



Gauge invariance of the background average effective action

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Abstract Using the background field method for the functional renormalization group approach in the case of a generic gauge theory, we study the background field symmetry and gauge dependence of the background average effective action, when the regulator action depends on external fields. The final result is that the symmetry of the average effective action can be maintained for a wide class of regulator functions, but in all cases the dependence of the gauge fixing remains on-shell. The Yang–Mills theory is considered as the main particular example.

1 Introduction

One of the most prospective non-perturbative approaches in quantum field theory (QFT) is the functional (or exact) renormalization group (FRG), which is based on the Wetterich equation for the average effective action [1–3] (see the reviews [4–7] and textbook [8] for an introduction to the subject). The application of FRG to gauge theories was extensively discussed, including in the recent work [9]. The considerations in the last and many other papers are based on the background field method, which enables one to maintain the gauge invariance for the Yang–Mills (or gravitational) field explicitly in the effective action. The background field method is, in general, a useful formalism in the theory of gauge fields, and that is why it attracted a very special attention recently, see e.g. [10–13]. The application of this method to the average effective action has been done long ago [14] (see also the recent work [15]), but in our opinion there are some important aspects of the problem which should be explored in more details.

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The main problem of FRG applied to the gauge theories is that the dependence on the choice of the gauge fixing condition does not disappear on-shell [16], as it is the case in the usual perturbative QFT. As a result of the on-shell gauge fixing dependence, the S -matrix of the theory is not well defined, except at the fixed point, where the effective average action coincides with the usual effective action. One can expect that the renormalization group flow in the Yang–Mills theory will also manifest a fundamental gauge dependence, and this certainly shadows the interpretation of the results obtained within the FRG approach in the gauge theories.

In order to better understand the situation with the gauge symmetry at the quantum level and with the gauge dependence, it is important to analyze the mentioned problems in the background field method, that is the main purpose of the present communication. In what follows we consider both gauge invariance and gauge fixing dependence for the effective average action.

The paper is organized as follows. In Sect. 2 we give a brief description of the background field formalism in non-Abelian gauge theories and the gauge independence of vacuum functional in this method. In Sect. 3 the background field symmetry is analyzed within the background field method-based functional renormalization group approach. The regulator functions are dependent on the external (background) fields but are chosen not to be invariant under gauge transformations of external vector field. In Sect. 4 we present a solution to the background field symmetry of background average effective action with regulator functions which are invariant under gauge transformations of the external field. In Sect. 5 the gauge dependence problem of the background average effective action is considered. Finally, we discuss the results and draw our conclusions in Sect. 6.

Our notations system mainly follows the DeWitt's book [17]. Also, the Grassmann parity of a quantity A is denoted $\varepsilon(A)$.

2 Background field formalism

We start by making a brief review of the background field formalism for a gauge theory describing by an initial action $S_0(A)$ of fields $A = \{A^i\}$, $\varepsilon(A^i) = \varepsilon_i$ invariant under gauge transformations

$$S_{0,i}(A)R_\alpha^i(A) = 0, \quad \delta A^i = R_\alpha^i(A)\xi^\alpha, \tag{1}$$

where $R_\alpha^i(A)$, $\varepsilon(R_\alpha^i(A)) = \varepsilon_i + \varepsilon_\alpha$ are the generators of gauge transformations, ξ^α , $\varepsilon(\xi^\alpha) = \varepsilon_\alpha$ are arbitrary functions. In general, a set of fields $A^i = (A^{\alpha k}, A^m)$ includes fields $A^{\alpha k}$ of the gauge sector and also fields A^m of the matter sector of a given theory. We assume that the generators $R_\alpha^i = R_\alpha^i(A)$ satisfy a closed algebra with structure coefficients $F_{\alpha\beta}^\gamma$ that do not depend on the fields,

$$R_{\alpha,j}^i R_\beta^j - (-1)^{\varepsilon_\alpha \varepsilon_\beta} R_{\beta,j}^i R_\alpha^j = -R_\gamma^i F_{\alpha\beta}^\gamma, \tag{2}$$

where we denote the right functional derivative by $\delta_r X / \delta A^i = X_{,i}$. The structure coefficients satisfy the symmetry properties $F_{\alpha\beta}^\gamma = -(-1)^{\varepsilon_\alpha \varepsilon_\beta} F_{\beta\alpha}^\gamma$. We assume as well that the generators are linear operators in A^i , $R_\alpha^i(A) = t_{\alpha j}^i A^j + r_\alpha^i$.

We apply the background field method (BFM) [18–21] replacing the field A^i by $A^i + \mathcal{B}^i$ in the classical action $S_0(A)$,

$$S_0(A) \longrightarrow S_0(A + \mathcal{B}). \tag{3}$$

Here \mathcal{B}^i are external (background) vector fields being not equal to zero only in the gauge sector. The action $S_0(A + \mathcal{B})$ obeys the gauge invariance in the form

$$\delta S_0(A + \mathcal{B}) = 0, \quad \delta A^i = R_\alpha^i(A + \mathcal{B})\xi^\alpha. \tag{4}$$

Through the Faddeev–Popov quantization [22] the field configuration space is extended to

$$\phi^A = (A^i, B^\alpha, C^\alpha, \bar{C}^\alpha), \quad \varepsilon(\phi^A) = \varepsilon_A, \tag{5}$$

where C^α , \bar{C}^α are the Faddeev–Popov ghost and antighost fields, respectively, and B^α is the auxiliary (Nakanishi–Lautrup) field. The Grassmann parities distribution are the following

$$\varepsilon(C^\alpha) = \varepsilon(\bar{C}^\alpha) = \varepsilon_\alpha + 1, \quad \varepsilon(B^\alpha) = \varepsilon_\alpha. \tag{6}$$

