



The Dirac equation in a class of topologically trivial flat Gödel-type space-time backgrounds

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Abstract In this work, we investigate the behaviour of Dirac particles in a class of Gödel-type space-time backgrounds in the presence of non-minimal coupling of the gravitational field with background curvature. We obtain the allowed energies for this relativistic system by solving analytically the Dirac equation in flat and curved space in a topologically trivial flat Gödel-type metric, and analyze the effects on the energy eigenvalues.

1 Introduction

Working on the basis of the Dirac and Klein–Gordon equations, spin-zero and spin-half particles have been extensively discussed to calculate the eigenvalues and corresponding wave functions via many analytical and numerical techniques. The particle motions are commonly described using either the Klein–Gordon or the Dirac equation [1, 2] depending on the spin character of the particle. The spin-zero particles like mesons are described by the Klein–Gordon equation and spin-half particles such as electron is described satisfactorily by the Dirac equation. The relativistic equations contain two objects : the four-vector linear momentum operator and the scalar rest mass. The Dirac and Klein–Gordon equations have been of interest for theoretical physicists in many branches of physics [3, 4]. Since the exact solutions of the Klein–Gordon and Dirac equations play an important role in relativistic quantum Physics as well as in various physical applications including those in nuclear and high energy physics [5, 6]. Thus there has been an increased interest in finding the exact solutions to these equations with physical potentials of various kind such as linear scalar, vector, Coulomb, Cornell, inverse square, Hulthén, harmonic oscillator-like, electromagnetic, Hartmann potentials etc. (e. g. [7–24]) using different methods. These method includes the Nikiforov–Uvarov method [25–

29], super-symmetry (SUSY) [30], Super-symmetric WKB approach [31], functional analysis method [32], Variational method [33], Asymptotic Iteration method [34–39] etc. To have a real bound state, the relationship between vector and scalar potential must be $S(r) \geq V(r)$ [40]. In recent years, many studies have been carried out to explore the relativistic energy eigenvalues and corresponding wave functions of the relativistic wave equations (e. g. [41–59]).

The investigations of relativistic quantum effects on scalar and spin-half particles in Gödel-type space-times have been addressed by several authors. Figueiredo et al. [60] first studied the Klein–Gordon and Dirac equations in Gödel-type space-times with positive, negative and zero curvatures of Gödel-type space-time. Furthermore, the investigation of the close relationship between the quantum dynamics of a scalar particle in backgrounds of the general relativity with Gödel-type solutions and the Landau levels in the different class of Gödel-type space-times such as flat, spherical and hyperbolic spaces, were studied in [61]. The authors there analyzed the similarity between the spectrum of energy of a scalar quantum particle in the known classes of Gödel-type space-times and the Landau levels in curved background. This similarity has also been investigated in Ref. [62] by solving the Klein–Gordon equation in Som–Raychaudhuri space-time (called flat Gödel-type metrics). The scalar quantum particle in a class of Gödel-type metrics with a cosmic string passing through the space-time has been investigated in Ref. [63]. The relativistic quantum dynamics of a scalar particle in a class of topologically trivial flat Gödel space-time, was investigated in Ref. [64]. The result obtained there suggest that the energy eigenvalues depend on the vorticity parameter Ω characterising the space-time. The linear and Coulomb confinement of a scalar particles in a class of topologically trivial flat Gödel space-time, was investigated in Ref. [65]. A Scalar quantum particle in the backgrounds of (1+2)-dimensions Gürses space-time, was investigated in Ref. [66] and analyzed the effects on eigen-

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values. The Klein–Gordon equation for the Klein–Gordon oscillator in the backgrounds of (1+2)-dimensions Gürses space-time, was investigated in Ref. [67]. In addition, the Klein–Gordon equation for the generalised Klein–Gordon oscillator subject to Coulomb-like scalar potential in the backgrounds of (1 + 2)-dimensions Gürses space-time, was investigated in Ref. [68]. There we have obtained compact expression of energy eigenvalues and wave functions, in details.

