The operator product expansions in the $\mathcal{N} = 4$ orthogonal Wolf space coset model

Changhyun Ahn^a, Man Hea Kim^b, Jinsub Paeng^c

Department of Physics, Kyungpook National University, Taegu 41566, Korea

Received: 15 May 2019 / Accepted: 30 June 2019 / Published online: 10 July 2019
 © The Author(s) 2019

Abstract Some of the operator product expansions (OPEs) between the lowest $SO(4)$ singlet higher spin-2 multiplet of spins $(2, \frac{5}{2}, \frac{5}{2}, \frac{5}{2}, \frac{5}{2}, 3, 3, 3, 3, 3, \frac{7}{2}, \frac{7}{2}, \frac{7}{2}, \frac{7}{2}, 4)$ in an extension of the large $\mathcal{N} = 4$ (non)linear superconformal algebra were constructed in the $\mathcal{N} = 4$ superconformal coset $\frac{SO(N+4)}{SO(N) \times SO(4)}$ theory with $N = 4$ previously. In this paper, by rewriting the above OPEs with $N = 5$, the remaining undetermined OPEs are completely determined. There exist additional $SO(4)$ singlet higher spin-2 multiplet, six $SO(4)$ adjoint higher spin-3 multiplets, four $SO(4)$ vector higher spin- $\frac{7}{2}$ multiplets, $SO(4)$ singlet higher spin-4 multiplet and four $SO(4)$ vector higher spin- $\frac{9}{2}$ multiplets in the right hand side of these OPEs. Furthermore, by introducing the arbitrary coefficients in front of the composite fields in the right hand sides of the above complete 136 OPEs, the complete structures of the above OPEs are obtained by using various Jacobi identities for generic N . Finally, we describe them as one single $\mathcal{N} = 4$ super OPE between the above lowest $SO(4)$ singlet higher spin-2 multiplet in the $\mathcal{N} = 4$ superspace.

Contents

1	Introduction	1
2	Review of $\mathcal{N} = 4$ orthogonal Wolf space coset model	3
2.1	The 11 currents of $\mathcal{N} = 4$ nonlinear superconformal algebra	3
2.2	The 16 currents of $\mathcal{N} = 4$ linear superconformal algebra	4
2.3	The 16 lowest higher spin currents	4
3	The OPE for $N = 4$	5
3.1	The known facts	5

3.2	The complete OPEs in components and $\mathcal{N} = 4$ superspace	5
4	The OPE for $N \geq 5$	6
4.1	What happens for $N = 5$?	6
4.2	The new higher spin current of spin 2	6
4.3	The new higher spin currents of spin 3	8
4.4	The new higher spin currents of spin $\frac{7}{2}$	10
4.5	The higher spin current of spin 4	11
4.6	The higher spin currents of spin $\frac{9}{2}$	11
4.7	The fundamental 16 OPEs	12
4.8	One single $\mathcal{N} = 4$ super OPE	12
4.9	The 136 OPEs between the 16 lowest higher spin currents for generic N	14
5	Conclusions and outlook	15
A	The OPEs between the 16 currents and the (non)singlet higher spin currents in the component approach	15
B	Partial expressions of quasi primary fields in the component approach	17
C	Partial expressions of quasi primary fields in the $\mathcal{N} = 4$ superspace	19
D	The single $\mathcal{N} = 4$ OPE in different order	20
E	The expression of $X_0^{(2)}$ for $N = 5$	21
	References	22

1 Introduction

The large $\mathcal{N} = 4$ holography [1] connects the unitary Wolf space coset conformal field theory in two dimensions and the matrix extended higher spin theory on AdS_3 space. The (large) $\mathcal{N} = 4$ supersymmetry plays an important role in this holography. One of the reasons why we want to understand the Wolf space coset construction is that this coset construction is a generalization of the free field construction. We obtain the latter by taking large level (the second order pole term of the OPE between spin-1 currents) k limit of the former. In other words, the large level k limit corresponds to the

^a e-mail: ahn@knu.ac.kr

^b e-mail: manhea@knu.ac.kr

^c e-mail: jdp2r@knu.ac.kr

vanishing of 't Hooft-like coupling constant [2]. In order to observe the behavior of finite 't Hooft-like coupling constant, the Wolf space coset construction is necessary to describe its nontrivial structure fully. Then we obtain the finite (N, k) behavior in the Wolf space coset construction and this will provide some hints for the higher spin theory on AdS_3 space at the quantum level. According to the results of [3], there exist different types of Wolf space cosets. One of them is given by orthogonal Wolf space coset we are interested in.¹ See also the relevant work in [7].

Contrary to the unitary Wolf space coset model [8–11], the orthogonal Wolf space coset model contains the lowest higher spin current of spin-2 rather than spin-1 [12]. This will make some calculations be rather involved. So far, the complete OPEs between the lowest 16 higher spin currents are not known. In [13], the lowest higher spin-2 current living in the lowest $\mathcal{N} = 4$ higher spin-2 multiplet (that is, there are 16 higher spin currents) in terms of orthogonal Wolf space coset fields for generic N was found. Then it is straightforward to obtain the remaining 15 higher spin currents from this higher spin-2 current by using the four supersymmetry generators of the large $\mathcal{N} = 4$ (non)linear superconformal algebra for fixed low values of N . For fixed $N = 4$, the three kinds of higher spin-3 currents were obtained from the OPEs between the higher spin currents of the above the lowest $\mathcal{N} = 4$ higher spin-2 multiplet. It was not clear how they appear in different $\mathcal{N} = 4$ multiplets at that time. We should look at the OPEs between the $\mathcal{N} = 4$ stress energy tensor and the possible $\mathcal{N} = 4$ multiplets by allowing the $SO(\mathcal{N} = 4)$ nonsinglet property to these $\mathcal{N} = 4$ multiplets.²

In this paper, we reconsider the complete OPEs between the lowest 16 higher spin currents in the $\mathcal{N} = 4$ orthogonal Wolf space coset model. After determining the complete OPEs for the particular $N = 4$ where all the higher spin currents can be written in terms of orthogonal Wolf space coset fields, we discuss the $N = 5$ case by adding more fields. The main idea to this purpose is that by using the fundamental orthogonal Wolf space coset fields, we compute the various OPEs. When the new higher spin primary fields arise in the right hand side of the OPEs, then we should reorganize them under the $SO(\mathcal{N} = 4)$ symmetry and observe how they transform under the $SO(4)$ symmetry.³ For the $SO(4)$ singlet $\mathcal{N} = 4$ multiplet in the unitary case, we can construct all the new higher spin currents living this multiplet once the

lowest new higher spin current is determined with the help of the four supersymmetry generators.

However, if we have a single $SO(4)$ nonsinglet $\mathcal{N} = 4$ multiplet which will appear in the right hand side of the our OPEs, then there are several elements on this multiplet. Each element transforms nontrivially under the 16 currents of the large $\mathcal{N} = 4$ linear superconformal algebra. The reason why we need to have this transformation is that we should calculate the OPEs between the 16 currents and the higher spin currents living in the $SO(4)$ nonsinglet $\mathcal{N} = 4$ multiplet in order to use the Jacobi identities. More explicitly, the several lowest higher spin currents can be determined by the six spin-1 currents of the above 16 currents after computing the OPEs between them and reading off the first order poles. Once these lowest higher spin currents are obtained completely, then we can repeat the procedure for the singlet case because we can act the four supersymmetry generators on each lowest higher spin current.

The most difficult part of the present work is to write down all the possible orthogonal Wolf space composite fields appearing in the right hand sides of the OPEs in terms of the known (higher spin) currents. As the spin at the specific pole increases, the number of composite fields becomes large. When the new higher spin current appears, the situation is more involved. Even for $N = 5$, when the spin at the particular pole is large, then it is not obvious to observe how to express that pole in terms of known (higher spin) currents and a new primary higher spin current. Due to the many independent terms, it is not possible to solve the linear equations for the undetermined coefficients coming from the vanishing of the sum of the particular pole (written in terms of coset fields), the possible known composite (higher spin) current terms with arbitrary coefficients and a new higher spin current. We can calculate the OPE between the spin- $\frac{1}{2}$ currents and the above particular pole term given in terms of coset fields. Then the first order pole of this OPE has spin which is less than the original spin by $\frac{1}{2}$. This will make some computations easier. At the same time, we can calculate the OPE between the spin- $\frac{1}{2}$ currents and the above sum of known composite terms and a new higher spin current. By comparing these two expressions, we can reduce the number of unknown coefficients and moreover, by using other conditions from other spin-1, $\frac{3}{2}$ and 2 currents from the above $\mathcal{N} = 4$ primary, we can eventually determine all the coefficients if there is no new higher spin current. If not, the new higher spin current can be written in terms of the known composite terms as well as the extra terms which can be written in terms of coset fields.

Instead of considering all the possible $136 (= \sum_{i=1}^{16} i)$ OPEs (which arise from the OPEs between 16 higher spin currents), we focus on the 16 OPEs among them (the OPEs between 16 higher spin currents and the lowest higher spin-2 current or their reversed OPEs) because the remaining 120

¹ There exists the large $\mathcal{N} = 4$ holography corresponding to a symplectic Wolf space coset conformal field theory [4] and its AdS_3 Vasiliev higher spin theory [5, 6].

² The $\mathcal{N} = 3$ supersymmetric example, where the $SO(\mathcal{N} = 3)$ nonsinglet structure plays an important role, can be found in [14] in the context of the $\mathcal{N} = 3$ Kazama-Suzuki model. See also the nonsinglet structure in [15].

³ We will use the notation $SO(4)$ for $SO(\mathcal{N} = 4)$ for simplicity.

OPEs can be extracted from the $\mathcal{N} = 4$ supersymmetry. That is, once the above 16 OPEs are obtained (this implies that we can write down the corresponding $\mathcal{N} = 4$ OPE explicitly by putting these five kinds of OPEs into the expansion of fermionic coordinates), then the $\mathcal{N} = 4$ superspace description allows us to write down the above 120 OPEs automatically by multiplying various super derivatives and putting the fermionic coordinates to zero in this OPE successively. Then after inserting the arbitrary coefficients which will depend on (N, k) explicitly (and possible other structure constants) in front of all the composite fields arising in the right hand side of 136 OPEs, the Jacobi identities can be used. Eventually, the complete OPEs can be determined and we will present them in a single OPE in $\mathcal{N} = 4$ superspace.

In Sect. 2, the $\mathcal{N} = 4$ orthogonal Wolf space coset model is reviewed.⁴

In Sect. 3, based on the findings in [13] which is valid for $N = 4$, the new observations will be added.

In Sect. 4, based on the results of section 3, we consider $N = 5$ case. We will find various new higher spin currents (some of them are not present for $N = 4$ case). The 136 OPEs will be obtained eventually.

In Appendices, some of the detailed expressions described in previous sections are given.

The Thielemans package [29] is used.

An ancillary (mathematica) file ancillary.nb, where the complete OPEs with the explicit structure constants appearing in Appendices B and C are given, is included.

2 Review of $\mathcal{N} = 4$ orthogonal Wolf space coset model

We consider the Wolf space coset in the ‘supersymmetric’ version with groups $G = SO(N + 4)$ and $H = SO(N) \times SO(4)$ as follows⁵:

$$\text{Wolf} = \frac{G}{H} = \frac{SO(N + 4)}{SO(N) \times SO(4)}. \quad (2.1)$$

The group indices are denoted by

$$\begin{aligned} G \text{ indices : } a, b, c, \dots \\ = 1, 2, \dots, \frac{1}{4}(N + 4) \\ (N + 3), 1^*, 2^*, \dots, \left(\frac{1}{4}(N + 4)(N + 3) \right)^* \\ \frac{G}{H} \text{ indices : } \bar{a}, \bar{b}, \bar{c}, \dots = 1, 2, \dots, 2N, 1^*, 2^*, \dots, 2N^*. \end{aligned} \quad (2.2)$$

⁴ There are related works in [9–11, 16] on this Wolf space coset model. There are also previous works on the orthogonal coset models in [17–25] along the line of [26–28].

⁵ After we divide $SU(2) \times U(1)$ in the coset of [3], we obtain the Wolf space coset.

In the bosonic version, there exist $4N$ free fermions living in the extra $SO(4N)$ group in the numerator of the coset at level 1.

The $\mathcal{N} = 1$ affine Kac–Moody algebra can be determined by the adjoint spin-1 current and the spin- $\frac{1}{2}$ current of group $G = SO(N + 4)$. By adding the quadratic term in the fermions to the above spin-1 current, the operator product expansions between the ‘modified’ spin-1 current $V^a(z)$ and the spin- $\frac{1}{2}$ current $Q^a(z)$ are described as

$$\begin{aligned} V^a(z) V^b(w) &= \frac{1}{(z - w)^2} k g^{ab} \\ &\quad - \frac{1}{(z - w)} f^{ab}_c V^c(w) + \dots, \\ Q^a(z) Q^b(w) &= -\frac{1}{(z - w)} (k + N + 2) g^{ab} + \dots. \end{aligned} \quad (2.3)$$

The level k is a positive integer. The metric can be obtained from $g_{ab} = \frac{1}{2c_g} f_{ac}^d f_{bd}^c$ where c_g is the dual Coxeter number of the Lie algebra $G = SO(N + 4)$. That is, $c_g = (N + 2)$. The metric g_{ab} is given by the generators of $SO(N + 4)$ in the complex basis, $g_{ab} = \frac{1}{2} \text{Tr}(T_a T_b)$. The commutation relation of generators is given by $[T_a, T_b] = f_{ab}^c T_c$.

For given $(N + 4) \times (N + 4)$ matrix, the above $4N$ coset indices (2.2) can be associated with the following locations with asterisk

$$\begin{pmatrix} & & & * & * & * & * \\ & & & * & * & * & * \\ & & & \vdots & \vdots & \vdots & \vdots \\ & & & * & * & * & * \\ & & & * & * & * & * \\ * & * & \dots & * & * & * & * \\ * & * & \dots & * & * & * & * \\ * & * & \dots & * & * & * & * \\ * & * & \dots & * & * & * & * \end{pmatrix}_{(N+4) \times (N+4)}. \quad (2.4)$$

That is, the generators with $2N$ coset indices have two nonzero elements located at the above $N \times 4$ and $4 \times N$ off diagonal matrices in (2.4). The remaining $2N$ coset generators can be obtained from the above coset generators by transposing. Note that the size of two block diagonals at $N = 4$ is equal to each other.

2.1 The 11 currents of $\mathcal{N} = 4$ nonlinear superconformal algebra

The four supersymmetry currents of spin- $\frac{3}{2}$, the six spin-1 currents of $\hat{S}\hat{U}(2)_k \times \hat{S}\hat{U}(2)_N$, and the spin-2 stress energy tensor [30–33] can be described as follows:

$$\begin{aligned} \hat{G}^0(z) &= \frac{i}{(k + N + 2)} Q_{\bar{a}} V^{\bar{a}}(z), \\ \hat{G}^i(z) &= \frac{i}{(k + N + 2)} h^i_{\bar{a}\bar{b}} Q^{\bar{a}} V^{\bar{b}}(z), \quad i = 1, 2, 3, \end{aligned}$$

$$\begin{aligned}
\hat{A}_i(z) &= (-1)^{i+1} \frac{1}{4N} f^{\bar{a}\bar{b}}_c h^i_{\bar{a}\bar{b}} V^c(z), \\
\hat{B}_i(z) &= -\frac{1}{4(k+N+2)} h^i_{\bar{a}\bar{b}} Q^{\bar{a}} Q^{\bar{b}}(z), \\
\hat{T}(z) &= \frac{1}{2(k+N+2)^2} \left[(k+N+2) V_{\bar{a}} V^{\bar{a}} + k Q_{\bar{a}} \partial Q^{\bar{a}} \right. \\
&\quad \left. + f^{\bar{a}\bar{b}}_c Q^{\bar{a}} Q^{\bar{b}} V^c \right](z) \\
&\quad - \frac{1}{(k+N+2)} \sum_{i=1}^3 ((-1)^i \hat{A}_i + \hat{B}_i)^2(z). \quad (2.5)
\end{aligned}$$

Here the three almost complex structures are given by

$$\begin{aligned}
h^1_{\bar{a}\bar{b}} &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad h^2_{\bar{a}\bar{b}} = \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \\
h^3_{\bar{a}\bar{b}} &= \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad (2.6)
\end{aligned}$$

where each entry is $N \times N$ matrix. Note that we introduce $h^0_{\bar{a}\bar{b}} \equiv g_{\bar{a}\bar{b}}$. The only coset indices associated with (2.4) appear in the fermionic fields.

We introduce the above 11 currents in different basis in order to describe them in the $\mathcal{N} = 4$ superspace as follows:

$$\begin{aligned}
\hat{T}(z) &\rightarrow \hat{L}(z), \\
\hat{G}^0(z) &\rightarrow \hat{G}^2_{ss}(z), \quad \hat{G}^1(z) \rightarrow \hat{G}^3_{ss}(z), \\
\hat{G}^2(z) &\rightarrow -\hat{G}^4_{ss}(z), \quad \hat{G}^3(z) \rightarrow \hat{G}^1_{ss}(z), \\
\hat{A}^{\pm 1}(z) &\rightarrow \frac{i}{2} (\hat{T}^{14} \mp T^{23})(z), \\
\hat{A}^{\pm 2}(z) &\rightarrow \frac{i}{2} (\hat{T}^{13} \pm \hat{T}^{24})(z), \quad \hat{A}^{\pm 3}(z) \rightarrow \pm \frac{i}{2} (\hat{T}^{12} \mp \hat{T}^{34})(z). \quad (2.7)
\end{aligned}$$

We use $\hat{A}_1(z) = -\hat{A}^{+1}(z)$, $\hat{A}_2(z) = \hat{A}^{+2}(z)$, $\hat{A}_3(z) = -\hat{A}^{+3}(z)$ and $\hat{B}_i(z) = \hat{A}^{-i}(z)$. Then the 11 currents in this new basis satisfy Appendix (A.1) of [34] by using (2.3).

2.2 The 16 currents of $\mathcal{N} = 4$ linear superconformal algebra

In the linear version [3, 35–38] of $\mathcal{N} = 4$ superconformal algebra, there are also four spin- $\frac{1}{2}$ currents and one spin-1 current

$$\begin{aligned}
F_{11}(z) &= \frac{i}{\sqrt{2}} Q^{(2N+3)}(z), \quad F_{22}(z) = -\frac{i}{\sqrt{2}} Q^{(2N+3)*}(z), \\
F_{12}(z) &= \frac{(1-i)}{2} Q^{(2N+2)*}(z), \\
F_{21}(z) &= \frac{(1+i)}{2} Q^{(2N+2)}(z),
\end{aligned}$$

$$\begin{aligned}
U(z) &= \frac{(1+i)}{2\sqrt{2}} V^{(2N+2)}(z) + \frac{(-1+i)}{2\sqrt{2}} V^{(2N+2)*}(z) \\
&\quad + \frac{i}{(N+k+2)} Q^{(2N+1)} Q^{(2N+1)*}(z) \\
&\quad - \frac{i}{2(N+k+2)} \left(\sum_{a=1}^N Q^a Q^{a*} - \sum_{a=N+1}^{2N} Q^a Q^{a*} \right)(z). \quad (2.8)
\end{aligned}$$

We use the following transformation in order to describe in the $\mathcal{N} = 4$ superspace

$$\begin{aligned}
\frac{i}{\sqrt{2}} (F_{12} + F_{21})(z) &\rightarrow \Gamma^1(z), \\
-\frac{1}{\sqrt{2}} (F_{12} + F_{21})(z) &\rightarrow \Gamma^2(z), \\
-\frac{i}{\sqrt{2}} (F_{11} + F_{22})(z) &\rightarrow \Gamma^3(z), \\
-\frac{1}{\sqrt{2}} (F_{11} - F_{22})(z) &\rightarrow \Gamma^4(z), \quad U(z) \rightarrow U(z). \quad (2.9)
\end{aligned}$$

Furthermore, the six spin-1 currents of $\hat{S}\hat{U}(2)_{k+1} \times \hat{S}\hat{U}(2)_{N+1}$, four supersymmetry currents of spin- $\frac{3}{2}$ and the spin-2 stress energy tensor can be described as follows [30]:

$$\begin{aligned}
T^{ij}(z) &= \hat{T}^{ij}(z) + \frac{2i}{(2+k+N)} \Gamma^i \Gamma^j(z), \\
G^i(z) &= \hat{G}^i_{ss}(z) + \frac{2i}{(2+k+N)} U \Gamma^i(z) \\
&\quad + \varepsilon_{ijkl} \left[\frac{4i}{3(2+k+N)^2} \Gamma^j \Gamma^k \Gamma^l \right. \\
&\quad \left. - \frac{1}{(2+k+N)} T^{jk} \Gamma^l \right](z), \\
L(z) &= \hat{L}(z) - \frac{1}{(2+k+N)} (UU - \partial \Gamma^i \Gamma^i)(z). \quad (2.10)
\end{aligned}$$

Here we should use (2.5), (2.6), (2.7) and (2.8). The $\mathcal{N} = 4$ linear superconformal algebra can be realized from (2.9) and (2.10) in the orthogonal Wolf space coset model (2.1). The 16 currents are given by $L(z)$, $G^i(z)$, $T^{ij}(z)$, $\Gamma^i(z)$ and $U(z)$ which can be written in terms of $Q^{\bar{a}}(z)$ and $V^a(z)$.

2.3 The 16 lowest higher spin currents

The lowest higher spin-2 current in the $\mathcal{N} = 4$ orthogonal Wolf space is found in [13] and it is given by

$$\begin{aligned}
\Phi_0^{(2)}(z) &= c_1 V_{\bar{a}} V^{\bar{a}}(z) + c_2 \sum_{a':so(N)} V_{a'} V^{a'}(z) \\
&\quad + c_3 \sum_{a'':so(4)} V_{a''} V^{a''}(z) + c_4 \sum_{i=1}^3 \hat{A}_i \hat{A}_i(z)
\end{aligned}$$

$$\begin{aligned}
& +c_5 \sum_{i=1}^3 \hat{B}_i \hat{B}_i(z) + c_6 Q_{\bar{a}} \partial Q^{\bar{a}}(z) \\
& +c_7 \sum_{\mu=0}^3 h_{\bar{a}\bar{b}}^{\mu} h_{\bar{c}\bar{d}}^{\mu} f^{\bar{a}\bar{c}}_e Q^{\bar{b}} Q^{\bar{d}} V^e(z), \quad (2.11)
\end{aligned}$$

where the coefficients are⁶

$$\begin{aligned}
c_1 &= -\frac{(2k^2N + k^2 + 4kN^2 + 6kN + 2k + 11N^2 - 2N - 24)}{2(k-1)N(k+N+2)^2}, \\
c_2 &= \frac{6(2kN + 3k + 3N + 4)}{(k-1)N(k+N+2)^2}, \\
c_3 &= \frac{3(k+N-2)(2kN + 3k + 3N + 4)}{2(k-1)(k+2)(k+N+2)^2}, \\
c_4 &= \frac{2(N+2)(2k+N)}{(k+2)(k+N+2)^2}, \quad c_5 = \frac{2k(2k+N)}{N(k+N+2)^2}, \\
c_6 &= \frac{k(N+2)(2k+N)}{N(k+N+2)^3}, \quad c_7 = \frac{(N+2)(2k+N)}{4N(k+N+2)^3}. \quad (2.12)
\end{aligned}$$

The OPE between this higher spin-2 current and itself is described as

$$\begin{aligned}
\Phi_0^{(2)}(z) \Phi_0^{(2)}(w) &= \frac{1}{(z-w)^4} c^0 + \frac{1}{(z-w)^2} Q_0^{(2)}(w) \\
&+ \frac{1}{(z-w)} \frac{1}{2} \partial Q_0^{(2)}(w) + \dots, \quad (2.13)
\end{aligned}$$

where the central term is

$$c_0^{0,4} = \frac{3k(2k+N)(2kN+3k+3N+4)(2k^2N+k^2+4kN^2+6kN+2k+11N^2-2N-24)}{(k-1)(k+2)N(k+N+2)^3}. \quad (2.14)$$

The quasi primary field $Q_0^{(2)}(w)$ depends on the higher spin-2 current itself $\Phi_0^{(2)}(w)$ and the spin- $\frac{1}{2}$, 1, 2 currents of the $\mathcal{N} = 4$ linear superconformal algebra. The explicit form is given Appendix B. The 16 higher spin currents can be combined into one single $\mathcal{N} = 4$ super field as follows:

$$\begin{aligned}
\Phi^{(s=2)}(Z) &\equiv \left(\Phi_0^{(2)}(z), \Phi_{\frac{1}{2}}^{(2),i}(z), \Phi_1^{(2),ij}(z), \Phi_{\frac{3}{2}}^{(2),i}(z), \Phi_2^{(2)}(z) \right). \quad (2.15)
\end{aligned}$$

In this paper, we construct 136 OPEs between the higher spin currents in (2.15) explicitly.⁷

⁶ The coefficients c_2 and c_3 in (2.12) are the same for $N = 4$.