The corresponding Faddeev–Popov action $S_{FP}(\phi, \mathcal{B})$ in the singular gauge fixing has the form [22]

$$S_{FP}(\phi, \mathcal{B}) = S_0(A + \mathcal{B}) + S_{gh}(\phi, \mathcal{B}) + S_{gf}(\phi, \mathcal{B}) \tag{7}$$

where

$$S_{gh}(\phi, \mathcal{B}) = \bar{C}^\alpha \chi_{\alpha,i}(A, \mathcal{B}) R_\beta^i(A + \mathcal{B}) C^\beta, \tag{8}$$

$$S_{gf}(\phi, \mathcal{B}) = B^\alpha \chi_\alpha(A, \mathcal{B}). \tag{9}$$

In the last expression $\chi_\alpha(A, \mathcal{B})$ are functions lifting the degeneracy for the action $S_0(A + \mathcal{B})$. The standard background field gauge condition in the BFM is linear in the quantum fields

$$\chi_\alpha(A, \mathcal{B}) = F_{\alpha i}(A, \mathcal{B}) A^i. \tag{10}$$

The action (7) is invariant under the BRST symmetry [23, 24]

$$\delta_B \phi^A = s^A(\phi, \mathcal{B})\mu, \quad \varepsilon(s^A(\phi, \mathcal{B})) = \varepsilon_A + 1, \tag{11}$$

where

$$s^A(\phi, \mathcal{B}) = \left(R_\alpha^i(A + \mathcal{B}) C^\alpha, 0, -\frac{1}{2} F_{\beta\gamma}^\alpha C^\gamma C^\beta (-1)^{\varepsilon_\beta}, (-1)^{\varepsilon_\alpha} B^\alpha \right) \tag{12}$$

and μ is a constant Grassmann parameter with $\varepsilon(\mu) = 1$. One can write (12) as generator of BRST transformations,

$$\hat{s}(\phi, \mathcal{B}) = \overleftarrow{\delta} \frac{\delta}{\delta \phi^A} s^A(\phi, \mathcal{B}). \tag{13}$$

Then, the action (7) can be written in the form

$$S_{FP}(\phi, \mathcal{B}) = S_0(A + \mathcal{B}) + \Psi(\phi, \mathcal{B}) \hat{s}(\phi, \mathcal{B}), \tag{14}$$

where

$$\Psi(\phi, \mathcal{B}) = \bar{C}^\alpha \chi_\alpha(A, \mathcal{B}), \tag{15}$$

is the gauge fixing functional. The transformation (11) is nilpotent, that means $\hat{s}^2 = 0$. Taking into account that $S_0(A + \mathcal{B}) \hat{s}(\phi, \mathcal{B}) = 0$, the BRST symmetry of $S_{FP}(\phi, \mathcal{B})$ follows immediately

$$S_{FP}(\phi, \mathcal{B}) \hat{s}(\phi, \mathcal{B}) = 0. \tag{16}$$

Due to the presence of external vector field \mathcal{B}^i , the Faddeev–Popov action obeys an additional local symmetry known as the background field symmetry,

$$\delta_\omega S_{FP}(\phi, \mathcal{B}) = 0, \tag{17}$$

which is related to the background field transformations

$$\begin{aligned} \delta_\omega^{(c)} \mathcal{B}^i &= R_\alpha^i(\mathcal{B}) \omega^\alpha, \\ \delta_\omega^{(q)} A^i &= \left[R_\alpha^i(A + \mathcal{B}) - R_\alpha^i(\mathcal{B}) \right] \omega^\alpha, \\ \delta_\omega^{(q)} B^\alpha &= -F_{\gamma\beta}^\alpha B^\beta \omega^\gamma, \\ \delta_\omega^{(q)} C^\alpha &= -F_{\gamma\beta}^\alpha C^\beta \omega^\gamma (-1)^{\varepsilon_\gamma}, \\ \delta_\omega^{(q)} \bar{C}^\alpha &= -F_{\gamma\beta}^\alpha \bar{C}^\beta \omega^\gamma (-1)^{\varepsilon_\gamma}. \end{aligned} \tag{18}$$

Here the subscript (c) is used to indicate the background field transformations in the sector of external (classical) fields while the (q) in the sector of quantum fields (integration variables in functional integral for generating functional of Green functions). The symbol δ_ω means the combined background field transformations $\delta_\omega = \delta_\omega^{(c)} + \delta_\omega^{(q)}$. Note that in deriving

(17) the transformation rule for the gauge fixing functions (10)

$$\delta_\omega \chi_\alpha(\phi, \mathcal{B}) = -\chi_\beta(\phi, \mathcal{B}) F_{\alpha\gamma}^\beta \omega^\gamma, \tag{19}$$

under the background field transformations (18) is assumed. It is useful to introduce the generator of the background field transformation $\hat{\mathcal{R}}_\omega(\phi, \mathcal{B})$,

$$\begin{aligned} \hat{\mathcal{R}}_\omega(\phi, \mathcal{B}) &= \int dx \left(\overleftarrow{\frac{\delta}{\delta \mathcal{B}_\mu^a}} \delta_\omega^{(c)} \mathcal{B}_\mu^a + \overleftarrow{\frac{\delta}{\delta \phi^i}} \delta_\omega^{(q)} \phi^i \right) \\ &= \hat{\mathcal{R}}_\omega^{(c)}(\mathcal{B}) + \hat{\mathcal{R}}_\omega^{(q)}(\phi), \end{aligned} \tag{20}$$

where $\phi^j \hat{\mathcal{R}}_\omega^{(q)}(\phi) = \hat{\mathcal{R}}_\omega^j(\phi)$ and

$$\hat{\mathcal{R}}_\omega^j(\phi) = \left(\mathcal{R}_w^{(q)}(A), \mathcal{R}_w^{(q)}(B), \mathcal{R}_w^{(q)}(C), \mathcal{R}_w^{(q)}(\bar{C}) \right). \tag{21}$$

Using the new notations (20), the background field invariance of the Faddeev–Popov action (17) rewrites as

$$S_{FP}(\phi, \mathcal{B}) \hat{\mathcal{R}}_\omega(\phi, \mathcal{B}) = 0. \tag{22}$$

The symmetries (16) and (22) of the Faddeev–Popov action lead to the two very important properties at the quantum level. In order to reveal these consequences we have to introduce the extended generating functional of Green functions in the background field method in the form of functional integral

$$\begin{aligned} Z(J, \phi^*, \mathcal{B}) &= \int \mathcal{D}\phi \exp \left\{ \frac{i}{\hbar} [S_{FP}(\phi, \mathcal{B}) + \phi^*(\phi\hat{s}) + J\phi] \right\} \\ &= \exp \left\{ \frac{i}{\hbar} W(J, \phi^*, \mathcal{B}) \right\}, \end{aligned} \tag{23}$$

where $W = W(J, \phi^*, \mathcal{B})$ is the extended generating functional of connected Green functions and

$$J_A = \left(J_i, J_\alpha^B, \bar{J}_\alpha, J_\alpha \right) \tag{24}$$

are the external sources to the fields ϕ^A ($\varepsilon(J_A) = \varepsilon_A$). Furthermore, the new quantities (antifields) ϕ_A^* , with $\varepsilon(\phi_A^*) = \varepsilon_A + 1$, are the sources of the BRST transformations.