The Dirac equation for a free particle of mass m in flat space is given by

$$(i \gamma^\mu \partial_\mu - m) \Psi = 0 \quad (\hbar = 1 = c), \quad (1)$$

where $\gamma^\mu = \gamma^a e_a^\mu$ ($a = 0, 1, 2, 3$) are the generalized gamma matrices which satisfies the following relation

$$\{\gamma^\mu(\mathbf{x}), \gamma^\nu(\mathbf{x})\} = 2 g^{\mu\nu}(\mathbf{x}). \quad (2)$$

The gamma (γ^a) matrices in flat space are

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \quad (3)$$

The general class of Gödel-type metrics without a cosmic string in polar coordinates (t, r, ϕ, z) can be written as [69–72]

$$ds^2 = -(dt + F(r) d\phi)^2 + H^2(r) d\phi^2 + dr^2 + dz^2. \quad (4)$$

The necessary and sufficient conditions for the Gödel-type solution (4) to be space-time homogeneous are given by [72, 73]

$$\frac{F'}{H} = 2\Omega, \quad \frac{H''}{H} = 4l^2, \quad (5)$$

where the prime denotes derivative with respect r , and the parameters (Ω, μ) are constants such that $\Omega^2 > 0$ and $-\infty \leq l^2 \leq \infty$. The variables (t, r, ϕ, z) can take, respectively the values : $-\infty < (t, z) < \infty, 0 \leq r, \phi \in [0, 2\pi)$. The parameter Ω characterizes the vorticity of the space-time. The Gödel-type geometries can be grouped into the following three classes.

Case 1. Hyperbolic : $l^2 = const > 0$.

In that case, the functions $F(r)$ and $H(r)$ can be express as

$$F(r) = \frac{\Omega}{l^2} \sinh^2(lr), \quad H(r) = \frac{1}{2l} \sinh(2lr). \quad (6)$$

The original Gödel solution [74] belong to this class and can be recovered for $l^2 = \frac{\Omega^2}{2}$.

Case 2. Trigonometric or spherical : $l^2 = -\mu^2 < 0$.

In that case, the functions $F(r)$ and $H(r)$ are can be express as

$$F(r) = \frac{\Omega}{\mu^2} \sin^2(\mu r), \quad H(r) = \frac{1}{2\mu} \sin(2\mu r). \quad (7)$$

Case 3. Linear or flat : $l^2 = 0$.

In that case, the functions $F(r)$ and $H(r)$ can be express as

$$F(r) = \Omega r, \quad H(r) = r. \quad (8)$$

The well-known Som–Raychaudhuri space-time [75] belong to this class of Gödel-type metrics which is also called flat or linear class of Gödel-type metrics. So the Som–Raychaudhuri metric can be written as

$$ds^2 = -(dt + \Omega r^2 d\phi)^2 + r^2 d\phi^2 + dr^2 + dz^2. \quad (9)$$

This solution attracted much attention in string theory and can be interpreted as flat Gödel-type solution. In Cartesian coordinate system, the Som–Raychaudhuri space-time can be written as

$$ds_{SR}^2 = -[dt + \Omega(x dy - y dx)]^2 + \delta_{ij} dx^i dx^j, \quad (10)$$

by transforming $r \rightarrow \sqrt{x^2 + y^2}$ and $\phi \rightarrow \tan^{-1}(\frac{y}{x})$ into (9).

In the present work, we are mainly interested on third case, that is, a flat or linear class of Gödel-type space-times. We investigate the behaviour of a Dirac particle in a class of flat Gödel-type space-time in the presence of non-minimal coupling of the gravitational field with the background curvature, and analyze the effects on energy eigenvalues. It is worth mentioning the studies of relativistic wave equations such as the Klein–Gordon and Dirac equation in all three classes of Gödel-type geometries with or without a cosmic string have been investigated by several authors. They obtained the solutions of the wave equations in all three classes of Gödel-type space-times (see, e. g., Refs. [63–65, 69–71, 76–79]) and observed the similarity of the energy eigenvalues with the Landau levels in flat space [61, 62, 80, 81]. The obtained eigenvalues of energy in these different classes of