⁷ In addition to (2.15), we will consider the following $\mathcal{N} = 4$ multiplets

$$\begin{aligned}
\mathbf{X}^{(2)} &\equiv \left(X_0^{(2)}, X_{\frac{1}{2}}^{(2),i}, X_1^{(2),ij}, X_{\frac{3}{2}}^{(2),i}, X_2^{(2)} \right), \\
\Phi^{(3),\alpha} &\equiv \left(\Phi_0^{(3),\alpha}, \Phi_{\frac{1}{2}}^{(3),i,\alpha}, \Phi_1^{(2),ij,\alpha}, \Phi_{\frac{3}{2}}^{(3),i,\alpha}, \Phi_2^{(3),\alpha} \right),
\end{aligned}$$

3 The OPE for $N = 4$

In this section, we continue to calculate the OPEs between the lowest higher spin-2 multiplet for $N = 4$. Some of the OPEs were found in [13]. We would like to obtain the general structure of these OPEs which will give us some hints for the general N in next section.

3.1 The known facts

For fixed $N = 4$, it was straightforward to calculate the various higher spin currents in $\mathcal{N} = 4$ orthogonal Wolf space starting from the above higher spin-2 current in (2.11). One of the main results in [13] was to obtain the new three higher spin-3 currents which live in different higher spin multiplet. It was not clear how they appear in the right hand side of the whole 136 OPEs. They will turn out to be the lowest components of three $SO(4)$ vector $\mathcal{N} = 4$ higher spin-3 multiplets in next subsection.

3.2 The complete OPEs in components and $\mathcal{N} = 4$ superspace

In order to observe the symmetry behind the presence of the new three higher spin-3 currents, we should go into the $\mathcal{N} = 4$ superspace. It is known that the above 16 currents can be combined into the following single $\mathcal{N} = 4$ super field [37]

$$\begin{aligned}
\mathbf{J}(Z) &= -\Delta(z) + i\theta^j \Gamma^j(z) - \frac{i}{2} \theta^{4-jk} T^{jk}(z) \\
&- \theta^{4-j} (G^j - 2\alpha i \partial \Gamma^j)(z) + \theta^{4-0} (2L - 2\alpha \partial^2 \Delta)(z) \\
&\equiv -\Delta(z) + i\theta^j \Gamma^j(z) - \frac{i}{2} \theta^{4-jk} T^{jk}(z) \\
&- \theta^{4-j} \tilde{G}^j(z) + \theta^{4-0} 2\tilde{L}(z). \quad (3.1)
\end{aligned}$$

Here we use the notation θ^{4-0} for the product of fermionic coordinates $\theta^{4-0} \equiv \theta^1 \theta^2 \theta^3 \theta^4$ and we have $U(z) \equiv$

Footnote 7 continued

$$\begin{aligned}
\Phi_{\left(\frac{7}{2}\right),\mu} &\equiv \left(\Phi_0^{\left(\frac{7}{2}\right),\mu}, \Phi_{\frac{1}{2}}^{\left(\frac{7}{2}\right),i,\mu}, \Phi_1^{(2),ij,\mu}, \Phi_{\frac{3}{2}}^{\left(\frac{7}{2}\right),i,\mu}, \Phi_2^{\left(\frac{7}{2}\right),\mu} \right), \\
\Phi_{\left(\frac{9}{2}\right),\mu} &\equiv \left(\Phi_0^{\left(\frac{9}{2}\right),\mu}, \Phi_{\frac{1}{2}}^{\left(\frac{9}{2}\right),i,\mu}, \Phi_1^{(2),ij,\mu}, \Phi_{\frac{3}{2}}^{\left(\frac{9}{2}\right),i,\mu}, \Phi_2^{\left(\frac{9}{2}\right),\mu} \right), \\
\Phi^{(4)} &\equiv \left(\Phi_0^{(4)}, \Phi_{\frac{1}{2}}^{(4),i}, \Phi_1^{(4),ij}, \Phi_{\frac{3}{2}}^{(4),i}, \Phi_2^{(4)} \right).
\end{aligned}$$

Note that there are $SO(4)$ nonsinglet representations denoted by α and μ . Sometimes the index μ is replaced by the index i because it is a $SO(4)$ vector index.

$-\partial \Delta(z)$. The parameter α appears in the above and is given by

$$\alpha \equiv \frac{1}{2} \frac{(k^+ - k^-)}{(k^+ + k^-)}, \quad k^+ \equiv k + 1, \quad k^- \equiv N + 1. \quad (3.2)$$

Then the explicit realization for these 16 currents described in previous section can be inserted into the above single $\mathcal{N} = 4$ $SO(4)$ singlet super field.

It is known that the $\mathcal{N} = 4$ higher spin multiplet, which transforms nontrivially under the $SO(4)$ [the index α stands for this representation which is nothing to do with (3.2)], of (conformal) (super)spin s has the following OPE with the above $\mathcal{N} = 4$ stress energy tensor as follows [37]:

$$\begin{aligned} \mathbf{J}(Z_1) \Phi^{(s),\alpha}(Z_2) &= \frac{\theta_{12}^{4-0}}{z_{12}^2} 2s \Phi^{(s),\alpha}(Z_2) + \frac{\theta_{12}^{4-i}}{z_{12}} D^i \Phi^{(s),\alpha}(Z_2) \\ &+ \frac{\theta_{12}^{4-0}}{z_{12}} 2 \partial \Phi^{(s),\alpha}(Z_2) \\ &- \frac{i}{2} \frac{\theta_{12}^{4-ij}}{z_{12}} (T^{ij})^{\alpha\beta} \Phi^{(s),\beta}(Z_2) + \dots \end{aligned} \quad (3.3)$$

Note that for the $SO(4)$ singlet higher spin multiplet the last term in (3.3) will disappear. We will see two kinds of T^{ij} which span the representation of the $SO(4)$ Lie algebra in this OPE of this paper.

By using the 16 component fields for fixed α as in

$$\begin{aligned} \Phi^{(s),\alpha}(Z) &\equiv \left(\Phi_0^{(s),\alpha}(z), \Phi_{\frac{1}{2}}^{(s),i,\alpha}(z), \Phi_1^{(s),ij,\alpha}(z), \Phi_{\frac{3}{2}}^{(s),i,\alpha}(z), \Phi_2^{(s),\alpha}(z) \right), \end{aligned} \quad (3.4)$$

the various complicated component results of (3.3) are presented in Appendix A.

We can show that the three higher spin-3 currents ($P^{(3)}(z)$, $\tilde{Q}^{(3)}(z)$ and $\tilde{R}^{(3)}(z)$) found in [13] with proper change of basis can be written in terms of the three lowest components of the three $SO(3)$ vector $\Phi^{(s=3),\alpha}(Z)$ where $\alpha = 1, 2, 3$ and this higher spin-3 multiplet transforms as in (3.3) with

$$\begin{aligned} T^{12} &= \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -T^{34}, & T^{13} &= \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} = -T^{24}, \\ T^{23} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} = -T^{14}. \end{aligned} \quad (3.5)$$

We can check that the above three $SO(3)$ (inside of $SO(4)$) generators in (3.5) satisfy $[T^i, T^j] = \varepsilon^{ijk} T^k$ where $T^i \equiv -\frac{i}{2} \varepsilon^{ijk} T^{jk}$.⁸ Each higher spin-3 multiplet $\Phi^{(s=3),\alpha}(Z)$ has

⁸ For example, the lowest component $\Phi_0^{(s=3),\alpha}$ has the nontrivial OPEs with the spin-1 currents A^{+i} associated with $\hat{SU}(2)_{k+1}$ and has trivial

16 components of higher spin currents according to (3.4). Totally, we have 48 higher spin currents.

Furthermore, there exist the $SO(4)$ singlet higher spin-4 multiplet $\Phi^{(s=4)}(Z)$ and the four $SO(4)$ vector higher spin- $\frac{9}{2}$ multiplets $\Phi^{(s=\frac{9}{2}),i}(Z)$ with $i = 1, 2, 3, 4$. They have their component fields as in (3.4).

It turns out that there will be a problem to generalize the OPEs for general N without introducing the new primary fields which will be discussed in next section.

4 The OPE for $N \geq 5$

We would like to construct the OPEs between the 16 higher spin currents for generic N in component approach and in $\mathcal{N} = 4$ superspace.

4.1 What happens for $N = 5$?

It is natural to take the OPEs found in previous section and introduce the arbitrary coefficients in front of the composite fields appearing in the right hand side of the OPEs. It is straightforward to apply the Jacobi identity in order to determine these coefficients completely. It turns out that there are no consistent solutions unless we introduce the new primary fields. Therefore, we focus on the case of $N = 5$ in order to understand the algebraic structures more clearly.

4.2 The new higher spin current of spin 2

Let us consider the OPE between the last component and the first component of (2.15). How we can obtain the last component in terms of orthogonal Wolf space coset fields from the first component? According to the OPEs in Appendix A with a singlet α , we obtain the second component using the OPE between $G^i(z)$ and the first component $\Phi_0^{(2)}(w)$. Then we can calculate the OPE between $G^i(z)$ and the second component $\Phi_{\frac{1}{2}}^{(2),j}(w)$ with $i \neq j$. Then the third component can be determined. The fourth component can be obtained by the OPE between $G^i(z)$ and the third component $\Phi_1^{(2),jk}(w)$ with $i = k$. Finally, the last component can be determined by the OPE between $G^i(z)$ and the fourth component $\Phi_{\frac{3}{2}}^{(2),j}(w)$ with $i = j$.

The reason for describing this particular OPE rather than others is that the structure of the right hand side of this OPE will be simple because these two higher spin currents are $SO(4)$ singlets rather than nonsinglets, although we have

Footnote 8 continued

OPEs with the spin-1 currents A^{-i} associated with $\hat{SU}(2)_{N+1}$. That is, $\Phi_0^{(s=3),\alpha}$ transforms in the representation $(3, 1)$ under the $\hat{SU}(2)_{k+1} \times \hat{SU}(2)_{N+1}$.

found the presence of this new higher spin-2 current in other OPEs.

Let us emphasize that the last component of (2.15) is not a quasi primary field. See also Appendix A. As in unitary case [8], we subtract the additional terms from the last component of (2.15) in order to make it to be primary field.⁹ It turns out that we have the following OPE with implicit notation

$$\begin{aligned} \Phi_2^{(2)}(z) \Phi_0^{(2)}(w) &= \frac{1}{(z-w)^6} 8\alpha c_0^{0,4} \\ &+ \frac{1}{(z-w)^5} Q_2^{(1)}(w) + \frac{1}{(z-w)^4} \left[\frac{3}{2} \partial Q_2^{(1)} + Q_2^{(2)} \right](w) \\ &+ \frac{1}{(z-w)^3} \left[\partial^2 Q_2^{(1)} + \partial Q_2^{(2)} + Q_2^{(3)} \right](w) \\ &+ \frac{1}{(z-w)^2} \left[\frac{5}{12} \partial^3 Q_2^{(1)} + \frac{1}{2} \partial^2 Q_2^{(2)} + \frac{5}{6} \partial Q_2^{(3)} + Q_2^{(4)} \right](w) \\ &+ \frac{1}{(z-w)} \left[\frac{1}{8} \partial^4 Q_2^{(1)} + \frac{1}{6} \partial^3 Q_2^{(2)} + \frac{5}{14} \partial^2 Q_2^{(3)} \right. \\ &\quad \left. + \frac{3}{4} \partial Q_2^{(4)} + Q_2^{(5)} \right](w) \\ &+ p_1 \sum_{n=3}^4 \frac{1}{(z-w)^n} \left\{ \partial^2 \Phi_0^{(2)} \Phi_0^{(2)} \right\}_n(w) \\ &+ p_2 \sum_{m=2}^4 \frac{1}{(z-w)^m} \left\{ (L \Phi_0^{(2)}) \Phi_0^{(2)} \right\}_m(w) \\ &+ \dots \end{aligned} \quad (4.1)$$

The central term is proportional to the previous central term in (2.14) together with (3.2).¹⁰ Note that the above central term comes from these p_1 and p_2 terms.

We will use the following quasi primary fields with their spins, $SO(4)$ indices i, j and the subscript indicating the number of fermionic coordinates

$$\begin{aligned} Q_0^{(2)}(z); Q_{\frac{1}{2}}^{(\frac{3}{2}),i}(z), Q_{\frac{1}{2}}^{(\frac{5}{2}),i}(z), \\ Q_{\frac{1}{2}}^{(\frac{7}{2}),i}(z); Q_1^{(1),ij}(z), Q_1^{(2),ij}(z), Q_1^{(3),ij}(z), Q_1^{(4),ij}(z); \\ Q_{\frac{3}{2}}^{(\frac{1}{2}),ij}(z), Q_{\frac{3}{2}}^{(\frac{3}{2}),ij}(z), Q_{\frac{3}{2}}^{(\frac{5}{2}),ij}(z), \\ Q_{\frac{3}{2}}^{(\frac{7}{2}),ij}(z), Q_{\frac{3}{2}}^{(\frac{9}{2}),ij}(z); Q_2^{(1)}(z), \\ Q_2^{(2)}(z), Q_2^{(3)}(z), Q_2^{(4)}(z), Q_2^{(5)}(z). \end{aligned} \quad (4.2)$$

⁹ That is, we have $\Phi_2^{(s=2)}(z) \equiv \tilde{\Phi}_2^{(2)}(z) - p_1 \partial^2 \Phi_0^{(2)}(z) - p_2 L \Phi_0^{(2)}(z)$ where $\tilde{\Phi}_2^{(2)}(z)$ is a primary field under the stress energy tensor. The coefficients p_1 and p_2 were given in [8] or in (4.13).

¹⁰ The last line of (4.1) with specific notations for the singular terms [39] comes from the subtracted terms as described above.

Note that the spin is given by the number inside the bracket and we do not add the subscript for the spin, contrary to the notation of (3.4).¹¹

First of all, the fifth order pole has the spin-1 current $U(w)$ of the $\mathcal{N} = 4$ linear superconformal algebra from Appendix B. The next fourth order pole contains the descendant field $\partial U(w)$ with the known coefficient and other terms.

We observe that there exists a new primary higher spin field of spin-2 denoted by $X_0^{(2)}(w)$ which cannot be written in terms of the known composite fields of the currents and higher spin currents as in Appendix B. That is,

$$Q_2^{(2)}(w) = w_{1,2} \Phi_0^{(2)}(w) + w_{2,2} X_0^{(2)}(w) + \dots, \quad (4.3)$$

where other remaining terms are given in Appendix B. See also Appendix E for explicit form for the $X_0^{(2)}(w)$ for $N = 5$. By considering the condition that the fourth order pole of the OPE between $\Phi_0^{(2)}(z)$ and $X_0^{(2)}(w)$ should vanish, we can determine the structure constant appearing the $\Phi_0^{(2)}(w)$ of (4.3). For $N = 5$, we obtain this particular structure constant as

$$\begin{aligned} w_{1,2} &\equiv C_{(4)(2)}^{(2)} \Big|_{N=5} \\ &= -\frac{36}{5(k-1)(k+2)(k+7)^2(28k+61)(11k^2+132k+241)} \\ &\quad \times (7227k^7 + 201718k^6 + 2017067k^5 \\ &\quad + 8606534k^4 + 13128257k^3 \\ &\quad - 11460814k^2 - 54096247k - 42478238). \end{aligned} \quad (4.4)$$

At the moment, it is rather difficult to determine the N generalization of (4.4) because although we can expect the N dependence for the denominator of (4.4) by increasing the N values, the numerical values appearing in front of k th power in the numerator are functions of N . Even if we can try to calculate (4.4) for seven (which is the maximum power of k) N values where $N = 5, 8, 9, 12, 13, 16$ and 17 , it will take too much time to extract the higher spin-4 $\Phi_2^{(2)}(z)$. In this paper, the above structure constant for generic N is not determined.

The next third order pole can be expressed in terms of the descendant fields and other known composite fields where there are two higher spin dependent terms $U \Phi_0^{(2)}(w)$ and $\Gamma^i \Phi_{\frac{1}{2}}^{(2),i}(w)$. The four component fields of spin- $\frac{5}{2}$ in (2.15) arise in this pole.

The second and first order poles appearing in the third and fourth lines of (4.1) will be described later subsection. We will observe that there will be a primary higher spin-4 current.

¹¹ The corresponding $\mathcal{N} = 4$ super fields can be denoted by the bold-face later.

We can easily see that the singular terms appearing in the last two terms of (4.1) can be rewritten as follows.¹²

Moreover, for the second term, by introducing

$$\left\{ (L\Phi_0^{(2)})\Phi_0^{(2)} \right\}_{n+2} \equiv E_2^{(4-n)}, \quad n = 0, 1, 2, \quad (4.5)$$

the following relations for three in (4.5) can be obtained from the OPE $(L\Phi_0^{(2)})(z)\Phi_0^{(2)}(w)$, where the previous relation (2.13) is used, in terms of $Q_0^{(2)}(w)$, $L(w)$ and $\Phi_0^{(2)}(w)$,

$$\begin{aligned} E_2^{(2)}(w) &= (4Q_0^{(2)} + c_0^{0,4}L)(w), \\ E_2^{(3)}(w) &= \left(\frac{5}{2} \partial Q_0^{(2)} + c_0^{0,4} \partial L \right)(w), \\ E_2^{(4)}(w) &= \left(\frac{1}{2} \partial^2 Q_0^{(2)} + LQ_0^{(2)} + \frac{1}{2} \partial^2 L + 2\Phi_0^{(2)}\Phi_0^{(2)} \right)(w), \end{aligned} \quad (4.6)$$

where $c_0^{0,4}$ is the central term of the OPE between the lowest higher spin-2 current in (2.14).

Therefore, we can present the above OPE (4.1), together with (4.6), in complete form as follows:

$$\begin{aligned} \Phi_2^{(2)}(z)\Phi_0^{(2)}(w) &= \frac{1}{(z-w)^6} 8\alpha c_0^{0,4} + \frac{1}{(z-w)^5} Q_2^{(1)}(w) \\ &+ \frac{1}{(z-w)^4} \left[\frac{3}{2} \partial Q_2^{(1)} + Q_2^{(2)} - 6p_1 Q_0^{(2)} - p_2 E_2^{(2)} \right](w) \\ &+ \frac{1}{(z-w)^3} \left[\partial^2 Q_2^{(1)} + \partial Q_2^{(2)} + Q_2^{(3)} \right. \\ &\quad \left. - p_1 \partial Q_0^{(2)} - p_2 E_2^{(3)} \right](w) \\ &+ \frac{1}{(z-w)^2} \left[\frac{5}{12} \partial^3 Q_2^{(1)} + \frac{1}{2} \partial^2 Q_2^{(2)} \right. \\ &\quad \left. + \frac{5}{6} \partial Q_2^{(3)} + Q_2^{(4)} - p_2 E_2^{(4)} \right](w) \\ &+ \frac{1}{(z-w)} \left[\frac{1}{8} \partial^4 Q_2^{(1)} + \frac{1}{6} \partial^3 Q_2^{(2)} + \frac{5}{14} \partial^2 Q_2^{(3)} \right. \\ &\quad \left. + \frac{3}{4} \partial Q_2^{(4)} + Q_2^{(5)} \right](w) \\ &+ \dots, \end{aligned} \quad (4.7)$$

where the relations (3.2), (2.14), (4.6) and (4.13) are used. All the singlet quasi primary fields appearing in this OPE are obtained and we present the partial expressions given in Appendix B.

¹² For the first term, there is a relation $\left\{ \partial^2 \Phi_0^{(2)} \Phi_0^{(2)} \right\}_{n+2}(w) = n(n+1) \left\{ \Phi_0^{(2)} \Phi_0^{(2)} \right\}_n(w)$ with $n = 1, 2$. That is, we have $6Q_0^{(2)}(w)$ for the fourth order pole ($n = 2$) and $\partial Q_0^{(2)}(w)$ for the third order pole ($n = 1$) from (2.13).

4.3 The new higher spin currents of spin 3

Let us describe the OPE between the higher spin-3 (primary) current transforming as the $SO(4)$ adjoint representation and the higher spin-2 current. We observe the following OPE

$$\begin{aligned} \Phi_1^{(2),ij}(z)\Phi_0^{(2)}(w) &= \frac{1}{(z-w)^4} Q_1^{(1),ij}(w) + \frac{1}{(z-w)^3} \left[\partial Q_1^{(1),ij} + Q_1^{(2),ij} \right](w) \\ &+ \frac{1}{(z-w)^2} \left[\frac{1}{2} \partial^2 Q_1^{(1),ij} + \frac{3}{4} \partial Q_1^{(2),ij} + Q_1^{(3),ij} \right](w) \\ &+ \frac{1}{(z-w)} \left[\frac{1}{6} \partial^3 Q_1^{(1),ij} + \frac{3}{10} \partial^2 Q_1^{(2),ij} \right. \\ &\quad \left. + \frac{2}{3} \partial Q_1^{(3),ij} + Q_1^{(4),ij} \right](w) \\ &- \sum_{n=1}^4 \frac{1}{(z-w)^n} (i \leftrightarrow j) + \dots \end{aligned} \quad (4.8)$$

Note that the higher spin-3 currents $\Phi_1^{(2),ij}(z)$ are antisymmetric under the interchange of the index i and the index j . The last line of (4.8) implies that we should take the three lines with $i \leftrightarrow j$ with minus sign. The quasi primary fields appearing in the fourth and third order poles, which are written in terms of the known composite fields are given in Appendix B.

After subtracting the descendant fields, the second order pole contains the six $SO(4)$ adjoint higher spin-3 currents $\Phi_0^{(3),\alpha}(w)$ with adjoint α (that is, there will be 96 higher spin currents in these six higher spin multiplets $\Phi^{(3),\alpha}(Z_2)$).¹³

By introducing the following six generators M^α which are 4×4 matrices with $\alpha = 1, 2, \dots, 6$ of $SO(4)$

$$\begin{aligned} M^1 \equiv L_1 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad M^2 \equiv L_2 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ M^3 \equiv L_3 &= \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \end{aligned}$$

¹³ One way to observe the presence of the higher spin-2 current $X_0^{(2)}(w)$ described in previous subsection is as follows. We can calculate the OPE between $G^i(z)$ and the quasi primary fields $Q_1^{(3),jk}(w)$ of spin 3 and focus on the second order pole which has spin $\frac{5}{2}$. Then we can compute the OPE between $G^i(z)$ and this second order pole and look at the particular second order pole which has spin-2. We can check whether this spin-2 field can be written in terms of the known composite fields, as usual. It turns out that we observe that there should be $X_0^{(2)}(w)$ -dependent terms.

$$\begin{aligned}
M^4 \equiv K_1 &= \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, & M^5 \equiv K_2 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \\
M^6 \equiv K_3 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, & T^{24} &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, & T^{34} &= \begin{pmatrix} 0 & 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},
\end{aligned}
\quad (4.9)$$

we can write down

$$Q_1^{(3),ij}(w) = w_{0,3} (M^\alpha)^{ij} \Phi_0^{(3),\alpha}(w) + \dots, \quad (4.10)$$

as the one in Appendix B.¹⁴ There are commutation relations $[L_i, L_j] = i \varepsilon_{ijk} L_k$, $[K_i, K_j] = i \varepsilon_{ijk} L_k$, and $[L_i, K_j] = i \varepsilon_{ijk} K_k$ [40]. The relative coefficients of (4.10) can be determined by using the defining nontrivial OPEs of $L(z) \Phi_0^{(3),\alpha}(w)$, $G^i(z) \Phi_0^{(3),\alpha}(w)$ and $T^{ij}(z) \Phi_0^{(3),\alpha}(w)$ in Appendix A.