The introduction of antifields enable one to simplify the use of the BRST symmetry at the quantum level. The next step is to introduce the extended effective action $\Gamma = \Gamma(\Phi, \phi^*, \mathcal{B})$ through the Legendre transformation of $W(J, \phi^*, \mathcal{B})$

$$\Gamma(\Phi, \phi^*, \mathcal{B}) = W(J, \phi^*, \mathcal{B}) - J\Phi, \tag{25}$$

where

$$\Phi^A = \frac{\delta_l W}{\delta J_A} \quad \text{and} \quad \frac{\delta_r \Gamma}{\delta \Phi^A} = -J_A. \tag{26}$$

From one hand, one can prove that the BRST symmetry (16) of S_{FP} results in the Slavnov–Taylor identity [25,26]

$$\frac{\delta_r \Gamma}{\delta \Phi^A} \frac{\delta_l \Gamma}{\delta \phi_A^*} = 0. \tag{27}$$

On the other hand, the background field symmetry (22) of S_{FP} leads to the symmetry of the effective action under the background field transformations,

$$\tilde{\Gamma}(\Phi, \mathcal{B}) \hat{\mathcal{R}}_\omega(\Phi, \mathcal{B}) = 0, \quad \tilde{\Gamma}(\Phi, \mathcal{B}) = \Gamma(\Phi, \phi^* = 0, \mathcal{B}). \tag{28}$$

The fundamental object of the background field method is the background effective action $\Gamma(\mathcal{B}) \equiv \tilde{\Gamma}(\Phi = 0, \mathcal{B})$. Thanks to the linearity of $\hat{\mathcal{R}}_\omega(\Phi, \mathcal{B})$ with respect to the mean fields Φ^i , from (28) it follows

$$\delta_\omega^{(c)} \Gamma(\mathcal{B}) = 0, \quad \delta_\omega^{(c)} \mathcal{B}^i = R_\omega^i(\mathcal{B}) \omega^\alpha, \tag{29}$$

i.e. the background effective action is a gauge invariant functional of the external field \mathcal{B}^i .

The last important feature of the Faddeev–Popov quantization is related to the universality of the S -matrix, that is independent on the choice of the gauge fixing. According to the well-known result [27], the universality of the S -matrix is equivalent to the gauge fixing independent vacuum functional. In the background field formalism this functional is defined starting from (23) as

$$\begin{aligned} Z_\Psi(\mathcal{B}) &= Z(\mathcal{B}, J = \phi^* = 0) \\ &= \int \mathcal{D}\phi \exp \left\{ \frac{i}{\hbar} S_{FP}(\phi, \mathcal{B}) \right\}. \end{aligned} \tag{30}$$

Regardless this object depends on the background field, it is constructed for a certain choice of gauge $\Psi(\phi, \mathcal{B})$. However, it can be shown to be independent on this choice. Without the presence of background field, the discussion of this issue in usual QFT and in the FRG approach can be found in Ref. [16]. Here we generalize it for the background field method case.

Taking an infinitesimal change of the gauge fixing functional, $\Psi(\phi, \mathcal{B}) \rightarrow \Psi(\phi, \mathcal{B}) + \delta\Psi(\phi, \mathcal{B})$, we get

$$\begin{aligned} Z_{\Psi+\delta\Psi}(\mathcal{B}) &= \int \mathcal{D}\phi \exp \left\{ \frac{i}{\hbar} \left[S_{FP}(\phi, \mathcal{B}) \right. \right. \\ &\quad \left. \left. + \delta\Psi(\phi, \mathcal{B}) \hat{s}(\phi, \mathcal{B}) \right] \right\}. \end{aligned} \tag{31}$$

Then, after a change of variables in the form of BRST transformation (11) but with replacement of the constant parameter μ by the functional

$$\mu(\phi, \mathcal{B}) = \frac{i}{\hbar} \delta\Psi(\phi, \mathcal{B}), \tag{32}$$

one can show that

$$Z_{\Psi+\delta\Psi}(\mathcal{B}) = Z_\Psi(\mathcal{B}), \tag{33}$$

which is the starting point for the proof of the gauge fixing independence of the S -matrix [27, 28]. In the next sections we shall see how this and other features for the case of the Yang–Mills theory look in the framework of the FRG approach.

3 Background average effective action

In this section we shall discuss the use of the BFM applied to the FRG, following the original publication on this subject by Reuter and Wetterich [14] for the case of pure Yang–Mills theory with the action

$$S_0(A) = -\frac{1}{4} F_{\mu\nu}^a(A) F_{\mu\nu}^a(A), \tag{34}$$

where $F_{\mu\nu}^a(A) = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc} A_\mu^b A_\nu^c$ is the field strength for the non-Abelian vector field A_μ and g is coupling constant. The correspondence with the notations used in Sect. 2 reads

$$\begin{aligned} A^i &\rightarrow A_\mu^a, & B^i &\rightarrow B_\mu^a, & F_{\beta\gamma}^\alpha &\rightarrow f^{abc}, \\ R_\alpha^i(A) &\rightarrow D_\mu^{ab}(A) = \delta^{ab} \partial_\mu + gf^{acb} A_\mu^c. \end{aligned} \tag{35}$$

Here the structure coefficients f^{abc} of the gauge group are constant. The action (34) is invariant under the gauge transformations defined by the generator $D_\mu^{ab}(A)$ with an arbitrary gauge function ω^a with $\varepsilon(\omega^a) = 0$. In the Faddeev–Popov quantization, the Grassmann parity of the fields $\phi^A = (A_\mu^a, B^a, C^a, \bar{C}^a)$ is, respectively, $\varepsilon_A = (0, 0, 1, 1)$.

The background field formalism for Yang–Mills theory comprises the definition of the background field transformation

$$\begin{aligned} \delta_\omega^{(c)} B_\mu^a &= D_\mu^{ab}(\mathcal{B}) \omega^b, & \delta_\omega^{(q)} A_\mu^a &= gf^{abc} A_\mu^b \omega^c, \\ \delta_\omega^{(q)} B^a &= gf^{abc} B^b \omega^c, & \delta_\omega^{(q)} C^a &= gf^{abc} C^b \omega^c, \\ \delta_\omega^{(q)} \bar{C}^a &= gf^{abc} \bar{C}^b \omega^c. \end{aligned} \tag{36}$$