Gödel-type space-times are found different and the results are enough significant [71,76]. Other works are the quantum dynamics of Klein–Gordon scalar field subject to Cornell potential [82], survey on the Klein–Gordon equation in a class of Gödel-type space-times [83], the Dirac–Weyl equation in graphene under a magnetic field [84], effects of cosmic string framework on thermodynamical properties of anharmonic oscillator [85], study of bosons for three special limits of Gödel-type space-times [86], the Klein–Gordon oscillator in the presence of Cornell potential in the cosmic string space-time [87], the covariant Duffin–Kemmer–Petiau (DKP) equation in the cosmic-string space-time with interaction of a DKP field with the gravitational field produced by topological defects investigated in [88], the Klein–Gordon field in spinning cosmic string space-time with the Cornell potential [89], the relativistic spin-zero bosons in a Som–Raychaudhuri space-time investigated in [90], investigation of the Dirac equation using the conformable fractional derivative [91], effect of the Wigner–Dunkl algebra on the Dirac equation and Dirac harmonic oscillator investigated in [92], investigation of the relativistic dynamics of a Dirac field in the Som–Raychaudhuri space-time, which is described by Gödel-type metric and a stationary cylindrical symmetric solution of Einstein’s field equations for a charged dust distribution in rigid rotation [93], investigation of relativistic free bosons in the Gödel-type space-times [94], investigation of relativistic quantum dynamics of a DKP oscillator field subject to a linear interaction in cosmic string space-time to understand the effects of gravitational fields produced by topological defects on the scalar field [95], the behaviour of relativistic spin-zero bosons in the space-time generated by a spinning cosmic string investigated in [96], relativistic spin-0 system in the presence of a Gödel-type background space-time investigated in [97], study of the Duffin–Kemmer–Petiau (DKP) equation for spin-zero bosons in the space-time generated by a cosmic string subject to a linear interaction of a DKP field with gravitational fields produced by topological defects investigated in [98], the information-theoretic measures of (1 + 1)-dimensional Dirac equation in both position and momentum spaces are investigated for the trigonometric Rosen–Morse and the Morse potentials investigated in [99], analytical bound and scattering state solutions of Dirac equation for the modified deformed Hylleraas potential with a Yukawa-type tensor interaction investigated in [100], investigation of the Dirac equation subject to a vector and scalar potentials in the space-time generated by a cosmic string investigated in [101], and the covariant Dirac equation in the space-time generated by a cosmic string in presence of vector and scalar potentials of electromagnetic field investigated in [102] and many more.

2 An example of a class of flat Gödel-type space-time

Consider the following stationary space-time [103] (see also Refs. [64,65]) in the coordinates $(x^0 = t, x^1 = x, x^2 = y, x^3 = z)$ given by

$$ds^2 = -dt^2 + dx^2 + (1 - \alpha_0^2 x^2) dy^2 - 2\alpha_0 x dt dy + dz^2 = -[dt + H(x) dy]^2 + dx^2 + D^2(x) dy^2 + dz^2, \tag{11}$$

where $\alpha_0 > 0$ is a real number, and the function $H(x) = \alpha_0 x$ and $D(x) = 1$. The ranges of the coordinates are

$$-\infty < t < \infty, \quad -\infty < x < \infty, \quad -\infty < y < \infty, \quad -\infty < z < \infty. \tag{12}$$

Using the conditions given by Eq. (5) into the metric (12), we have

$$\frac{H'}{D} = \alpha_0 = 2\Omega, \quad \frac{D''}{D} = 0, \tag{13}$$

where prime denotes ordinary differentiation w. r. t. x . From the above condition Eq. (13), it is clear that the study space-time (11) belong to a linear or flat class of Gödel-type metrics.

The determinant of the metric tensor $\det(g_{\mu\nu}) = -1$ with the scalar curvature $R = 2\Omega^2$ and the metric tensor for the space-time (11) to be

$$g_{\mu\nu}(\mathbf{x}) = \begin{pmatrix} -1 & 0 & -\alpha_0 x & 0 \\ 0 & 1 & 0 & 0 \\ -\alpha_0 x & 0 & 1 - \alpha_0^2 x^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{14}$$

with its inverse

$$g^{\mu\nu}(\mathbf{x}) = \begin{pmatrix} \alpha_0^2 x^2 - 1 & 0 & -\alpha_0 x & 0 \\ 0 & 1 & 0 & 0 \\ -\alpha_0 x & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{15}$$

