p -Branes with AdS_{p+1} vacuum as models of R^2 gravity

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Abstract Branes with *constant mean curvature* of their hyper-worldsheets of codimension 1 are treated as the Nambu-Goldstone fields of the broken Poincare symmetry. Mapping of their action into quadratic curvature gravity action with spontaneously generated gravity, is shown. Equation for the brane potential extremals and its solution describing hyper-ws of *constant curvature* are found. For membranes in $\mathbf{R}^{1,3}$ this extremum is shown to be a *saddle* 3-dim. hypersurface which defines classically instable vacuum.

1 Introduction

Invention of the Green–Schwarz superstring and kappa symmetry led to the construction of a renormalizable ten-dimensional theory of all fundamental interactions including quantum gravity [1]. The local kappa symmetry acting on two-dimensional worldsheet of superstring is a key element of the theory which provides the balance between the bosonic and fermionic degrees of freedom. The very existence of this symmetry imposes essential restrictions on the dimension of the ambient supersymmetric space-time in which strings move. The admissible dimensions of the space-time are $D = 3, 4, 6, 10$, but only $D = 10$ turns out to be consistent in the quantum picture resulting in five possible renormalizable theories [2]. The latter are known as type I, type IIA, type IIB supplemented by two theories originating from heterotic strings invariant under $SO(32)$ and $E8 \times E8$ gauge symmetries, respectively. These five theories are based on different types of strings: type I theory includes both open and closed strings, while types IIA and IIB include only closed strings. All these theories are entangled by S and T duality transformations. In particular, S -duality transforms type I theory to the $SO(32)$ heterotic one, and type IIB theory into itself. The latter is transformed into type IIA theory under T -duality

transformations [3]. These theories are believed to be limiting cases of one unknown theory which consistently describes quantum gravity. Herewith, the $D = 10, N = 2a$ supergravity, in particular, arises as low energy limit of Type IIA superstring. In this limit which corresponds to the infinitely large string tension the discussed five theories yield different 10-dimensional versions of supergravity. The later belongs to field theories associated with point-like particles treated as idealized geometric objects with the dimension $p = 0$. So, the transition from point-like particles to one-dimensional extended objects ($p = 1$) cures $D = 10$ supergravity from ultraviolet divergencies. However, this scenario works only for the dimension $D = 10$ that implies the Kaluza-Klein compactification of six excess coordinates in order to restore the physical dimension $D = 4$. But such a compactification creates the Landscape problem [4] that means loss of uniqueness in the choice of the physically relevant universes describing $SU(3) \times SU(2) \times U(1)$ invariant interactions and three families of quarks and leptons [5]. Unfortunately, the list of such universes contains billions of candidates. Another problem is the existence of unique $N = 1$ supergravity in eleven dimensions which cannot be derived from the superstring theories in 10-dimensional space-time. At the same time the 10-dimensional IIA superstring is derived from 11-dimensional supermembrane [6] through simultaneous dimensional reduction in both $D = 11$ space-time and 3-dimensional ($p = 2$) supermembrane worldsheet [7]. In view of this observation, the $N = 2a, D = 10$ supergravity is also derived from the $N = 1, D = 11$ supermembrane. As a result, supermembranes and superstrings get equal status in the frame of mysterious 11-dimensional M -theory [8]. But the story does not end there, in view of the tendency to further growth of the dimension p of relevant extended objects as shown in [9]. Therein, the set of the numbers p and D , ensuring the presence of the kappa symmetry on a $(p + 1)$ -dimensional hyper-worldsheet embedded into D -dimensional Minkowski space, was revealed. This set includes, in particular, the values $p = 2, D = 4, 5, 7, 11$;

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$p = 3, D = 6, 8$; $p = 4, D = 9$ and $p = 5, D = 10$ corresponding to the embeddings of membranes and 3-, 4-, 5-branes into the mentioned higher-dimensional Minkowski spaces. The pair $p = 5, D = 10$ describes the heterotic 5-brane associated with the soliton solution of the heterotic string equations found in [10]. In M -theory the supermembrane and super 5-brane are called $M2$ and $M5$ branes. M -branes together with superstrings are treated as fundamental constituents of supersymmetric 11-dimensional M -theory unifying gravity with other fundamental forces. Thus, both branes and strings play key roles in theoretical understanding of fundamental interactions, phenomenology of elementary particle physics, black holes, dark matter and AdS/CFT correspondence. A comprehensive review of these and other problems of contemporary physics in which branes play a vital role may be found in [11]. The breakthrough role of branes in attempts to reveal new physics sharpens the problem of their quantum consistency. Solution of this problem has to shed a new light on quantization of gravity. Until now quantization of extended objects remains an open problem even for the case of (super)membranes ($p = 2$) [12]. This uncertainty in the quantum status of the brane paradigm requires new tools for analyzing its viability. A weak progress in this matter is explained by a complicated non-linear character of brane equations and constraints. The non-linearity problem is typical of the theories unifying the Yang-Mills, gravitational and other fields covariant under diffeomorphisms, gauge and internal symmetries. To solve the problem, we need to deepen understanding of the structure of brane non-linearities.

Here we assume that the non-linearities can be converted into the geometric structures known from gravity and gauge theories. For branes, the proof of this assumption allows to implement the BRST-BFV quantization and others methods used for general non-linear systems. We expect that the non-linearities of ordinary p -branes can be described by a combination of scalars built from the curvature tensor of $(p+1)$ -dimensional hyper-worldsheets (h-ws).¹ It implies a map of non-linear terms into those known from $(p+1)$ -dim. $f(R)$ gravity which generalizes the Starobinsky 4-dim. model [13]. Construction of such a map will also give an additional motivation for the hypothesis that our world is a 3-brane embedded into higher-dimensional spaces [14–17].

The inherent non-linearities prevent the use of string harmonic oscillators for brane quantization. On the other hand, it is known that for some non-linear systems the transition to new dynamical variables, similar to the action-angle variables, permits to linearize equations of motion and to solve them. Then, the general solution makes it possible to perform quantization in terms of the initial data. Such a possibility

necessitates search for variables alternative to the world vectors of brane h-ws. In string theory Regge and Lund [18] used the first and the second fundamental forms of string worldsheets embedded into 4-dim. Minkowski space as new variables (see also [19,20]). Generalization of this idea for strings in D -dim. Minkowski space, accompanied with treatment of these variables as gauge multiplets, allowed to transform string equations into an exactly integrable system of PDEs [21,22]. So, it is natural to use such geometric variables for p -branes to verify whether brane non-linearities are integrable. For this purpose we apply the renowned differential-geometric approach describing hypersurfaces embedded into Riemannian spaces based on the Maurer-Cartan structure equations [23–26]. This approach was represented in terms of gauge theory and spontaneous symmetry breaking in [27] for the Dirac (fundamental) p -branes sweeping minimal h-ws. The spontaneous symmetry breaking (see [28] and refs. there) appears due to embedding of hypersurfaces in space-time. In the case a h-ws Σ_{p+1}^{min} swept by a fundamental p -brane [29] in D -dim. Minkowski space causes spontaneous breaking of its Poincare symmetry $ISO(1, D-1)$ to the subgroup $ISO(1, p) \times SO(D-p-1)$ [30–33]. The breaking yields the Nambu-Goldstone (N-G) fields identified with the coefficients $l_{\mu\nu}^a$ of the second fundamental form of Σ_{p+1}^{min} with arbitrary codimension. These coefficients form a N-G tensor multiplet of the local group $SO(D-p-1)$. The Y-M multiplet B_{μ}^{ab} of this group, together with the N-G multiplet and the metric $g_{\mu\nu}$, compose a set of $(p+1)$ -dim. h-ws fields. Their invariant action S_{Dir} and the potential $V_{Dir}(l, g)$ built in [27] are given by Eqs. (4) and (5) from Sect. 2. The potential encodes the non-linearities of fundamental p -branes in terms of $l_{\mu\nu}^a$ and $g_{\mu\nu}$. So, $V_{Dir}(l, g)$ expressed through the curvature tensor of Σ_{p+1}^{min} has to uncover the required geometric structure of brane non-linearities.