Note that the generator of the transformation in the sector of fields A_μ^a reads

$$D_\mu^{ab}(A + \mathcal{B}) - D_\mu^{ab}(\mathcal{B}) = gf^{acb} A_\mu^c, \tag{37}$$

and thus all the quantum fields transform according the same rule. The standard choice of the gauge-fixing function is

$$\chi^a(A, \mathcal{B}) = D_\mu^{ab}(\mathcal{B}) A_\mu^b. \tag{38}$$

It leads to the tensor transformation rule for $\chi^a(A, \mathcal{B})$ under the background field transformation,

$$\delta_\omega \chi^a(A, \mathcal{B}) = gf^{abc} \chi^b(A, \mathcal{B}) \omega^c. \tag{39}$$

The main point of the FRG approach is the introduction of the scale-dependent regulator action $S_k(\phi, \mathcal{B})$, in the framework of the background field method. Let us choose the regulator action for the quantum fields A_μ^a and C^a, \bar{C}^a in the

form

$$\begin{aligned} S_k(\phi, \mathcal{B}) &= \frac{1}{2} A_\mu^a R_k^{(1)ab}(D_T(\mathcal{B})) A_\mu^b \\ &\quad + \bar{C}^a R_k^{(2)ab}(D_S(\mathcal{B})) C^b. \end{aligned} \tag{40}$$

The regulator functions depend on the external field through the covariant derivatives of tensor D_T and scalar D_S fields

$$(D_T(\mathcal{B}))_{\mu\nu}^{ab} = -\eta_{\mu\nu} (D^2)^{ab} + 2gf^{acb} F_{\mu\nu}^c(\mathcal{B}), \tag{41}$$

$$(D^2)^{ab} = D_\rho^{ac}(\mathcal{B}) D_\rho^{cb}(\mathcal{B}),$$

$$(D_S(\mathcal{B}))^{ab} = -(D^2)^{ab}. \tag{42}$$

The form of these functions can be chosen e.g. as in [14],

$$R_k(z) = Z_k \frac{ze^{-z/k^2}}{1 - e^{-z/k^2}}, \tag{43}$$

with Z_k corresponding to the wave function renormalization.

Let us consider the variation of the regulator action (40) under the background field transformations (18) in the first order approximation, $R_k(z) = Z_k z$. The first term in (41) can be rewritten through integration by parts, as follows

$$-A_\mu^a \eta_{\mu\nu} (D^2)^{ab} A_\nu^b = \chi_{\rho\mu}^c(A, \mathcal{B}) \chi_{\rho\mu}^c(A, \mathcal{B}), \tag{44}$$

where

$$\chi_{\rho\mu}^a(A, \mathcal{B}) \equiv D_\rho^{ab}(\mathcal{B}) A_\mu^b. \tag{45}$$

The transformation rule for $\chi_{\rho\mu}^a(A, \mathcal{B})$ under the background field transformation is very close to (19). It has the form

$$\delta_\omega \chi_{\rho\mu}^a(A, \mathcal{B}) = gf^{acb} \chi_{\rho\mu}^c(A, \mathcal{B}) \omega^b. \tag{46}$$

As consequence, we find the first term invariance

$$\begin{aligned} \delta_\omega (-A_\mu^a \eta_{\mu\nu} (D^2)^{ab} A_\nu^b) &= \delta_\omega (\chi_{\rho\mu}^a(A, \mathcal{B}) \chi_{\rho\mu}^a(A, \mathcal{B})) \\ &= 2gf^{acb} \chi_{\rho\mu}^a(A, \mathcal{B}) \chi_{\rho\mu}^c(A, \mathcal{B}) \omega^b \\ &= 0. \end{aligned} \tag{47}$$

Furthermore, taking into account that

$$\delta_\omega^{(c)} F_{\mu\nu}^a(\mathcal{B}) = gf^{acb} F_{\mu\nu}^c(\mathcal{B}) \omega^b, \tag{48}$$

for the second term in (41), we have

$$\begin{aligned} \delta_\omega (f^{acb} A_\mu^a F_{\mu\nu}^c(\mathcal{B}) A_\nu^b) &= g A_\mu^a A_\nu^b F_{\mu\nu}^c (f^{ace} f^{ebd} + \\ &\quad + f^{abe} f^{edc} + f^{ade} f^{ecb}) = 0, \end{aligned}$$

because of the Jacobi identity. The invariance holds also for the ghost regulator, as one can easily verify. In this approximation the scale-dependent action $S_k(\phi, \mathcal{B})$ obeys the background field symmetry, $\delta_\omega S_k(\phi, \mathcal{B}) = 0$.

The same consideration can be done for the terms of the higher orders in z . Thus, we can ensure that the invariance is maintained in all orders. With these results the action (40) is invariant under the background field transformations,

$$\delta_\omega S_k(\phi, \mathcal{B}) = 0. \tag{49}$$

The full action $S_{k\,FP} = S_{k\,FP}(\phi, \mathcal{B})$ is constructed by the rule

$$S_{k\,FP}(\phi, \mathcal{B}) = S_{FP}(\phi, \mathcal{B}) + S_k(\phi, \mathcal{B}), \tag{50}$$

where $S_{FP}(\phi, \mathcal{B})$ is the Faddeev–Popov action (7). Using the action (50), the generating functional of Green function is given by the following functional integral¹:

$$\begin{aligned} Z_k(J, \mathcal{B}) &= \int \mathcal{D}\phi \exp \left\{ i[S_{FP}(\phi, \mathcal{B}) + S_k(\phi, \mathcal{B}) + J\phi] \right\} \\ &= \exp \{ iW_k(J, \mathcal{B}) \}, \end{aligned} \tag{51}$$

where $W_k = W_k(J, \mathcal{B})$ is the generating functional of connected Green functions. The main object of the FRG approach in the background field method is the background average effective action $\Gamma_k = \Gamma_k(\Phi, \mathcal{B})$, defined through the Legendre transform of W_k ,

$$\Gamma_k(\Phi, \mathcal{B}) = W_k(J, \mathcal{B}) - J\Phi, \tag{52}$$

where

$$\Phi^A = \frac{\delta_l W_k}{\delta J_A}$$

and

$$\frac{\delta_r \Gamma_k}{\delta \Phi^A} = -J_A.$$

The effective average action can be presented as a sum of the regulator action of the mean field and the quantum correction,

$$\Gamma_k(\Phi, \mathcal{B}) = S_k(\Phi, \mathcal{B}) + \bar{\Gamma}_k(\Phi, \mathcal{B}). \tag{53}$$

The functional $\bar{\Gamma}_k$ satisfies the flow equation, or the Wetterich equation [1, 2, 14],

$$\partial_t \bar{\Gamma}_k(\Phi, \mathcal{B}) = \frac{i}{2} \text{sTr} \left\{ \frac{\partial_t R_k(\mathcal{B})}{\bar{\Gamma}_k''(\Phi, \mathcal{B}) + R_k(\mathcal{B})} \right\}. \tag{54}$$

In (54) $\partial_t = k \frac{d}{dk}$ and the symbol sTr means the functional supertrace, this last is necessary due to the presence of quantum fields A_μ^a and C^a, \bar{C}^a , with different Grassmann parity. Another important notation is

$$\left(\bar{\Gamma}_k''(\Phi, \mathcal{B}) \right)_{AB} = \frac{\delta_l}{\delta \Phi^A} \left(\frac{\delta_r \bar{\Gamma}_k(\Phi, \mathcal{B})}{\delta \Phi^B} \right) \tag{55}$$

for the matrix of the second order functional derivatives with respect to the mean fields Φ .