By recalling that from the relation (3.3) or Appendix A, the OPE $T^{ij}(z) \Phi_0^{(3),\alpha}(w)$ between the spin-1 currents of the $\mathcal{N} = 4$ linear superconformal algebra and the lowest higher spin-3 currents contains the nontrivial singular terms $(T^{ij})^{\alpha\beta} \Phi_0^{(3),\beta}(w)$ where the generators T^{ij} in the $SO(4)$ adjoint representation are given by

$$\begin{aligned}
T^{12} &= \begin{pmatrix} 0 & -i & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, & T^{13} &= \begin{pmatrix} 0 & 0 & -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 & 0 \end{pmatrix}, \\
T^{14} &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 \end{pmatrix}, & T^{23} &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & i & 0 \end{pmatrix},
\end{aligned}$$

where they satisfy $[T^{ij}, T^{kl}] = i(\delta^{ik} T^{jl} - \delta^{il} T^{jk} - \delta^{jk} T^{il} + \delta^{jl} T^{ik})$.

Sometimes we need to have the (quasi)primary fields in order to express the OPE with known coefficients appearing in the descendant fields. Then we should introduce each primary field at the third, fourth and fifth components of (3.4). The point here is to consider the possible terms with correct spins, $SO(4)$ vector index and $SO(4)$ adjoint index. It turns out that we obtain the following decomposition where the primary field has a tilde

$$\begin{aligned}
\Phi^{(s),\alpha}(Z) &= \Phi_0^{(s),\alpha}(z) + \theta^i \Phi_{\frac{1}{2}}^{(s),i,\alpha}(z) \\
&\quad + \frac{1}{2} \theta^{4-i} \Phi_1^{(s),ij,\alpha}(z) + \theta^{4-i} \Phi_{\frac{3}{2}}^{(s),i,\alpha}(z) + \theta^{4-0} \Phi_2^{(s),\alpha}(z) \\
&\equiv \Phi_0^{(s),\alpha}(z) + \theta^i \Phi_{\frac{1}{2}}^{(s),i,\alpha}(z) \\
&\quad + \frac{1}{2} \theta^{4-i} \left[\tilde{\Phi}_1^{(s),ij,\alpha} + \frac{i}{s} (T_R^{ij})^{\alpha\beta} \partial \Phi_0^{(s),\beta} \right](z) \\
&\quad + \theta^{4-i} \left[\tilde{\Phi}_{\frac{3}{2}}^{(s),i,\alpha} + \frac{2\alpha}{(2s+1)} \partial \Phi_{\frac{1}{2}}^{(s),i,\alpha} \right. \\
&\quad \left. - \frac{2i}{(2s+1)} (T_R^{ij})^{\alpha\beta} \partial \Phi_{\frac{1}{2}}^{(s),j,\beta} \right](z) \\
&\quad + \theta^{4-0} \left[\tilde{\Phi}_2^{(s),\alpha} - p_1 \partial^2 \Phi_0^{(s),\alpha} - p_2 L \Phi_0^{(s),\alpha} \right. \\
&\quad \left. + \frac{i}{2(2s+1)} (T_L^{ij})^{\alpha\beta} \partial \Phi_1^{(s),ij,\beta} \right. \\
&\quad \left. + (T_L^{ij} T_R^{ij})^{\alpha\beta} (p_3 \partial^2 \Phi_0^{(s),\beta} + p_4 L \Phi_0^{(s),\beta}) \right](z), \quad (4.12)
\end{aligned}$$

where the coefficients in the last component depend on N , k and s (together with (3.2)) and are given by

¹⁴ For given the higher spin current $\Phi_0^{(3),\alpha}(w)$ for fixed α , the other component $\Phi_0^{(3),\beta}(w)$ with $\beta \neq \alpha$ can be obtained, for example, from the OPE $T^{ij}(z) \Phi_0^{(3),\alpha}(w) = \frac{1}{(z-w)} (T^{ij})^{\alpha\beta} \Phi_0^{(3),\beta}(w) + \dots$ in Appendix A. From the explicit form of 6×6 nondiagonal matrix (4.11), the remaining higher spin currents can be determined completely.

$$\begin{aligned}
p_1 &= -\frac{2\alpha(3+3k+3N+3kN+26s+13ks+13Ns)}{(3+3k+3N+3kN-4s+ks+Ns+6kNs+16s^2+8ks^2+8Ns^2)}, \\
p_2 &= \frac{12(k-N)s(1+s)}{(3+3k+3N+3kN-4s+ks+Ns+6kNs+16s^2+8ks^2+8Ns^2)}, \\
p_3 &= \frac{(-15-6k-6N+3kN+8s+4ks+4Ns)}{2(1+s)(3+3k+3N+3kN-4s+ks+Ns+6kNs+16s^2+8ks^2+8Ns^2)}, \\
p_4 &= \frac{3(2+k+N)}{(3+3k+3N+3kN-4s+ks+Ns+6kNs+16s^2+8ks^2+8Ns^2)}. \quad (4.13)
\end{aligned}$$

Note that the values p_1 and p_2 also appear in the corresponding higher spin- s multiplet in the unitary Wolf space coset model [8]. The expression (4.12) holds for $SO(4)$ nonsinglet higher spin multiplets in the unitary Wolf space coset. Moreover, the following quantities with (3.2) are introduced

$$\begin{aligned}
\tilde{T}^{ij} &\equiv \frac{1}{2} \varepsilon_{ijkl} T^{kl}, \quad T_L^{ij} \equiv \frac{1}{2} T^{ij} + \alpha \tilde{T}^{ij}, \\
T_R^{ij} &\equiv \alpha T^{ij} + \frac{1}{2} \tilde{T}^{ij}. \quad (4.14)
\end{aligned}$$

For the $SO(4)$ vector representation α , we will see similar construction in next subsection.

Then we can check that the $SO(4)$ adjoint higher spin multiplet (4.12) satisfies the relation (3.3) with (4.11). Its component relations are given in Appendix A.

The first order pole of (4.8) contains other higher spin currents in various way. For example, the other components of the $\mathcal{N} = 4$ multiplets $\Phi^{(3),\alpha}(Z)$ can arise.

4.4 The new higher spin currents of spin $\frac{7}{2}$

We consider the OPE between the four higher spin- $\frac{5}{2}$ currents and the higher spin-2 current. It turns out that we obtain

$$\begin{aligned}
\Phi_{\frac{1}{2}}^{(2),i}(z) \Phi_0^{(2)}(w) &= \frac{1}{(z-w)^3} Q_{\frac{1}{2}}^{(\frac{3}{2}),i}(w) \\
&+ \frac{1}{(z-w)^2} \left[\frac{2}{3} \partial Q_{\frac{1}{2}}^{(\frac{3}{2}),i} + Q_{\frac{1}{2}}^{(\frac{5}{2}),i} \right](w) \\
&+ \frac{1}{(z-w)} \left[\frac{1}{4} \partial^2 Q_{\frac{1}{2}}^{(\frac{3}{2}),i} \right. \\
&\left. + \frac{3}{5} \partial Q_{\frac{1}{2}}^{(\frac{5}{2}),i} + Q_{\frac{1}{2}}^{(\frac{7}{2}),i} \right](w) + \dots \quad (4.15)
\end{aligned}$$

The quasi primary fields appearing in the above poles of (4.15) are given in Appendix B. In particular, the quasi primary field $Q_{\frac{1}{2}}^{(\frac{3}{2}),i}(w)$ contains $\Phi_{\frac{1}{2}}^{(2),i}(w)$ which is the second component of the lowest higher spin-2 multiplet in (2.15). In the first order pole, there exist four new primary

fields $\Phi_0^{(\frac{7}{2}),i}(w)$ ¹⁵ as well as the composite fields containing the $SO(4)$ adjoint $\Phi_0^{(3),\alpha}(w)$ (and $\Phi_{\frac{1}{2}}^{(3),i,\alpha}(w)$), the $SO(4)$ singlet $\Phi_0^{(2)}(w)$ ($\Phi_{\frac{1}{2}}^{(2),i}(w)$, $\Phi_1^{(2),ij}(w)$ and $\Phi_{\frac{3}{2}}^{(2),i}(w)$) and the other $SO(4)$ singlet $X_0^{(2)}(w)$ ($X_{\frac{1}{2}}^{(2),i}(w)$, $X_1^{(2),ij}(w)$ and $X_{\frac{3}{2}}^{(2),i}(w)$). In other words, we have

$$Q_{\frac{1}{2}}^{(\frac{7}{2}),i}(w) = w_{0,\frac{7}{2}} \Phi_0^{(\frac{7}{2}),i}(w) + \dots \quad (4.16)$$

The abbreviated part is given in Appendix B.

The $\mathcal{N} = 4$ four $SO(4)$ vector higher spin- $\frac{7}{2}$ multiplets transform under the stress energy tensor as follows [37]. The OPE looks like (3.3) with $s = \frac{7}{2}$:

$$\begin{aligned}
\mathbf{J}(Z_1) \Phi^{(s),\mu}(Z_2) &= \frac{\theta_{12}^4}{z_{12}^2} 2s \Phi^{(s),\mu}(Z_2) + \frac{\theta_{12}^{4-i}}{z_{12}} D^i \Phi^{(s),\mu}(Z_2) \\
&+ \frac{\theta_{12}^4}{z_{12}} 2 \partial \Phi^{(s),\mu}(Z_2) \\
&- \frac{i}{2} \frac{\theta_{12}^{4-ij}}{z_{12}} (T^{ij})^{\mu\nu} \Phi^{(s),\nu}(Z_2) + \dots, \quad (4.17)
\end{aligned}$$

where the T^{ij} matrix is the generator of the $SO(4)$ vector representation (4.9)

$$\begin{aligned}
T^{12} &= \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad T^{13} = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
T^{14} &= \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad T^{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
T^{24} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \quad T^{34} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \quad (4.18)
\end{aligned}$$

¹⁵ We can obtain each component by following the procedure described in the footnote 14.

where the commutators $[T^{ij}, T^{kl}]$ satisfy the previous relations described in (4.11). The corresponding component OPEs of (4.17) can be obtained from Appendix A by considering the T^{ij} matrix as the ones in (4.18). Moreover, we should use the corresponding primary fields (if we need them) according to (4.12) by substituting the 4×4 matrices in (4.18).

4.5 The higher spin current of spin 4

In the second order pole of (4.1), the quasi primary field $Q_2^{(4)}(w)$ contains the $SO(4)$ singlet higher spin-4 current $\Phi_0^{(4)}(w)$. The other part of the quasi primary field $Q_2^{(4)}(w)$ is given in Appendix B. Note that the composite field $X_0^{(2)} X_0^{(2)}(w)$ (as well as other dependent terms) is absorbed in $\Phi_0^{(4)}(w)$ such that $\Phi_0^{(4)}(w)$ should transform as the primary field under the stress energy tensor. If the composite field $X_0^{(2)} X_0^{(2)}(w)$ is not included in the $SO(4)$ singlet higher spin-4 current, then we should calculate the OPE $X_0^{(2)}(z) X_0^{(2)}(w)$ (and their $\mathcal{N} = 4$ version) in order to use its normal ordered product for the Jacobi identity. We observe that all the components of the $\mathcal{N} = 4$ multiplets $\Phi^{(2)}(Z)$ and $X^{(2)}(Z)$ appear in the quasi primary field $Q_2^{(4)}(w)$.

Moreover, once we combine the above higher spin-4 current with $\Phi_0^{(2)} \Phi_0^{(2)}(w)$ term as well as other terms as in

$$\begin{aligned} \tilde{\Phi}_0^{(4)}(w) = & w_{11,4} \Phi_0^{(2)} \Phi_0^{(2)}(w) + (M^\alpha)^{ij} (c_1 \Phi_1^{(3),ij,\alpha} \\ & + \dots + \varepsilon^{ijkl} c_8 \Gamma^k \Gamma^l \Phi_0^{(3),\alpha})(w) + \delta_\mu^i c_9 \Phi_{\frac{1}{2}}^{(7),i,\mu}(w) \\ & + (c_{10} \Phi_2^{(2)} + c_{11} \tilde{L} \Phi_0^{(2)} \\ & + \dots + c_{32} \varepsilon^{ijkl} T^{ij} T^{kl} \Phi_0^{(2)})(w) \\ & + c_{33} X_2^{(2)}(w) + c_{34} \tilde{L} X_0^{(2)}(w) \\ & + \dots + c_{55} \varepsilon^{ijkl} T^{ij} T^{kl} X_0^{(2)}(w) \\ & + c_{56} \tilde{L} \tilde{L}(w) + c_{57} \tilde{L} U U(w) \\ & + \dots + c_{155} \varepsilon^{ijkl} \Gamma^i \Gamma^j \partial \Gamma^k \partial \Gamma^l(w), \end{aligned} \quad (4.19)$$

then the structure constant $w_{11,4} \equiv C_{(4)(2)}^{(4)}$ does not appear in the remaining OPEs. The coefficients $c_{10}, \dots, c_{32}$ in (4.19) also depend on the structure constant $C_{(4)(2)}^{(2)}$ appeared in the Sect. 4.2. We have checked that this feature arises also in the unitary case [8].

The final first order pole of (4.1) can be obtained and it turns out that there is no new primary field. All the terms after subtracting the descendant fields can be written in terms of the known composite fields (including the higher spin- $\frac{9}{2}$ currents which will be described in next subsection). We expect that the $\mathcal{N} = 4$ higher spin-5 multiplets (we have not found in this

paper) will appear by considering the other OPEs between the $\mathcal{N} = 4$ multiplets we have found in this paper.¹⁶

4.6 The higher spin currents of spin $\frac{9}{2}$

We consider the OPE between the four higher spin- $\frac{7}{2}$ currents and the higher spin-2 current. It turns out that we obtain

$$\begin{aligned} \Phi_{\frac{3}{2}}^{(2),i}(z) \Phi_0^{(2)}(w) = & \frac{1}{(z-w)^5} Q_{\frac{3}{2}}^{(\frac{1}{2}),i}(w) \\ & + \frac{1}{(z-w)^4} \left[2 \partial Q_{\frac{3}{2}}^{(\frac{1}{2}),i} + Q_{\frac{3}{2}}^{(\frac{3}{2}),i} \right](w) \\ & + \frac{1}{(z-w)^3} \left[\frac{3}{2} \partial^2 Q_{\frac{3}{2}}^{(\frac{1}{2}),i} + \partial Q_{\frac{3}{2}}^{(\frac{3}{2}),i} + Q_{\frac{3}{2}}^{(\frac{5}{2}),i} \right](w) \\ & + \frac{1}{(z-w)^2} \left[\frac{2}{3} \partial^3 Q_{\frac{3}{2}}^{(\frac{1}{2}),i} + \frac{1}{2} \partial^2 Q_{\frac{3}{2}}^{(\frac{3}{2}),i} \right. \\ & \left. + \frac{4}{5} \partial Q_{\frac{3}{2}}^{(\frac{5}{2}),i} + Q_{\frac{3}{2}}^{(\frac{7}{2}),i} \right](w) \\ & + \frac{1}{(z-w)} \left[\frac{5}{24} \partial^4 Q_{\frac{3}{2}}^{(\frac{1}{2}),i} + \frac{1}{6} \partial^3 Q_{\frac{3}{2}}^{(\frac{3}{2}),i} + \frac{1}{3} \partial^2 Q_{\frac{3}{2}}^{(\frac{5}{2}),i} \right. \\ & \left. + \frac{5}{7} \partial Q_{\frac{3}{2}}^{(\frac{7}{2}),i} + Q_{\frac{3}{2}}^{(\frac{9}{2}),i} \right](w) \\ & + \frac{2\alpha}{5} \sum_{n=2}^4 \frac{1}{(z-w)^n} \left\{ \partial \Phi_{\frac{1}{2}}^{(2),i} \Phi_0^{(2)} \right\}_n(w) + \dots \end{aligned} \quad (4.20)$$

Again, the OPE (4.20) consists of two parts. The first four lines comes from the corresponding four primary higher spin- $\frac{7}{2}$ currents $\tilde{\Phi}_{\frac{3}{2}}^{(2),i}(z)$ and the last line comes from the additional term described in (4.12).¹⁷

¹⁶ Let us emphasize that in this case (together with the higher spin- $\frac{9}{2}$ current case which will be described in next subsection), due to the too many number of composite fields, we should go into the nonlinear version where there are no spin- $\frac{1}{2}$ and spin-1 currents. Without these currents, the possible composite terms are reduced significantly. We can obtain the higher spin currents in the nonlinear version, by following the work of [41], in terms of the ones we have found in the linear version (and vice versa). Then by using the OPEs between them we can rewrite the right hand sides of these OPEs in terms of the fields in the nonlinear version. After that, we can go into the linear version (by changing the composite terms in the linear basis) together with the known field contents of the composite terms.

¹⁷ By decoupling of four spin- $\frac{1}{2}$ currents and spin-1 current of the $\mathcal{N} = 4$ linear superconformal algebra, we obtain the 11 currents of the $\mathcal{N} = 4$ nonlinear superconformal algebra. The explicit expressions are given in (2.5). Then we can determine the remaining higher spin currents in the nonlinear version starting from the lowest higher spin-2 current (2.11) as in the footnote 16. After obtaining the composite fields in the nonlinear version, we can use them in the linear version. Note that we saw the higher spin-4, $\frac{9}{2}$ currents for $N = 4$ in previous section.

By using the relation (by using the notation of [39])

$$\left\{ \partial \Phi_{\frac{1}{2}}^{(2),i} \Phi_0^{(2)} \right\}_{n+1}(w) = -n \left\{ \Phi_{\frac{1}{2}}^{(2),i} \Phi_0^{(2)} \right\}_n(w), \quad n = 1, 2, 3,$$

where the previous relation (4.15) can be used, we can present the above OPE as follows:

$$\begin{aligned} \Phi_{\frac{3}{2}}^{(2),i}(z) \Phi_0^{(2)}(w) &= \frac{1}{(z-w)^5} Q_{\frac{3}{2}}^{(\frac{1}{2}),i}(w) \\ &+ \frac{1}{(z-w)^4} \left[2 \partial Q_{\frac{3}{2}}^{(\frac{1}{2}),i} + Q_{\frac{3}{2}}^{(\frac{3}{2}),i} + R_{\frac{3}{2}}^{(\frac{3}{2}),i} \right](w) \\ &+ \frac{1}{(z-w)^3} \left[\frac{3}{2} \partial^2 Q_{\frac{3}{2}}^{(\frac{1}{2}),i} + \partial Q_{\frac{3}{2}}^{(\frac{3}{2}),i} + Q_{\frac{3}{2}}^{(\frac{5}{2}),i} + R_{\frac{3}{2}}^{(\frac{5}{2}),i} \right](w) \\ &+ \frac{1}{(z-w)^2} \left[\frac{2}{3} \partial^3 Q_{\frac{3}{2}}^{(\frac{1}{2}),i} + \frac{1}{2} \partial^2 Q_{\frac{3}{2}}^{(\frac{3}{2}),i} \right. \\ &\left. + \frac{4}{5} \partial Q_{\frac{3}{2}}^{(\frac{5}{2}),i} + Q_{\frac{3}{2}}^{(\frac{7}{2}),i} + R_{\frac{3}{2}}^{(\frac{7}{2}),i} \right](w) \\ &+ \frac{1}{(z-w)} \left[\frac{5}{24} \partial^4 Q_{\frac{3}{2}}^{(\frac{1}{2}),i} + \frac{1}{6} \partial^3 Q_{\frac{3}{2}}^{(\frac{3}{2}),i} + \frac{1}{3} \partial^2 Q_{\frac{3}{2}}^{(\frac{5}{2}),i} \right. \\ &\left. + \frac{5}{7} \partial Q_{\frac{3}{2}}^{(\frac{7}{2}),i} + Q_{\frac{3}{2}}^{(\frac{9}{2}),i} \right](w) \\ &+ \cdots, \end{aligned} \quad (4.21)$$

where we introduce

$$\begin{aligned} R_{\frac{3}{2}}^{(\frac{7}{2}),i}(w) &\equiv -\frac{2\alpha}{5} \left(\frac{1}{4} \partial^2 Q_{\frac{3}{2}}^{(\frac{3}{2}),i} + \frac{3}{5} \partial Q_{\frac{3}{2}}^{(\frac{5}{2}),i} + Q_{\frac{3}{2}}^{(\frac{7}{2}),i} \right)(w), \\ R_{\frac{3}{2}}^{(\frac{5}{2}),i}(w) &\equiv -\frac{4\alpha}{5} \left(\frac{2}{3} \partial Q_{\frac{3}{2}}^{(\frac{3}{2}),i} + Q_{\frac{3}{2}}^{(\frac{5}{2}),i} \right)(w), \\ R_{\frac{3}{2}}^{(\frac{3}{2}),i}(w) &\equiv -\frac{6\alpha}{5} Q_{\frac{3}{2}}^{(\frac{3}{2}),i}(w), \end{aligned} \quad (4.22)$$

from the last line of (4.20). We can see that the fifth and fourth order terms of (4.21) consist of the composite fields from the 16 currents as in Appendix B. The third and second order poles of (4.21) have the higher spin currents (found before) dependent terms. The first order pole has the following form

$$Q_{\frac{3}{2}}^{(\frac{9}{2}),i}(w) = w_{0,\frac{9}{2}} \Phi_0^{(\frac{9}{2}),i}(w) + \cdots, \quad (4.23)$$

where the other remaining terms are given in Appendix B. Each component of the higher spin- $\frac{9}{2}$ current can be obtained by following the procedure described in the footnote 14. We see that all kinds of higher spin currents appear in this quasi primary field. We can use (4.17) together with the above 4×4 matrices in (4.18) in order to obtain the OPEs between this higher spin multiplet and the $\mathcal{N} = 4$ stress energy tensor. Its component results are given in Appendix A.

4.7 The fundamental 16 OPEs

Therefore, the fundamental 16 OPEs (five kinds of OPEs) are given by (2.13), (4.7), (4.8), (4.15) and (4.21). The structure constants are written in terms of N , k and $C_{(4)(2)}^{(2)}$ which will be given in the ancillary.nb. These will determine the remaining 120 OPEs by using the $\mathcal{N} = 4$ supersymmetry soon.

We summarize the higher spin currents appearing in various quasi primary fields (4.2) of these OPEs in Table 1. Some of the higher spin currents are not present in this Table 1 and they will arise in the remaining 120 OPEs. For simplicity, we do not include the dependence of 16 currents of the large $\mathcal{N} = 4$ linear superconformal algebra.

4.8 One single $\mathcal{N} = 4$ super OPE

From the fundamental 16 OPEs (2.13), (4.15), (4.8), (4.21) and (4.7) (that is, five different kinds of OPEs), we can generalize them in $\mathcal{N} = 4$ superspace by taking [8]

$$\begin{aligned} U(w) &\rightarrow \partial \mathbf{J}(Z_2), \\ \Gamma^i(w) &\rightarrow -i D^i \mathbf{J}(Z_2) \equiv -i \mathbf{J}^i(Z_2), \\ T^{ij}(w) &\rightarrow -\frac{i}{2!} \varepsilon^{ijkl} D^k D^l \mathbf{J}(Z_2) \equiv -\frac{i}{2!} \varepsilon^{ijkl} \mathbf{J}^{kl}(Z_2), \\ \tilde{G}^i(w) &\rightarrow \frac{1}{3!} \varepsilon^{ijkl} D^j D^k D^l \mathbf{J}(Z_2) \equiv \frac{1}{3!} \varepsilon^{ijkl} \mathbf{J}^{jkl}(Z_2), \\ \tilde{L}(w) &\rightarrow \frac{1}{2 \cdot 4!} \varepsilon^{ijkl} D^i D^j D^k D^l \mathbf{J}(Z_2) \\ &\equiv \frac{1}{2 \cdot 4!} \varepsilon^{ijkl} \mathbf{J}^{ijkl}(Z_2), \\ \Phi_0^{(s),\alpha}(w) &\rightarrow \Phi^{(s),\alpha}(Z_2), \\ \Phi_{\frac{1}{2}}^{(s),i,\alpha}(w) &\rightarrow D^i \Phi^{(s),\alpha}(Z_2), \\ \Phi_1^{(s),ij,\alpha}(w) &\rightarrow -\frac{1}{2!} \varepsilon^{ijkl} D^k D^l \Phi^{(s),\alpha}(Z_2), \\ \Phi_{\frac{3}{2}}^{(s),i,\alpha}(w) &\rightarrow -\frac{1}{3!} \varepsilon^{ijkl} D^j D^k D^l \Phi^{(s),\alpha}(Z_2), \\ \Phi_2^{(s),\alpha}(w) &\rightarrow \frac{1}{4!} \varepsilon^{ijkl} D^i D^j D^k D^l \Phi^{(s),\alpha}(Z_2), \\ &\alpha = \text{singlet, adjoint, vector}, \end{aligned} \quad (4.24)$$

where $\tilde{G}^i(w)$ and $\tilde{L}(w)$ are given in (3.1) and putting the relevant fermionic coordinates. For the singlet, adjoint and vector representations, we substitute the corresponding indices into the α . In doing this, there are additional terms arising from the summation over the same indices. We present the quasi primary fields in $\mathcal{N} = 4$ superspace in Appendix C. The total number of terms in Appendix B and Appendix C is little different from each other.