This scheme was realized in [34] for fundamental p -branes sweeping minimal h-ws with codimension 1 (i.e. $D = p+2$). In this case the gauge field B_{μ}^{ab} is absent. The Dirac membrane $p = 2, D = 4$ is an example of such branes. It describes the bosonic sector of the supermembrane from the above discussed list of branes invariant under the kappa-symmetry [9]. The case $p = 3, D = 5$ selects fundamental 3-branes living in 5-dim. Minkowski space. The 3-brane potential follows from Eq. (5)

$$V_{Dir}|_{p=3} = \left[-\frac{1}{2} S p(l^2) S p(l^2) + c_p \right] |_{p=3}. \quad (1)$$

As shown in [34], V_{Dir} is expressed through the squared scalar curvature R of the h-ws Σ_{p+1}^{min} by Eq. (10) from Sect. 3.² It means that action (4) with codim 1 encodes a particular case of $(p+1)$ -dim. gravity theories quadratic

¹ To be short we will further use the abbreviation h-ws for hyper-worldsheet(s).

² The formula (1) is the same for any $(p+1)$ -dim. minimal h-ws Σ_{p+1}^{min} with codim 1.

in curvature [35–38]. So, the transition to new geometric variables uncovers the structure of non-linearities of fundamental branes with codim 1 and proves their similarity to non-linearities of $f(R)$ theories of gravity.

For $c_p = 0$ action S_{Dir} (4) contains only one coupling constant $\kappa_p \sim T_p^{\frac{3-p}{2(p+1)}}$, where T_p is the tension of p -branes. The power law shows three different regimes of the behavior of κ_p as a function of the tension and, consequently, the energy corresponding to the cases $p < 3$, $p > 3$ and $p = 3$. These regimes testify to the presence of the phases of asymptotic freedom and confinement and their links with inflation and collapse of the branes.³

The phase with $p = 3$ is peculiar because the coupling constant κ_3 becomes dimensionless, so that $S_{Dir}|_{p=3}$ is scale-invariant. The latter forbids the Hilbert–Einstein (H–E) term because it breaks the scale symmetry. To hold the desired connection of 3-branes with our 4-dim. world, we have to modify the potential of fundamental branes to activate the H–E term [42].

Here we find that the same problem interferes with 4-dim. scale-invariant and renormalizable theories of gravity quadratic in curvature (see [43–49] and refs. there). The latter are converted to gravity theories containing the H–E term by implementing the idea of spontaneously generated gravity. This idea is illustrated by a simple example using the scale-invariant action [45]

$$A = \int d^4x \sqrt{|g|} \left\{ \frac{\alpha}{2} \varphi^2 R + \frac{1}{2} \nabla_\mu \varphi \nabla_\nu \varphi g^{\mu\nu} - V(\varphi, g) \right\}. \quad (2)$$

Action (2) contains a scalar φ and a dimensionless constant α . The potential $V(\varphi, g)$ is assumed to have a deep minimum at $\varphi_o = v$ which provides vacuum expectation value (vev) v for φ . Expansion of the first term in (2) around the minimum generates the H–E term with the Newton constant $G_N \approx \frac{1}{\alpha v^2}$. So, the scale symmetry of (2) is spontaneously broken that results in a 4-dim. gravity in the low energy limit. On the contrary, in the early universe, v as a function of the temperature, is expected to vanish resulting in a scale-invariant R^2 action. This model prevents the appearance of a cosmological constant usually arising in such cases [50]. The replacement of φ by a scalar $\bar{\psi}\psi$ proposed by Adler does not improve the situation [44].

Here we show that this problem is overcome by the substitution of the tensor field $l_{\mu\nu}$ for the scalar φ and the use of the brane action $S_{Dir}|_{p=3}$ (4), comprising potential (1), instead of action (2). As a result, we find a modified brane potential and the corresponding new models of gravity quadratic in curvature. These models include the tensor $l_{\mu\nu}$ playing

the role of the Brans–Dicke scalars [51] commonly used in extended models of gravity.

These results are based on the following observations. First we note that the fundamental 3-brane potential (1) has the extremal at $l_{\mu\nu} = 0$. This means the absence of spontaneous breaking of the scaling, because we choose the vev associated with $l_{\mu\nu}$ equal to the value of the diff-invariant trace $Spl \equiv l_{\mu\nu} g^{\mu\nu}$ at the extremal. So, the vev turns out to be zero at $l_{\mu\nu} = 0$. Secondly, keeping in mind that Spl is a fundamental invariant of embedded hypersurfaces, called mean curvature, we resume that $l_{\mu\nu} = 0$ corresponds to a flat h-ws. This makes us search for a deformation of potential (1) to a new one which has its extremal at $l_{\mu\nu} = l_{o\mu\nu} \neq 0$, so that $Spl_o = \mu$, where the constant μ has the dimension $[\mu] = [L^{-1}]$. From this we conclude that the required deformation implies the transition from minimal h-ws Σ_4^{min} to constant mean curvature (cmc) h-ws Σ_4^{cmc} .⁴ This observation reduces the discussed problem to the construction of a potential $V_{cmc}(l, g)$ which encodes hypersurfaces with the constant mean curvature $Spl = \mu$.

The required potential $V_{cmc}(l, g, \mu)$ is constructed in Sect. 3 extending the discussed mapping to $(p + 1)$ -dim. hypersurfaces Σ_{p+1}^{cmc} with codim 1 and constant mean curvature. As a result, we find action S_{cmc} (23) encoding the H–E action with a cosmological constant and equation of motion (25) self-consistent with the cmc condition. We note that the presence of the coupling constants T_p and μ in the deformed p -brane action controls the dynamical regimes produced by the elastic and gravitational forces.⁵

Studied in Sect. 4 is the Cauchy problem for the EOM (25) which are the second order PDEs. We prove that the Peterson–Codazzi (P–C) embedding conditions (14) and the cmc conditions, spontaneously breaking the scale symmetry for 3-branes, can be chosen as the initial data constraints preserved by the PDEs (25). Then the Cauchy–Kowalevskaya theorem of uniqueness ensures that the solution of Eq. (25) describes the cmc hyper-worldsheet encoded by deformed potential $V_{cmc}(l, g, \mu)$ (22).