¹ From here we adopt units in which $\hbar = 1$.

As we have seen above, because of the invariance of the scale-dependent regulator term (40), the full action (50) is invariant under the background field transformations (17),

$$\delta_\omega S_{k\,FP}(\phi, \mathcal{B}) = \delta_\omega S_k(\phi, \mathcal{B}) = S_k(\phi, \mathcal{B}) \hat{\mathcal{R}}_\omega(\phi, \mathcal{B}) = 0. \tag{56}$$

At the quantum level (56) provides the invariance of the background average effective action $\Gamma_k(\Phi, \mathcal{B})$. Indeed, variation of $Z_k(J, \mathcal{B})$ with respect to the external field \mathcal{B}_μ^a reads

$$\delta_\omega^{(c)} Z_k(J, \mathcal{B}) = i J_A \mathcal{R}_\omega^A \left(\frac{\delta_l Z_k}{i \delta J} \right). \tag{57}$$

In terms of the functional $W_k(J, \mathcal{B})$ the relation (57) rewrites

$$\delta_\omega^{(c)} W_k(J, \mathcal{B}) = J_A \mathcal{R}_\omega^A \left(\frac{\delta_l W_k}{\delta J} \right). \tag{58}$$

As a consequence of (58), the background average effective action is invariant under the background field transformations,

$$\delta_\omega \Gamma_k(\Phi, \mathcal{B}) = 0. \tag{59}$$

In terms of the functional $\bar{\Gamma}_k(\Phi, \mathcal{B})$ the relation (59) becomes

$$\delta_\omega \bar{\Gamma}_k(\Phi, \mathcal{B}) = 0. \tag{60}$$

Thus, the background field symmetry is preserved for the background average effective action $\bar{\Gamma}_k(\Phi, \mathcal{B})$, confirming the main statement of the paper [14].

For the functional $\bar{\Gamma}_k(\mathcal{B}) = \bar{\Gamma}_k(\Phi = 0, \mathcal{B})$, the background field symmetry is preserved as well due to linearity of the background field symmetry

$$\delta_\omega^{(c)} \bar{\Gamma}_k(\mathcal{B}) = 0, \tag{61}$$

in agreement with (29). In particular this means that the flow equation for $\bar{\Gamma}_k(\mathcal{B})$,

$$\partial_t \bar{\Gamma}_k(\mathcal{B}) = \frac{i}{2} \text{sTr} \left\{ \frac{\partial_t R_k(\mathcal{B})}{\bar{\Gamma}_k''(\Phi, \mathcal{B})|_{\Phi=0} + R_k(\mathcal{B})} \right\}, \tag{62}$$

maintains the background field symmetry.

4 Background invariant regulator functions

The prove of invariance of S_k under background field transformations (49) is based on the certain form of the regulator functions and its arguments. In particular, the regulator functions (43) with argument (41) by itself are not invariant under background field transformations $\delta_\omega^{(c)} R_{k\,\mu\nu}^{(1)ab}(D_T(\mathcal{B})) \neq 0$, $\delta_\omega^{(c)} R_k^{(2)ab}(D_S(\mathcal{B})) \neq 0$. In this section we shall discuss the background field symmetry of the background average effective action and formulate a possible restriction on the regulator functions in the scale-dependent action S_k in the general settings that allow us to arrive at the invariance of the

background average effective action under background field transformations.

Consider the scale-dependent regulator action $S_k = S_k(\phi, \mathcal{B})$ in the background field formalism, including the ghost sector,

$$S_k(\phi, \mathcal{B}) = \frac{1}{2} A_\mu^a R_k^{(1)ab}(\mathcal{B}) A_\nu^b + \bar{C}^a R_k^{(2)ab}(\mathcal{B}) C^b, \quad (63)$$

where $R_k^{(1)ab}(\mathcal{B})$ and $R_k^{(2)ab}(\mathcal{B})$ are the regulator functions. We assume that they are local functions of external fields \mathcal{B}_μ^a and their partial derivatives. The full action has a standard FRG form

$$S_{kFP}(\phi, \mathcal{B}) = S_{FP}(\phi, \mathcal{B}) + S_k(\phi, \mathcal{B}). \quad (64)$$

Due to the background field symmetry of the Faddeev–Popov action (17), the full action (64) will be invariant under the background field transformations (18), if the scale-dependent regulator action $S_k = S_k(\phi, \mathcal{B})$ satisfies the equation

$$\delta_\omega S_k(\phi, \mathcal{B}) = 0. \quad (65)$$

Using the explicit form of the background field transformations (18) the variation of $S_k(\phi, \mathcal{B})$ reads

$$\begin{aligned} \delta_\omega S_k(\phi, \mathcal{B}) = & \frac{1}{2} A_\mu^a \left[g \left(f^{adc} \omega^d R_k^{(1)cb}(\mathcal{B}) \right. \right. \\ & \left. \left. - R_k^{(1)ac}(\mathcal{B}) f^{cdb} \omega^d \right) + \delta_\omega^{(c)} R_k^{(1)ab}(\mathcal{B}) \right] A_\nu^b \\ & + \int dx \bar{C}^a \left[g \left(f^{adc} \omega^d R_k^{(2)cb}(\mathcal{B}) \right. \right. \\ & \left. \left. - R_k^{(2)ac}(\mathcal{B}) f^{cdb} \omega^d \right) + \delta_\omega^{(c)} R_k^{(2)ab}(\mathcal{B}) \right] C^b. \end{aligned} \quad (66)$$

From Eq. (66) follows that (65) is satisfied if

$$g \left(f^{adc} \omega^d R_k^{(1)cb}(\mathcal{B}) - R_k^{(1)ac}(\mathcal{B}) f^{cdb} \omega^d \right) + \delta_\omega^{(c)} R_k^{(1)ab}(\mathcal{B}) = 0, \quad (67)$$

$$g \left(f^{adc} \omega^d R_k^{(2)cb}(\mathcal{B}) - R_k^{(2)ac}(\mathcal{B}) f^{cdb} \omega^d \right) + \delta_\omega^{(c)} R_k^{(2)ab}(\mathcal{B}) = 0. \quad (68)$$

