} {}^*\Psi^{\beta\mu} \\ = -\kappa \left(P^{-1} \right)_{\alpha\beta} \Psi^\beta{}_\nu \check{\Theta}_-^{\nu\mu}, \end{aligned} \tag{2.32}$$

$$\left({}^*P^{-1} \right)_{\alpha\beta} = \left(P^{-1} \right)_{\alpha\beta} - 4\kappa \left(P^{-1} \right)_{\alpha\gamma} \Psi^\gamma{}_\mu \check{\Theta}_-^{\mu\nu} \check{\Psi}^\delta{}_\nu \left(P^{-1} \right)_{\delta\beta}. \tag{2.33}$$

By direct calculation, solving the above four equations, we finally get

$${}^*\Pi_+^{\mu\nu} = \frac{\kappa}{2} \Theta_-^{\mu\nu}, \tag{2.34}$$

$${}^*\Psi^{\alpha\mu} = -\kappa \Psi^\alpha{}_\nu \Theta_-^{\nu\mu}, \quad {}^*\check{\Psi}^{\mu\alpha} = \kappa \Theta_-^{\mu\nu} \check{\Psi}^\alpha{}_\nu, \tag{2.35}$$

$${}^*P^{\alpha\beta} = \check{P}^{\alpha\beta}, \tag{2.36}$$

which is in full agreement with the case where we T-dualize the same model in the form of the first-order theory.

Combining the equations of motion for the Lagrange multiplier (2.17) with the equations of motion for the gauge fields, (2.28) and (2.28), we obtain the relation between the initial x^μ and T-dual coordinates y_μ

$$\partial_\pm x^\mu \cong -\kappa \check{\Theta}_\pm^{\mu\nu} \left[\partial_\pm y_\nu + 4\Psi^\alpha{}_\nu \left(P_{\mp}^{-1} \right)_{\alpha\beta} \partial_\pm \theta^\beta \right]. \tag{2.37}$$

The inverse of this relation is also useful and it is of the form

$$\partial_\pm y_\mu \cong -2\check{\Pi}_{\mp\mu\nu} \partial_\pm x^\nu - 4\Psi^\alpha{}_\mu \left(P_{\mp}^{-1} \right)_{\alpha\beta} \partial_\pm \theta^\beta. \tag{2.38}$$

Here we use the notation

$$\theta_+^\alpha \equiv \theta^\alpha, \quad \theta_-^\alpha \equiv \bar{\theta}^\alpha, \tag{2.39}$$

$$P_+^{\alpha\beta} \equiv P^{\alpha\beta}, \quad P_-^{\alpha\beta} \equiv P^{\beta\alpha}, \tag{2.40}$$

$$\Psi_{+\mu}^\alpha \equiv \Psi^\alpha_\mu, \quad \Psi_{-\mu}^\alpha \equiv \bar{\Psi}^\alpha_\mu, \tag{2.41}$$

$$\check{\Theta}_+^{\mu\nu} \equiv -\check{\Theta}_-^{\nu\mu}. \tag{2.42}$$

As we see, the two chirality sectors transform differently under T-dualization. The form of the T-dualization transformation laws is of the same form as in [1]. Consequently, in accordance with the results of Refs. [1, 50, 51], we introduce the proper fermionic coordinates

$$\bullet\theta_+^\alpha = \theta_+^\alpha, \quad \bullet\bar{\theta}_-^\alpha = -(\Gamma_{11}\bar{\theta}_-^\alpha), \tag{2.43}$$

and the correct form of the T-dual fields is

$$\star\Pi_+^{\mu\nu} = \frac{\kappa}{2}\Theta_-^{\mu\nu}, \tag{2.44}$$

$$\star\Psi^{\alpha\mu} = -\kappa\Psi^\alpha_\nu\Theta_-^{\nu\mu}, \quad \star\bar{\Psi}^{\mu\alpha} = -\kappa\Theta_-^{\mu\nu}(\Gamma_{11}\bar{\Psi})^\alpha_\nu, \tag{2.45}$$

$$\star P^{\alpha\beta} = -(\check{P}\Gamma_{11})^{\alpha\beta}. \tag{2.46}$$

3 T-dualization of type II superstring in double space

In this section we will demonstrate another framework in which we can perform the T-dualization procedure. Unlike the case of the T-dualization of type II superstring theory in the form of the first-order theory [1] where the T-dual R-R field strength is not obtained within double space framework, here we will see that, when fermionic momenta are integrated out, the double space formalism gives all T-dual background fields. Before the T-dualization procedure we will introduce double space and the corresponding quantities.