In Sect. 5 potential $V_{cmc}(l, g, \mu)$ is converted to the Lagrangian of R^2 gravity (58) including the H–E term in addition to pure R^2 . Nowadays there is a great interest to such Lagrangians extending the model [13] to scale-invariant models (see e.g. [55–62]). The latter comprise scale-invariant combinations built from R , R^2 , an interaction potential of massless scalar field(s), and a few of dimensionless parameters. The spontaneous breaking of the scale-symmetry follows from the solutions of the equations of motion including a special scalar potential. Interest in such scale-invariant models is due to the fact that they describe inflation and

³ Exact solutions for p -brane collapse were earlier obtained in [39–41].

⁴ The string model with constant mean curvature worldsheet embedded into $\mathbf{R}^{1,2}$ was studied in [52] by the extension of the Nambu–Goto action.

⁵ A similar dynamics is observed for string in curved space [53, 54].

reheating in our 4-dim. Universe and fit the new experimental data from Planck [62]. A remarkable feature of the modern stage of experimental cosmology is the possibility to select the relevant models to understand the structure of inhomogeneities in the mass distribution in very small time scales. The above mentioned R^2 scale-invariant models study scalar perturbations of the gravitational potential. In the case of p -branes we introduce the tensor perturbations associated with the massless symmetric tensor field $l_{\mu\nu}$, and have no any free dimensionless parameters. A massless scalar field ϕ naturally appears in the brane scenario in the form of the diff invariant $\phi \equiv Sp l$ treated as a dynamical mean curvature field. Spontaneous breaking of the scaling for 3-branes is achieved by imposing the condition $\langle \phi \rangle_0 = \langle Sp l \rangle_0 = \mu$ which generates a fundamental mass scale similarly to the Higgs effect in QFT. So, the use of $l_{\mu\nu}$ proposes new scale-invariant tensor models of R^2 gravity alternative to the well-known models with scalar fields. This motivates application of the considered brane models to analysis of experimental cosmological data.

In Sect. 6 we derive the equation for extremals of $V_{cmc}(l, g, \mu)$ and find its general solution. Consistency of the solution with the Gauss and P-C embedding conditions results in the matrix equation for null vectors of the $(p+1)$ -dim. Einstein tensor (matrix) $G_{\mu\nu}$. Its solution reveals the presence of the extremal $l_{o\mu\nu}(\xi) = \frac{\mu}{p+1} g_{\mu\nu}(\xi)$. This critical point corresponds to the AdS_{p+1} hyper-ws Σ_{p+1}^o of constant negative curvature. Next, the behavior of V_{cmc} in the vicinity of $l_{o\mu\nu}(\xi)$ is analysed. The second partial derivatives of V_{cmc} with respect to l at $l_o(\xi)$ have different signs depending, in particular, on the brane dimension p . So, for string the extremum is the minimum that shows classical stability of the string potential in $\mathbf{R}^{1,2}$. But for cmc membrane ($p = 2$) the critical point $l_o(\xi)$ is proved to be the saddle 3-dim. hypersurface. It points to the classical instability of the cmc membrane potential in 4-dim. Minkowski space-time. The classical instability of the fundamental (super)membrane [63], described by the potential $V(x(\xi))$ [64], is well known. Thus, the transition to the hyper-worldsheets with constant mean curvature embedded into $\mathbf{R}^{1,3}$ does not remove the instability.

2 Branes with minimal hyper-worldsheets

In the geometrical approach Dirac p -brane embedded into $\mathbf{R}^{1,D-1}$ with the signature $(+ - \dots -)$ are described as dynamical systems with the Poincare symmetry $ISO(1, D-1)$ spontaneously broken to $ISO(1, p) \times SO(D-p-1)$. The p -branes sweep minimal h-ws Σ_{p+1}^{min} with zero mean curvature

$$Sp l^a := g^{\mu\nu} l_{\nu\mu}^a = 0, \quad (3)$$

where $g_{\mu\nu}$ is the induced metric in Σ_{p+1}^{min} . The symmetric tensor $l_{\mu\nu}^a(\xi)$ is the second fundamental form of the h-ws treated as a constrained multiplet of the local group $SO(D-p-1)$. We explain the constraints (3) as the inverse Higgs effect. The indexes $a, b = p+1, p+2, \dots, D-1$ enumerate the orts $\mathbf{n}_a(\xi^\rho)$ of an orthonormal moving frame attached to Σ_{p+1}^{min} and orthogonal to it. The internal coordinates $\xi^\mu = (\tau, \sigma^r)$, $r = 1, 2, \dots, p$ parametrize the h-ws world vector $\mathbf{x}(\xi^\rho)$. The mapping of the Dirac p -brane action to the diff and $SO(D-p-1)$ invariant h-ws action is realized by the field action [27]

$$S_{Dir} = \frac{1}{k_p^2} \int d^{p+1} \xi \sqrt{|g|} \left\{ -\frac{1}{4} Sp(H_{\mu\nu} H^{\nu\mu}) + \frac{1}{2} \nabla_\mu^\perp l_{\nu\rho a} \nabla^\perp \{ \mu l^{\nu\} \rho a - \nabla_\mu^\perp l_{\rho a}^\mu \nabla_v^\perp l^{\nu\rho a} + V(l, g) \right\}, \quad (4)$$

where the brackets $\{\mu\nu\}$, $[\mu\nu]$ imply the μ, ν symmetrization and antisymmetrization, respectively. The potential energy term $V(l, g)$ is equal to

$$V(l, g) = V_{Dir} := -\frac{1}{2} Sp(l_a l_b) Sp(l^a l^b) + Sp(l_a l_b l^a l^b) - Sp(l_a l^a l_b l^b) + c_p. \quad (5)$$

The kinetic term for the $SO(D-p-1)$ gauge field $B_\mu^{ab}(\xi) = -B_\mu^{ba}(\xi)$ in the fundamental representation is given by the trace $Sp(H_{\mu\nu} H^{\nu\mu}) := H_{\mu\nu}^{ab} H_{ba}^{\nu\mu}$

$$H_{\mu\nu}{}^b{}_a := (\partial_{[\mu} B_{\nu]} + [B_\mu, B_\nu])_a{}^b. \quad (6)$$

The metric and Y-M covariant derivative $\nabla_\mu^\perp l_{\nu\rho}{}^a$ is defined by the expression

$$\nabla_\mu^\perp l_{\nu\rho}{}^a := \partial_\mu l_{\nu\rho}{}^a - \Gamma_{\mu\nu}^\lambda l_{\lambda\rho}{}^a - \Gamma_{\mu\rho}^\lambda l_{\nu\lambda}{}^a + B_\mu^{ab} l_{\nu\rho b}. \quad (7)$$

The hyper-worldsheet metric $g_{\mu\nu}(\xi)$ in (4) is treated as a background field since its equations are encoded by the Gauss's Theorema Egregium

$$R_{\mu\nu}{}^\gamma{}_\lambda = l_{[\mu}{}^\gamma{}^a l_{\nu] \lambda a} \quad (8)$$

used under construction of action (4). The Gauss Eq. (8) together with the Peterson-Codazzi equations for the covariant derivatives (7)

$$\nabla_{[\mu}^\perp l_{\nu]\rho}^a = 0 \quad (9)$$

form the consistency conditions for embedding of h-ws into the Minkowski space. Below we apply the discussed gauge approach to description of hyper-ws with constant mean curvature and derive their potential V_{cmc} (22).