Any solution of these equations provides the invariance of S_k under background field transformations. Let us consider the case when regulator functions are invariant under background transformations of external field \mathcal{B}_μ^a ,

$$\delta_\omega^{(c)} R_k^{(1)ab}(\mathcal{B}) = 0, \quad \delta_\omega^{(c)} R_k^{(2)ab}(\mathcal{B}) = 0. \quad (69)$$

Due to the arbitrariness in the choice of the functions $\omega^a(x)$, from (67)–(69) follow the relations

$$[t^d, R_k^{(1)\mu\nu}(\mathcal{B})]_{ab} = 0, \quad [t^d, R_k^{(2)\mu\nu}(\mathcal{B})]_{ab} = 0, \quad (70)$$

for the generators $(t^a)_{bc} = f^{bac}$ of the Lie group. Therefore, we see that the regulator functions commute with all the generators of Lie group. Then, applying the Shur’s lemma we find

$$\begin{aligned} R_k^{(1)ab}(\mathcal{B}) &= \delta^{ab} R_k^{(1)}(D(\mathcal{B})), \\ R_k^{(2)ab}(\mathcal{B}) &= \delta^{ab} R_k^{(2)}(D(\mathcal{B})), \end{aligned} \quad (71)$$

where the quantities $R_k^{(1)}(D(\mathcal{B}))$ and $R_k^{(2)}(D(\mathcal{B}))$ are scalars with respect to the background transformations of external field \mathcal{B}_μ^a . It means that the arguments of these quantities should be scalars as well. It is easy to construct an example of such kind of a scalar argument, $D(\mathcal{B}) = F_{\mu\nu}^a(\mathcal{B}) D_\mu^{ab}(\mathcal{B}) \mathcal{B}_\nu^b$, where $F_{\mu\nu}^a$ is defined in (34).

So, in the case under consideration, the scale-dependent regulator action has the form

$$\begin{aligned} S_k(\phi, \mathcal{B}) = & \frac{1}{2} A_\mu^a R_k^{(1)}(D(\mathcal{B})) A_\nu^a \\ & + \bar{C}^a(x) R_k^{(2)}(D(\mathcal{B})) C^a, \end{aligned} \quad (72)$$

maintaining the background field symmetry $\delta_\omega S_k(\phi, \mathcal{B}) = 0$.

5 Gauge dependence of background average effective action

Here the problem of gauge dependence of background average effective action will be discussed in general setting of Sect. 2. The regulator action S_k is invariant under the background transformations (49), but not under the BRST transformations,

$$S_k(\phi, \mathcal{B}) \hat{s}(\phi, \mathcal{B}) \neq 0. \quad (73)$$

Let us discuss the implications of this fact for the gauge dependence problem of the background average effective action. Consider the extended generating functional of Green functions $Z_k(J, \phi^*, \mathcal{B})$, and the extended generating functional of connected Green functions $W_k(J, \phi^*, \mathcal{B})$,

$$\begin{aligned} Z_k(J, \phi^*, \mathcal{B}) = & \int \mathcal{D}\phi \exp\{i[S_{FP}(\phi, \mathcal{B}) + S_k(\phi, \mathcal{B}) + J\phi \\ & + \phi^*(\hat{s}\phi)]\} = \exp\{iW_k(J, \phi^*, \mathcal{B})\}, \end{aligned} \quad (74)$$

As the first step we derive the modified Ward identity for the FRG in the BFM which is a consequence of the BRST invariance of the action $S_{FP}(\phi, \mathcal{B})$ (16). Making use the change of variables in the form of the BRST transformations in the functional integral (74), $\phi^A \rightarrow \varphi^A(\phi) = \phi^A + (\hat{s}\phi^A)\mu$, and taking into account the triviality of the corresponding Jacobian if the conditions

$$(-1)^{\varepsilon_i} \frac{\delta_l R_\alpha^i}{\delta A^i} + (-1)^{(\varepsilon_\alpha+1)} F_{\beta\alpha}^\beta = 0 \quad (75)$$

are satisfied (for detailed discussion of this point see [29]), we arrive at the relation

$$\begin{aligned}
 0 &= \int D\phi (J_A + S_{k,A}(\phi, \mathcal{B})) (\hat{s}\phi^A) \\
 &\quad \times \exp\{i[S_{FP}(\phi, \mathcal{B}) + S_k(\phi, \mathcal{B}) + J\phi + \phi^*(\hat{s}\phi)]\} \\
 &= \left[J_A + S_{k,A} \left(\frac{\delta_l}{i\delta J}, \mathcal{B} \right) \right] \int D\phi (\hat{s}\phi^A) \exp\{i[S_{FP}(\phi, \mathcal{B}) \\
 &\quad + S_k(\phi, \mathcal{B}) + J\phi + \phi^*(\hat{s}\phi)]\}. \tag{76}
 \end{aligned}$$

From (76) it follows the modified Ward identity for the extended generating functional of Green functions $Z_k(J, \phi^*, \mathcal{B})$

$$\left[J_A + S_{k,A} \left(\frac{\delta_l}{i\delta J}, \mathcal{B} \right) \right] \frac{\delta_l}{\delta\phi_A^*} Z_k(J, \phi^*, \mathcal{B}) = 0. \tag{77}$$

This identity in terms of the extended generating functional of connected Green functions $W_k(J, \phi^*, \mathcal{B})$ reads

$$\left[J_A + S_{k,A} \left(\frac{\delta_l W_k}{\delta J} + \frac{\delta_l}{i\delta J}, \mathcal{B} \right) \right] \frac{\delta_l}{\delta\phi_A^*} W_k(J, \phi^*, \mathcal{B}) = 0. \tag{78}$$