Then the single $\mathcal{N} = 4$ super OPE between the $SO(4)$ singlet higher spin-2 multiplet can be summarized by (after rearranging (D.1))

Table 1 The structure constant $w_{1,2} \equiv C_{(4)(2)}^{(2)}$ appears in front of the above quasi primaries appearing in the OPEs except $Q_{\frac{1}{2}}^{(\frac{5}{2}),i}(z)$ and $Q_2^{(3)}(z)$, which do not have the components of $\mathcal{N} = 4$ multiplet $\mathbf{X}^{(2)}(Z)$. The quasi primaries, which are not in this list, have the composite fields

consisting of 16 currents. Although the $X_0^{(2)}(z)$ dependence in $Q_2^{(3)}(z)$ does not appear, its dependence appears in $\partial Q_2^{(2)}(z)$ in (4.7). Similarly, the $\Phi_0^{(4)}(z)$ dependence in $Q_2^{(5)}(z)$ does not appear and its dependence appears via $\partial Q_2^{(4)}(z)$ in (4.7)

Quasi primaries	Higher spin currents
$Q_{\frac{1}{2}}^{(\frac{5}{2}),i}(z)$	$\Phi_{\frac{1}{2}}^{(2),j}$
$Q_{\frac{1}{2}}^{(\frac{7}{2}),i}(z)$	$\Phi_0^{(\frac{7}{2}),j}; \Phi_{\frac{1}{2}}^{(3),j,\alpha}, \Phi_0^{(3),\alpha}; \Phi_{\frac{3}{2}}^{(2),j}, \Phi_1^{(2),jk}, \Phi_{\frac{1}{2}}^{(2),j}, \Phi_0^{(2)}; X_{\frac{3}{2}}^{(2),j}, X_1^{(2),jk}, X_{\frac{1}{2}}^{(2),j}, X_0^{(2)}$
$Q_1^{(3),ij}(z)$	$\Phi_0^{(3),\alpha}; \Phi_1^{(2),kl}, \Phi_{\frac{1}{2}}^{(2),k}, \Phi_0^{(2)}; X_1^{(2),kl}, X_{\frac{1}{2}}^{(2),k}, X_0^{(2)}$
$Q_1^{(4),ij}(z)$	$\Phi_1^{(3),kl,\alpha}, \Phi_{\frac{1}{2}}^{(3),k,\alpha}, \Phi_0^{(3),\alpha}; \Phi_{\frac{3}{2}}^{(2),k}, \Phi_1^{(2),kl}, \Phi_{\frac{1}{2}}^{(2),k}, \Phi_0^{(2)}; X_{\frac{3}{2}}^{(2),k}, X_1^{(2),kl}, X_{\frac{1}{2}}^{(2),k}, X_0^{(2)}$
$Q_{\frac{3}{2}}^{(\frac{5}{2}),i}(z)$	$\Phi_{\frac{1}{2}}^{(2),j}, \Phi_0^{(2)}; X_{\frac{1}{2}}^{(2),j}$
$Q_{\frac{3}{2}}^{(\frac{7}{2}),i}(z)$	$\Phi_0^{(\frac{7}{2}),j}; \Phi_{\frac{1}{2}}^{(3),j,\alpha}, \Phi_0^{(3),\alpha}; \Phi_{\frac{3}{2}}^{(2),j}, \Phi_1^{(2),jk}, \Phi_{\frac{1}{2}}^{(2),j}, \Phi_0^{(2)}; X_{\frac{3}{2}}^{(2),j}, X_1^{(2),jk}, X_{\frac{1}{2}}^{(2),j}, X_0^{(2)}$
$Q_{\frac{3}{2}}^{(\frac{9}{2}),i}(z)$	$\Phi_0^{(\frac{9}{2}),j}; \Phi_{\frac{1}{2}}^{(4),j}; \Phi_1^{(\frac{7}{2}),jk,l}, \Phi_{\frac{1}{2}}^{(\frac{7}{2}),j,k}, \Phi_0^{(\frac{7}{2}),j}; \Phi_{\frac{3}{2}}^{(3),j,\alpha}, \Phi_1^{(3),jk,\alpha}, \Phi_{\frac{1}{2}}^{(3),j,\alpha}, \Phi_0^{(3),\alpha};$ $\Phi_{\frac{3}{2}}^{(2),j}, \Phi_1^{(2),jk}, \Phi_{\frac{1}{2}}^{(2),j}, \Phi_0^{(2)}; X_{\frac{3}{2}}^{(2),j}, X_1^{(2),jk}, X_{\frac{1}{2}}^{(2),j}, X_0^{(2)}$
$Q_2^{(2)}(z)$	$\Phi_0^{(2)}; X_0^{(2)}$
$Q_2^{(3)}(z)$	$\Phi_{\frac{1}{2}}^{(2),i}; \Phi_0^{(2)}$
$Q_2^{(4)}(z)$	$\Phi_0^{(4)}; \Phi_{\frac{1}{2}}^{(\frac{7}{2}),i,\mu}, \Phi_0^{(\frac{7}{2}),i}; \Phi_1^{(3),jk,\alpha}, \Phi_{\frac{1}{2}}^{(3),j,\alpha}, \Phi_0^{(3),\alpha}; \Phi_2^{(2)}, \Phi_{\frac{3}{2}}^{(2),j}, \Phi_1^{(2),jk}, \Phi_{\frac{1}{2}}^{(2),j}, \Phi_0^{(2)};$ $X_2^{(2)}, X_{\frac{3}{2}}^{(2),j}, X_1^{(2),jk}, X_{\frac{1}{2}}^{(2),j}, X_0^{(2)}$
$Q_2^{(5)}(z)$	$\Phi_{\frac{1}{2}}^{(\frac{9}{2}),i,j}; \Phi_{\frac{3}{2}}^{(\frac{7}{2}),i,j}, \Phi_1^{(\frac{7}{2}),ij,k}, \Phi_{\frac{1}{2}}^{(\frac{7}{2}),i,j}, \Phi_0^{(\frac{7}{2}),i}; \Phi_1^{(3),ij,\alpha}, \Phi_{\frac{1}{2}}^{(3),i,\alpha}, \Phi_0^{(3),\alpha};$ $\Phi_2^{(2)}, \Phi_{\frac{3}{2}}^{(2),i}, \Phi_1^{(2),ij}, \Phi_{\frac{1}{2}}^{(2),i}, \Phi_0^{(2)}; X_2^{(2)}, X_{\frac{3}{2}}^{(2),i}, X_1^{(2),ij}, X_{\frac{1}{2}}^{(2),i}, X_0^{(2)}$

$$\begin{aligned}
\Phi^{(2)}(Z_1) \Phi^{(2)}(Z_2) = & \frac{1}{z_{12}^4} c_0^{0,4} + \frac{\theta_{12}^{4-0}}{z_{12}^6} 8\alpha c_0^{0,4} + \frac{\theta_{12}^{4-i}}{z_{12}^5} Q_{\frac{3}{2}}^{(\frac{1}{2}),i}(Z_2) \\
& + \frac{\theta_{12}^{4-0}}{z_{12}^5} Q_2^{(1)}(Z_2) + \frac{\theta_{12}^{4-ij}}{z_{12}^4} Q_1^{(1),ij}(Z_2) \\
& + \frac{\theta_{12}^{4-i}}{z_{12}^4} \left[2\partial Q_{\frac{3}{2}}^{(\frac{1}{2}),i} + Q_{\frac{3}{2}}^{(\frac{3}{2}),i} + R_{\frac{3}{2}}^{(\frac{3}{2}),i} \right](Z_2) \\
& + \frac{\theta_{12}^{4-0}}{z_{12}^4} \left[\frac{3}{2} \partial Q_2^{(1)} + Q_2^{(2)} + R_2^{(2)} \right](Z_2) \\
& + \frac{\theta_{12}^i}{z_{12}^3} Q_{\frac{1}{2}}^{(\frac{3}{2}),i}(Z_2) + \frac{\theta_{12}^{4-ij}}{z_{12}^3} \left[\partial Q_1^{(1),ij} + Q_1^{(2),ij} \right](Z_2) \\
& + \frac{\theta_{12}^{4-i}}{z_{12}^3} \left[\frac{3}{2} \partial^2 Q_{\frac{3}{2}}^{(\frac{1}{2}),i} + \partial Q_{\frac{3}{2}}^{(\frac{3}{2}),i} + Q_{\frac{3}{2}}^{(\frac{5}{2}),i} + R_{\frac{3}{2}}^{(\frac{5}{2}),i} \right](Z_2) \\
& + \frac{\theta_{12}^{4-0}}{z_{12}^3} \left[\partial^2 Q_2^{(1)} + \partial Q_2^{(2)} + Q_2^{(3)} + R_2^{(3)} \right](Z_2) + \frac{1}{z_{12}^2} Q_0^{(2)}(Z_2) \\
& + \frac{\theta_{12}^i}{z_{12}^2} \left[\frac{2}{3} \partial Q_{\frac{1}{2}}^{(\frac{3}{2}),i} + Q_{\frac{1}{2}}^{(\frac{5}{2}),i} \right](Z_2) \\
& + \frac{\theta_{12}^{4-ij}}{z_{12}^2} \left[\frac{1}{2} \partial^2 Q_1^{(1),ij} + \frac{3}{4} \partial Q_1^{(2),ij} + Q_1^{(3),ij} \right](Z_2) \\
& + \frac{\theta_{12}^{4-i}}{z_{12}^2} \left[\frac{5}{24} \partial^4 Q_{\frac{3}{2}}^{(\frac{1}{2}),i} + \frac{1}{6} \partial^3 Q_{\frac{3}{2}}^{(\frac{3}{2}),i} + \frac{1}{3} \partial^2 Q_{\frac{3}{2}}^{(\frac{5}{2}),i} \right. \\
& \quad \left. + \frac{5}{7} \partial Q_{\frac{3}{2}}^{(\frac{7}{2}),i} + Q_{\frac{3}{2}}^{(\frac{9}{2}),i} \right](Z_2) \\
& + \frac{\theta_{12}^{4-0}}{z_{12}^2} \left[\frac{1}{8} \partial^4 Q_2^{(1)} + \frac{1}{6} \partial^3 Q_2^{(2)} + \frac{5}{14} \partial^2 Q_2^{(3)} \right. \\
& \quad \left. + \frac{3}{4} \partial Q_2^{(4)} + Q_2^{(5)} \right](Z_2) + \dots
\end{aligned}
\tag{4.25}$$

The central term is presented in (2.14). The maximum number of super spin is given by 5 and the corresponding (composite) higher spin currents appear in the last line of (4.25). Due to the space of the paper, we cannot write down all the operators in the right hand side of (4.25). The partial expressions of quasi primary super fields corresponding to the component fields in (4.2) are given in Appendix C (together with ancillary.nb).

Here we introduce the following quantities ($\mathcal{N} = 4$ expressions of (4.22), (4.6) and the one in the footnote 12)¹⁸ with (3.2)

$$\begin{aligned} \mathbf{R}_{\frac{3}{2}}^{(\frac{7}{2}),i}(Z_2) &\equiv -\frac{2\alpha}{5} \left(\frac{1}{4} \partial^2 \mathbf{Q}_{\frac{1}{2}}^{(\frac{3}{2}),i} + \frac{3}{5} \partial \mathbf{Q}_{\frac{1}{2}}^{(\frac{5}{2}),i} + \mathbf{Q}_{\frac{1}{2}}^{(\frac{7}{2}),i} \right) (Z_2), \\ \mathbf{R}_{\frac{3}{2}}^{(\frac{5}{2}),i}(Z_2) &\equiv -\frac{4\alpha}{5} \left(\frac{2}{3} \partial \mathbf{Q}_{\frac{1}{2}}^{(\frac{3}{2}),i} + \mathbf{Q}_{\frac{1}{2}}^{(\frac{5}{2}),i} \right) (Z_2), \\ \mathbf{R}_{\frac{3}{2}}^{(\frac{3}{2}),i}(Z_2) &\equiv -\frac{6\alpha}{5} \mathbf{Q}_{\frac{1}{2}}^{(\frac{3}{2}),i}(Z_2), \\ \mathbf{R}_2^{(4-n)}(Z_2) &\equiv -p_2 \mathbf{E}_2^{(4-n)}(Z_2) \\ &\quad - p_1 n(n+1) \mathbf{Q}_0^{(4-n)}(Z_2), \quad n = 0, 1, 2. \end{aligned}$$

Note that $\mathbf{R}_{\frac{3}{2}}^{(\frac{7}{2}),i}(Z_2)$, $\mathbf{R}_{\frac{3}{2}}^{(\frac{5}{2}),i}(Z_2)$, $\mathbf{R}_2^{(3)}(Z_2)$ and $\mathbf{R}_2^{(4)}(Z_2)$ are not quasi primary. From (4.3), (4.10), (4.16), (4.19) and (4.23), the new primary higher spin $\mathcal{N} = 4$ multiplets in (4.25) arise in the following quasi primary ones

$$\begin{aligned} \mathbf{Q}_2^{(2)}(Z_2) &= c_1^{2,4} \Phi^{(2)}(Z_2) + c_2^{2,4} \mathbf{X}^{(2)}(Z_2) + \dots, \\ \mathbf{Q}_1^{(3),ij}(Z_2) &= c_0^{1,2} (M^\alpha)^{ij} \Phi^{(3),\alpha}(Z_2) + \dots, \\ \mathbf{Q}_{\frac{1}{2}}^{(\frac{7}{2}),i}(Z_2) &= c_0^{\frac{1}{2},1} \Phi^{(\frac{7}{2}),i}(Z_2) + \dots, \\ \mathbf{Q}_2^{(4)}(Z_2) &= c_0^{2,2} \Phi^{(4)}(Z_2) + \dots, \\ \mathbf{Q}_{\frac{3}{2}}^{(\frac{9}{2}),i}(Z_2) &= c_0^{\frac{3}{2},1} \Phi^{(\frac{9}{2}),i}(Z_2) + \dots, \end{aligned} \quad (4.26)$$

where some of the abbreviated parts are given in Appendix C. Its component relations can be found in Appendix B. We can easily figure out the $SO(4)$ indices and it is rather nontrivial to observe the second quasi primary field of (4.26) where $SO(4)$ adjoint index is contracted with the one in the matrix M (4.9).

¹⁸ The p_1 and p_2 are given in (4.13) and we also use, together with (2.14),

$$\begin{aligned} \mathbf{E}_2^{(2)}(Z_2) &\equiv (4\mathbf{Q}_0^{(2)} + \frac{c_0^{0,4}}{2} \mathbf{J}^{4-0})(Z_2), \\ \mathbf{E}_2^{(3)}(Z_2) &\equiv \left(\frac{5}{2} \partial \mathbf{Q}_0^{(2)} + \frac{c_0^{0,4}}{2} \partial \mathbf{J}^{4-0} \right) (Z_2), \\ \mathbf{E}_2^{(4)}(Z_2) &\equiv \left(\frac{1}{2} \partial^2 \mathbf{Q}_0^{(2)} + \frac{1}{4} \mathbf{J}^{4-0} \mathbf{Q}_0^{(2)} + \frac{1}{4} \partial^2 \mathbf{J}^{4-0} + 2 \Phi^{(2)} \Phi^{(2)} \right) (Z_2), \end{aligned}$$

which are the supersymmetric extension of (4.6). Note that there are some different numerical factors according to (4.24).

Schematically, we can present the above OPE as follows:

$$\begin{aligned} [\Phi^{(2)} \cdot \Phi^{(2)}] &= [\mathbf{I}] + \theta^{4-0} ([\Phi^{(2)}] \\ &\quad + [\mathbf{X}^{(2)}]) + \theta^{4-ij} (M^\alpha)^{ij} [\Phi^{(3),\alpha}] \\ &\quad + \theta^i [\Phi^{(\frac{7}{2}),i}] + \theta^{4-0} [\tilde{\Phi}^{(4)}] \\ &\quad + \theta^{4-i} [\Phi^{(\frac{9}{2}),i}], \end{aligned} \quad (4.27)$$

where $[\mathbf{I}]$ stands for the large $\mathcal{N} = 4$ linear superconformal family of identity operator and the various composite fields consisting of $\mathcal{N} = 4$ multiplet $\mathbf{J}$ up to the super spin 5 can appear. We also insert the $SO(4)$ vector indices i, j and the $SO(4)$ matrix M^α in (4.9). Note that in the component approach described in previous subsections, the $SO(4)$ indices are present in the two higher spin currents of the left hand side of the given OPE. In the $\mathcal{N} = 4$ superspace description (4.27), all the $SO(4)$ indices are contracted with the ones in the fermionic coordinates. In order to obtain the component results, we can act various super derivatives both sides of the $\mathcal{N} = 4$ OPE to restore the $SO(4)$ indices. Here the $\mathcal{N} = 4$ multiplet $\tilde{\Phi}^{(4)}(Z)$ has its lowest component given in (4.19) and contains the quadratic $\mathcal{N} = 4$ multiplet $\Phi^{(2)}(Z)$. Compared with the $\mathcal{N} = 3$ example [14], there are more higher spin multiplets contracted with fermionic coordinates.

4.9 The 136 OPEs between the 16 lowest higher spin currents for generic N

We can calculate the remaining $136 - 16 = 120$ OPEs from (4.25) by taking the super derivatives D_1^i or D_2^j both sides of (4.25) and putting $\theta_1^k = 0 = \theta_2^l$. By introducing the various coefficients in front of composite fields appearing in 136 OPEs (the number of coefficients is 2000 or so and coefficients are denoted by $w_{1,s}, \dots, w_{2043,s}$ in Appendix B) and using the Jacobi identities, we obtain the 136 OPEs between the 16 lowest higher spin currents (2.15) for generic N . Note that this number of coefficients is huge compared to the unitary case in [8]. Furthermore, after substituting these coefficients into (4.25) back, we obtain the final single OPE with fixed coefficients which depend on N and k (as well as $C_{(4)(2)}^{(2)}$).¹⁹ Because there are new 16 $\mathcal{N} = 4$ multiplets ($\mathbf{X}^{(2)}(Z)$, $\Phi^{(3),\alpha}(Z)$, $\Phi^{(\frac{7}{2}),\mu}(Z)$, $\Phi^{(4)}(Z)$ and $\Phi^{(\frac{9}{2}),\mu}(Z)$) in the right hand side of the 136 OPEs, we have $16 \times 16 = 256$

¹⁹ The field contents appearing in the right hand side of (4.25) are taken from the results for $N = 5$ case. We also have checked that for $N = 8, 9$, the six $SO(4)$ adjoint higher spin-3 multiplets appear in the corresponding OPEs similarly and there are no new higher spin multiplets having super spins 2 and $\frac{5}{2}$. If there exist the extra $\mathcal{N} = 4$ higher spin multiplets in the right hand side of (4.25) for large N , we expect that they will appear linearly without spoiling its algebraic structure. We may try to calculate the OPEs from the closed forms written in terms of the orthogonal Wolf space fields but this will be rather involved.

higher spin currents (in components) totally. Under the 16 currents of the large $\mathcal{N} = 4$ superconformal algebra, they transform nontrivially as in Appendix A. It is an open problem to determine the OPEs between these 256 higher spin currents (or the OPEs between the lowest 16 higher spin currents and those 256 higher spin currents) systematically.

5 Conclusions and outlook

We have described one single $\mathcal{N} = 4$ super OPE (4.25) between the lowest higher spin-2 multiplet in the $\mathcal{N} = 4$ superspace. As in the abstract, there exist several $\mathcal{N} = 4$ higher spin multiplets in the right hand side of this OPE.

There are open problems we can consider in the future as follows:

- Higher spin algebra in the bulk theory

In [42], the free field construction at $\lambda = 0$ by using the bosons and fermions is presented. Maybe at this particular $\lambda = 0$ case, the full higher spin algebra can be described. In other words, the commutators and anticommutators for the higher spin currents (including the 16 currents) can be determined with complete structure constants. The final goal is to obtain the higher spin algebra at finite λ which will provide the corresponding algebra in the dual conformal field theory at the classical level. Contrary to the unitary case, the orthogonal case needs to obtain the appropriate truncation on the matrix elements observed in [42].

- Three-point functions

One way to check the dual relation between the orthogonal Wolf space coset model and the higher spin theory on AdS_3 space is to compute the three-point functions of the two scalars and the higher spin currents. According to the results of this paper, there are many higher spin currents from the single OPE (4.25). It is an open problem to obtain the remaining 15 higher spin currents in terms of the orthogonal Wolf space coset fields explicitly and to calculate the eigenvalues of the zero modes, by following the procedure studied in [43–47]. Although this will be rather involved, once we obtain them, then it is straightforward to compute the three-point functions at finite N and k .

- $\mathcal{N} = 2$ superspace description

In principle, we can rewrite the above $\mathcal{N} = 4$ superspace OPE in terms of various 10 OPEs in $\mathcal{N} = 2$ superspace. One merit for this description is that contrary to the $\mathcal{N} = 4$ superspace OPE we have described so far, the $\mathcal{N} = 2$ superspace description enables us to write down in terms of quasi (super) primary fields completely. In doing this, it is rather nontrivial to obtain the correct component fields for the $SO(4)$ nonsinglet $\mathcal{N} = 4$ multiplets. The relevant work in this direction appeared in [14].

- The large k limit

We can examine the behavior of large k limit (for example, see the work of [48]) from the results we have obtained in this paper. We take the large k limit in the structure constants appearing in the right hand sides of the OPEs we have found. We can read off the leading behavior of k of the right hand sides. Even the $N = 5$ results are enough to analyze this large k limit. We expect to observe the realization of vanishing of 't Hooft-like coupling constant $\lambda = \frac{(N+1)}{(N+k+2)} \rightarrow 0$ for fixed N [2]. It is also interesting to observe whether there is an extension of the small $\mathcal{N} = 4$ superconformal algebra along the line of [2] or not.

- The nonlinear version

It is an open problem to obtain the above single OPE (4.25) in the context of nonlinear version which is an extension of the large $\mathcal{N} = 4$ nonlinear algebra. Due to the fact that we do not know its $\mathcal{N} = 4$ superspace version, we need to present the whole 136 OPEs. Although the lowest 16 higher spin currents are primary under the corresponding stress energy tensor, they do transform nontrivially with respect to other 10 currents. In principle, because we do have the complete OPE in the linear version, it is straightforward to obtain them in the nonlinear version although the careful analysis should be done. In this paper, we applied some computations in this nonlinear basis, although we did not present them explicitly (some OPEs are rather complicated).

Acknowledgements CA acknowledges warm hospitality from the School of Liberal Arts (and Institute of Convergence Fundamental Studies), Seoul National University of Science and Technology. This research was supported by Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education (no. 2017R1D1A1A09079512).

Data Availability Statement This manuscript has associated data in a data repository. [Authors' comment: All analyzed data have been published in arXiv.org.]

Open Access This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (<http://creativecommons.org/licenses/by/4.0/>), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made. Funded by SCOAP³.