Using the definition of $e_{(a)}^\mu$ and $e_\mu^{(a)}$ in space-time (11) and $\alpha_0 = 2\Omega$, we have

$$e_{(a)}^\mu(\mathbf{x}) = \begin{pmatrix} 1 & 0 & \alpha_0 x & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{16}$$

$$e_\mu^{(a)}(\mathbf{x}) = \begin{pmatrix} 1 & 0 & -\alpha_0 x & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{17}$$

which must satisfy

$$e_{(a)}^\mu(\mathbf{x}) e_{(b)}^\nu(\mathbf{x}) = \delta_b^\mu, \quad e_{(a)}^\mu(\mathbf{x}) e_{(b)}^\nu(\mathbf{x}) = \delta_b^a, \\ g_{\mu\nu}(\mathbf{x}) = e_{(a)}^\mu(\mathbf{x}) e_{(b)}^\nu(\mathbf{x}) \eta_{(a)(b)}, \tag{18}$$

where $\eta_{(a)(b)} = \text{diag}(-1, 1, 1, 1)$ is the Minkowski flat space metric.

The next step is a derivation of the Christoffel symbols for the study space-time. It is define by

$$\Gamma_{\mu\nu}^\lambda = \frac{1}{2} g^{\lambda\sigma} (g_{\nu\sigma,\mu} - g_{\mu\sigma,\nu} - g_{\mu\nu,\sigma}), \tag{19}$$

where comma denotes ordinary derivative. For the space-time (11), we get the following non-zero components of the Christoffel symbols

$$\Gamma_{01}^0 = \Gamma_{10}^0 = \frac{\alpha_0^2 x}{2}, \\ \Gamma_{12}^0 = \Gamma_{21}^0 = \frac{\alpha_0}{2} (1 + \alpha_0^2 x^2), \\ \Gamma_{02}^1 = \Gamma_{20}^1 = \frac{\alpha_0}{2} = -\Gamma_{01}^2 = -\Gamma_{10}^2, \\ \Gamma_{22}^1 = \alpha_0^2 x, \quad \Gamma_{12}^2 = \Gamma_{21}^2 = -\frac{\alpha_0^2 x}{2}. \tag{20}$$

The spin connections can be determined using Christoffel symbols with the definition

$$\omega_{\mu(a)(b)}(\mathbf{x}) = \eta_{(a)(c)} e_{(b)}^{(c)} e_{(b)}^\tau e_{\tau\mu}^\nu - \eta_{(a)(c)} e_{(b)}^\nu \partial_\mu e_{(b)}^{(c)}. \tag{21}$$

And these are

$$\omega_t(\mathbf{x}) = \frac{\alpha_0}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ \omega_x(\mathbf{x}) = -\frac{3\alpha_0}{2} \begin{pmatrix} \alpha_0 x & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ \omega_y(\mathbf{x}) = \alpha_0 \begin{pmatrix} 0 & -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & \alpha_0 x & 0 \\ 0 & -\alpha_0 x & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ \omega_z(\mathbf{x}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{22}$$

The spinorial affine connection $\Gamma_\mu(\mathbf{x})$, according to the definition $\Gamma_\mu(\mathbf{x}) = -\frac{1}{8} \omega_{\mu ab}(\mathbf{x}) [\gamma^a, \gamma^b]$, can be identified as

$$\Gamma_t(\mathbf{x}) = \frac{i\alpha_0}{4} \begin{pmatrix} \sigma^3 & \mathbf{0} \\ \mathbf{0} & \sigma^3 \end{pmatrix}, \\ \Gamma_x(\mathbf{x}) = \frac{3\alpha_0}{4} \begin{pmatrix} \mathbf{0} & \sigma^2 \\ \sigma^2 & \mathbf{0} \end{pmatrix},$$

$$\Gamma_y(\mathbf{x}) = \frac{\alpha_0}{4} \begin{pmatrix} 2i\alpha_0 x \sigma^3 & \sigma^1 \\ \sigma^1 & 2i\alpha_0 x \sigma^3 \end{pmatrix}, \\ \Gamma_z(\mathbf{x}) = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}. \tag{23}$$

The generalized gamma matrices $\gamma^\mu(\mathbf{x}) = e_{(a)}^\mu(\mathbf{x}) \gamma^a$ in curved space-time are

$$\gamma^t(\mathbf{x}) = \begin{pmatrix} \mathbf{I} & -\alpha_0 x \sigma^2 \\ \alpha_0 x \sigma^2 & -\mathbf{I} \end{pmatrix}, \\ \gamma^x(\mathbf{x}) = \begin{pmatrix} \mathbf{0} & \sigma^1 \\ -\sigma^1 & \mathbf{0} \end{pmatrix}, \\ \gamma^y(\mathbf{x}) = \begin{pmatrix} \mathbf{0} & \sigma^2 \\ -\sigma^2 & \mathbf{0} \end{pmatrix}, \\ \gamma^z(\mathbf{x}) = \begin{pmatrix} \mathbf{0} & \sigma^3 \\ -\sigma^3 & \mathbf{0} \end{pmatrix}. \tag{24}$$