3 Brane h-ws with constant mean curvature

For hyper- ws of codim 1 action (4) with zero gauge field B_μ^{ab} (since $a = b = p + 1$) encodes quadratic curvature gravity on the hyper- ws with zero mean curvature. The reduced potential (5) found in [34] is expressed through R

$$V_{Dir} = -\frac{1}{2}Sp(l^2)Sp(l^2) + c_p = -\frac{1}{2}R^2 + c_p, \\ R := g^{\mu\nu}R_{\mu\nu}. \quad (10)$$

In the case $p = 3$, $D = 5$ the coupling k_3 is dimensionless and reduced action (4) including V_{Dir} (10) is invariant under the rigid scale transformations

$$xi^{\mu'} = e^{-\lambda}\xi^\mu \quad g'_{\mu\nu}(\xi') = g_{\mu\nu}(\xi), \quad l'_{\mu\nu}(\xi') = e^{\frac{p+1}{4}\lambda}l_{\mu\nu}(\xi) \quad (11)$$

if a h- ws cosmological constant $c_3 = 0$. Minimal h- ws correspond to the vev $v \equiv Spl_0 = 0$. Here we derive the deformed potential V_{cmc} generating a non-zero $v = \mu$. This potential encodes constant mean curvature hyper- ws Σ_{p+1}^{cmc} of codim 1. To find V_{cmc} we explore the hyper- ws action

$$S = \frac{1}{k_p^2} \int d^{p+1}\xi \sqrt{|g|} \left(\frac{1}{2} \nabla_\mu l_{\nu\rho} \nabla^{\{\mu} l^{\nu\}\rho} - \nabla_\mu l_\rho^\mu \nabla_\nu l^{\nu\rho} - V(l, g) \right), \quad (12)$$

where the reduced field $l_{\mu\nu} := -l_{\mu\nu}^{(p+1)}$, and obtain the following EOM

$$\frac{1}{2} \nabla_\mu \nabla^{\{\mu} l^{\nu\}\rho\}} = -[\nabla^\mu, \nabla^{\{\nu} l_\mu^{\rho\}}] - \frac{\partial V}{\partial l_{\nu\rho}}. \quad (13)$$

Taking into account the Peterson-Codazzi (P-C) conditions (9) with $B_\mu = 0$

$$\nabla_{[\mu} l_{\nu]\rho} = 0 \quad (14)$$

in (13), vanishes their l.h.s. and yields the consistency condition for $V(l, g)$

$$\frac{\partial V}{\partial l_{\nu\rho}} = -[\nabla^\mu, \nabla^{\{\nu} l_\mu^{\rho\}}]. \quad (15)$$

Using the Bianchi identities for the commutator in (15)

$$[\nabla_\mu, \nabla_\nu] l^{\gamma\rho} = R_{\mu\nu}{}^\gamma{}_\lambda l^{\lambda\rho} + R_{\mu\nu}{}^\rho{}_\lambda l^{\gamma\lambda} \quad (16)$$

and the Gauss Eq. (8) we present the r.h.s. of Eq. (15) in the form

$$-\frac{1}{2}[\nabla^\mu, \nabla^{\{\nu} l_\mu^{\rho\}}] = (l^2)^{\nu\rho} Spl - l^{\nu\rho} Sp(l^2), \quad (17)$$

where $Sp(l^2) = l_{\mu\rho} l_\nu^\rho g^{\mu\nu}$. Substitution of (17) into (15) yields the PDE's

$$\frac{1}{2} \frac{\partial V}{\partial l_{\nu\rho}} = (l^2)^{\nu\rho} Spl - l^{\nu\rho} Sp(l^2). \quad (18)$$

for V which have the general solution including a cosmological constant c_p

$$V(l, g) = \frac{2}{3} Spl Sp(l^3) - \frac{1}{2} Sp(l^2) Sp(l^2) + c_p \quad (19)$$

as an integration constant. Action (12) with V (19) takes the form

$$S_{df} = \frac{1}{k_p^2} \int d^{p+1}\xi \sqrt{|g|} \left(\frac{1}{2} \nabla_\mu l_{\nu\rho} \nabla^{\{\mu} l^{\nu\}\rho} - \nabla_\mu l_\rho^\mu \nabla_\nu l^{\nu\rho} - \frac{2}{3} Spl Sp(l^3) + \frac{1}{2} Sp(l^2) Sp(l^2) - c_p \right) \quad (20)$$

which is scale-invariant for $p = 3$ and $c_3 = 0$. Then the scale-symmetry of S_{df} , treated as the gravity action, explains the observed smallness of the cosmological constant of our 4-dim. world interpreted as a 3-brane. In other words, the exact scaling requires $c_3 = 0$. Thereat, a soft breaking of the scale symmetry corresponds to the consideration of c_3 as a small parameter. Action (20) generalizes well-known scale-invariant models of gravity extended by scalar fields. The latter are supported by current experimental data. Alternatively, the scale symmetry can be broken by imposing the condition

$$Spl \equiv l_{\nu\rho} g^{\nu\rho} = \mu, \quad (21)$$

including a dimensionfull constant μ . As a result, we obtain the potential

$$V = V_{cmc} := \frac{2}{3} \mu Sp(l^3) - \frac{1}{2} Sp(l^2) Sp(l^2) + c_p, \\ Spl = \mu(p) \quad (22)$$

which encodes h- ws with the constant mean curvature (cmc) equal to $\mu(p)$. Then the action (12) is represented in the following form

$$S_{cmc} = \frac{1}{k_p^2} \int d^{p+1}\xi \sqrt{|g|} \left(\frac{1}{2} \nabla_\mu l_{\nu\rho} \nabla^{\{\mu} l^{\nu\}\rho} - \nabla_\mu l_\rho^\mu \nabla_\nu l^{\nu\rho} - \frac{2}{3} \mu Sp(l^3) + \frac{1}{2} Sp(l^2) Sp(l^2) - c_p \right). \quad (23)$$

This action is consistent with the Gauss (8) and P-C (14) embedding conditions. From the viewpoint of the variational principle, the hyper-worldsheet metric $g_{\mu\nu}$ works as a background metric, since its evolution is defined by the Gauss Eq. (8). The Euler-Lagrange equations for $l_{\mu\nu}$ following from (23) are

$$\frac{1}{2} \nabla_\mu \nabla^{\{\mu} l^{\nu\}\rho\}} = 2(l^2)^{\nu\rho} (Spl - \mu) \quad (24)$$

and their consistency with (14) demands the condition $(l^2)^{\nu\rho} (Spl - \mu) = 0$ equivalent to $(Spl - \mu) = 0$.⁶ Then

⁶ The second solution $(l^2)^{\nu\rho} = 0$ means $Sp(l^3) = Sp(l^2) = 0$ that corresponds to $V = c_p$.