A The OPEs between the 16 currents and the (non)singlet higher spin currents in the component approach

We present the component results [37] for the $\mathcal{N} = 4$ primary condition for the $SO(4)$ nonsinglet field in (3.3)

$$L(z) \Phi_2^{(s),\alpha}(w) = -\frac{1}{(z-w)^4} 12 s \alpha \Phi_0^{(s),\alpha}(w)$$

$$\begin{aligned}
& + \frac{1}{(z-w)^3} \left[4\alpha \partial \Phi_0^{(s),\alpha} + i (T_L^{ij})^{\alpha\beta} \Phi_1^{(s),ij,\beta} \right] (w) \\
& + \frac{1}{(z-w)^2} (s+2) \Phi_2^{(s),\alpha}(w) + \frac{1}{(z-w)} \partial \Phi_2^{(s),\alpha}(w) + \dots, \\
L(z) \Phi_{\frac{3}{2}}^{(s),i,\alpha}(w) &= \frac{1}{(z-w)^3} 2 \left[\alpha \Phi_{\frac{1}{2}}^{(s),i,\alpha} - i (T_R^{ij})^{\alpha\beta} \Phi_{\frac{1}{2}}^{(s),j,\beta} \right] (w) \\
& + \frac{1}{(z-w)^2} (s + \frac{3}{2}) \Phi_{\frac{3}{2}}^{(s),i,\alpha}(w) \\
& + \frac{1}{(z-w)} \partial \Phi_{\frac{3}{2}}^{(s),i,\alpha}(w) + \dots, \\
L(z) \Phi_1^{(s),ij,\alpha}(w) &= \frac{1}{(z-w)^3} 2i (T_R^{ij})^{\alpha\beta} \Phi_0^{(s),\beta}(w) \\
& + \frac{1}{(z-w)^2} (s+1) \Phi_1^{(s),ij,\alpha}(w) \\
& + \frac{1}{(z-w)} \partial \Phi_1^{(s),ij,\alpha}(w) + \dots, \\
L(z) \Phi_{\frac{1}{2}}^{(s),i,\alpha}(w) &= \frac{1}{(z-w)^2} (s + \frac{1}{2}) \Phi_{\frac{1}{2}}^{(s),i,\alpha}(w) \\
& + \frac{1}{(z-w)} \partial \Phi_{\frac{1}{2}}^{(s),i,\alpha}(w) + \dots, \\
L(z) \Phi_0^{(s),\alpha}(w) &= \frac{1}{(z-w)^2} s \Phi_0^{(s),\alpha}(w) + \frac{1}{(z-w)} \partial \Phi_0^{(s),\alpha}(w) + \dots, \\
G^i(z) \Phi_2^{(s),\alpha}(w) &= -\frac{1}{(z-w)^3} 4 \left[(1+2s) \alpha \Phi_{\frac{1}{2}}^{(s),i,\alpha} - i (T_R^{ij})^{\alpha\beta} \Phi_{\frac{1}{2}}^{(s),j,\beta} \right] (w) \\
& - \frac{1}{(z-w)^2} \left[(3+2s) \Phi_{\frac{3}{2}}^{(s),i,\alpha} - 2\alpha \partial \Phi_{\frac{1}{2}}^{(s),i,\alpha} \right. \\
& \left. - 2i (T_L^{ij})^{\alpha\beta} \Phi_{\frac{3}{2}}^{(s),j,\beta} \right] (w) \\
& - \frac{1}{(z-w)} \partial \Phi_{\frac{3}{2}}^{(s),i,\alpha}(w) + \dots, \\
G^i(z) \Phi_{\frac{3}{2}}^{(s),j,\alpha}(w) &= \frac{1}{(z-w)^3} \left[8s \alpha \delta_{ij} \Phi_0^{(s),\alpha} - 4i (T_R^{ij})^{\alpha\beta} \Phi_0^{(s),\beta} \right] (w) \\
& - \frac{1}{(z-w)^2} \left[2(1+s) \Phi_1^{(s),ij,\alpha} + \varepsilon_{ijkl} (\alpha \Phi_1^{(s),kl,\alpha} \right. \\
& \left. - 2i (T_L^{jk})^{\alpha\beta} \Phi_1^{(s),jl,\beta} \right. \\
& \left. + 2\delta_{ij} (\alpha \partial \Phi_0^{(s),\alpha} - i (T_R^{ik})^{\alpha\beta} \Phi_1^{(s),ik,\beta}) \right] (w) \\
& - \frac{1}{(z-w)} \left[\partial \Phi_1^{(s),ij,\alpha} + \delta_{ij} \Phi_2^{(s),\alpha} \right] (w) \\
& + \dots, \\
G^i(z) \Phi_1^{(s),jk,\alpha}(w) &
\end{aligned}$$

$$\begin{aligned}
& = -\frac{1}{(z-w)^2} \left[-2\delta_{ij} (\alpha \Phi_{\frac{1}{2}}^{(s),k,\alpha} + \varepsilon^{ikpq} (T_L^{ip})^{\alpha\beta} \Phi_{\frac{1}{2}}^{(s),q,\beta}) \right. \\
& \left. + 2\delta_{ik} (j \leftrightarrow k) + \varepsilon^{ijkl} ((1+2s) \Phi_{\frac{1}{2}}^{(s),l,\alpha} \right. \\
& \left. + 2i (T_L^{il})^{\alpha\beta} \Phi_{\frac{1}{2}}^{(s),i,\beta}) \right] (w) \\
& - \frac{1}{(z-w)} \left[(\delta_{ij} \Phi_{\frac{3}{2}}^{(s),k,\alpha} - \delta_{ik} \Phi_{\frac{3}{2}}^{(s),j,\alpha}) \right. \\
& \left. + \partial \Phi_{\frac{1}{2}}^{(s),4-ijk,\alpha} \right] (w) + \dots, \\
G^i(z) \Phi_{\frac{1}{2}}^{(s),j,\alpha}(w) &= -\frac{1}{(z-w)^2} \left[2s \delta_{ij} \Phi_0^{(s),\alpha} - 2i (T_L^{ij})^{\alpha\beta} \Phi_0^{(s),\beta} \right] (w) \\
& - \frac{1}{(z-w)} \left[\delta_{ij} \partial \Phi_0^{(s),\alpha} - \Phi_1^{(s),4-ij,\alpha} \right] (w) + \dots, \\
G^i(z) \Phi_0^{(s),\alpha}(w) &= -\frac{1}{(z-w)} \Phi_{\frac{1}{2}}^{(s),i,\alpha}(w) + \dots, \\
T^{ij}(z) \Phi_2^{(s),\alpha}(w) &= -\frac{1}{(z-w)^3} 2 (\tilde{T}^{ij})^{\alpha\beta} \Phi_0^{(s),\beta}(w) \\
& + \frac{1}{(z-w)^2} \left[2i (s+1) \Phi_1^{(s),ij,\alpha} - (T^{ik})^{\alpha\beta} \Phi_1^{(s),jk,\beta} \right. \\
& \left. + (T^{jk})^{\alpha\beta} \Phi_1^{(s),ik,\beta} \right] (w) \\
& + \frac{1}{(z-w)} (T^{ij})^{\alpha\beta} \Phi_2^{(s),\beta}(w) + \dots, \\
T^{ij}(z) \Phi_{\frac{3}{2}}^{(s),k,\alpha}(w) &= \frac{1}{(z-w)^2} \left[\varepsilon_{ijkl} (-i (2s+1) \Phi_{\frac{1}{2}}^{(s),l,\alpha} \right. \\
& \left. - (T^{li})^{\alpha\beta} \Phi_{\frac{1}{2}}^{(s),i,\beta} - (T^{lj})^{\alpha\beta} \Phi_{\frac{1}{2}}^{(s),j,\beta}) \right. \\
& \left. + \varepsilon_{ijpq} (\delta_{ik} (T^{ip})^{\alpha\beta} \Phi_{\frac{1}{2}}^{(s),q,\beta} + \delta_{jk} (T^{jp})^{\alpha\beta} \Phi_{\frac{1}{2}}^{(s),q,\beta}) \right] (w) \\
& - \frac{1}{(z-w)} \left[\delta_{ik} (i \Phi_{\frac{3}{2}}^{(s),j,\alpha} - (T^{ij})^{\alpha\beta} \Phi_{\frac{3}{2}}^{(s),i,\beta}) \right. \\
& \left. - \delta_{jk} (i \Phi_{\frac{3}{2}}^{(s),i,\alpha} - (T^{ij})^{\alpha\beta} \Phi_{\frac{3}{2}}^{(s),j,\beta}) \right. \\
& \left. - \varepsilon_{ijkl} (\tilde{T}^{kl})^{\alpha\beta} \Phi_{\frac{3}{2}}^{(s),k,\beta} \right] (w) + \dots, \\
T^{ij}(z) \Phi_1^{(s),kl,\alpha}(w) &= \frac{1}{(z-w)^2} \left[2i s \varepsilon_{ijkl} \Phi_0^{(s),\alpha} \right. \\
& \left. + i \delta_{ik} (\tilde{T}^{jl})^{\alpha\beta} \Phi_0^{(s),\beta} - i \delta_{il} (\tilde{T}^{jk})^{\alpha\beta} \Phi_0^{(s),\beta} \right. \\
& \left. - i \delta_{jk} (\tilde{T}^{il})^{\alpha\beta} \Phi_0^{(s),\beta} + i \delta_{jl} (\tilde{T}^{ik})^{\alpha\beta} \Phi_0^{(s),\beta} \right] (w) \\
& - \frac{1}{(z-w)} \left[i \delta_{ik} \Phi_1^{(s),jl,\alpha} \right.
\end{aligned}$$

$$\begin{aligned}
& -i\delta_{il}\Phi_1^{(s),jk,\alpha} - i\delta_{jk}\Phi_1^{(s),il,\alpha} + i\delta_{jl}\Phi_1^{(s),ik,\alpha} \\
& - (T^{ij})^{\alpha\beta}\Phi_1^{(s),kl,\beta} \Big] (w) + \dots, \\
T^{ij}(z)\Phi_{\frac{1}{2}}^{(s),k,\alpha}(w) &= \frac{1}{(z-w)} \left[-i\delta^{ik}\Phi_{\frac{1}{2}}^{(s),j,\alpha} + i\delta^{jk}\Phi_{\frac{1}{2}}^{(s),i,\alpha} \right. \\
& \left. + (T^{ij})^{\alpha\beta}\Phi_{\frac{1}{2}}^{(s),k,\beta} \right] (w) + \dots, \\
T^{ij}(z)\Phi_0^{(s),\alpha}(w) &= \frac{1}{(z-w)} (T^{ij})^{\alpha\beta}\Phi_0^{(s),\beta}(w) + \dots, \\
U(z)\Phi_2^{(s),\alpha}(w) &= -\frac{1}{(z-w)^3} 4s\Phi_0^{(s),\alpha}(w) + \frac{1}{(z-w)^2} \left[2\partial\Phi_0^{(s),\alpha} \right. \\
& \left. + \frac{i}{2}(\tilde{T}^{ij})^{\alpha\beta}\Phi_1^{(s),ij,\beta} \right] (w) \\
& + \dots, \\
U(z)\Phi_{\frac{3}{2}}^{(s),i,\alpha}(w) &= \frac{1}{(z-w)^2} \left[\Phi_{\frac{1}{2}}^{(s),i,\alpha} + i(T^{ij})^{\alpha\beta}\Phi_{\frac{1}{2}}^{(s),j,\beta} \right] (w) + \dots, \\
U(z)\Phi_1^{(s),ij,\alpha}(w) &= \frac{1}{(z-w)^2} i(T^{ij})^{\alpha\beta}\Phi_0^{(s),\beta}(w) + \dots, \\
\Gamma^i(z)\Phi_2^{(s),\alpha}(w) &= -\frac{1}{(z-w)^2} \left[i(2s+1)\Phi_{\frac{1}{2}}^{(s),i,\alpha} + (T^{ij})^{\alpha\beta}\Phi_{\frac{1}{2}}^{(s),j,\beta} \right] (w) \\
& + \frac{1}{(z-w)} \left[i\partial\Phi_{\frac{1}{2}}^{(s),i,\alpha} - (\tilde{T}^{ij})^{\alpha\beta}\Phi_{\frac{3}{2}}^{(s),j,\beta} \right] (w) + \dots, \\
\Gamma^i(z)\Phi_{\frac{3}{2}}^{(s),j,\alpha}(w) &= \frac{1}{(z-w)^2} \left[2is\delta_{ij}\Phi_0^{(s),\alpha} + (T^{ij})^{\alpha\beta}\Phi_0^{(s),\beta} \right] (w) \\
& - \frac{1}{(z-w)} \left[\delta_{ij}(i\partial\Phi_0^{(s),\alpha} - (\tilde{T}^{ik})^{\alpha\beta}\Phi_1^{(s),ik,\beta}) \right. \\
& \left. + \varepsilon_{ijkl} \left(\frac{i}{2}\Phi_1^{(s),kl,\alpha} - (T^{jk})^{\alpha\beta}\Phi_1^{(s),jl,\beta} \right) \right] (w) + \dots, \\
\Gamma^i(z)\Phi_1^{(s),jk,\alpha}(w) &= -\frac{1}{(z-w)} \left[\delta_{ij}(i\Phi_{\frac{1}{2}}^{(s),k,\alpha} \right. \\
& \left. - \varepsilon_{ikl}(\tilde{T}^{il})^{\alpha\beta}\Phi_{\frac{1}{2}}^{(s),q,\beta}) - \delta_{ik}(j \leftrightarrow k) \right. \\
& \left. + \varepsilon_{ijkl}(\tilde{T}^{il})^{\alpha\beta}\Phi_{\frac{1}{2}}^{(s),i,\beta} \right] (w) + \dots, \\
\Gamma^i(z)\Phi_{\frac{1}{2}}^{(s),j,\alpha}(w) &= -\frac{1}{(z-w)} (\tilde{T}^{ij})^{\alpha\beta}\Phi_0^{(s),\beta}(w) \\
& + \dots, \quad \alpha = \text{singlet, adjoint, vector.}
\end{aligned}$$

We use the following simplified notations (3.2) and (4.14)

$$\begin{aligned}
\tilde{T}^{ij} &\equiv \frac{1}{2!} \varepsilon_{ijkl} T^{kl}, \quad T_L^{ij} \equiv \frac{1}{2} (T^{ij}) + \alpha (\tilde{T}^{ij}), \\
T_R^{ij} &\equiv \alpha (T^{ij}) + \frac{1}{2} (\tilde{T}^{ij}), \\
\alpha &\equiv \frac{1}{2} \frac{(k^+ - k^-)}{(k^+ + k^-)}, \quad k^+ \equiv k + 1, \quad k^- \equiv N + 1.
\end{aligned} \tag{A.1}$$

Note that there are trivial OPEs $U(z)\Phi_{\frac{1}{2}}^{(s),\alpha,i}(w) = + \dots = U(z)\Phi_0^{(s),\alpha}(w) = \Gamma^i(z)\Phi_0^{(s),\alpha}(w)$. For fixed indices i, j, k, l appearing in the left hand side, we do not sum over those in the right hand side but we sum over other indices. For example, in the OPE $G^i(z)\Phi_{\frac{3}{2}}^{(s),j,\alpha}(w)$, the third term of the second order pole has indices i, j, k, l . Among them, the only indices k, l are summed.

B Partial expressions of quasi primary fields in the component approach

We present the various quasi primary fields appearing in section 4 (the complete expressions can be found in ancillary.nb file)

$$\begin{aligned}
Q_0^{(2)} &= w_{1,2}\Phi_0^{(2)} + w_{2,2}\tilde{L} + w_{3,2}\partial U + w_{4,2}UU \\
&+ w_{5,2}T^{ij}T^{ij} + w_{6,2}T^{ij}\tilde{T}^{ij} + w_{7,2}\Gamma^i\Gamma^jT^{ij} \\
&+ w_{8,2}\Gamma^i\Gamma^j\tilde{T}^{ij} + w_{9,2}\Gamma^i\partial\Gamma^i + \varepsilon^{ijkl}w_{10,2}\Gamma^i\Gamma^j\Gamma^k\Gamma^l, \\
Q_{\frac{1}{2}}^{(\frac{3}{2})^i} &= w_{1,\frac{3}{2}}\tilde{G}^i + w_{2,\frac{3}{2}}\partial\Gamma^i \\
&+ w_{3,\frac{3}{2}}\Gamma^iU + \varepsilon^{ijkl}(w_{4,\frac{3}{2}}\Gamma^jT^{kl} + w_{5,\frac{3}{2}}\Gamma^j\Gamma^k\Gamma^l), \\
Q_{\frac{1}{2}}^{(\frac{5}{2})^i} &= w_{1,\frac{5}{2}}\Phi_{\frac{1}{2}}^{(2),i} + w_{2,\frac{5}{2}}\partial\tilde{G}^i + w_{3,\frac{5}{2}}\partial^2\Gamma^i \\
&+ w_{4,\frac{5}{2}}\Gamma^i\partial U + w_{5,\frac{5}{2}}\partial\Gamma^iU + w_{6,\frac{5}{2}}T^{ij}\tilde{G}^i \\
&+ w_{7,\frac{5}{2}}\Gamma^i\Gamma^j\tilde{G}^j + w_{8,\frac{5}{2}}\partial\Gamma^jT^{ij} + w_{9,\frac{5}{2}}\Gamma^j\partial T^{ij} \\
&+ w_{10,\frac{5}{2}}\Gamma^jT^{ij}U + w_{11,\frac{5}{2}}\Gamma^i\Gamma^j\partial\Gamma^j \\
&+ w_{12,\frac{5}{2}}\Gamma^iT^{jk}T^{jk} + w_{13,\frac{5}{2}}\Gamma^i\Gamma^j\Gamma^kT^{jk} \\
&+ w_{14,\frac{5}{2}}\Gamma^jT^{ik}T^{jk} \\
&+ \varepsilon^{ijkl}(w_{15,\frac{5}{2}}T^{jk}\tilde{G}^l + w_{16,\frac{5}{2}}\Gamma^j\Gamma^k\tilde{G}^l \\
&+ w_{17,\frac{5}{2}}\partial\Gamma^jT^{kl} + w_{18,\frac{5}{2}}\Gamma^jT^{kl}U + w_{19,\frac{5}{2}}\Gamma^iT^{ij}T^{kl} \\
&+ w_{20,\frac{5}{2}}\partial(\Gamma^j\Gamma^k\Gamma^l) + w_{21,\frac{5}{2}}\Gamma^j\Gamma^k\Gamma^lU), \\
Q_{\frac{1}{2}}^{(\frac{7}{2})^i} &= (M^\alpha)^{ij}(w_{1,\frac{7}{2}}\Phi_{\frac{1}{2}}^{(3),j,\alpha} + w_{2,\frac{7}{2}}\Gamma^j\Phi_0^{(3),\alpha}) \\
&+ \varepsilon^{ijkl}(M^\alpha)^{jk}(w_{3,\frac{7}{2}}\Phi_{\frac{1}{2}}^{(3),l,\alpha} + w_{4,\frac{7}{2}}\Gamma^l\Phi_0^{(3),\alpha}) \\
&+ w_{0,\frac{7}{2}}\delta_\mu^i\Phi_0^{(\frac{7}{2}),\mu} + w_{5,\frac{7}{2}}\Phi_{\frac{3}{2}}^{(2),i} + w_{6,\frac{7}{2}}\partial\Phi_{\frac{1}{2}}^{(2),i} \\
&+ w_{7,\frac{7}{2}}U\Phi_{\frac{1}{2}}^{(2),i} + w_{8,\frac{7}{2}}\tilde{G}^i\Phi_0^{(2)} + \dots
\end{aligned}$$

$$\begin{aligned}
& +w_{19,\frac{7}{2}} X_{\frac{3}{2}}^{(2),i} + w_{20,\frac{7}{2}} \partial X_{\frac{3}{2}}^{(2),i} \\
& +w_{21,\frac{7}{2}} U X_{\frac{3}{2}}^{(2),i} + w_{22,\frac{7}{2}} \tilde{G}^i X_0^{(2)} + \dots \\
& +w_{32,\frac{7}{2}} \varepsilon^{ijkl} \Gamma^j \Gamma^k \Gamma^l X_0^{(2)} + w_{33,\frac{7}{2}} \tilde{G}^i \tilde{L} \\
& + \dots + w_{122,\frac{7}{2}} \varepsilon^{ijkl} \Gamma^j T^{ij} T^{ik} T^{lj} \\
& +w_{123,\frac{7}{2}} (\varepsilon^{ijkl})^2 \Gamma^j \Gamma^k \Gamma^l T^{ij} T^{kl}, \\
Q_1^{(1),ij} &= w_{1,1} T^{ij} \\
& +w_{2,1} \Gamma^i \Gamma^j + \varepsilon^{ijkl} (w_{3,1} T^{kl} + w_{4,1} \Gamma^k \Gamma^l) - (i \leftrightarrow j), \\
Q_1^{(2),ij} &= w_{1,2} \Gamma^i \tilde{G}^j + w_{2,2} \Gamma^i \Gamma^j U \\
& +w_{3,2} \partial(\Gamma^i \Gamma^j) + w_{4,2} \Gamma^i \Gamma^k \tilde{T}^{jk} - (i \leftrightarrow j), \\
Q_1^{(3),ij} &= w_{0,3} (M^\alpha)^{ij} \Phi_0^{(3),\alpha} + w_{1,3} \Phi_1^{(2),ij} + w_{5,3} \tilde{\Phi}_1^{(2),ij} \\
& +w_{2,3} \Gamma^i \Phi_{\frac{1}{2}}^{(2),j} + w_{3,3} T^{ij} \Phi_0^{(2)} \\
& +w_{4,3} \Gamma^i \Gamma^j \Phi_0^{(2)} + \dots + w_{9,3} X_1^{(2),ij} + w_{13,3} \tilde{X}_1^{(2),ij} \\
& +w_{10,3} \Gamma^i X_{\frac{1}{2}}^{(2),j} + w_{11,3} T^{ij} X_0^{(2)} + \dots \\
& +w_{16,3} \tilde{T}^{ij} X_0^{(2)} + w_{17,3} \tilde{G}^i \tilde{G}^j \\
& +w_{18,3} \Gamma^i \Gamma^j \tilde{L} + \dots + w_{91,3} (\varepsilon^{ijkl})^2 \Gamma^k \Gamma^l T^{ik} T^{jl} - (i \leftrightarrow j), \\
Q_1^{(4),ij} &= (M^\alpha)^{ik} (w_{1,4} \Phi_1^{(3),jk,\alpha} + w_{2,4} T^{jk} \Phi_0^{(3),\alpha} \\
& +w_{3,4} \tilde{T}^{jk} \Phi_0^{(3),\alpha} + w_{4,4} \Gamma^j \Phi_{\frac{1}{2}}^{(3),k,\alpha} + \dots) \\
& +\varepsilon^{ijkl} ((M^\alpha)^{kq} w_{5,4} \Phi_1^{(3),lq,\alpha} + (M^\alpha)^{ik} w_{6,4} \Gamma^i \Phi_{\frac{1}{2}}^{(3),l,\alpha} \\
& + (M^\alpha)^{kl} w_{7,4} \Gamma^k \Phi_{\frac{1}{2}}^{(3),k,\alpha} + \dots) \\
& +\delta_\mu^j w_{13,4} \Phi_{\frac{1}{2}}^{(\frac{7}{2}),i,\mu} + w_{14,4} \partial \Phi_1^{(2),ij} \\
& +w_{15,4} \partial(\Gamma^i \Gamma^j \Phi_0^{(2)}) + \dots + w_{53,4} \partial X_1^{(2),ij} \\
& +w_{54,4} \partial(\Gamma^i \Gamma^j X_0^{(2)}) + \dots + w_{87,4} \partial \tilde{X}_1^{(2),ij} \\
& +w_{88,4} \tilde{L} U \Gamma^i \Gamma^j + \dots \\
& +w_{303,4} \varepsilon^{ijkl} (\partial^2 T^{ik} T^{il} + \partial^2 T^{jk} T^{jl}) \\
& +w_{304,4} \varepsilon^{ijkl} \varepsilon^{mnpq} (T^{mn} T^{pq} U \Gamma^k \Gamma^l) - (i \leftrightarrow j), \\
Q_{\frac{3}{2}}^{(\frac{1}{2}),i} &= w_{1,\frac{3}{2}} \Gamma^i, \\
Q_{\frac{3}{2}}^{(\frac{3}{2}),i} &= w_{1,\frac{3}{2}} \tilde{G}^i + w_{2,\frac{3}{2}} \partial \Gamma^i + w_{3,\frac{3}{2}} \Gamma^i U + w_{4,\frac{3}{2}} \Gamma^j T^{ij} \\
& +\varepsilon^{ijkl} (w_{5,\frac{3}{2}} \Gamma^j T^{kl} + w_{6,\frac{3}{2}} \Gamma^j \Gamma^k \Gamma^l), \\
Q_{\frac{3}{2}}^{(\frac{5}{2}),i} &= w_{1,\frac{5}{2}} \Phi_{\frac{1}{2}}^{(2),i} + w_{2,\frac{5}{2}} \Gamma^i \Phi_0^{(2)} + w_{3,\frac{5}{2}} X_{\frac{1}{2}}^{(2),i} \\
& +w_{4,\frac{5}{2}} \tilde{G}^i + \dots + w_{27,\frac{5}{2}} \varepsilon^{ijkl} T^{ji} \Gamma^k \Gamma^l \Gamma^i, \\
Q_{\frac{3}{2}}^{(\frac{7}{2}),i} &= (M^\alpha)^{ij} (w_{1,\frac{7}{2}} \Phi_{\frac{1}{2}}^{(3),j,\alpha} + w_{2,\frac{7}{2}} \Gamma^j \Phi_0^{(3),\alpha}) \\
& +\varepsilon^{ijkl} (M^\alpha)^{jk} (w_{3,\frac{7}{2}} \Phi_{\frac{1}{2}}^{(3),l,\alpha} + w_{4,\frac{7}{2}} \Gamma^l \Phi_0^{(3),\alpha}) \\
& +w_{5,\frac{7}{2}} \delta_\mu^i \Phi_0^{(\frac{7}{2}),\mu} + w_{6,\frac{7}{2}} \Phi_{\frac{3}{2}}^{(2),i} + w_{7,\frac{7}{2}} \partial \Phi_{\frac{1}{2}}^{(2),i} \\
& +w_{8,\frac{7}{2}} U \Phi_{\frac{1}{2}}^{(2),i} + w_{9,\frac{7}{2}} \tilde{G}^i \Phi_0^{(2)} + \dots \\
& +w_{22,\frac{7}{2}} X_{\frac{3}{2}}^{(2),i} + w_{23,\frac{7}{2}} \partial X_{\frac{1}{2}}^{(2),i} + w_{24,\frac{7}{2}} U X_{\frac{1}{2}}^{(2),i} \\
& +w_{25,\frac{7}{2}} \tilde{G}^i X_0^{(2)} + \dots
\end{aligned}$$