Introducing the generating functional of vertex functions $\Gamma_k = \Gamma_k(\Phi, \phi^*, \mathcal{B})$ with the help of Legendre transformation of $W_k = W_k(J, \phi^*, \mathcal{B})$

$$\begin{aligned}
 \Gamma_k(\Phi, \phi^*, \mathcal{B}) &= W_k(J, \phi^*, \mathcal{B}) - J_A \Phi^A, \\
 \Phi^A &= \frac{\delta_l W_k}{\delta J_A}, \\
 \frac{\delta_r \Gamma_k}{\delta \Phi^A} &= -J_A, \quad \frac{\delta_l \Gamma_k}{\delta \phi_A^*} = \frac{\delta_l W_k}{\delta \phi_A^*}, \tag{79}
 \end{aligned}$$

the modified Ward identity rewrites in the form

$$\frac{\delta_r \Gamma_k}{\delta \Phi^A} \frac{\delta_l \Gamma_k}{\delta \phi_A^*} = S_{k,A}(\hat{\Phi}, \mathcal{B}) \frac{\delta_l \Gamma_k}{\delta \phi_A^*}, \tag{80}$$

where the notation

$$\hat{\Phi}^A = \Phi^A + i (\Gamma_k''^{-1})^{AB} \frac{\delta_l}{\delta \Phi^B}, \tag{81}$$

has been used. The matrix $(\Gamma_k''^{-1})$ is inverse to the matrix Γ_k'' , the last has elements

$$\begin{aligned}
 (\Gamma_k'')_{AB} &= \frac{\delta_l}{\delta \Phi^A} \left(\frac{\delta_r \Gamma_k}{\delta \Phi^B} \right), \quad \text{i.e.,} \\
 (\Gamma_k''^{-1})^{AC} \cdot (\Gamma_k'')_{CB} &= \delta_B^A. \tag{82}
 \end{aligned}$$

Now consider the variation of the extended generating functional of Green functions under infinitesimal variation of the gauge fixing functional, $\Psi(\phi, \mathcal{B}) \rightarrow \Psi(\phi, \mathcal{B}) +$

$\delta\Psi(\phi, \mathcal{B})$. We find

$$\begin{aligned}
 &\delta Z_k(J, \phi^*, \mathcal{B}) \\
 &= i \int D\phi (\delta\Psi(\phi, \mathcal{B}) \hat{s}(\phi, \mathcal{B})) \exp\{i[S_{FP}(\phi, \mathcal{B}) \\
 &\quad + S_k(\phi, \mathcal{B}) + J\phi + \phi^*(\hat{s}\phi)]\}. \tag{83}
 \end{aligned}$$

Now take into account that the functional integral of total variational derivative is zero we have the relation

$$\begin{aligned}
 0 &= \int D\phi \frac{\delta_r}{\delta \phi^A} \left[(\delta\Psi s^A) \exp\{i[S_{FP}(\phi, \mathcal{B}) + S_k(\phi, \mathcal{B}) \\
 &\quad + J\phi + \phi^*(\hat{s}\phi)]\} \right] = \int D\phi \left[i\delta\Psi s^A (J_A + S_{k,A}) + (\delta\Psi \hat{s}) \right] \\
 &\quad \times \exp\{i[S_{FP}(\phi, \mathcal{B}) + S_k(\phi, \mathcal{B}) + J\phi + \phi^*(\hat{s}\phi)]\}, \tag{84}
 \end{aligned}$$

where the BRST invariance of S_{FP} action, the nilpotency of BRST transformations and the relations (75) have been used. From (84) one has

$$\begin{aligned}
 i \int D\phi (\delta\Psi(\phi, \mathcal{B}) \hat{s}(\phi, \mathcal{B})) \exp\{i[S_{FP}(\phi, \mathcal{B}) + S_k(\phi, \mathcal{B}) \\
 + J\phi + \phi^*(\hat{s}\phi)]\} &= \int D\phi (J_A + S_{k,A} \phi, \mathcal{B}) s^A(\phi, \mathcal{B}) \\
 &\quad \times \delta\Psi(\phi, \mathcal{B}) \exp\{i[S_{FP}(\phi, \mathcal{B}) \\
 &\quad + S_k(\phi, \mathcal{B}) + J\phi + \phi^*(\hat{s}\phi)]\}, \tag{85}
 \end{aligned}$$

which allows to present the Eq. (83) in the form closed with respect to $Z_k(J, \phi^*, \mathcal{B})$,

$$\begin{aligned}
 \delta Z_k(J, \phi^*, \mathcal{B}) &= -i \left[J_A + S_{k,A} \left(\frac{\delta_l}{i\delta J}, \mathcal{B} \right) \right] \\
 &\quad \times \frac{\delta_l}{\delta \phi_A^*} \delta\Psi \left(\frac{\delta_l}{i\delta J}, \mathcal{B} \right) Z_k(J, \phi^*, \mathcal{B}), \tag{86}
 \end{aligned}$$

or, in terms of $W_k(J, \mathcal{B})$,

$$\begin{aligned}
 \delta W_k(J, \phi^*, \mathcal{B}) &= - \left[J_A + S_{k,A} \left(\frac{\delta_l W_k}{\delta J} + \frac{\delta_l}{i\delta J}, \mathcal{B} \right) \right] \\
 &\quad \times \frac{\delta_l}{\delta \phi_A^*} \delta\Psi \left(\frac{\delta_l W_k}{\delta J} + \frac{\delta_l}{i\delta J}, \mathcal{B} \right). \tag{87}
 \end{aligned}$$

In deriving (87) the modified Slavnov–Taylor identity (78) has been used. The last equation can be rewritten for the background average effective action, $\Gamma_k(\Phi, \phi^*, \mathcal{B})$, in the form

$$\begin{aligned}
 \delta \Gamma_k(\Phi, \phi^*, \mathcal{B}) &= \frac{\delta_r \Gamma_k}{\delta \Phi^A} \frac{\delta_l}{\delta \phi_A^*} \delta\Psi(\hat{\Phi}, \mathcal{B}) \\
 &\quad - S_{k,A}(\hat{\Phi}, \mathcal{B}) \frac{\delta_l}{\delta \phi_A^*} \delta\Psi(\hat{\Phi}, \mathcal{B}), \tag{88}
 \end{aligned}$$

where $\hat{\Phi}$ was introduced in (81). From Eq. (88) follows that

$$\delta \Gamma_k(\Phi, \phi^*, \mathcal{B}) \Big|_{\frac{\delta \Gamma_k}{\delta \Phi} = 0} \neq 0. \tag{89}$$

As result, the average effective action depends on gauge fixing even on the equations of motion (on-shell) and the S -matrix defined in the framework of the FRG approach is gauge dependent.

6 Conclusions

We considered several aspects of background average effective action in the FRG framework. At the first place we confirmed the well-known classical result of [14] concerning the background invariance of the regulator actions and background average effective action in the framework of the background field method for a wide class of regulator functions which include (43), but can be generalized to any other functions of the arguments z . As a new technical result we formulated general conditions of regulator actions being invariant with respect to the purely background transformations.