Eq. (24) are reduced to the Eq.

$$\frac{1}{2}\nabla_\mu\nabla^{[\mu}l^{v]\rho}=0, \quad Spl=\mu. \quad (25)$$

Using the identity $\nabla^\perp[\mu l^{v]\rho] = -\nabla^\perp[v l^{\rho]\mu}$ transforms (25) into the equation

$$\frac{1}{2}\nabla_\mu\nabla^{[\mu}l^{v]\rho]}\equiv\nabla_\mu\left(\nabla^{[\mu}l^{v]\rho}+\frac{1}{2}\nabla^{[v}l^{\rho]\mu}\right)=0. \quad (26)$$

Contracting (26) with $g_{\rho v}$ and using the condition $Spl=\mu$ we obtain the covariant conservation law for the divergence $\nabla^\rho l_\rho^\mu$

$$\nabla_\mu(\nabla^\rho l_\rho^\mu)=0. \quad (27)$$

Now it is suitable to rewrite EOM (26) in the equivalent form

$$\nabla_\mu\nabla^{[\mu}l^{v]\rho}=-\frac{1}{2}\{[\nabla^\mu,\nabla^{[v}l^{\rho]}_\mu+\nabla^{[v}\nabla_\mu l^{\rho]\mu}\}.$$

Observing that the (v,ρ) -antisymmetric part of the commutator is presented in the form

$$\frac{1}{2}[\nabla^\perp_\mu,\nabla^\perp^{[v}l^{\rho]}_\mu]^a=-\frac{1}{2}([l^a,l^b])^{v\rho}Spl_b, \quad (28)$$

as it follows from the Bianchi identities and Gauss conditions, we obtain vanishing of the l.h.s. of (28) for codim 1. Then EOM (25) are reduced to

$$\nabla_\mu\nabla^{[\mu}l^{v]\rho}+\frac{1}{2}\nabla^{[v}\nabla_\mu l^{\rho]\mu}=0, \quad Spl=\mu. \quad (29)$$

Equation (29) are consistent with Eq. (14) in view of the identity

$$\nabla^\mu l_{\mu\lambda}=\nabla_\lambda Spl\equiv\partial_\lambda Spl=0 \quad (30)$$

following from (14) after contraction of μ with ρ and using $Spl=\mu$.

We note that Eq. (29) may have other solutions that do not satisfy the P-C Eq. (14) and violate the embedding conditions for hyper-worldsheets. To select the relevant solutions describing brane h-ws, we have to add the appropriate initial data conditions (IDC). The latter are chosen to be the first order PDEs (14) and the condition $\nabla^\mu l_{\mu\lambda}=0$ satisfying the covariant conservation law (27). In the next section we prove that the IDC form a remarkable set of constraints preserved by the second order PDEs (29).

4 Cauchy constraints

The Cauchy problem for PDEs (29) is studied here treating all Eq. (14) as the initial data constraints fixed at the moment $\tau=0$

$$\nabla^{[\mu}l^{v]\rho}(0,\sigma^r)=0, \quad \nabla_\mu l^{\mu\rho}(0,\sigma^r)=0. \quad (31)$$

Using the results [27] we prove that the constraints (31) are always fulfilled during the evolution prescribed by EOM (29). Thereat we take into account vanishing of the space covariant derivatives of the IDC (31)

$$\begin{aligned} \nabla_r\nabla_\mu l^{\mu\rho}|_{\tau=0}=0, \quad \nabla_r\nabla^{[\mu}l^{v]\rho}|_{\tau=0}=0, \\ (r=1,2,\dots,p). \end{aligned} \quad (32)$$

Consider the power series expansion of $\nabla^{[\mu}l^{v]\rho}$ and $\chi^\rho\equiv\nabla_\mu l^{\mu\rho}$

$$\begin{aligned} \chi^\rho(\delta\tau,\sigma^r)=\chi^\rho|_{\tau=0}+\partial_\tau\chi^\rho|_{\tau=0}\delta\tau \\ +\dots=\nabla_\tau\chi^\rho|_{\tau=0}\delta\tau+\dots, \end{aligned} \quad (33)$$

$$\begin{aligned} \nabla^{[\tau}l^{v]\rho}(\delta\tau,\sigma^r)=\partial_\tau\nabla^{[\tau}l^{v]\rho}|_{\tau=0}\delta\tau \\ +\dots=\nabla_\tau\nabla^{[\tau}l^{v]\rho}|_{\tau=0}\delta\tau+\dots. \end{aligned} \quad (34)$$

The use of Eq. (27) $\nabla_\tau\chi^\tau(\xi)=-\nabla_r\chi^r(\xi)$ and (32) $\nabla_r\chi^r|_{\tau=0}=0$ gives

$$\chi^\tau(\delta\tau,\sigma^r)=\chi^\tau|_{\tau=0}=0. \quad (35)$$

To prove that $\chi^r(\delta\tau,\sigma^r)=0$ we consider EOM (29) for $v=\tau$ and $\rho=r$

$$\nabla_s\nabla^{[s}l^{\tau]r}+\frac{1}{2}\nabla^\tau\nabla_\tau\chi^r=0, \quad (36)$$

and taking into account (32), (35) find

$$\nabla_\tau\chi^r|_{\tau=0}=\partial_\tau\chi^r|_{\tau=0}=0\longrightarrow\chi^r(\delta\tau,\sigma^r)=\chi^r|_{\tau=0}=0. \quad (37)$$

Equation $\partial_\tau\chi^\rho|_{\tau=0}=0$ and (35) complete the proof of the desired statement

$$\nabla_\mu l^{\mu\rho}(\xi)=0. \quad (38)$$

Equation (38) prove equivalence of EOM (29) to the system

$$\nabla_\mu\nabla^{[\mu}l^{v]\rho}=0, \quad \nabla_\mu l^{\mu\rho}(\xi)=0 \quad (39)$$

showing the local conservation of the P-C conditions (14).

Next we prove that the IDC (31) together with the local conservation laws (39) provide fulfillment of the P-C conditions (14) at any τ

$$\nabla^{[\mu}l^{v]\rho}(\xi)=0. \quad (40)$$

To this objective consider $v=r$ and $\rho=\tau$ in (39) and obtain the relations

$$\nabla_\tau\nabla^{[\tau}l^{r]\tau}=-\nabla_s\nabla^{[s}l^{r]\tau} \quad (41)$$

which in combination with the IDC (32) show that

$$\nabla_\tau\nabla^{[\tau}l^{r]\tau}|_{\tau=0}=\partial_\tau\nabla^{[\tau}l^{r]\tau}|_{\tau=0}=0. \quad (42)$$

The substitution of (42) into the expansion (34) proves that

$$\nabla^{[\tau}l^{r]\tau}(\xi)=0. \quad (43)$$

The choice of both indices ν, ρ in (39) as space ones $\nu = r, \rho = s$ gives

$$\nabla_\tau \nabla^{[\tau} l^{r]s} = -\nabla_q \nabla^{[q} l^{r]s}. \quad (44)$$

Equations (44) and (32) show that $\nabla_\tau \nabla^{[\tau} l^{r]s}|_{\tau=0} = \partial_\tau \nabla^{[\tau} l^{r]s}|_{\tau=0} = 0$ that provides

$$\nabla^{[\tau} l^{r]s}(\xi) = 0. \quad (45)$$

Equations (45) and (43) testify to fulfillment of the subset

$$\nabla^{[\tau} l^{r]\rho}(\xi) = 0 \quad (46)$$

of the P-C equations. Antisymmetrization in the ρ, r indices in (46) shows fulfillment of the next subset of Eq. (14)

$$\nabla^{[r} l^{s]\tau}(\xi) = 0. \quad (47)$$

The remaining subset of the P-C eqs. to be proved

$$\nabla^{[r} l^{s]q}(\xi) = 0 \quad (48)$$

contains only space-like indices r, s, q . Therefore, we consider the expansion

$$\begin{aligned} \nabla^{[r} l^{s]q}(\delta\tau, \sigma^r) &= \partial_\tau \nabla^{[r} l^{s]q}|_{\tau=0} \delta\tau \\ &+ \dots = g_{\tau\tau} \nabla^\tau \nabla^{[r} l^{s]q}|_{\tau=0} \delta\tau + \dots, \end{aligned} \quad (49)$$

where the relation $\nabla_\tau \nabla^{[r} l^{s]q}|_{\tau=0} = g_{\tau\tau} \nabla^\tau \nabla^{[r} l^{s]q}|_{\tau=0}$ together with (31), (32) are used. The proof of (48) reduces to the that of the relation

$$\nabla^\tau \nabla^{[r} l^{s]q}|_{\tau=0} \equiv ([\nabla^\tau, \nabla^{[r} l^{s]q} + \nabla^{[r} l^{s]q} \nabla^\tau]|_{\tau=0}) = 0. \quad (50)$$