$$\begin{aligned}
& +w_{37,\frac{7}{2}} \varepsilon^{ijkl} \Gamma^j \Gamma^k \Gamma^l X_0^{(2)} + w_{38,\frac{7}{2}} \partial^2 \tilde{G}^i \\
& +w_{39,\frac{7}{2}} \partial^3 \Gamma^i + w_{40,\frac{7}{2}} U U \tilde{G}^i + \dots \\
& +(\varepsilon^{ijkl})^2 (w_{143,\frac{7}{2}} T^{ij} T^{kl} \Gamma^j \Gamma^k \Gamma^l \\
& +w_{144,\frac{7}{2}} T^{ij} \tilde{T}^{kl} \Gamma^j \Gamma^k \Gamma^l), \\
Q_{\frac{3}{2}}^{(\frac{9}{2}),i} &= w_{0,\frac{9}{2}} \delta_\mu^i \Phi_0^{(\frac{9}{2}),\mu} + w_{1,\frac{9}{2}} \Phi_{\frac{1}{2}}^{(4),i} + (M^\alpha)^{ij} (w_{2,\frac{9}{2}} \Phi_{\frac{3}{2}}^{(3),\alpha} \\
& +w_{3,\frac{9}{2}} \partial \Phi_{\frac{1}{2}}^{(3),\alpha} + w_{4,\frac{9}{2}} \Gamma^i \Phi_1^{(3),ij,\alpha} \\
& + \dots + w_{22,\frac{9}{2}} \varepsilon^{ijkl} \Gamma^i \Gamma^k \Gamma^l \Phi_0^{(3),\alpha}) + \dots \\
& +\delta_\mu^l (\varepsilon^{ijkl} w_{49,\frac{9}{2}} \Phi_1^{(\frac{7}{2}),jk,\mu} + w_{50,\frac{9}{2}} \varepsilon^{ijkl} \tilde{\Phi}_1^{(\frac{7}{2}),jk,\mu} + \dots \\
& +w_{59,\frac{9}{2}} \partial \Phi_0^{(\frac{7}{2}),\mu}) + w_{60,\frac{9}{2}} \Phi_0^{(2)} \Phi_{\frac{1}{2}}^{(2),i} \\
& +w_{61,\frac{9}{2}} \tilde{L} \Phi_{\frac{1}{2}}^{(2),i} + w_{62,\frac{9}{2}} U \Phi_{\frac{3}{2}}^{(2),i} + \dots + w_{149,\frac{9}{2}} \tilde{L} X_{\frac{1}{2}}^{(2),i} \\
& +w_{150,\frac{9}{2}} U X_{\frac{3}{2}}^{(2),i} + \dots + w_{236,\frac{9}{2}} \varepsilon^{ijkl} \partial T^{jk} \Gamma^l X_0^{(2)} \\
& +w_{237,\frac{9}{2}} \tilde{L} U U \Gamma^i + w_{238,\frac{9}{2}} \tilde{L} U \partial \Gamma^i \\
& +w_{239,\frac{9}{2}} \tilde{L} \tilde{G}^i U + \dots + \varepsilon^{ijkl} (w_{379,\frac{9}{2}} \tilde{L} U \Gamma^j \Gamma^k \Gamma^l \\
& +w_{380,\frac{9}{2}} \tilde{L} \tilde{G}^j \Gamma^k \Gamma^l + \dots \\
& +w_{668,\frac{9}{2}} \partial^2 \Gamma^i \Gamma^j \Gamma^k \Gamma^l + w_{669,\frac{9}{2}} \partial^3 \Gamma^j \Gamma^k \Gamma^l \\
& +w_{670,\frac{9}{2}} \partial^2 \Gamma^j \partial \Gamma^k \partial \Gamma^l + w_{671,\frac{9}{2}} \partial \Gamma^j \partial \Gamma^k \partial \Gamma^l), \\
Q_2^{(1)} &= w_{1,1} U, \\
Q_2^{(2)} &= w_{1,2} \Phi_0^{(2)} + w_{2,2} X_0^{(2)} + w_{3,2} \tilde{L} + w_{4,2} \partial U \\
& +w_5 U U + w_{6,2} T^{ij} T^{ij} + w_{7,2} T^{ij} \tilde{T}^{ij} \\
& +w_{8,2} \Gamma^i \Gamma^j T^{ij} + w_{9,2} \Gamma^i \Gamma^j \tilde{T}^{ij} \\
& +w_{10,2} \Gamma^i \tilde{G}^i + w_{11,2} \Gamma^i \partial \Gamma^i \\
& +\varepsilon^{ijkl} w_{12,2} \Gamma^i \Gamma^j \Gamma^k \Gamma^l, \\
Q_2^{(3)} &= w_{1,3} U \Phi_0^{(2)} + w_{2,3} \Gamma^i \Phi_{\frac{1}{2}}^{(2),i} + w_{3,3} U \tilde{L} \\
& +w_{4,3} U \partial U + w_{5,3} U U U + w_{6,3} \Gamma^i \partial \tilde{G}^i \\
& +w_{7,3} \partial \Gamma^i \tilde{G}^i + w_{8,3} \Gamma^i \partial \Gamma^i U + w_{9,3} \Gamma^i \partial^2 \Gamma^i \\
& +w_{10,3} \partial \Gamma^i \partial \Gamma^i + w_{11,3} \Gamma^i T^{ij} \tilde{G}^j \\
& +w_{12,3} \Gamma^i \Gamma^j T^{ij} U \\
& +w_{13,3} T^{ij} T^{ij} U + w_{14,3} \Gamma^i \Gamma^j \partial T^{ij} \\
& +w_{15,3} \partial \Gamma^i \Gamma^j T^{ij} + w_{16,3} \partial^2 T^{ij} + \varepsilon^{ijkl} (w_{17,3} \Gamma^i T^{jk} \tilde{G}^l \\
& +w_{18,3} \Gamma^i \Gamma^j \Gamma^k \tilde{G}^l + w_{19,3} \Gamma^i \Gamma^j T^{kl} U + w_{20,3} T^{ij} T^{kl} U \\
& +w_{21,3} \Gamma^i \Gamma^j \partial T^{kl} + w_{22,3} \partial \Gamma^i \Gamma^j T^{kl} \\
& +w_{23,3} \partial(\Gamma^i \Gamma^j \Gamma^k \Gamma^l)), \\
Q_2^{(4)} &= w_{0,4} \Phi_0^{(4)} + (M^\alpha)^{ij} (w_{1,4} \Phi_1^{(3),ij,\alpha} + w_{2,4} \tilde{\Phi}_1^{(3),ij,\alpha} + \dots \\
& +\varepsilon^{ijkl} w_{8,4} \Gamma^k \Gamma^l \Phi_0^{(3),\alpha}) \\
& +\delta_\mu^i (w_{9,4} \Phi_{\frac{1}{2}}^{(\frac{7}{2}),i,\mu} + w_{10,4} \Gamma^i \Phi_0^{(\frac{7}{2}),\mu}) \\
& +w_{11,4} \Phi_0^{(2)} \Phi_0^{(2)} + w_{12,4} \Phi_2^{(2)} + w_{13,4} \tilde{L} \Phi_0^{(2)} + \dots \\
& +w_{36,4} \varepsilon^{ijkl} T^{ij} T^{kl} \Phi_0^{(2)} + w_{37,4} X_2^{(2)} \\
& +w_{38,4} \tilde{L} X_0^{(2)} + \dots + w_{61,4} \varepsilon^{ijkl} T^{ij} T^{kl} X_0^{(2)} \\
& +w_{62,4} \tilde{L} \tilde{L} + w_{63,4} \tilde{L} U U + \dots \\
& +w_{168,4}^2 \varepsilon^{ijkl} \Gamma^i \Gamma^j \partial \Gamma^k \partial \Gamma^l,
\end{aligned}$$

$$\begin{aligned}
Q_2^{(5)} = & w_{1,5} \delta_\mu^i \Phi_{\frac{1}{2}}^{(9),i,\mu} + (M^\alpha)^{ij} (w_{2,5} \partial \Phi_1^{(3),ij,\alpha} \\
& + w_{3,5} \partial \tilde{\Phi}_1^{(3),ij,\alpha} + \dots + w_{38,5} \varepsilon^{ijkl} \partial T^{kl} \Phi_0^{(3),\alpha} \\
& + w_{39,5} \varepsilon^{ijkl} T^{jk} \Gamma^l \Phi_{\frac{1}{2}}^{(3)j,\alpha}) \\
& + \delta_\mu^i (w_{40,5} \Phi_{\frac{3}{2}}^{(7),i,\mu} + w_{41,5} \partial \Phi_{\frac{1}{2}}^{(7),i,\mu} + \dots \\
& + w_{52,5} \varepsilon^{ijk\mu} T^{ij} \Gamma^k \Phi_0^{(7),\mu} \\
& + w_{53,5} \varepsilon^{ijk\mu} \Gamma^i \Gamma^j \Phi_{\frac{1}{2}}^{(7),k,\mu}) + w_{54,5} \partial \Phi_2^{(2)} \\
& + w_{55,5} \tilde{L} \partial \Phi_0^{(2)} + \dots + w_{129,5} \partial X_2^{(2)} \\
& + w_{130,5} \tilde{L} \partial X_0^{(2)} + \dots + w_{203,5} \varepsilon^{ijkl} \partial \Gamma^i \Gamma^j \Gamma^k \Gamma^l X_0^{(2)} \\
& + w_{204,5} \partial^3 \tilde{L} + w_{205,5} \partial \tilde{L} \partial U + \dots \\
& + w_{348,5} \partial \tilde{G}^i \tilde{T}^{ij} T^{jk} \Gamma^k + \varepsilon^{ijkl} (w_{349,5} \tilde{L} \partial \Gamma^i \Gamma^j \Gamma^k \Gamma^l + \dots \\
& + w_{426,5} \tilde{G}^i T^{jk} \partial \Gamma^i \Gamma^j \Gamma^l \\
& + w_{427,5} \tilde{G}^i T^{ij} \partial \Gamma^i \Gamma^k \Gamma^l + w_{428,5} \varepsilon^{ijkl} \partial \tilde{G}^i T^{ij} \Gamma^k \Gamma^l),
\end{aligned} \quad (B.1)$$

where the fields with tilde are defined in (3.1), (4.12) and (A.1). Let us explain the notations in (B.1). Because the total number of coefficients is greater than 2000, we present some of the full composite fields. In the beginning of these expressions, the higher spin currents appear and then the composite fields made of 16 currents appear. For example, in the expression of $Q_{\frac{1}{2}}^{(\frac{7}{2},i)}(w)$, there are 123 terms where the higher spin dependent terms arise until the thirty second term and from the thirty third term to the last term, the higher spin independent terms arise.

C Partial expressions of quasi primary fields in the $\mathcal{N} = 4$ superspace

We also present the various quasi primary super fields appeared in section 4 (the complete expressions can be found in ancillary.nb file)

$$\begin{aligned}
Q_0^{(2)} = & c_1^{0,2} \Phi^{(2)} + c_2^{0,2} \mathbf{J}^{4-0} + c_3^{0,2} \partial^2 \mathbf{J} \\
& + c_4^{0,2} \partial \mathbf{J} \partial \mathbf{J} + c_5^{0,2} \mathbf{J}^i \mathbf{J}^j + c_6^{0,2} \mathbf{J}^i \mathbf{J}^{4-ij} \\
& + c_7^{0,2} \mathbf{J}^i \mathbf{J}^j \mathbf{J}^{4-ij} + c_8^{0,2} \mathbf{J}^i \mathbf{J}^j \mathbf{J}^i + c_9^{0,2} \mathbf{J}^i \partial \mathbf{J}^i \\
& + c_{10}^{0,2} \varepsilon^{ijkl} \mathbf{J}^i \mathbf{J}^j \mathbf{J}^k \mathbf{J}^l, \\
Q_{\frac{1}{2}}^{(\frac{3}{2},i)} = & c_1^{\frac{1}{2},3} \mathbf{J}^{4-i} + c_2^{\frac{1}{2},3} \partial \mathbf{J}^i + c_3^{\frac{1}{2},3} \mathbf{J}^i \partial \mathbf{J} \\
& + \varepsilon^{ijkl} (c_4^{\frac{1}{2},3} \mathbf{J}^j \mathbf{J}^{4-kl} + c_5^{\frac{1}{2},3} \mathbf{J}^j \mathbf{J}^k \mathbf{J}^l), \\
Q_{\frac{1}{2}}^{(\frac{5}{2},i)} = & c_1^{\frac{1}{2},2} D^i \Phi^{(2)} + c_2^{\frac{1}{2},2} \partial \mathbf{J}^{4-i} + c_3^{\frac{1}{2},2} \partial^2 \mathbf{J}^i + c_4^{\frac{1}{2},2} \mathbf{J}^i \partial^2 \mathbf{J} \\
& + c_5^{\frac{1}{2},2} \partial \mathbf{J}^i \partial \mathbf{J} + c_6^{\frac{1}{2},2} \mathbf{J}^{4-ij} \mathbf{J}^{4-j} \\
& + c_7^{\frac{1}{2},2} \mathbf{J}^i \mathbf{J}^j \mathbf{J}^{4-j} + c_8^{\frac{1}{2},2} \partial \mathbf{J}^j \mathbf{J}^{4-ij} + c_9^{\frac{1}{2},2} \mathbf{J}^j \partial \mathbf{J}^{4-ij} \\
& + c_{10}^{\frac{1}{2},2} \mathbf{J}^j \mathbf{J}^{4-ij} \partial \mathbf{J} + c_{11}^{\frac{1}{2},2} \mathbf{J}^i \mathbf{J}^j \partial \mathbf{J}^j
\end{aligned}$$

$$\begin{aligned}
& + c_{12}^{\frac{1}{2},2} \mathbf{J}^i \mathbf{J}^{jk} \mathbf{J}^{jk} + c_{13}^{\frac{1}{2},2} \mathbf{J}^i \mathbf{J}^j \mathbf{J}^k \mathbf{J}^{4-jk} \\
& + c_{14}^{\frac{1}{2},2} \mathbf{J}^j \mathbf{J}^{4-ik} \mathbf{J}^{jk} + \varepsilon^{ijkl} (c_{15}^{\frac{1}{2},2} \mathbf{J}^{4-jk} \mathbf{J}^{4-l} + c_{16}^{\frac{1}{2},2} \mathbf{J}^j \mathbf{J}^k \mathbf{J}^{4-l} \\
& + c_{17}^{\frac{1}{2},2} \partial \mathbf{J}^j \mathbf{J}^{4-kl} + c_{18}^{\frac{1}{2},2} \mathbf{J}^j \mathbf{J}^{4-kl} \partial \mathbf{J} + c_{19}^{\frac{1}{2},2} \mathbf{J}^i \mathbf{J}^j \mathbf{J}^{kl} \\
& + c_{20}^{\frac{1}{2},2} \partial (\mathbf{J}^i \mathbf{J}^k \mathbf{J}^l) + c_{21}^{\frac{1}{2},2} \mathbf{J}^i \mathbf{J}^k \mathbf{J}^l \partial \mathbf{J}), \\
Q_{\frac{1}{2}}^{(\frac{7}{2},i)} = & (M^\alpha)^{ij} (c_1^{\frac{1}{2},1} D^j \Phi^{(3),\alpha} + c_2^{\frac{1}{2},1} \mathbf{J}^j \Phi^{(3),\alpha}) \\
& + \varepsilon^{ijkl} (M^\alpha)^{jk} (c_3^{\frac{1}{2},1} D^l \Phi^{(3),\alpha} \\
& + c_4^{\frac{1}{2},1} \mathbf{J}^l \Phi^{(3),\alpha}) + c_0^{\frac{1}{2},1} \delta_\mu^i \Phi_{\frac{1}{2}}^{(\frac{7}{2},\mu)} + c_5^{\frac{1}{2},1} D^{4-i} \Phi^{(2)} \\
& + c_6^{\frac{1}{2},1} \partial D^i \Phi^{(2)} + c_7^{\frac{1}{2},1} \partial \mathbf{J} D^i \Phi^{(2)} \\
& + c_8^{\frac{1}{2},1} \mathbf{J}^{4-i} \Phi^{(2)} + \dots + c_{19}^{\frac{1}{2},1} D^{4-i} \mathbf{X}^{(2)} + c_{20}^{\frac{1}{2},1} \partial D^i \mathbf{X}^{(2)} \\
& + c_{21}^{\frac{1}{2},1} \partial \mathbf{J} D^i \mathbf{X}^{(2)} + c_{22}^{\frac{1}{2},1} \mathbf{J}^{4-i} \mathbf{X}^{(2)} \\
& + \dots + c_{32}^{\frac{1}{2},1} \varepsilon^{ijkl} \mathbf{J}^j \mathbf{J}^k \mathbf{J}^l \mathbf{X}^{(2)} + c_{33}^{\frac{1}{2},1} \mathbf{J}^{4-i} \mathbf{J}^{4-0} + \dots \\
& + c_{122}^{\frac{1}{2},1} \varepsilon^{ijkl} \mathbf{J}^j \mathbf{J}^{4-ij} \mathbf{J}^{4-ik} \mathbf{J}^{4-lj} \\
& + c_{123}^{\frac{1}{2},1} (\varepsilon^{ijkl})^2 \mathbf{J}^j \mathbf{J}^k \mathbf{J}^l \mathbf{J}^{4-ij} \mathbf{J}^{4-kl}, \\
Q_1^{(1),ij} = & c_1^{1,4} \mathbf{J}^{4-ij} + c_2^{1,4} \mathbf{J}^i \mathbf{J}^j + c_3^{1,4} \mathbf{J}^i \mathbf{J}^j \\
& + \varepsilon^{ijkl} c_4^{1,4} \mathbf{J}^k \mathbf{J}^l - (i \leftrightarrow j), \\
Q_1^{(2),ij} = & c_1^{1,3} \mathbf{J}^i \mathbf{J}^{4-j} + c_2^{1,3} \mathbf{J}^i \mathbf{J}^j \partial \mathbf{J} + c_3^{1,3} \partial (\mathbf{J}^i \mathbf{J}^j) \\
& + \varepsilon^{ijkl} c_4^{1,3} (\mathbf{J}^i \mathbf{J}^j \mathbf{J}^{4-l} + \mathbf{J}^j \mathbf{J}^k \mathbf{J}^{4-jl}) - (i \leftrightarrow j), \\
Q_1^{(3),ij} = & c_0^{1,2} (M^\alpha)^{ij} \Phi^{(3),\alpha} + c_1^{1,2} D^{4-ij} \Phi^{(2)} \\
& + c_5^{1,2} D^{ij} \Phi^{(2)} + c_2^{1,2} \mathbf{J}^i \mathbf{J}^j \Phi^{(2)} \\
& + c_3^{1,2} \mathbf{J}^{4-ij} \Phi^{(2)} + \dots + c_9^{1,2} D^{4-ij} \mathbf{X}^{(2)} \\
& + c_{13}^{1,2} D^{ij} \mathbf{X}^{(2)} + c_{10}^{1,2} \mathbf{J}^i D^j \mathbf{X}^{(2)} \\
& + c_{11}^{1,2} \mathbf{J}^{4-ij} \mathbf{X}^{(2)} + \dots + c_{16}^{1,2} \mathbf{J}^i \mathbf{J}^j \mathbf{X}^{(2)} \\
& + c_{17}^{1,2} \mathbf{J}^{4-i} \mathbf{J}^{4-j} + c_{18}^{1,2} \mathbf{J}^i \mathbf{J}^j \mathbf{J}^{4-0} + \dots \\
& + c_{91}^{1,2} (\varepsilon^{ijkl})^2 \mathbf{J}^k \mathbf{J}^l \mathbf{J}^{4-ik} \mathbf{J}^{4-jl} - (i \leftrightarrow j), \\
Q_1^{(4),ij} = & (M^\alpha)^{ik} (c_1^{1,1} D^{4-jk} \Phi^{(3),\alpha} + c_2^{1,1} \mathbf{J}^{4-jk} \Phi^{(3),\alpha} \\
& + c_3^{1,1} \mathbf{J}^{jk} \Phi^{(3),\alpha} + c_4^{1,1} \mathbf{J}^j D^k \Phi^{(3),\alpha} + \dots) \\
& + \varepsilon^{ijkl} ((M^\alpha)^{kq} c_5^{1,1} D^{4-lq} \Phi^{(3),\alpha} \\
& + (M^\alpha)^{ik} c_6^{1,1} \mathbf{J}^i D^l \Phi^{(3),\alpha} \\
& + (M^\alpha)^{kl} c_7^{1,1} \mathbf{J}^k D^l \Phi^{(3),\alpha} + \dots) \\
& + c_{12}^{1,1} \delta_\mu^j D^i \Phi_{\frac{1}{2}}^{(\frac{7}{2},\mu)} + c_{13}^{1,1} \partial D^{4-ij} \Phi^{(2)} \\
& + c_{14}^{1,1} \partial (\mathbf{J}^i \mathbf{J}^j \Phi^{(2)}) + \dots + c_{52}^{1,1} \partial D^{4-ij} \mathbf{X}^{(2)} \\
& + c_{53}^{1,1} \partial (\mathbf{J}^i \mathbf{J}^j \mathbf{X}^{(2)}) + \dots + c_{86}^{1,1} \partial D^{ij} \mathbf{X}^{(2)} \\
& + c_{87}^{1,1} \mathbf{J}^{4-0} \partial \mathbf{J}^i \mathbf{J}^j + \dots \\
& + c_{302}^{1,1} \varepsilon^{ijkl} (\partial^2 \mathbf{J}^{ik} \mathbf{J}^{jl} + \partial^2 \mathbf{J}^{jk} \mathbf{J}^{il}) \\
& + c_{303}^{1,1} \varepsilon^{ijkl} \varepsilon^{mnpq} \mathbf{J}^{4-mn} \mathbf{J}^{4-pq} \partial \mathbf{J}^k \mathbf{J}^l - (i \leftrightarrow j), \\
Q_{\frac{3}{2}}^{(\frac{1}{2},i)} = & c_1^{\frac{3}{2},5} \mathbf{J}^i, \\
Q_{\frac{3}{2}}^{(\frac{3}{2},i)} = & c_1^{\frac{3}{2},4} \mathbf{J}^{4-i} + c_2^{\frac{3}{2},4} \partial \mathbf{J}^i + c_3^{\frac{3}{2},4} \mathbf{J}^i \partial \mathbf{J} \\
& + c_4^{\frac{3}{2},4} \mathbf{J}^j \mathbf{J}^{4-ij} + \varepsilon^{ijkl} (c_5^{\frac{3}{2},4} \mathbf{J}^j \mathbf{J}^{4-kl} + c_6^{\frac{3}{2},4} \mathbf{J}^j \mathbf{J}^k \mathbf{J}^l),
\end{aligned}$$