3.1 T-dual transformation law in double space

Let us introduce the double space coordinate

$$Z^M = \begin{pmatrix} x^\mu \\ y_\mu \end{pmatrix}, \tag{3.1}$$

and rewrite the T-dual transformation laws (2.37)–(2.38) in the form

$$\begin{aligned} \pm\partial_\pm y_\mu &\cong \check{G}_{\mu\nu}^E \partial_\pm x^\nu + \kappa \check{G}_{\mu\rho}^E \check{\Theta}^{\rho\nu} \partial_\pm y_\nu \\ &+ 4\kappa G_{\mu\rho}^E \Theta_\pm^{\rho\lambda} \Psi_{\pm\lambda}^\alpha (P_\mp^{-1})_{\alpha\beta} \partial_\pm \theta_\mp^\beta, \end{aligned} \tag{3.2}$$

$$\begin{aligned} \pm\partial_\pm x^\mu &\cong (\check{G}^{-1})^{\mu\nu} \partial_\pm y_\nu + 2(\check{G}^{-1}\check{B})^\mu_\nu \partial_\pm x^\nu \\ &+ 4(\check{G}^{-1})^{\mu\rho} \Psi_{\pm\rho}^\alpha (P_\mp^{-1})_{\alpha\beta} \partial_\pm \theta_\mp^\beta. \end{aligned} \tag{3.3}$$

These two equations can be rewritten in double space as

$$\pm\Omega_{MN}\partial_\pm Z^N \cong \check{\mathcal{H}}_{MN}\partial_\pm Z^N + \check{J}_{\pm M}, \tag{3.4}$$

where the generalized metric is of the form

$$\check{\mathcal{H}}_{MN} = \begin{pmatrix} \check{G}_{\mu\nu}^E & \kappa \check{G}_{\mu\rho}^E \check{\Theta}^{\rho\nu} \\ 2(\check{G}^{-1})^{\mu\rho} \check{B}_{\rho\nu} & (\check{G}^{-1})^{\mu\nu} \end{pmatrix}, \tag{3.5}$$

and the double current is

$$\check{J}_{\pm M} = 4 \begin{pmatrix} \kappa \check{G}_{\mu\rho}^E \check{\Theta}^{\rho\nu} \\ (\check{G}^{-1})^{\mu\nu} \end{pmatrix} J_{\pm\mu}, \quad J_{\pm\mu} = \Psi_{\pm\mu}^\alpha (P_\mp^{-1})_{\alpha\beta} \partial_\pm \theta_\mp^\beta. \tag{3.6}$$

The matrix

$$\Omega = \begin{pmatrix} 0 & 1_D \\ 1_D & 0 \end{pmatrix}, \tag{3.7}$$

where 1_D denotes the unity matrix in D dimensions, known in double field theory (DFT) as the invariant $SO(D, D)$ metric.

Note that generalized metric is not of the standard form because its components contain the improved Kalb–Ramond field $\check{B}_{\mu\nu}$ and the improved metric $\check{G}_{\mu\nu}$. Those additional factors in $\check{B}_{\mu\nu}$ and $\check{G}_{\mu\nu}$ have a bilinear form in the NS-R fields Ψ^α_μ and $\bar{\Psi}^\alpha_\mu$. Still, we have

$$\check{\mathcal{H}}^T \Omega \check{\mathcal{H}} = \Omega, \quad \Omega^2 = 1, \quad \det \check{\mathcal{H}} = 1, \tag{3.8}$$

which means that $\check{\mathcal{H}} \in SO(D, D)$.