Using the identities (16) we present the first term in the r.h.s. of (50) as

$$[\nabla^\tau, \nabla^{[r} l^{s]q}] = R^{\tau[r} l^{s]q}{}_\mu l^{\mu q} + R^{\tau[r} l^{q]s}{}_\mu l^{\mu s]}. \quad (51)$$

The conditions (45) permit to rewrite the second term in the form

$$\begin{aligned} \nabla^{[r} l^{q]s]} &= \nabla^{[r} l^{q] \nabla^{s]} = [\nabla^{[r} l^{q] \nabla^{s]} + \nabla^{[r} l^{q] \nabla^{s]} \\ &= R^{[r} l^{q] \nabla^{s]} + R^{[r} l^{q] \nabla^{s]} + \nabla^{[r} l^{q] \nabla^{s]}. \end{aligned} \quad (52)$$

The sum of (51) and (52) transformss (50) into

$$\begin{aligned} \nabla^\tau \nabla^{[r} l^{s]q}|_{\tau=0} &= (R^{\tau[r} l^{s]q}{}_\mu l^{\mu q} + R^{\tau[r} l^{q]s}{}_\mu l^{\mu s]} \\ &+ R^{[r} l^{q] \nabla^{s]} + R^{[r} l^{q] \nabla^{s]}|_{\tau=0} \end{aligned} \quad (53)$$

after using (46) and (32). Further, we apply the Gauss theorem (8)

$$R^{\gamma\lambda\nu}{}_\mu l^{\mu r} = (l^2)^{r[\gamma} l^{\lambda]\nu} \quad (54)$$

and obtain the following expressions for the terms in (53)

$$R^{\tau[r} l^{s]q}{}_\mu l^{\mu q} = -(l^2)^{q[r} l^{s]\tau}, \quad R^{[r} l^{q]s}{}_\mu l^{\mu \tau} = (l^2)^{\tau[r} l^{s]q}, \quad (55)$$

$$R^{\tau[r} l^{q] \nabla^{s]} + R^{[r} l^{q] \nabla^{s]} = (l^2)^{\tau[s} l^{r]q} - (l^2)^{q[s} l^{r]\tau}. \quad (56)$$

Here we observe mutual cancellation of the corresponding terms in the sum of (55) and (56). This proves the validity of Eqs. (48) and (40). Then the cmc condition $Sp l = \mu$ emerges as a consequence of Eqs. (38) and (40).

So, we show that EOM (25) together with Eq. (8) provide conservation of IDC (38) and (40). The latter select a *closed* sector of the solutions describing constant mean curvature hyper-*ws* of codim 1.

Below, in Sect. 5, we prove that the p -brane action S_{cmc} (23) encodes the gravity action quadratic in curvature and including the desired H-E term.

5 Hilbert–Einstein action with R^2 term

The Gauss's theorem (8) permits to express V_{cmc} (22) associated with the constant mean curvature hyper-*ws* Σ_{p+1}^{cmc} in terms of the Riemann tensor

$$\begin{aligned} R_{\nu\lambda} &= (l^2)_{\nu\lambda} - \mu l_{\nu\lambda}, \quad R_{\nu\lambda} l^{\lambda\nu} = Sp(l^3) - \mu Sp(l^2), \\ R &= Sp(l^2) - \mu^2, \end{aligned} \quad (57)$$

where $R_{\nu\rho}$ is the Ricci tensor. Using (57) we obtain the R -reps. for V_{cmc}

$$V_{cmc} = -\frac{1}{2}R^2 - \frac{\mu^2}{3}R + \frac{2\mu}{3}R_{\nu\lambda} l^{\lambda\nu} + \frac{\mu^4}{6} + c_p \quad (58)$$

which reduces to V_{Dir} (10) when $\mu = 0$

$$V_{cmc} = V_{Dir} - \frac{\mu^2}{3} \left(R - \frac{\mu^2}{2} - \frac{2}{\mu} R_{\nu\lambda} l^{\lambda\nu} \right). \quad (59)$$

Representation (58) can be rewritten in the equivalent form as

$$V_{cmc} = -\frac{\mu^2}{3} \left\{ (R - 2\Lambda) + \frac{3}{2\mu^2} R^2 - \frac{2}{\mu} R_{\nu\lambda} l^{\lambda\nu} \right\} \quad (60)$$

after unification of two independent cosmological terms into cosmological constant $\Lambda(p) := (\frac{\mu^2(p)}{4} + \frac{3c_p}{2\mu^2(p)})$. Due to arbitrariness of c_p the constant Λ is additional dimensionfull phenomenological parameter of the quadratic gravity derived from p -branes. In particular, one can choose $\Lambda = 0$. The potential $V_{cmc} = \frac{2}{3}\mu^2 \Lambda(p)$ when $R_{\nu\rho} = 0$. Then (57) yields the relations

$$\begin{aligned} (l^n)_{\nu\rho} &= \mu^{n-1} l_{\nu\rho}, \quad (n = 2, 3, \dots) \implies l_{\nu\rho} \\ &= \mu g_{\nu\rho}, \quad \det l_\nu^\alpha \neq 0 \end{aligned} \quad (61)$$

which show that $Spl = (p+1)\mu$, otherwise $l_{\nu\rho} = 0$. Solution (61) is compatible with the cmc condition $Spl = \mu$ only if $p = 0$ that corresponds to a generate case of a worldline swept by point-like 0-brane.

We note that the H-E Lagrangian in (59) is generated simultaneously with the cosmological constant dependent on μ and the new bilinear invariant including the Ricchi tensor. This deformation of V_{Dir} is a result of the transition from minimal h-ws to those with a constant mean curvature. For $p = 3$, $c_3 = 0$, emergence of the H-E term and others in (59) breaks the scale symmetry of the four dimensional R^2 gravity encoded by $V_{Dir}|_{p=3}$ (1). The generated Newton constant turns out to be equal to $G_N \approx \frac{1}{\mu^2}$ similarly to the case of spontaneously generated gravity in scale-invariant action (2). This shows that μ is identified with the vev of Spl considered as a dynamical scalar field $\phi \equiv Spl$. To see this effect we come back to action (20) and potential (19) represented in the form of the gravity action Lagrangian

$$V_\phi = -\frac{1}{2}R^2 - \frac{1}{3}R\phi^2 + \frac{2}{3}R_{\nu\lambda}l^{\lambda\nu}\phi + \frac{1}{6}\phi^4 + c_p \quad (62)$$

after using (8). Comparison of (62) with (58) shows that $V_\phi = V_{cmc}|_{\mu=\phi}$. For $p = 3$, $c_3 = 0$ the 3-brane action with V_ϕ (62) is scale-invariant. Action Lagrangian (60) with broken scale-symmetry occurs since potential V (19) and (62), respectively, have the critical point $\phi = \mu$, as shown in Sect. 6. Potential (62) can be compared with that from the model [60,61] describing inflation and reheating in the presence of a massless scalar field φ . The latter can be identified with the composite scalar ϕ in (62) associated with brane matter. So, the massless tensor perturbations $l_{\mu\nu}$ result in creation of new models of R^2 gravity which can be used for analyzing the of current experiments.