$$\begin{aligned}
Q_{\frac{3}{2}}^{(\frac{5}{2})} &= c_1^{\frac{3}{2},3} D^i \Phi^{(2)} + c_2^{\frac{3}{2},3} J^i \Phi^{(2)} + c_3^{\frac{3}{2},3} D^i X^{(2)} \\
&\quad + c_4^{\frac{3}{2},3} \partial J^{4-i} + \dots + c_{27}^{\frac{3}{2},3} \varepsilon^{ijkl} J^{4-ji} J^k J^l J^l, \\
Q_{\frac{3}{2}}^{(\frac{7}{2})} &= (M^\alpha)^{ij} (c_1^{\frac{3}{2},2} D^j \Phi^{(3),\alpha} + c_2^{\frac{3}{2},2} J^j \Phi^{(3),\alpha}) \\
&\quad + \varepsilon^{ijkl} (M^\alpha)^{jk} (c_3^{\frac{3}{2},2} D^l \Phi^{(3),\alpha} + c_4^{\frac{3}{2},2} J^l \Phi^{(3),\alpha}) \\
&\quad + c_5^{\frac{3}{2},2} \delta_\mu^i \Phi^{(\frac{7}{2})}{}^\mu + c_6^{\frac{3}{2},2} D^{4-i} \Phi^{(2)} \\
&\quad + c_7^{\frac{3}{2},2} \partial D^i \Phi^{(2)} + c_8^{\frac{3}{2},2} \partial J D^i \Phi^{(2)} + c_9^{\frac{3}{2},2} J^{4-i} \Phi^{(2)} + \dots \\
&\quad + c_{22}^{\frac{3}{2},2} D^{4-i} X^{(2)} + c_{23}^{\frac{3}{2},2} \partial D^i X^{(2)} \\
&\quad + c_{24}^{\frac{3}{2},2} \partial J D^i X^{(2)} + c_{25}^{\frac{3}{2},2} J^{4-i} X^{(2)} + \dots \\
&\quad + c_{37}^{\frac{3}{2},2} \varepsilon^{ijkl} J^j J^k J^l X^{(2)} + c_{38}^{\frac{3}{2},2} \partial^2 J^{4-i} + c_{39}^{\frac{3}{2},2} \partial^3 J^i \\
&\quad + c_{40}^{\frac{3}{2},2} \partial J \tilde{G}^i + \dots + \varepsilon^{ijkl} (\dots + c_{141}^{\frac{3}{2},2} J^j J^k J^l J^l J^{4-kl} \\
&\quad + w_{142}^2 J^j J^k J^l J^i J^j), \\
Q_{\frac{3}{2}}^{(\frac{9}{2})} &= c_0^{\frac{3}{2},1} \delta_\mu^i \Phi^{(\frac{9}{2})}{}^\mu + c_1^{\frac{3}{2},1} D^i \Phi^{(4)} \\
&\quad + (M^\alpha)^{ij} (c_2^{\frac{3}{2},1} D^{4-j} \Phi^{(3),\alpha} + c_3^{\frac{3}{2},1} \partial D^j \Phi^{(3),\alpha}) \\
&\quad + c_4^{\frac{3}{2},1} J^i D^{4-ij} \Phi^{(3),\alpha} + \dots + c_{22}^{\frac{3}{2},1} \varepsilon^{ijkl} J^i J^k J^l \Phi^{(3),\alpha} + \dots \\
&\quad + \delta_\mu^l (\varepsilon^{ijkl} c_{49}^{\frac{3}{2},1} D^{4-jk} \Phi^{(\frac{7}{2})}{}^\mu \\
&\quad + c_{50}^{\frac{3}{2},1} \varepsilon^{ijkl} D^{jk} \Phi^{(\frac{7}{2})}{}^\mu + \dots + c_{59}^{\frac{3}{2},1} \partial \Phi^{(\frac{7}{2})}{}^\mu) \\
&\quad + c_{60}^{\frac{3}{2},1} \Phi^{(2)} D^i \Phi^{(2)} + c_{61}^{\frac{3}{2},1} J^{4-0} D^i \Phi^{(2)} \\
&\quad + c_{62}^{\frac{3}{2},1} \partial J D^{4-i} \Phi^{(2)} + \dots + c_{149}^{\frac{3}{2},1} J^{4-0} D^i X^{(2)} \\
&\quad + c_{150}^{\frac{3}{2},1} \partial J D^{4-i} X^{(2)} + \dots + c_{236}^{\frac{3}{2},1} \partial J^{ij} J^j X^{(2)} \\
&\quad + c_{237}^{\frac{3}{2},1} J^{4-0} \partial J \partial J^i + c_{238}^{\frac{3}{2},1} J^{4-0} \partial J \partial J^i \\
&\quad + c_{239}^{\frac{3}{2},1} J^{4-0} J^{4-i} \partial J + \dots + \varepsilon^{ijkl} (c_{379}^{\frac{3}{2},1} J^{4-0} \partial J J^j J^k J^l \\
&\quad + c_{380}^{\frac{3}{2},1} J^{4-0} J^{4-j} J^k J^l + \dots + c_{668}^{\frac{3}{2},1} \partial^2 J^i J^j J^k J^l \\
&\quad + c_{669}^{\frac{3}{2},1} \partial^3 J^j J^k J^l + c_{670}^{\frac{3}{2},1} \partial^2 J^j \partial J^k \partial J^l \\
&\quad + c_{671}^{\frac{3}{2},1} \partial J^j \partial J^k \partial J^l), \\
Q_2^{(1)} &= c_1^{2,5} \partial J, \\
Q_2^{(2)} &= c_1^{2,4} \Phi^{(2)} + c_2^{2,4} X^{(2)} + c_3^{2,4} J^{4-0} + c_4^{2,4} \partial^2 J \\
&\quad + c_5^{2,4} \partial J \partial J + c_6^{2,4} J^i J^{4-i} \\
&\quad + c_7^{2,4} J^i \partial J^i + c_8^{2,4} J^{ij} J^{ij} + c_9^{2,4} J^{ij} J^{4-ij} \\
&\quad + c_{10}^{2,4} J^i J^j J^{4-ij} + c_{11}^{2,4} J^i J^j J^{ij} + c_{12}^{2,4} \varepsilon^{ijkl} J^i J^j J^k J^l, \\
Q_2^{(3)} &= c_1^{2,3} \partial J \Phi^{(2)} + c_2^{2,3} J^i D^i \Phi^{(2)} + c_3^{2,3} \partial J J^{4-0} \\
&\quad + c_4^{2,3} \partial J \partial^2 J + c_5^{2,3} \partial J \partial J \partial J \\
&\quad + c_6^{2,3} J^i \partial J^{4-i} + c_7^{2,3} \partial J^i J^{4-i} + c_8^{2,3} J^i \partial J^i \partial J \\
&\quad + c_9^{2,3} J^i \partial^2 J^i + c_{10}^{2,3} \partial J^i \partial J^i + c_{11}^{2,3} J^i J^{4-ij} J^{4-j} \\
&\quad + c_{12}^{2,3} J^i J^j J^{4-ij} \partial J + c_{13}^{2,3} J^{4-ij} J^{4-ij} \partial J + c_{14}^{2,3} J^i J^j \partial J^{4-ij} \\
&\quad + c_{15}^{2,3} \partial J^i J^j J^{4-ij} + c_{16}^{2,3} \partial^2 J^{ij} \\
&\quad + \varepsilon^{ijkl} (c_{17}^{2,3} J^i J^{4-jk} J^{4-l} + c_{18}^{2,3} J^i J^j J^k J^{4-l} \\
&\quad + c_{19}^{2,3} J^i J^j J^{4-kl} \partial J + c_{20}^{2,3} J^{4-ij} J^{4-kl} \partial J \\
&\quad + c_{21}^{2,3} J^i J^j \partial J^{4-kl} + c_{22}^{2,3} \partial J^i J^j J^{4-kl} + c_{23}^{2,3} \partial (J^i J^j J^k J^l)), \\
Q_2^{(4)} &= c_0^{2,2} \Phi^{(4)} + (M^\alpha)^{ij} (c_1^{2,2} D^{ij} \Phi^{(3),\alpha} \\
&\quad + c_2^{2,2} D^{4-ij} \Phi^{(3),\alpha} + \dots + \varepsilon^{ijkl} c_8^{2,2} J^k J^l \Phi^{(3),\alpha}) \\
&\quad + \delta_\mu^i (c_9^{2,2} D^i \Phi^{(\frac{7}{2})}{}^\mu + c_{10}^{2,2} J^i \Phi^{(\frac{7}{2})}{}^\mu) \\
&\quad + c_{11}^{2,2} \Phi^{(2)} \Phi^{(2)} + c_{12}^{2,2} D^{4-0} \Phi^{(2)} + c_{13}^{2,2} J^{4-0} \Phi^{(2)} + \dots \\
&\quad + c_{36}^{2,2} \varepsilon^{ijkl} J^{4-ij} J^{4-kl} \Phi^{(2)} + c_{37}^{2,2} D^{4-0} X^{(2)} \\
&\quad + c_{38}^{2,2} J^{4-0} X^{(2)} + \dots + c_{61}^{2,2} \varepsilon^{ijkl} J^{4-ij} J^{4-kl} X^{(2)} \\
&\quad + c_{62}^{2,2} J^{4-0} J^{4-0} + c_{63}^{2,2} J^{4-0} \partial J \partial J + \dots \\
&\quad + c_{169}^{2,2} \varepsilon^{ijkl} J^i J^j \partial J^k \partial J^l, \\
Q_2^{(5)} &= c_1^{2,1} \delta_\mu^i D^i \Phi^{(\frac{9}{2})}{}^\mu + (M^\alpha)^{ij} (c_2^{2,1} \partial D^{4-ij} \Phi^{(3),\alpha} \\
&\quad + c_3^{2,1} \partial D^{ij} \Phi^{(3),\alpha} + \dots \\
&\quad + c_{38}^{2,1} \varepsilon^{ijkl} \partial J^{4-kl} \Phi^{(3),\alpha} + c_{39}^{2,1} \varepsilon^{ijkl} J^{4-jk} J^l D^j \Phi^{(3),\alpha}) \\
&\quad + \delta_\mu^i (c_{40}^{2,1} D^{4-i} \Phi^{(\frac{7}{2})}{}^\mu + c_{41}^{2,1} \partial D^i \Phi^{(\frac{7}{2})}{}^\mu \\
&\quad + \dots + c_{52}^{2,1} \varepsilon^{jjk\mu} J^{4-ij} J^k \Phi^{(\frac{7}{2})}{}^\mu \\
&\quad + c_{53}^{2,1} \varepsilon^{ijk\mu} \Gamma^i \Gamma^j D^k \Phi^{(\frac{7}{2})}{}^\mu) + c_{54}^{2,1} \partial D^{4-0} \Phi^{(2)} \\
&\quad + c_{55}^{2,1} J^{4-0} \partial \Phi^{(2)} + \dots + c_{129}^{2,1} \partial D^{4-0} X^{(2)} \\
&\quad + c_{130}^{2,1} J^{4-0} \partial X^{(2)} + \dots \\
&\quad + c_{204}^{2,1} \varepsilon^{ijkl} \partial J^i J^j J^k J^l X^{(2)} \\
&\quad + c_{205}^{2,1} \partial^3 J^{4-0} + c_{206}^{2,1} \partial J^{4-0} \partial J + \dots \\
&\quad + c_{351}^{2,1} \partial J^{4-i} J^{ij} J^{4-jk} J^k \\
&\quad + \varepsilon^{ijkl} (c_{352}^{2,1} J^{4-0} \partial J^i J^j J^k J^l + \dots \\
&\quad + c_{428}^{2,1} J^{4-i} J^{ij} \partial J^j J^k J^l + c_{429}^{2,1} J^{4-i} J^{4-ij} \partial J^i J^k J^l \\
&\quad + c_{430}^{2,1} \varepsilon^{ijkl} \partial J^{4-i} J^{4-ij} J^j J^k J^l). \tag{C.1}
\end{aligned}$$

These can be obtained from Appendix B by taking the procedure in (4.24). The coefficients appearing in (C.1) are proportional to the ones in (B.1) according to (4.24) in most of the terms. The number of terms are little different from each other. For example, see $Q_1^{(4),ij}(w)$ and $Q_1^{(4),ij}(Z_2)$ and other cases.

D The single $\mathcal{N} = 4$ OPE in different order

By applying the replacements in (4.24), we obtain the following intermediate expression

$$\begin{aligned}
\Phi^{(2)}(Z_1) \Phi^{(2)}(Z_2) &= \frac{\theta_{12}^{4-0}}{z_{12}^6} 8\alpha c_0^{0,4} + \frac{\theta_{12}^{4-0}}{z_{12}^5} Q_2^{(1)}(Z_2) \\
&\quad + \frac{\theta_{12}^{4-0}}{z_{12}^4} \left[\frac{3}{2} \partial Q_2^{(1)} + Q_2^{(2)} + R_2^{(2)} \right] (Z_2) \\
&\quad + \frac{\theta_{12}^{4-0}}{z_{12}^3} \left[\partial^2 Q_2^{(1)} + \partial Q_2^{(2)} + Q_2^{(3)} + R_2^{(3)} \right] (Z_2) \\
&\quad + \frac{\theta_{12}^{4-0}}{z_{12}^2} \left[\frac{5}{12} \partial^3 Q_2^{(1)} + \frac{1}{2} \partial Q_2^{(2)} + \frac{5}{6} Q_2^{(3)} + Q_2^{(4)} + R_2^{(4)} \right] (Z_2)
\end{aligned}$$

$$\begin{aligned}
& + \frac{\theta_{12}^{4-0}}{z_{12}} \left[\frac{1}{8} \partial^4 \mathbf{Q}_2^{(1)} + \frac{1}{6} \partial^3 \mathbf{Q}_2^{(2)} + \frac{5}{14} \partial^2 \mathbf{Q}_2^{(3)} \right. \\
& + \left. \frac{3}{4} \partial \mathbf{Q}_2^{(4)} + \mathbf{Q}_2^{(5)} \right] (Z_2) \\
& + \frac{\theta_{12}^{4-i}}{z_{12}} \mathbf{Q}_{\frac{3}{2}}^{(\frac{1}{2}),i} (Z_2) + \frac{\theta_{12}^{4-i}}{z_{12}} \left[2 \partial \mathbf{Q}_{\frac{3}{2}}^{(\frac{1}{2}),i} + \mathbf{Q}_{\frac{3}{2}}^{(\frac{3}{2}),i} + \mathbf{R}_{\frac{3}{2}}^{(\frac{3}{2}),i} \right] (Z_2) \\
& + \frac{\theta_{12}^{4-i}}{z_{12}} \left[\frac{3}{2} \partial^2 \mathbf{Q}_{\frac{3}{2}}^{(\frac{1}{2}),i} + \partial \mathbf{Q}_{\frac{3}{2}}^{(\frac{3}{2}),i} + \mathbf{Q}_{\frac{3}{2}}^{(\frac{5}{2}),i} + \mathbf{R}_{\frac{3}{2}}^{(\frac{5}{2}),i} \right] (Z_2) \\
& + \frac{\theta_{12}^{4-i}}{z_{12}} \left[\frac{2}{3} \partial^3 \mathbf{Q}_{\frac{3}{2}}^{(\frac{1}{2}),i} + \frac{1}{2} \partial^2 \mathbf{Q}_{\frac{3}{2}}^{(\frac{3}{2}),i} \right. \\
& + \left. \frac{4}{5} \partial \mathbf{Q}_{\frac{3}{2}}^{(\frac{5}{2}),i} + \mathbf{Q}_{\frac{3}{2}}^{(\frac{7}{2}),i} + \mathbf{R}_{\frac{3}{2}}^{(\frac{7}{2}),i} \right] (Z_2) \\
& + \frac{\theta_{12}^{4-i}}{z_{12}} \left[\frac{5}{24} \partial^4 \mathbf{Q}_{\frac{3}{2}}^{(\frac{1}{2}),i} + \frac{1}{6} \partial^3 \mathbf{Q}_{\frac{3}{2}}^{(\frac{3}{2}),i} + \frac{1}{3} \partial^2 \mathbf{Q}_{\frac{3}{2}}^{(\frac{5}{2}),i} \right. \\
& + \left. \frac{5}{7} \partial \mathbf{Q}_{\frac{3}{2}}^{(\frac{7}{2}),i} + \mathbf{Q}_{\frac{3}{2}}^{(\frac{9}{2}),i} \right] (Z_2) \\
& + \frac{\theta_{12}^{4-ij}}{z_{12}} \mathbf{Q}_1^{(1),ij} (Z_2) + \frac{\theta_{12}^{4-ij}}{z_{12}} \left[\partial \mathbf{Q}_1^{(1),ij} + \mathbf{Q}_1^{(2),ij} \right] (Z_2) \\
& + \frac{\theta_{12}^{4-ij}}{z_{12}} \left[\frac{1}{2} \partial^2 \mathbf{Q}_1^{(1),ij} + \frac{3}{4} \partial \mathbf{Q}_1^{(2),ij} + \mathbf{Q}_1^{(3),ij} \right] (Z_2) \\
& + \frac{\theta_{12}^{4-ij}}{z_{12}} \left[\frac{1}{6} \partial^3 \mathbf{Q}_1^{(1),ij} + \frac{3}{10} \partial^2 \mathbf{Q}_1^{(2),ij} \right. \\
& + \left. \frac{2}{3} \partial \mathbf{Q}_1^{(3),ij} + \mathbf{Q}_1^{(4),ij} \right] (Z_2) + \frac{\theta_{12}^i}{z_{12}} \mathbf{Q}_{\frac{1}{2}}^{(\frac{3}{2}),i} (Z_2) \\
& + \frac{\theta_{12}^i}{z_{12}} \left[\frac{2}{3} \partial \mathbf{Q}_{\frac{1}{2}}^{(\frac{3}{2}),i} + \mathbf{Q}_{\frac{1}{2}}^{(\frac{5}{2}),i} \right] (Z_2) + \frac{\theta_{12}^i}{z_{12}} \left[\frac{1}{4} \partial^2 \mathbf{Q}_{\frac{1}{2}}^{(\frac{3}{2}),i} \right. \\
& + \left. \frac{3}{5} \partial \mathbf{Q}_{\frac{1}{2}}^{(\frac{5}{2}),i} + \mathbf{Q}_{\frac{1}{2}}^{(\frac{7}{2}),i} \right] (Z_2) \\
& + \frac{1}{z_{12}} c_0^{0,4} + \frac{1}{z_{12}} \mathbf{Q}_0^{(2)} (Z_2) + \frac{1}{z_{12}} \frac{1}{2} \partial \mathbf{Q}_0^{(2)} (Z_2) + \dots \quad (D.1)
\end{aligned}$$

By rearranging this (D.1) in the order of increasing spin, it is easy to see that we can write down this expression as (4.25).

E The expression of $X_0^{(2)}$ for $N = 5$

We present the higher spin-2 current (the coset indices are given by 1, 2, ..., 10, the $SO(4)$ indices are 11, 12 and 13 and the remaining indices 14, ..., 18 are $SO(5)$ ones) appearing in (4.3)

$$\begin{aligned}
X_0^{(2)} = & c_1 (Q^1 Q^2 Q^{19} Q^{20} + Q^1 Q^3 Q^{19} Q^{21} + Q^1 Q^4 Q^{19} Q^{21} \\
& + Q^1 Q^5 Q^{19} Q^{23} + Q^2 Q^3 Q^{20} Q^{21} \\
& + Q^2 Q^4 Q^{20} Q^{22} + Q^2 Q^5 Q^{20} Q^{23} + Q^3 Q^4 Q^{21} Q^{22} \\
& + Q^3 Q^5 Q^{21} Q^{23} + Q^4 Q^5 Q^{22} Q^{23} \\
& + Q^6 Q^7 Q^{24} Q^{25} + Q^6 Q^8 Q^{24} Q^{26} + Q^6 Q^9 Q^{24} Q^{27} \\
& + Q^6 Q^{10} Q^{24} Q^{28} + Q^7 Q^8 Q^{25} Q^{26} \\
& + Q^7 Q^9 Q^{25} Q^{27} + Q^7 Q^{10} Q^{25} Q^{28} + Q^8 Q^9 Q^{26} Q^{27}
\end{aligned}$$

$$\begin{aligned}
& + Q^8 Q^{10} Q^{26} Q^{28} + Q^9 Q^{10} Q^{27} Q^{28}) \\
& + c_2 (Q^1 Q^6 Q^{20} Q^{25} + Q^1 Q^6 Q^{21} Q^{26} + Q^1 Q^6 Q^{22} Q^{27} \\
& + Q^1 Q^6 Q^{23} Q^{28} + Q^2 Q^7 Q^{19} Q^{24} \\
& + Q^2 Q^7 Q^{22} Q^{27} + Q^2 Q^7 Q^{23} Q^{28} + Q^3 Q^8 Q^{19} Q^{24} \\
& + Q^3 Q^8 Q^{22} Q^{27} + Q^3 Q^8 Q^{23} Q^{28} \\
& + Q^4 Q^9 Q^{19} Q^{24} + Q^4 Q^9 Q^{20} Q^{25} + Q^4 Q^9 Q^{21} Q^{26} \\
& + Q^5 Q^{10} Q^{19} Q^{24} + Q^5 Q^{10} Q^{20} Q^{25} \\
& + Q^5 Q^{10} Q^{21} Q^{26}) \\
& + c_3 (Q^1 Q^7 Q^{19} Q^{25} + Q^1 Q^8 Q^{19} Q^{26} + Q^1 Q^9 Q^{19} Q^{27} \\
& + Q^1 Q^{10} Q^{19} Q^{28} + Q^2 Q^6 Q^{20} Q^{24} \\
& + Q^2 Q^9 Q^{20} Q^{27} + Q^2 Q^{10} Q^{20} Q^{28} + Q^3 Q^6 Q^{21} Q^{24} \\
& + Q^3 Q^9 Q^{21} Q^{27} + Q^3 Q^{10} Q^{21} Q^{28} \\
& + Q^4 Q^6 Q^{22} Q^{24} + Q^4 Q^7 Q^{22} Q^{25} + Q^4 Q^8 Q^{22} Q^{26} \\
& + Q^5 Q^6 Q^{23} Q^{24} + Q^5 Q^7 Q^{23} Q^{25} \\
& + Q^5 Q^8 Q^{23} Q^{26}) \\
& + c_4 (Q^1 Q^7 Q^{21} Q^{24} + Q^1 Q^8 Q^{20} Q^{24} + Q^1 Q^9 Q^{23} Q^{24} \\
& + Q^1 Q^{10} Q^{22} Q^{24} + Q^2 Q^6 Q^{19} Q^{26} \\
& + Q^2 Q^9 Q^{23} Q^{26} + Q^2 Q^{10} Q^{22} Q^{26} + Q^3 Q^6 Q^{19} Q^{25} \\
& + Q^3 Q^9 Q^{23} Q^{25} + Q^3 Q^{10} Q^{22} Q^{25} \\
& + Q^4 Q^6 Q^{19} Q^{28} + Q^4 Q^7 Q^{21} Q^{28} + Q^4 Q^8 Q^{20} Q^{28} \\
& + Q^5 Q^6 Q^{19} Q^{27} + Q^5 Q^7 Q^{21} Q^{27} \\
& + Q^5 Q^8 Q^{20} Q^{27}) \\
& + c_5 (Q^2 Q^7 Q^{20} Q^{25} + Q^3 Q^8 Q^{21} Q^{26} \\
& + Q^4 Q^9 Q^{22} Q^{27} + Q^5 Q^{10} Q^{23} Q^{28}) \\
& + c_6 (Q^2 Q^7 Q^{21} Q^{26} + Q^3 Q^8 Q^{20} Q^{25} \\
& + Q^4 Q^9 Q^{23} Q^{28} + Q^5 Q^{10} Q^{22} Q^{27}) \\
& + c_7 (Q^3 Q^7 Q^{21} Q^{25} + Q^2 Q^8 Q^{20} Q^{26} \\
& + Q^4 Q^{10} Q^{22} Q^{28} + Q^5 Q^9 Q^{23} Q^{27}) \\
& + c_8 Q^1 Q^6 Q^{19} Q^{24} \\
& + c_9 (Q^1 Q^{19} V^{12} + Q^2 Q^{20} V^{12} + Q^3 Q^{21} V^{12} \\
& + Q^4 Q^{22} V^{12} + Q^5 Q^{23} V^{12} \\
& - Q^6 Q^{24} V^{12} - Q^7 Q^{25} V^{12} - Q^8 Q^{26} V^{12} \\
& - Q^9 Q^{27} V^{12} - Q^{10} Q^{28} V^{12}) \\
& + c_{10} (Q^1 Q^{19} V^{30} + Q^2 Q^{20} V^{30} + Q^3 Q^{21} V^{30} \\
& + Q^4 Q^{22} V^{30} + Q^5 Q^{23} V^{30}) \\
& + c_{11} (Q^1 Q^{20} V^{17} + Q^1 Q^{22} V^{18} + Q^2 Q^{19} V^{35} \\
& + Q^2 Q^{23} V^{34} + Q^3 Q^{23} V^{32} + Q^4 Q^{19} V^{36} \\
& + Q^5 Q^{20} V^{16} + Q^5 Q^{21} V^{14} + Q^6 Q^{26} V^{17} + Q^6 Q^{28} V^{18} \\
& + Q^7 Q^{27} V^{32} + Q^8 Q^{24} V^{35} \\
& + Q^8 Q^{27} V^{34} + Q^9 Q^{25} V^{14} + Q^9 Q^{26} V^{16} + Q^{10} Q^{24} V^{36} \\
& - Q^1 Q^{21} V^{35} - Q^1 Q^{23} V^{36} \\
& - Q^2 Q^{22} V^{14} - Q^3 Q^{19} V^{17} - Q^3 Q^{22} V^{16} - Q^4 Q^{20} V^{32} \\
& - Q^4 Q^{21} V^{34} - Q^5 Q^{19} V^{18} \\
& - Q^6 Q^{25} V^{35} - Q^6 Q^{27} V^{36} - Q^7 Q^{24} V^{17} - Q^7 Q^{28} V^{16} \\
& - Q^8 Q^{28} V^{14} - Q^9 Q^{24} V^{18} \\
& - Q^{10} Q^{26} V^{32} - Q^{10} Q^{25} V^{34}) \\
& + c_{12} (Q^1 Q^{24} V^{11} + Q^2 Q^{26} V^{11} + Q^3 Q^{25} V^{11} \\
& + Q^4 Q^{28} V^{11} + Q^5 Q^{27} V^{11} + Q^6 Q^{19} V^{29}
\end{aligned}$$