The main motivation of this work was to check whether the on-shell dependence of the average effective action [16] holds within the background field method formalism. The answer to this question is given by the relation (89) and is strictly positive. This output does not contradict the recent works [9, 15] because in these publications the subject of study was the gauge invariance of background average effective action, and the question of gauge fixing dependence was not investigated. From our viewpoint, the on-shell gauge dependence of the average effective action is a fundamental principal difficulty of the FRG approach applied to the Yang–Mills theories. We have confirmed that the situation does not improve in the background field method, regardless of the different structure of lifting the degeneracy of the classical action.

It is unclear whether one can achieve a reasonable physical interpretation of the results obtained within the FRG formalism applied to Yang–Mills theories, and therefore it makes sense to discuss the possible ways out from this difficult situation.

Certainly the simplest way is to ignore the problem e.g. by deciding that one special gauge fixing is “physical” or “correct”, such that changing the gauge should be strictly forbidden. As far as FRG provides valuable nonperturbative results, the theoretically inconsistent formulation is the price to pay for going beyond the well-defined perturbative framework.

Another possibility is to look for some observables that may be gauge-fixing invariant. For instance, in the fixed point the background average effective action boils down to the standard QFT effective action and then S -matrix, amplitudes and all related observables are well-defined. Unfortunately, even in the vicinity of the fixed point this is not true due to the relation (89). Since the search of the nonperturbative fixed point is based on the renormalization group flows and the last

are supposed to be gauge-fixing dependent, it is unclear how the fixed-point invariance can be actually used.

Finally, there is an alternative formulation of the FRG in gauge theories which is gauge-fixing independent, exactly as a conventional perturbative QFT is [16]. This scheme is technically more difficult, since the regulator actions are constructed in a more complicated way, that includes composite fields. At least by now, the disadvantage of this approach is that there is no method to perform practical calculations.

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References

1. C. Wetterich, Average action and the renormalization group equations. Nucl. Phys. B **352**, 529 (1991)
2. C. Wetterich, Exact evolution equation for the effective potential. Phys. Lett. B **301**, 90 (1993)
3. T.R. Morris, The exact renormalization group and approximate solutions. Int. J. Mod. Phys. A **9**, 2411 (1994)
4. J. Berges, N. Tetradis, C. Wetterich, Non-perturbative renormalization flow in quantum field theory and statistical physics. Phys. Rep. **363**, 223 (2002)
5. C. Bagnuls, C. Bervillier, Exact renormalization group equations: an introductory review. Phys. Rep. **348**, 91 (2001)
6. H. Gies, *Introduction to the Functional RG and Applications to Gauge Theories*, vol. 62, Lecture Notes in Physics (Springer, Berlin, 2012)
7. J.M. Pawłowski, Aspects of the functional renormalisation group. Ann. Phys. **322**, 2831 (2007)
8. A. Wipf, *Statistical Approach to Quantum Field Theory: An Introduction*, vol. 864, Lecture Notes in Physics (Springer, Berlin, 2013)
9. C. Wetterich, Gauge-invariant fields and flow equations for Yang–Mills theories. Nucl. Phys. B **934**, 265 (2018). [arXiv:1710.02494](https://arxiv.org/abs/1710.02494)
10. A.O. Barvinsky, D. Blas, M. Herrero-Valea, S.M. Sibiryakov, C.F. Steinwachs, Renormalization of gauge theories in the background-field approach. JHEP **1807**, 035 (2018). [arXiv:1705.03480](https://arxiv.org/abs/1705.03480)
11. I.A. Batalin, P.M. Lavrov, I.V. Tyutin, Multiplicative renormalization of Yang–Mills theories in the background-field formalism. Eur. Phys. J. C **78**, 570 (2018)

12. J. Frenkel, J.C. Taylor, Background gauge renormalization and BRST identities. *Ann. Phys.* **389**, 234 (2018)
13. P.M. Lavrov, Gauge (in)dependence and background field formalism. *Phys. Lett. B* **791**, 293 (2019). [arXiv:1805.02149](https://arxiv.org/abs/1805.02149)
14. M. Reuter, C. Wetterich, Effective average action for gauge theories and exact evolution equations. *Nucl. Phys. B* **417**, 181 (1994)
15. A. Codello, Renormalization group flow equations for the proper vertices of the background effective average action. *Phys. Rev. D* **91**, 065032 (2015). [arXiv:1304.2059](https://arxiv.org/abs/1304.2059)
16. P.M. Lavrov, I.L. Shapiro, On the functional renormalization group approach for Yang–Mills fields. *JHEP* **1306**, 086 (2013). [arXiv:1212.2577](https://arxiv.org/abs/1212.2577)
17. B.S. DeWitt, *Dynamical Theory of Groups and Fields* (Gordon and Breach, New York, 1965)
18. B.S. De Witt, Quantum theory of gravity. II. The manifestly covariant theory. *Phys. Rev.* **162**, 1195 (1967)
19. I.Ya. Arefeva, L.D. Faddeev, A.A. Slavnov, Generating functional for the s matrix in gauge theories. *Theor. Math. Phys.* **21**, 1165 (1975)
20. I.Ya. Arefeva, L.D. Faddeev, A.A. Slavnov, Generating functional for the s matrix in gauge theories. *Teor. Mat. Fiz.* **21**, 311–321 (1974)
21. L.F. Abbott, The background field method beyond one loop. *Nucl. Phys. B* **185**, 189–203 (1981)
22. L.D. Faddeev, V.N. Popov, Feynman diagrams for the Yang–Mills field. *Phys. Lett. B* **25**, 29 (1967)
23. C. Becchi, A. Rouet, R. Stora, The abelian Higgs–Kibble model, unitarity of the S-operator. *Phys. Lett. B* **52**, 344 (1974)
24. I.V. Tyutin, *Gauge Invariance in Field theory and Statistical Physics in Operator Formalism*, vol. 39 (Lebedev Institute, Moscow, 1975). (**preprint**)
25. J.C. Taylor, Ward identities and charge renormalization of the Yang–Mills field. *Nucl. Phys. B* **33**, 436 (1971)
26. A.A. Slavnov, Ward identities in gauge theories. *Theor. Math. Phys.* **10**, 99 (1972)
27. R.E. Kallosh, I.V. Tyutin, The equivalence theorem and gauge invariance in renormalizable theories. *Sov. J. Nucl. Phys.* **17**, 98 (1973)
28. I.V. Tyutin, Once again on the equivalence theorem. *Phys. Atom. Nucl.* **65**, 194 (2002)
29. B.L. Giacchini, P.M. Lavrov, I.L. Shapiro, Background field method and nonlinear gauges. [arXiv:1906.04767](https://arxiv.org/abs/1906.04767) [hep-th]