3.2 Full T-dualization in double space

Let us introduce the permutation matrix

$$\mathcal{T}^M_N = \begin{pmatrix} 0 & 1_D \\ 1_D & 0 \end{pmatrix}, \tag{3.9}$$

and define the T-dual double coordinate $\star Z^M$ as

$$\star Z^M = \mathcal{T}^M_N Z^N. \tag{3.10}$$

We require that the T-dual transformation law for the T-dual double coordinate $\star Z^M$ has the same form as for initial double coordinate Z^M (3.4)

$$\pm\Omega_{MN}\partial_\pm \star Z^N \cong \star\check{\mathcal{H}}_{MQ}\partial_\pm \star Z^Q + \star\check{J}_{\pm M}, \tag{3.11}$$

which implies that T-dual generalized metric and double current are of the form, respectively,

$$\star\check{\mathcal{H}}_{MN} = \mathcal{T}_M^P \check{\mathcal{H}}_{PQ} \mathcal{T}^Q_N, \quad \star\check{J}_{\pm M} = \mathcal{T}_M^N \check{J}_{\pm N}. \tag{3.12}$$

Let us make explicit the first equation in (3.12):

$$\begin{pmatrix} \star\check{G}_{\mu\nu}^E & \kappa \star\check{G}_{\mu\rho}^E \star\check{\Theta}^{\rho\nu} \\ 2(\star\check{G}^{-1})_{\mu\rho} \star\check{B}^{\rho\nu} & (\star\check{G}^{-1})_{\mu\nu} \end{pmatrix}$$

$$= \begin{pmatrix} (\check{G}^{-1})^{\mu\nu} & 2(\check{G}^{-1})^{\mu\rho} \check{B}_{\rho\nu} \\ \kappa \check{G}_{\mu\rho}^E \check{\Theta}^{\rho\nu} & \check{G}_{\mu\nu}^E \end{pmatrix}. \tag{3.13}$$

Equating the (2, 2) block components we get

$$(*\check{G}^{-1})_{\mu\nu} = \check{G}_{\mu\nu}^E, \tag{3.14}$$

which produces

$$*\check{G}^{\mu\nu} = \left(\check{G}_E^{-1}\right)^{\mu\nu}. \tag{3.15}$$

Using (2, 1) the block components equation

$$2(*\check{G}^{-1})_{\mu\rho} *\check{B}^{\rho\nu} = \kappa \check{G}_{\mu\rho}^E \check{\Theta}^{\rho\nu}, \tag{3.16}$$

combining with (3.14), we obtain

$$*\check{B}^{\mu\nu} = \frac{\kappa}{2} \check{\Theta}^{\mu\nu}. \tag{3.17}$$

Using these two results we have

$$*\check{\Pi}_{\pm}^{\mu\nu} = *B^{\mu\nu} \pm \frac{1}{2} *\check{G}^{\mu\nu} = \frac{\kappa}{2} \left[\check{\Theta}^{\mu\nu} \pm \frac{1}{\kappa} \left(\check{G}_E^{-1}\right)^{\mu\nu} \right] = \frac{\kappa}{2} \check{\Theta}_{\mp}^{\mu\nu}. \tag{3.18}$$

The obtained equation coincides with the equation obtained by the standard Buscher procedure (2.31). The block components (1, 1) and (1, 2) give, respectively,

$$*\check{G}_E^{\mu\nu} = (\check{G}^{-1})^{\mu\nu}, \quad *\check{G}_E^{\mu\rho} *\check{\Theta}_{\rho\nu} = \frac{2}{\kappa} (\check{G}^{-1})^{\mu\rho} \check{B}_{\rho\nu}. \tag{3.19}$$

Combining the last two equations produces

$$*\check{\Theta}_{\mu\nu} = \frac{2}{\kappa} \check{B}_{\mu\nu}, \tag{3.20}$$

and, furthermore, we have

$$*\check{\Theta}_{-\mu\nu} = \frac{2}{\kappa} \check{\Pi}_{+\mu\nu}. \tag{3.21}$$