Here $\mathbf{0}, \mathbf{I}$ are 2×2 null and unit matrices and σ^i are the Pauli matrices given by

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{25}$$

2.1 The Dirac equation in flat space

If one includes non-minimal coupling of the gravitational field with the background curvature, then the Dirac equation becomes

$$(i \gamma^\mu(\mathbf{x}) \partial_\mu - \xi R - m) \Psi = 0, \tag{26}$$

where R is the scalar curvature and ξ is the non-minimal coupling constant.

The z-translation symmetry of Eq. (10) allows us to reduce the four-component Dirac Eq. (26) to two two-component spinor equations. For the positive energy solution we can choose the following ansatz

$$\Psi(t, x, y, z) = e^{i(-Et + ly + kz)} \begin{pmatrix} \psi(x) \\ \chi(x) \end{pmatrix}, \tag{27}$$

where $E = i \partial_t$ is the total energy, $l = -i \partial_y$, and $k = -i \partial_z$ are constants. Substituting Eq. (27) into the Eq. (26), we have the following first order differential equations

$$(E - \bar{m}) \psi = [(\alpha_0 x E + l) \sigma^2 + k \sigma^3] \chi - i \sigma^1 \chi', \tag{28}$$

$$(E + \bar{m}) \chi = [(\alpha_0 x E + l) \sigma^2 + k \sigma^3] \psi - i \sigma^1 \psi', \tag{29}$$

where $\bar{m} = m + \frac{\xi \alpha_0^2}{2}$ and prime denotes derivative w. r. to x .

Now substituting Eq. (29) into the Eqs. (28) and (28) into the Eq. (29) and after decoupled, we get the following second order differential equation :

$$\frac{d^2 \psi_i}{dx^2} + [\lambda - \beta^2 x^2 - \eta x] \psi_i = 0, \quad i = 1, 2, \tag{30}$$

where $\psi_1 = \psi, \psi_2 = \chi$ and

$$\lambda = E^2 - \bar{m}^2 - \alpha_0 E s - k^2 - l^2, \quad \beta = \alpha_0 E, \quad \eta = 2\beta l. \tag{31}$$

Note that $\psi(x)$ and $\chi(x)$ is an eigenfunction of σ^3 with eigenvalues ± 1 , so we can write $\psi_s = (\psi_+, \psi_-)^T$ with $\sigma^3 \psi_s = s \psi_s, s = \pm 1$. This is the reason why we replaced σ^3 by s in Eqs. (28)–(29) and obtained Eqs. (30)–(31) (see, refs. [93,104,105]).

Transforming a new variable $r = \sqrt{\beta} x$ into the equation (30), we get

$$\psi_i''(r) + \left[\frac{\lambda}{\beta} - r^2 - \frac{\eta}{\beta^{\frac{3}{2}}} r \right] \psi_i(r) = 0. \tag{32}$$

The asymptotic behaviour of the possible solution to the Eq. (32) are to be determined for $r \rightarrow 0$ and $r \rightarrow \infty$. These conditions are necessary since the wave functions must be well-behaved in this limits, and thus, bound states of energy eigenvalues can be obtained. Let us impose requirement that the function $\psi_i(r)$ is well-behaved at $r \rightarrow 0$ and vanish at $r \rightarrow \infty$. Let the solution to Eq. (32) is given by

$$\psi_i(r) = r^A e^{-(Br+Dr^2)} H(r). \tag{33}$$

Substituting the above solution (33) into the Eq. (32), we get

$$\begin{aligned} H''(r) + \left[\frac{2A}{r} - 2B - 4Dr \right] H'(r) \\ + \left[\frac{A^2 - A}{r^2} - \frac{2AB}{r} - 4AD - 2D \right. \\ \left. + \frac{\lambda}{\beta} + B^2 + \left(4BD - \frac{\eta}{\beta^{\frac{3}{2}}} \right) r + (4D^2 - 1)r^2 \right] H(r) = 0. \end{aligned} \tag{34}$$