$$\begin{aligned}
& +Q^7 Q^{21} V^{29} + Q^8 Q^{20} V^{29} \\
& +Q^9 Q^{23} V^{29} + Q^{10} Q^{22} V^{29} \\
& +c_{13} (Q^2 Q^{20} V^{15} + Q^2 Q^{20} V^{33} + Q^8 Q^{26} V^{15} \\
& +Q^8 Q^{26} V^{33} - Q^3 Q^{21} V^{15} - Q^3 Q^{21} V^{33} \\
& -Q^7 Q^{25} V^{15} - Q^7 Q^{25} V^{33}) \\
& +c_{14} (Q^4 Q^{22} V^{15} - Q^4 Q^{22} V^{33} \\
& -Q^5 Q^{23} V^{15} + Q^5 Q^{23} V^{33} \\
& -Q^9 Q^{27} V^{15} + Q^9 Q^{27} V^{33} \\
& +Q^{10} Q^{28} V^{15} - Q^{10} Q^{28} V^{33}) \\
& +c_{15} (Q^6 Q^{24} V^{30} + Q^7 Q^{25} V^{30} + Q^8 Q^{26} V^{30} \\
& +Q^9 Q^{27} V^{30} + Q^{10} Q^{28} V^{30}) \\
& +c_{16} (Q^1 \partial Q^{19} + Q^2 \partial Q^{20} + Q^3 \partial Q^{21} + Q^4 \partial Q^{22} \\
& +Q^5 \partial Q^{23} + Q^6 \partial Q^{24} + Q^7 \partial Q^{25} \\
& +Q^8 \partial Q^{26} + Q^9 \partial Q^{27} + Q^{10} \partial Q^{28} \\
& -\partial Q^1 Q^{19} - \partial Q^2 Q^{20} \\
& -\partial Q^3 Q^{21} - \partial Q^4 Q^{22} - \partial Q^5 Q^{23} \\
& -\partial Q^6 Q^{24} - \partial Q^7 Q^{25} - \partial Q^8 Q^{26} \\
& -\partial Q^9 Q^{27} - \partial Q^{10} Q^{28}) \\
& +c_{17} V^{11} V^{29} + c_{18} (V^{12} V^{12} - V^{30} V^{30}) \\
& +c_{19} V^{12} V^{30} \\
& +c_{20} (V^{14} V^{32} + V^{15} V^{33} + V^{16} V^{34} \\
& +V^{17} V^{35} + V^{18} V^{36}) \\
& +c_{21} \partial V^{12} + c_{22} \partial V^{30} \\
& +c_{23} \partial V^{15} + c_{24} \partial V^{33},
\end{aligned}$$

where the k dependent coefficients are

$$\begin{aligned}
c_1 &= \frac{126k(5+2k)(19+13k)^2(23+12k+k^2)}{25(-1+k)(2+k)(7+k)^6(241+132k+11k^2)}, \\
c_2 &= \frac{54k(3+k)(9+k)(5+2k)(19+13k)^2}{25(-1+k)(2+k)(7+k)^6(241+132k+11k^2)}, \\
c_3 &= -\frac{72k(10+k)(5+2k)(19+13k)^2}{25(-1+k)(7+k)^6(241+132k+11k^2)}, \\
c_4 &= -\frac{18k(5+2k)(19+13k)^2}{25(-1+k)(2+k)(7+k)^6}, \\
c_5 &= -\frac{18k(5+2k)(19+13k)^2(-1+12k+k^2)}{25(-1+k)(2+k)(7+k)^6(241+132k+11k^2)}, \\
c_6 &= -\frac{(144k(10+k)(5+2k)(19+13k)^2)}{25(-1+k)(7+k)^6(241+132k+11k^2)}, \\
c_7 &= -\frac{54k(5+2k)(19+13k)^2(107+60k+5k^2)}{25(-1+k)(2+k)(7+k)^6(241+132k+11k^2)}, \\
c_8 &= -\frac{216k(10+k)(5+2k)(19+13k)^2}{25(-1+k)(7+k)^6(241+132k+11k^2)}, \\
c_9 &= -\frac{\left(\frac{252}{25} - \frac{252i}{25}\right) \sqrt{2}(10+k)(5+2k)(19+13k)^2}{(-1+k)(7+k)^5(241+132k+11k^2)}, \\
c_{10} &= -\frac{\left(\frac{252}{25} + \frac{252i}{25}\right) \sqrt{2}(10+k)(5+2k)(19+13k)^2}{(-1+k)(7+k)^5(241+132k+11k^2)},
\end{aligned}$$

$$\begin{aligned}
c_{11} &= -\frac{378(3+k)(9+k)(5+2k)(19+13k)^2}{25(-1+k)(2+k)(7+k)^5(241+132k+11k^2)}, \\
c_{12} &= -\frac{1008(10+k)(5+2k)(19+13k)^2}{25(-1+k)(7+k)^5(241+132k+11k^2)}, \\
c_{13} &= \frac{189\sqrt{2}(3+k)(9+k)(5+2k)(19+13k)^2}{25(-1+k)(2+k)(7+k)^5(241+132k+11k^2)}, \\
c_{14} &= \frac{189i\sqrt{2}(3+k)(9+k)(5+2k)(19+13k)^2}{25(-1+k)(2+k)(7+k)^5(241+132k+11k^2)}, \\
c_{15} &= \frac{\left(\frac{252}{25} + \frac{252i}{25}\right) \sqrt{2}(10+k)(5+2k)(19+13k)^2}{(-1+k)(7+k)^5(241+132k+11k^2)}, \\
c_{16} &= \frac{756k(5+2k)(19+13k)^2}{25(-1+k)(2+k)(7+k)^5(241+132k+11k^2)}, \\
c_{17} &= -\frac{1008(10+k)(5+2k)(19+13k)^2}{5(-1+k)(2+k)(7+k)^4(241+132k+11k^2)}, \\
c_{18} &= \frac{252i(10+k)(5+2k)(19+13k)^2}{5(-1+k)(2+k)(7+k)^4(241+132k+11k^2)}, \\
c_{19} &= -\frac{504(10+k)(5+2k)(19+13k)^2}{5(-1+k)(2+k)(7+k)^4(241+132k+11k^2)}, \\
c_{20} &= \frac{1512(9+k)(5+2k)(19+13k)^2}{25(-1+k)(2+k)(7+k)^4(241+132k+11k^2)}, \\
c_{21} &= -\frac{\left(\frac{252}{5} - \frac{252i}{5}\right) \sqrt{2}(10+k)(5+2k)(19+13k)^2}{(-1+k)(2+k)(7+k)^4(241+132k+11k^2)}, \\
c_{22} &= -\frac{\left(\frac{252}{5} + \frac{252i}{5}\right) \sqrt{2}(10+k)(5+2k)(19+13k)^2}{(-1+k)(2+k)(7+k)^4(241+132k+11k^2)}, \\
c_{23} &= -\frac{\left(\frac{378}{25} + \frac{1134i}{25}\right) \sqrt{2}(9+k)(5+2k)(19+13k)^2}{(-1+k)(2+k)(7+k)^4(241+132k+11k^2)}, \\
c_{24} &= \frac{\left(\frac{378}{25} - \frac{1134i}{25}\right) \sqrt{2}(9+k)(5+2k)(19+13k)^2}{(-1+k)(2+k)(7+k)^4(241+132k+11k^2)}.
\end{aligned}$$

The lowest power of $\frac{1}{k}$ of $X_0^{(2)}$ under the large k limit is given by 4 while the corresponding value in (2.11) is given by 1, 2 or 0. Therefore, the higher spin-2 current $X_0^{(2)}$ will vanish under the large k limit for fixed N .

References

1. M.R. Gaberdiel, R. Gopakumar, Large $N = 4$ holography. JHEP **1309**, 036 (2013). [https://doi.org/10.1007/JHEP09\(2013\)036](https://doi.org/10.1007/JHEP09(2013)036). [arXiv:1305.4181](https://arxiv.org/abs/1305.4181) [hep-th]
2. M.R. Gaberdiel, R. Gopakumar, Higher spins and strings. JHEP **1411**, 044 (2014). [https://doi.org/10.1007/JHEP11\(2014\)044](https://doi.org/10.1007/JHEP11(2014)044). [arXiv:1406.6103](https://arxiv.org/abs/1406.6103) [hep-th]
3. A. Sevrin, G. Theodoridis, $N = 4$ superconformal coset theories. Nucl. Phys. B **332**, 380 (1990). [https://doi.org/10.1016/0550-3213\(90\)90100-R](https://doi.org/10.1016/0550-3213(90)90100-R)
4. K. Ferreira, Even spin $\mathcal{N} = 4$ holography. JHEP **1709**, 110 (2017). [https://doi.org/10.1007/JHEP09\(2017\)110](https://doi.org/10.1007/JHEP09(2017)110). [arXiv:1702.02641](https://arxiv.org/abs/1702.02641) [hep-th]

5. S.F. Prokushkin, M.A. Vasiliev, Higher spin gauge interactions for massive matter fields in 3-D AdS space-time. *Nucl. Phys. B* **545**, 385 (1999). [https://doi.org/10.1016/S0550-3213\(98\)00839-6](https://doi.org/10.1016/S0550-3213(98)00839-6). [arXiv:hep-th/9806236](https://arxiv.org/abs/hep-th/9806236)
6. S. Prokushkin, M.A. Vasiliev, 3-d higher spin gauge theories with matter. [arXiv:hep-th/9812242](https://arxiv.org/abs/hep-th/9812242)
7. M.R. Gaberdiel, R. Gopakumar, String theory as a higher spin theory. *JHEP* **1609**, 085 (2016). [https://doi.org/10.1007/JHEP09\(2016\)085](https://doi.org/10.1007/JHEP09(2016)085). [arXiv:1512.07237](https://arxiv.org/abs/1512.07237) [hep-th]
8. C. Ahn, M.H. Kim, The operator product expansion between the 16 lowest higher spin currents in the $\mathcal{N} = 4$ superspace. *Eur. Phys. J. C* **76**(7), 389 (2016). <https://doi.org/10.1140/epjc/s10052-016-4234-2>. [arXiv:1509.01908](https://arxiv.org/abs/1509.01908) [hep-th]
9. C. Ahn, Higher spin currents in wolf space. Part I. *JHEP* **1403**, 091 (2014). [https://doi.org/10.1007/JHEP03\(2014\)091](https://doi.org/10.1007/JHEP03(2014)091). [arXiv:1311.6205](https://arxiv.org/abs/1311.6205) [hep-th]
10. C. Ahn, Higher spin currents in wolf space: Part II. *Class. Quantum Gravity* **32**(1), 015023 (2015). <https://doi.org/10.1088/0264-9381/32/1/015023>. [arXiv:1408.0655](https://arxiv.org/abs/1408.0655) [hep-th]
11. C. Ahn, Higher spin currents in Wolf space: III. *Class. Quantum Gravity* **32**(18), 185001 (2015). <https://doi.org/10.1088/0264-9381/32/18/185001>. [arXiv:1504.00070](https://arxiv.org/abs/1504.00070) [hep-th]
12. C. Ahn, J. Paeng, Higher spin currents in orthogonal wolf space. *Class. Quantum Gravity* **32**(4), 045011 (2015). <https://doi.org/10.1088/0264-9381/32/4/045011>. [arXiv:1410.0080](https://arxiv.org/abs/1410.0080) [hep-th]
13. C. Ahn, H. Kim, J. Paeng, Three-point functions in the $\mathcal{N}=4$ orthogonal coset theory. *Int. J. Mod. Phys. A* **31**(16), 1650090 (2016). <https://doi.org/10.1142/S0217751X16500901>. [arXiv:1510.03139](https://arxiv.org/abs/1510.03139) [hep-th]
14. C. Ahn, H. Kim, Higher spin currents in the enhanced $\mathcal{N} = 3$ Kazama-Suzuki model. *JHEP* **1612**, 001 (2016). [https://doi.org/10.1007/JHEP12\(2016\)001](https://doi.org/10.1007/JHEP12(2016)001). [arXiv:1607.00728](https://arxiv.org/abs/1607.00728) [hep-th]
15. M.R. Gaberdiel, C. Peng, I.G. Zadeh, Higgsing the stringy higher spin symmetry. *JHEP* **1510**, 101 (2015). [https://doi.org/10.1007/JHEP10\(2015\)101](https://doi.org/10.1007/JHEP10(2015)101). [arXiv:1506.02045](https://arxiv.org/abs/1506.02045) [hep-th]
16. C. Ahn, H. Kim, Higher spin currents in wolf space for generic $\mathcal{N}$. *JHEP* **1412**, 109 (2014). [https://doi.org/10.1007/JHEP12\(2014\)109](https://doi.org/10.1007/JHEP12(2014)109). [arXiv:1411.0356](https://arxiv.org/abs/1411.0356) [hep-th]
17. C. Ahn, J. Paeng, A supersymmetric enhancement of $\mathcal{N} = 1$ holographic minimal model. *JHEP* **1905**, 135 (2019). [https://doi.org/10.1007/JHEP05\(2019\)135](https://doi.org/10.1007/JHEP05(2019)135). [arXiv:1902.03699](https://arxiv.org/abs/1902.03699) [hep-th]
18. C. Ahn, Higher spin currents in the orthogonal coset theory. *Eur. Phys. J. C* **77**(6), 394 (2017). <https://doi.org/10.1140/epjc/s10052-017-4956-9>. [arXiv:1701.02410](https://arxiv.org/abs/1701.02410) [hep-th]
19. C. Ahn, The large $\mathcal{N}$ 't Hooft limit of coset minimal models. *JHEP* **1110**, 125 (2011). [https://doi.org/10.1007/JHEP10\(2011\)125](https://doi.org/10.1007/JHEP10(2011)125). [arXiv:1106.0351](https://arxiv.org/abs/1106.0351) [hep-th]
20. M.R. Gaberdiel, C. Vollenweider, Minimal model holography for $SO(2N)$. *JHEP* **1108**, 104 (2011). [https://doi.org/10.1007/JHEP08\(2011\)104](https://doi.org/10.1007/JHEP08(2011)104). [arXiv:1106.2634](https://arxiv.org/abs/1106.2634) [hep-th]
21. C. Ahn, The primary spin-4 casimir operators in the holographic $SO(N)$ coset minimal models. *JHEP* **1205**, 040 (2012). [https://doi.org/10.1007/JHEP05\(2012\)040](https://doi.org/10.1007/JHEP05(2012)040). [arXiv:1202.0074](https://arxiv.org/abs/1202.0074) [hep-th]
22. T. Creutzig, Y. Hikida, P.B. Rønne, $\mathcal{N} = 1$ supersymmetric higher spin holography on AdS_3 . *JHEP* **1302**, 019 (2013). [https://doi.org/10.1007/JHEP02\(2013\)019](https://doi.org/10.1007/JHEP02(2013)019). [arXiv:1209.5404](https://arxiv.org/abs/1209.5404) [hep-th]
23. C. Candu, M.R. Gaberdiel, M. Kelm, C. Vollenweider, Even spin minimal model holography. *JHEP* **1301**, 185 (2013). [https://doi.org/10.1007/JHEP01\(2013\)185](https://doi.org/10.1007/JHEP01(2013)185). [arXiv:1211.3113](https://arxiv.org/abs/1211.3113) [hep-th]
24. C. Ahn, J. Paeng, The OPEs of spin-4 casimir currents in the holographic $SO(N)$ coset minimal models. *Class. Quantum Gravity* **30**, 175004 (2013). <https://doi.org/10.1088/0264-9381/30/17/175004>. [arXiv:1301.0208](https://arxiv.org/abs/1301.0208) [hep-th]
25. C. Ahn, J. Paeng, Higher spin currents in the holographic $\mathcal{N} = 1$ coset minimal model. *JHEP* **1401**, 007 (2014). [https://doi.org/10.1007/JHEP01\(2014\)007](https://doi.org/10.1007/JHEP01(2014)007). [arXiv:1310.6185](https://arxiv.org/abs/1310.6185) [hep-th]
26. M.R. Gaberdiel, R. Gopakumar, An AdS_3 dual for minimal model CFTs. *Phys. Rev. D* **83**, 066007 (2011). <https://doi.org/10.1103/PhysRevD.83.066007>. [arXiv:1011.2986](https://arxiv.org/abs/1011.2986) [hep-th]
27. M.R. Gaberdiel, R. Gopakumar, Triality in minimal model holography. *JHEP* **1207**, 127 (2012). [https://doi.org/10.1007/JHEP07\(2012\)127](https://doi.org/10.1007/JHEP07(2012)127). [arXiv:1205.2472](https://arxiv.org/abs/1205.2472) [hep-th]
28. M.R. Gaberdiel, R. Gopakumar, Minimal model holography. *J. Phys. A* **46**, 214002 (2013). <https://doi.org/10.1088/1751-8113/46/21/214002>. [arXiv:1207.6697](https://arxiv.org/abs/1207.6697) [hep-th]
29. K. Thielemans, A Mathematica package for computing operator product expansions. *Int. J. Mod. Phys. C* **2**, 787 (1991). <https://doi.org/10.1142/S0129183191001001>
30. P. Goddard, A. Schwimmer, Factoring out free fermions and superconformal algebras. *Phys. Lett. B* **214**, 209 (1988). [https://doi.org/10.1016/0370-2693\(88\)91470-0](https://doi.org/10.1016/0370-2693(88)91470-0)
31. A. Van Proeyen, Realizations of $N = 4$ superconformal algebras on wolf spaces. *Class. Quantum Gravity* **6**, 1501 (1989). <https://doi.org/10.1088/0264-9381/6/10/018>
32. M. Gunaydin, J.L. Petersen, A. Taormina, A. Van Proeyen, On the unitary representations of a class of $N = 4$ superconformal algebras. *Nucl. Phys. B* **322**, 402 (1989). [https://doi.org/10.1016/0550-3213\(89\)90421-5](https://doi.org/10.1016/0550-3213(89)90421-5)
33. S.J. Gates Jr., S.V. Ketov, No $N = 4$ strings on wolf spaces. *Phys. Rev. D* **52**, 2278 (1995). <https://doi.org/10.1103/PhysRevD.52.2278>. [arXiv:hep-th/9501140](https://arxiv.org/abs/hep-th/9501140)
34. C. Ahn, D.G. Kim, M.H. Kim, The next 16 higher spin currents and three-point functions in the large $\mathcal{N} = 4$ holography. *Eur. Phys. J. C* **77**(8), 523 (2017). <https://doi.org/10.1140/epjc/s10052-017-5064-6>. [arXiv:1703.01744](https://arxiv.org/abs/1703.01744) [hep-th]
35. A. Sevrin, W. Troost, A. Van Proeyen, Superconformal algebras in two-dimensions with $N = 4$. *Phys. Lett. B* **208**, 447 (1988). [https://doi.org/10.1016/0370-2693\(88\)90645-4](https://doi.org/10.1016/0370-2693(88)90645-4)
36. A. Sevrin, W. Troost, A. Van Proeyen, P. Spindel, Extended supersymmetric sigma models on group manifolds. 2. Current algebras. *Nucl. Phys. B* **311**, 465 (1988). [https://doi.org/10.1016/0550-3213\(88\)90070-3](https://doi.org/10.1016/0550-3213(88)90070-3)
37. K. Schoutens, $O(n)$ extended superconformal field theory in superspace. *Nucl. Phys. B* **295**, 634 (1988). [https://doi.org/10.1016/0550-3213\(88\)90539-1](https://doi.org/10.1016/0550-3213(88)90539-1)
38. N. Saulina, Geometric interpretation of the large $N = 4$ index. *Nucl. Phys. B* **706**, 491 (2005). <https://doi.org/10.1016/j.nuclphysb.2004.11.049>. [arXiv:hep-th/0409175](https://arxiv.org/abs/hep-th/0409175)
39. P. Bouwknegt, K. Schoutens, W symmetry in conformal field theory. *Phys. Rep.* **223**, 183 (1993). [https://doi.org/10.1016/0370-1573\(93\)90111-P](https://doi.org/10.1016/0370-1573(93)90111-P). [arXiv:hep-th/9210010](https://arxiv.org/abs/hep-th/9210010)
40. E. van Beveren, Lecture note of group theory. <http://cft.fis.uc.pt/eef/evbggroups.pdf>
41. M. Beccaria, C. Candu, M.R. Gaberdiel, The large $N = 4$ superconformal W_∞ algebra. *JHEP* **1406**, 117 (2014). [https://doi.org/10.1007/JHEP06\(2014\)117](https://doi.org/10.1007/JHEP06(2014)117). [arXiv:1404.1694](https://arxiv.org/abs/1404.1694) [hep-th]
42. L. Eberhardt, M.R. Gaberdiel, I. Rienacker, Higher spin algebras and large $\mathcal{N} = 4$ holography. *JHEP* **1803**, 097 (2018). [https://doi.org/10.1007/JHEP03\(2018\)097](https://doi.org/10.1007/JHEP03(2018)097). [arXiv:1801.00806](https://arxiv.org/abs/1801.00806) [hep-th]
43. C. Ahn, Higher spin currents with manifest $SO(4)$ symmetry in the large $\mathcal{N} = 4$ holography. *Int. J. Mod. Phys. A* **33**(35), 1850208 (2018). <https://doi.org/10.1142/S0217751X18502081>. [arXiv:1805.02298](https://arxiv.org/abs/1805.02298) [hep-th]
44. C. Ahn, Wolf space coset spectrum in the large $\mathcal{N} = 4$ holography. *J. Phys. A* **51**(43), 435402 (2018). <https://doi.org/10.1088/1751-8121/aae15d>. [arXiv:1711.07599](https://arxiv.org/abs/1711.07599) [hep-th]
45. C. Ahn, H. Kim, Three point functions in the large $\mathcal{N} = 4$ holography. *JHEP* **1510**, 111 (2015). [https://doi.org/10.1007/JHEP10\(2015\)111](https://doi.org/10.1007/JHEP10(2015)111). [arXiv:1506.00357](https://arxiv.org/abs/1506.00357) [hep-th]
46. C. Ahn, H. Kim, Spin-5 Casimir operator its three-point functions with two scalars, *JHEP* **1401**, 012 (2014) (Erratum: [*JHEP* **1401**,

- 174 (2014)). [https://doi.org/10.1007/JHEP01\(2014\)012](https://doi.org/10.1007/JHEP01(2014)012). [https://doi.org/10.1007/JHEP01\(2014\)174](https://doi.org/10.1007/JHEP01(2014)174). [arXiv:1308.1726](https://arxiv.org/abs/1308.1726) [hep-th]
47. C. Ahn, The coset spin-4 casimir operator and its three-point functions with scalars. JHEP **1202**, 027 (2012). [https://doi.org/10.1007/JHEP02\(2012\)027](https://doi.org/10.1007/JHEP02(2012)027). [arXiv:1111.0091](https://arxiv.org/abs/1111.0091) [hep-th]
48. K. Ferreira, M.R. Gaberdiel, The so -Kazama-Suzuki models at large level. JHEP **1504**, 017 (2015). [https://doi.org/10.1007/JHEP04\(2015\)017](https://doi.org/10.1007/JHEP04(2015)017). [arXiv:1412.7213](https://arxiv.org/abs/1412.7213) [hep-th]