Using the relations between the initial and T-dual NS-NS background fields obtained above and the second equation in (3.12), we get

$$*J_{\pm}^{\mu} = \kappa \check{\Theta}_{\pm}^{\mu\nu} J_{\pm\nu}, \tag{3.22}$$

where the T-dual current $*J_{\pm}^{\mu}$ has the same form as the initial one but in terms of the T-dual background fields and proper fermionic coordinates (for details see [1])

$$*J_{\pm}^{\mu} \equiv *\Psi_{\pm}^{\alpha\mu} (*P_{\mp}^{-1})_{\alpha\beta} \partial_{\pm} * \theta_{\mp}^{\beta}. \tag{3.23}$$

The proper fermionic coordinates are defined as

$$*\theta_{+}^{\alpha} = \theta_{+}^{\alpha}, \quad *\theta_{-}^{\alpha} = -(\Gamma_{11}\theta_{-})^{\alpha}. \tag{3.24}$$

From Eq. (3.23) we have

$$*\check{\Pi}_{+}^{\mu\nu} = *\Pi_{+}^{\mu\nu} + 2*\bar{\Psi}^{\alpha\mu} (*P^{-1})_{\alpha\beta} *\Psi^{\beta\nu}, \tag{3.25}$$

$$*\check{\Theta}_{-\mu\nu} = *\Theta_{-\mu\nu} - 4\kappa *\Theta_{-\mu\rho} *\bar{\Psi}^{\rho\alpha} (*\tilde{P}^{-1})_{\alpha\beta} *\Psi^{\beta\lambda} *\Theta_{-\lambda\nu}, \tag{3.26}$$

and, solving these equations, we get

$$*\Psi^{\alpha\mu} = \pm\kappa \Psi^{\alpha}_{\nu} \Theta_{-}^{\nu\mu}, \quad *\bar{\Psi}^{\alpha\mu} = \pm\kappa \Theta_{-}^{\mu\nu} (\Gamma_{11} \bar{\Psi})_{\nu}^{\alpha}, \tag{3.27}$$

$$*P^{\alpha\beta} = - (P^{\alpha\gamma} + 4\kappa \Psi^{\alpha}_{\mu} \Theta^{\mu\nu} \bar{\Psi}^{\gamma}_{\nu}) (\Gamma_{11})_{\gamma}^{\beta}. \tag{3.27}$$

Here the double space formalism produces (3.18) and (3.27) i.e. all relations (2.44)–(2.46), up to the sign of the T-dual NS-R background fields. This uncertainty in sign is a consequence of the fact that in both equations, (3.25) and (3.26), the NS-R fields, $*\Psi^{\alpha\mu}$ and $*\bar{\Psi}^{\alpha\mu}$, appear in a bilinear combination. In comparison with the case where the fermionic momenta are not integrated out [1], this is an improvement because the double formalism gives all T-dual background fields. In Ref. [1] we did not obtain the relation for the T-dual R-R background field and we had to impose additional conditions. Here the calculation is slightly more complicated, but we obtain all fields within one formalism. The reason is that in [1] the R-R field is coupled by fermionic momenta which are not T-dualized. Consequently, after integration of the fermionic momenta, there appears a coupling between the R-R field strength with bosonic coordinates x^{μ} which results in Eq. (3.27).

4 Conclusion

In this article we considered the type II superstring theory in a pure spinor formulation with constant background fields. We integrated out the fermionic momenta and obtained the theory quadratic in world-sheet derivatives of bosonic and fermionic coordinates. Our goal was to show the advantage of the T-dualization within the double space formalism comparing to the first-order theory [1].

At the beginning we explained how we obtained the action with constant background field from the general one derived in [49]. The assumed shift symmetry along the bosonic directions x^{μ} means that the background fields do not depend on x^{μ} . On the other hand, for technical simplicity of the calculations, we take just the first terms in the expansions of background fields in powers of θ^{α} and $\bar{\theta}^{\alpha}$. All these assumptions result in constant background fields. In the final form of the action, just physical superfields are present, while the auxiliary fields and field strengths are zero.