Equating the coefficients of r^{-2}, r, r^2 equals to zero in the above differential equation, we get

$$\begin{aligned} A^2 - A = 0 &\Rightarrow A = 1, \quad A \neq 0, \\ 4BD - \frac{\eta}{\beta^{\frac{3}{2}}} = 0 &\Rightarrow B = \frac{1}{4D} \frac{\eta}{\beta^{\frac{3}{2}}}, \\ 4D^2 - 1 = 0 &\Rightarrow D = \frac{1}{2}. \end{aligned} \tag{35}$$

With these the above Eq. (34) can be express as

$$H''(r) + \left[\frac{\gamma}{r} - \zeta - \delta r \right] H'(r) + \left[-\frac{q}{r} + \theta \right] H(r) = 0, \tag{36}$$

where

$$\begin{aligned} \gamma &= 2A, \\ \zeta &= 2B, \end{aligned}$$

$$\begin{aligned} \delta &= 4D, \\ q &= 2AB, \\ \theta &= B^2 + \frac{\lambda}{\beta} - 4AD - 2D. \end{aligned} \tag{37}$$

Equation (36) is the biconfluent Heun's differential equation [106–113] and $H(r)$ is the Heun polynomials.

Equation (36) can be easily solve by using the Frobenius method as follow :

$$H(r) = \sum_{i=0}^{\infty} c_i r^i. \tag{38}$$

Substituting Eq. (38) into the Eq. (36), we get the following recurrence relation for the coefficient:

$$c_{n+2} = \frac{1}{(n+2)(n+1+\gamma)} [(q+\zeta(n+1))c_{n+1} - (\theta-2n)c_n]. \tag{39}$$

And the various coefficients are

$$c_1 = \frac{q}{\gamma} c_0, \quad c_2 = \frac{1}{2(1+\gamma)} [(q+\zeta)c_1 - \theta c_0]. \tag{40}$$

The power series becomes a polynomial of degree n by imposing the following two conditions [106–113]

$$c_{n+1} = 0, \quad (\theta - 2n) = 0, \quad n = 1, 2, \dots \tag{41}$$

Using the above energy quantization condition we get the following eigenvalues equation :

$$B^2 + \frac{\lambda}{\beta} - 4AD - 2D = 2n. \tag{42}$$

Substituting A, B, D into the above eigenvalue equation, we get the following eigenvalues of energy

$$\begin{aligned} E_n &= \frac{1}{2} [\alpha_0(2n+3+s) \\ &\pm \sqrt{\alpha_0^2(2n+3+s)^2 + 4 \left\{ k^2 + \left(m + \frac{\xi \alpha_0^2}{2} \right)^2 \right\}}] \\ &= \Omega(2n+3+s) \\ &\pm \sqrt{\Omega^2(2n+3+s)^2 + k^2 + (m+2\xi\Omega^2)^2}. \end{aligned} \tag{43}$$

The corresponding eigen functions

$$\psi_{in}(r) = r e^{-\frac{|l|}{\sqrt{2\Omega E_n}} r} e^{-\frac{r^2}{2}} H(r), \tag{44}$$

where $l = 0, \pm 1, \pm 2, \dots \in \mathbf{Z}$ is integer.

In the absence of non-minimal coupling constant $\xi = 0$, we have the following energy eigenvalues

$$E_n = \Omega (2n+3+s) \pm \sqrt{\Omega^2 (2n+3+s)^2 + k^2 + m^2}. \quad (45)$$

For $m = 0 = k$ and considering the non-minimal coupling constant ($\xi \neq 0$), the energy eigenvalues becomes

$$E_n = \Omega [(2n+3+s) \pm \sqrt{(2n+3+s)^2 + 4\xi^2 \Omega^2}]. \quad (46)$$

which shows that the presence of the term $4\xi^2 \Omega^2$ causes an asymmetry in the energy levels. The energy levels are equally spaced by 4Ω in absence of the non-minimal coupling constant ($\xi = 0$), and the allowed energies are not analogous to the relativistic Landau levels for a Dirac particle in a cosmic string space-time [106].