6 Extremals of the brane potential V_{cmc}

The above mentioned connection of the brane potential V_{cmc} with inflation models puts forward the question on its extremals which are defined by solutions of the equation

$$\frac{1}{2} \frac{\partial V_{cmc}}{\partial l_{\nu\rho}} = \mu(l^2)^{\nu\rho} - l^{\nu\rho} Sp(l^2) = 0. \quad (63)$$

In the dual R -representation this equation takes the form

$$Rl_{\nu\rho}(\xi) = \mu R_{\nu\rho} \quad (64)$$

after using relations (57). The general solution of (64) is

$$l_{\nu\rho}(\xi) = \mu \frac{R_{\nu\rho}}{R}, \quad R \neq 0. \quad (65)$$

The substitution of this extremal in the Gauss Eq. (8) yields the relation

$$R_{\mu\nu\gamma\lambda} = -\left(\frac{\mu}{R}\right)^2 (R_{\mu\gamma}R_{\nu\lambda} - R_{\nu\gamma}R_{\mu\lambda}). \quad (66)$$

The extremal (65) should obey the Peterson–Codazzi Eq. (14)

$$\nabla_{[\mu} R_{\nu]\rho} - \frac{1}{R} R_{\rho[\nu} \nabla_{\mu]} R = 0. \quad (67)$$

and Eq. (30) which follows from (14) and has the form

$$\nabla_\nu R^{\nu\rho} - \frac{1}{R} R^{\rho\nu} \partial_\nu R = 0 \quad (68)$$

after substitution of (65) in (30). The use of the relation

$$\nabla_\nu R^\nu{}_\rho = \frac{1}{2} \partial_\rho R, \quad (69)$$

which follows from the Bianchi identities

$$\nabla_\alpha R^\mu{}_{\nu\gamma\lambda} + \nabla_\gamma R^\mu{}_{\nu\lambda\alpha} + \nabla_\lambda R^\mu{}_{\nu\alpha\gamma} = 0, \quad (70)$$

transforms Eq. (68) into the equation

$$\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right)\partial_\nu R = 0. \quad (71)$$

The latter means that $\partial_\mu R$ should be a null vector for the Einstein matrix $G^{\mu\nu} := R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R$ associated with Σ_{p+1}^{extr} . In the R -reps. the extremal h-ws is fixed by Eqs. (66), (71). Their solution for the case when $\det G^{\mu\nu} \neq 0$ is

$$\partial_\nu R = 0 \implies R = R_o = constant. \quad (72)$$

This solution defines a h-ws Σ_{p+1}^o of *constant curvature* (cc) characterized by the Riemann tensor and the Gauss curvature K_o

$$R_{o\mu\nu\gamma\lambda} = K_o(g_{\mu\gamma}g_{\nu\lambda} - g_{\nu\gamma}g_{\mu\lambda}). \quad (73)$$

The substitution of (73) in the Gauss Eq. (66) results in the relation

$$K_o(g_{\mu\gamma}g_{\nu\lambda} - g_{\nu\gamma}g_{\mu\lambda}) = -\left(\frac{\mu}{R}\right)^2 (R_{\mu\gamma}R_{\nu\lambda} - R_{\nu\gamma}R_{\mu\lambda}) \quad (74)$$

which shows proportionality of $R_{o\mu\nu}$ to $g_{\mu\nu}$ and defines K_o

$$R_{o\mu\nu} = \Lambda_o g_{\mu\nu}, \quad R_o = (p+1)\Lambda_o, \quad K_o = -\left(\frac{\mu}{p+1}\right)^2. \quad (75)$$

The latter completely fixes the Riemann tensor of Σ_{p+1}^o

$$R_{o\mu\nu\gamma\lambda} = -\left(\frac{\mu}{p+1}\right)^2 (g_{\mu\gamma}g_{\nu\lambda} - g_{\nu\gamma}g_{\mu\lambda}) \quad (76)$$

and the function $\Lambda_o = pK_o$, respectively. Then we obtain

$$R_{o\mu\nu} = -\frac{p}{(p+1)^2} \mu^2 g_{\mu\nu}, \quad R_o := g^{\mu\nu} R_{o\mu\nu} = -\frac{p}{p+1} \mu^2. \quad (77)$$

As shows the substitution of (77) in (65), this extremal acquires the form

$$l_{o\mu\nu} = \frac{\mu}{p+1} g_{\mu\nu} \implies R_{o\mu\nu} = -\frac{\mu p}{p+1} l_{o\mu\nu}. \quad (78)$$

Solutions (78) are proportional to $g_{\mu\nu}(\xi)$, and therefore they are solutions of the P-C Eq. (14) in the l -reps. and Eq. (67) in the R -reps., respectively. Note that the extremal of $V(l, g)$ (19) is $l_{o\mu\nu} = \frac{Sp l_o}{p+1} g_{\mu\nu}$, which satisfies Eq. (14) for $Sp l_o = \mu$.

Coming back to Eq. (63) in the l -reps. we rewrite it in the form

$$l_\alpha^v (\mu l^{\alpha\rho} - g^{\alpha\rho} Sp(l^2)) = 0$$

which shows that there exist two sets of the solutions. For the first of them $\det l_{\mu\nu} = 0$, for the other it is nonzero. In the latter case (63) reduces to

$$l^{\alpha\rho} = \frac{Sp(l^2)}{\mu} g^{\alpha\rho}, \quad \det l_{\mu\nu} \neq 0 \quad (79)$$

and corresponds to the above-considered solution (78). Indeed, in this case contraction of (79) with $g^{\rho\alpha}$ and $l^{\rho\alpha}$ yields the conditions

$$Sp l_o = \mu, \quad Sp(l_o^2) = \frac{\mu}{p+1} Sp l_o = \frac{\mu^2}{p+1}, \quad (80)$$

respectively. Then substitution of (80) in (79) reproduces solution (78) for $l_{o\mu\nu}$. Substitution of $l_{o\mu\nu}$ in the Gauss Eq. (8) yields solution (76) for the Riemann tensor of the extremal h- ws Σ_{p+1}^o . The h- ws Σ_{p+1}^o is a hyperbolic one with the negative constant curvature K_o . The use of the relation

$$R_{o\mu\nu} = \frac{R_o}{p+1} g_{\mu\nu}, \quad \Lambda_o = -\frac{p}{(p+1)^2} \mu^2$$

shows that Σ_{p+1}^o is a $(p+1)$ -dim. Einstein space with the negative cosmological constant Λ_o which can be treated as the anti-de Sitter space AdS_{p+1} .