The main mathematical difference from Ref. [1] is that the fermionic momenta are integrated out. In this way we obtained a theory which is quadratic in the world-sheet derivatives of the coordinates, $\partial_{\pm}x^{\mu}$, $\partial_{\pm}\theta^{\alpha}$ and $\partial_{\pm}\bar{\theta}^{\alpha}$. It is important to emphasize that in such a formulation R-R field strength $P^{\alpha\beta}$ is coupled with the derivatives of the bosonic coordinates $\partial_{\pm}x^{\mu}$. Then we applied the Buscher procedure and, beside some slightly more complicated mathematical calculations, we obtained the same result as in the case for the first-order theory [1].

Our contribution was to show the benefit in performing a T-dualization procedure in double space using the action (2.14), to be compared with the results obtained for the action (2.11) in Ref. [1].

The double space is spanned by the coordinates $Z^M = (x^{\mu}, y_{\mu})$, where x^{μ} are initial bosonic coordinates and y_{μ} are corresponding the T-dual ones. The T-dual transformation laws are rewritten in terms of the double space coordinates introducing the generalized metric $\check{\mathcal{H}}_{MN}$ and the current $\check{J}_{\pm M}$. Note that their components are expressed in terms of the improved Kalb–Ramond field and the metric containing additional terms bilinear in the NS-R background fields Ψ_{μ}^{α} and $\bar{\Psi}_{\mu}^{\alpha}$. Requiring that T-dual double space coordinates $*Z^M = \mathcal{T}^M_N Z^N$ satisfy a transformation law of the same form as the initial coordinates Z^M we found the T-dual generalized metric $*\check{\mathcal{H}}_{MN}$ and the T-dual current $*J_{\pm M}$. The T-dual generalized metric should have the same form as the initial ones, so, in this way we obtain relations which produce the expressions for T-dual background fields in terms of the initial ones, which agrees with that obtained applying the Buscher procedure.

In Ref. [1] we obtained the expressions for the T-dual NS-NS background fields as well as for the NS-R fields. But because we T-dualized along the bosonic directions which are not coupled with R-R field strength, the double space formalism did not give us the expression for the T-dual R-R field strength. These expressions were obtained under some additional assumptions out of the double space formalism.

Here we succeeded in obtaining the expressions for all T-dual background fields, which showed that there is an advantage in performing double space T-dualization in the second-order theory where the fermionic momenta are integrated out.

After having summarized the results of this article it is interesting to discuss their significance and relation to the results of other articles addressing the same or similar subjects. For example, in Ref. [52] the authors construct the type II model with T-duality as a manifest symmetry. The way of construction is partially similar to the one from [49] used in this paper. In [49] they used (anti)holomorphicity and nilpotency conditions, while in [52] instead of nilpotency conditions they used conditions originating from κ symmetry.

But in [52] they do not study the problem of the RR field strength as well as an interchange between types IIA/B in the T-dualization process, which are the subjects addressed in this article. Furthermore, in [53] one version of the doubled superspace is discussed. It is pretty similar to the space we used in this article, but it is obtained by multiplication of the left and right chiral sectors with $\mathcal{N} = 1$ supersymmetry in $D = 10$. The space which is obtained is spanned by the initial bosonic coordinates, their T-dual ones and two fermionic coordinates. Our doubled coordinate contains just the bosonic part of this doubled supercoordinate because we consider here just bosonic T-dualization and, consequently, we do not consider a fermionic sector. The obvious difference is that in [53] they obtained a type II action in the Green–Schwarz formalism, while we study here a pure spinor action. In [54] the geometry of superspace is developed with a type II model as the main example. One of the things discussed is the relation of some sectors with pure spinor fields. Finally, it is useful to mention also Ref. [55]. In this article a reduction of type IIA/B superstring theory from $10d$ to $9d$ is done, which effectively could be T-dualization. The L_{∞} isomorphisms relate two coefficient L_{∞} algebras. Also one derived the Buscher rules for the RR field strength, which is done in this article using simpler mathematical methods.

Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors' comment: This paper does not include any additional information.]

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