In Ref. [104], the energy eigenvalues of Dirac oscillator in the cosmic string space-time is given by

$$E^2 = M^2 + 4M\omega \left\{ n + \left| \frac{l + \frac{1}{2}(1 \mp \alpha)}{2\alpha} \right| - \left[\frac{l + \frac{1}{2}(1 \mp \alpha)}{2\alpha} \right] + \frac{1-s}{2} \right\} + k^2, \quad (47)$$

where $s = \pm 1$, and ω is the oscillator frequency.

In Ref. [105], the energy eigenvalue equation of Dirac oscillator in the spinning cosmic string space-time is given by

$$E^2 - M^2 = 4M\omega \left[n + \frac{1}{2} \left| \frac{m}{\alpha} + \frac{1}{2\alpha} (1-\alpha)s + \frac{aE}{\alpha} \right| - \frac{s}{2} \left(\frac{m}{\alpha} + \frac{1}{2\alpha} (1-\alpha)s + \frac{aE}{\alpha} \right) \right]. \quad (48)$$

By imposing different condition, the authors there obtained the various energy eigenvalues of the system (Please see Eqs. (46)–(50) in Ref. [105]).

For $\alpha_0 = 0$, the space-time (8) reduces to Minkowski metric

$$ds^2 = -dt^2 + \delta_{ij} dx^i dx^j. \quad (49)$$

Therefore, we have the following differential equations

$$\frac{d^2 \psi_i}{dx^2} + (E^2 - k^2 - l^2 - m^2) \psi_i = 0, \quad i = 1, 2, \quad (50)$$

which is the Schrödinger time-independent wave equation with zero potential and whose solution are well-known in quantum mechanics. The energy eigenvalues are given by

$$E_n^2 = k^2 + l^2 + m^2 + \frac{n^2 \pi^2}{L^2}, \quad n = 1, 2, 3 \dots \quad (51)$$

which are not the same the result obtained for the Dirac oscillator in the Minkowski space-time in cylindrical symmetry system with a cosmic string [77] for zero oscillator frequency. Here L is the length of a box where the particle is assuming moves in 1-d with zero potential inside it.

2.2 The Dirac equation in curved space

For the Dirac equation in curved space, partial derivatives ∂_μ are replaced by the covariant derivatives $\nabla_\mu = \partial_\mu + \Gamma_\mu$ where, Γ_μ are called the spinorial affine connection. Therefore the Dirac equation in curved space with non-minimal coupling of the field becomes

$$[i \gamma^\mu(\mathbf{x}) (\partial_\mu + \Gamma_\mu(\mathbf{x})) - m - \xi R] \Psi = 0, \quad (52)$$

where we have derived $\gamma^\mu(\mathbf{x})$ and $\Gamma_\mu(\mathbf{x})$ earlier. We have

$$i \gamma^\mu(\mathbf{x}) \Gamma_\mu(\mathbf{x}) = \frac{\alpha_0}{4} \begin{pmatrix} -3\sigma^3 & -i\alpha_0 x \sigma^1 \\ i\alpha_0 x \sigma^1 & 3\sigma^3 \end{pmatrix}. \quad (53)$$

Choosing the wave functions given by ansatz (27) into the Eq. (52), we get the following first order differential equations :

$$\begin{aligned} & \left(E - \bar{m} - \frac{3\alpha_0}{4} \sigma^3 \right) \psi \\ & = \left[(\alpha_0 E x + l) \sigma^2 + k \sigma^3 + i \frac{\alpha_0^2 x}{4} \sigma^1 \right] \chi - i \sigma^1 \chi', \end{aligned} \quad (54)$$

$$\begin{aligned} & \left(E + \bar{m} - \frac{3\alpha_0}{4} \sigma^3 \right) \chi \\ & = \left[(\alpha_0 E x + l) \sigma^2 + k \sigma^3 + i \frac{\alpha_0^2 x}{4} \sigma^1 \right] \psi - i \sigma^1 \psi'. \end{aligned} \quad (55)$$

Substituting Eq. (55) into the Eq. (54) and (54) into the Eq. (55) and after decoupled, we get the following second order differential equations :

$$\psi_i''(x) - \frac{\alpha_0^2}{2} x \psi_i' + [\lambda_0 - \bar{\omega}^2 x^2 - \eta x] \psi_i = 0, \quad (56)$$

where

$$\begin{aligned} \lambda_0 &= E^2 - \bar{m}^2 - \frac{5}{2} \alpha_0 E s - k^2 - l^2 + \frac{5}{16} \alpha_0^2, \\ \bar{\omega}^2 &= \beta^2 - \frac{\alpha_0^4}{16}. \end{aligned} \quad (57)$$