The vacuum value of V_{cmc} (22) is the constant given by

$$V_o := V_{cmc}(l = l_o, g, \mu) = \frac{\mu^4(p)}{6(p+1)^2} - c_p. \quad (81)$$

The action (23) at the extremum $l_o(\xi)$ is also constant equal to

$$S_o := S_{cmc}|_{vac} = \frac{1}{k_p^2} \int d^{p+1} \xi \sqrt{|g|} \left(\frac{\mu^4(p)}{6(p+1)^2} - c_p \right), \quad (82)$$

due to vanishing of the kinetic term of $l_o(\xi)$ because $l_{o\mu\nu} \sim g_{\mu\nu}$.

The differential of Eq. (18) yields the second partial derivative of V_{cmc}

$$\begin{aligned} \frac{\partial^2 V_{cmc}}{\partial l_{\alpha\beta} \partial l_{\nu\rho}} &= [g^{\beta\{v} l^{\rho\}\alpha} + g^{\alpha\{v} l^{\rho\}\beta}] Sp l - g^{\alpha\{v} g^{\rho\}\beta} Sp(l^2) \\ &\quad + [(l^2)^{\nu\rho} g^{\alpha\beta} + (l^2)^{\alpha\beta} g^{\nu\rho}] - 4 l^{\nu\rho} l^{\alpha\beta}. \end{aligned} \quad (83)$$

The substitution of $l_{\mu\nu} = l_{o\mu\nu}$ (78) in Eq. (83) gives its value

$$\frac{\partial^2 V_{cmc}}{\partial l_{\alpha\beta} \partial l_{\nu\rho}}|_{vac} = \left(\frac{\mu}{p+1} \right)^2 [(p+1) g^{\beta\{v} g^{\rho\}\alpha} - 2 g^{\nu\rho} g^{\alpha\beta}] \quad (84)$$

at the critical point $l_{o\mu\nu}(\xi)$. This value is not positively defined that hints on possible classical instability of the vacuum. The latter is easily checked for strings and membranes where the diagonal gauge for $g_{\mu\nu}$ can be chosen. For strings we obtain two non-zero components $l_{\tau\tau} \equiv l_\tau$, $l_{\sigma\sigma} \equiv l_\sigma$ that gives

$$\begin{aligned} \frac{\partial^2 V_{cmc}}{\partial l_\tau \partial l_\tau}|_{vac} &= \frac{(\mu g^\tau)^2}{2}, \quad \frac{\partial^2 V_{cmc}}{\partial l_\sigma \partial l_\sigma}|_{vac} = \frac{(\mu g^\sigma)^2}{2}, \\ \frac{\partial^2 V_{cmc}}{\partial l_\tau \partial l_\sigma}|_{vac} &= -\left(\frac{\mu}{2} \right)^2 g^\tau g^\sigma. \end{aligned} \quad (85)$$

Since $g^\tau g^\sigma < 0$, we find the extremum to be the *minimum* defining classically stable *string* vacuum. For membrane $g_{\tau\tau} = g_{\sigma\sigma} = 0$, where $\sigma^1 \equiv \sigma$ and $\sigma^2 \equiv \eta$. Then we obtain three non-zero diagonal components: $l_{\tau\tau} \equiv l_\tau$, $l_{\sigma\sigma} \equiv l_\sigma$, $l_{\eta\eta} \equiv l_\eta$, and Eq. (85) take the form

$$\begin{aligned} \frac{\partial^2 V_{cmc}}{\partial l_\tau \partial l_\tau}|_{vac} &= \frac{(2\mu g^\tau)^2}{2}, \quad \frac{\partial^2 V_{cmc}}{\partial l_\sigma \partial l_\sigma}|_{vac} = \frac{(2\mu g^\sigma)^2}{2}, \\ \frac{\partial^2 V_{cmc}}{\partial l_\eta \partial l_\eta}|_{vac} &= \frac{(2\mu g^\eta)^2}{2} \end{aligned} \quad (86)$$

with all *positive* derivatives. The same we see for the mixed derivatives

$$\begin{aligned} \frac{\partial^2 V_{cmc}}{\partial l_\tau \partial l_\sigma}|_{vac} &= -2 \left(\frac{\mu}{3} \right)^2 g^\tau g^\sigma, \\ \frac{\partial^2 V_{cmc}}{\partial l_\tau \partial l_\eta}|_{vac} &= -2 \left(\frac{\mu}{3} \right)^2 g^\tau g^\eta, \end{aligned} \quad (87)$$

since $g^\tau g^\sigma < 0$ and $g^\tau g^\eta < 0$ due to the time-like character of Σ_3^o .

However, the mixed derivative with respect to l_σ and l_η is *negative*

$$\frac{\partial^2 V_{cmc}}{\partial l_\sigma \partial l_\eta}|_{vac} = -2 \left(\frac{\mu}{3} \right)^2 g^\sigma g^\eta, \quad (88)$$

since $g^\sigma g^\eta > 0$. It means that the extremal of the potential $V_{cmc}(l, g, \mu)$ is a 3-dim. saddle hypersurface that shows classical *instability* of the membrane vacuum. So, the potential of cmc membrane with $R_{\mu\nu} \neq 0$ in 4-dim. Minkowski

space turns out to be instable. This statement generalizes the famous theorem [63] on the classical instability of fundamental (super)membranes to the case of membranes with constant mean curvature of their 3-dim. worldsheets. Our result based on the sign indefiniteness of the second derivatives in (84) hints that the extremals of $V_{cmc}(l, g, \mu)$ for p -branes with codim 1 and higher $p > 2$ may also be saddle hypersurfaces.

7 Conclusion

We studied p -branes sweeping hyper-worldsheets (h-ws) of codimension 1 with a constant mean curvature μ , and constructed their action expressed through the h-ws metric $g_{\mu\nu}$ and Nambu-Goldstone tensor field $l_{\mu\nu}$ associated with the broken Poincare symmetry. The interaction potential for $l_{\mu\nu}$ and $g_{\mu\nu}$ was shown to encode a generalized h-ws action of quadratic curvature gravity with a cosmological constant, where $l_{\mu\nu}$ turned out to play the role of a scalar field φ similar to the Brans-Dicke one. The latter fulfills a key function in the Adler-Zee mechanism of spontaneously generated gravity arising from breaking of the scale symmetry [45]. We reopened this scenario and its generalization realized by branes, where $\langle Spl \rangle_0 \equiv \langle l_{\mu\nu} g^{\mu\nu} \rangle_0 = \mu$ replaces the vacuum expectation value $\langle \varphi \rangle_0$. This result follows from the equation for the brane potential extremals and its solution describing h-ws of negative constant curvature. These h-ws are maximally symmetric spaces identified with the anti-de Sitter spaces AdS_{p+1} . The found extremal for the membrane potential was proved to be realized by a saddle 3-dim. hypersurface. This points to the classical instability of the membranes with constant mean curvature h-ws in $\mathbf{R}^{1,3}$. This result generalizes the well-known theorem on the classical instability of fundamental (super)membranes [63]. It is interesting to connect the saddle-like instability of the deformed membranes with the inflation and reheating mechanisms studied in quadratic curvature models of gravity. The obtained results propose new models of R^2 gravity with spontaneously broken scale symmetry implementing the symmetric tensor field $l_{\mu\nu}$ instead of well-studied scalar fields.

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