Note that here we have used

$$\begin{aligned} \left(E - \frac{3\alpha_0}{4}\sigma^3\right)^2 \psi_s &= E^2 \psi_s - \frac{3}{2}\alpha_0 E \sigma^3 \psi_s + \frac{9}{16}\alpha_0^2 \psi_s \\ &= \left(E^2 - \frac{3}{2}\alpha_0 E s + \frac{9}{16}\alpha_0^2\right) \psi_s. \end{aligned} \tag{58}$$

where $\sigma^3 \psi_s = \pm 1 \psi_s = s \psi_s, s = \pm 1$ (see, Refs. [93, 104, 105]).

Now we do the following transformation

$$\psi_i(x) = e^{\frac{\alpha_0^2}{8}x^2} \Phi(x). \tag{59}$$

into the Eq. (56), we get

$$\Phi''(x) + [\tilde{\lambda} - \beta^2 x^2 - \eta x] \Phi(x) = 0, \tag{60}$$

where

$$\tilde{\lambda} = \lambda_0 + \frac{1}{4}\alpha_0^2. \tag{61}$$

Replacing the variable $r = \sqrt{\beta} x$ into the Eq. (60), we get

$$\Phi''(r) + \left[\frac{\tilde{\lambda}}{\beta} - r^2 - \frac{\eta}{\beta^{\frac{3}{2}}}r\right] \Phi(r) = 0. \tag{62}$$

Let us impose the requirement that the function $\Phi(r)$ is well-behaved at $r \rightarrow 0$ and vanish at $r \rightarrow \infty$. Suppose the solution to Eq. (62) is given by

$$\Phi(r) = r^A e^{-(Br+Dr^2)} H(r), \tag{63}$$

where $H(r)$ is an unknown function. Substituting the above solution into the Eq. (62), we get

$$\begin{aligned} H''(r) + \left[\frac{2A}{r} - 2B - 4Dr\right] H'(r) \\ + \left[\frac{A^2 - A}{r} - \frac{2AB}{r} - 4AD - 2D \right. \\ \left. + \frac{\tilde{\lambda}}{\beta} + B^2 + \left(4BD - \frac{\eta}{\beta^{\frac{3}{2}}}\right)r + (4D^2 - 1)r^2\right] H(r) = 0, \end{aligned} \tag{64}$$

Equating the coefficients of r^{-2}, r, r^2 equals to zero into the above equation, we get

$$\begin{aligned} A^2 - A = 0 &\Rightarrow A = 1, \quad A \neq 0, \\ 4D^2 - 1 = 0 &\Rightarrow D = \frac{1}{2}, \\ 4BD - \frac{\eta}{\beta^{\frac{3}{2}}} = 0 &\Rightarrow B = \frac{1}{4D} \frac{\eta}{\beta^{\frac{3}{2}}}. \end{aligned} \tag{65}$$

With these the above Eq. (64) can be express as

$$H''(r) + \left[\frac{\gamma}{r} - \zeta - \delta r\right] H'(r) + \left[-\frac{q}{r} + \theta_0\right] H(r) = 0. \tag{66}$$

where

$$\theta_0 = B^2 + \frac{\tilde{\lambda}}{\beta} - 2D - 4AD. \tag{67}$$

Equation (66) is the bi-confluent Heun's differential equation [106–113] and $H(r)$ is the Hen polynomials.

The above Eq. (66) can be solved using the power series method as done in the previous section. Substituting the power series solution (38) into the Eq. (66), we get the following recurrence relation for the coefficients :

$$\begin{aligned} c_{n+2} &= \frac{1}{(n+2)(n+1+\gamma)} [(q + \zeta(n+1)) \\ &\quad c_{n+1} - (\theta_0 - 2n)c_n]. \end{aligned} \tag{68}$$

And the various coefficients are

$$c_1 = \frac{q}{\gamma} c_0, \quad c_2 = \frac{1}{2(1+\gamma)} [(q + \zeta)c_1 - \theta_0 c_0]. \tag{69}$$

Bound state solution can be achieved by imposing that the power series expansion becomes a polynomial of degree n . Thereby we guarantee that Φ is finite at the origin $r \rightarrow 0$ and vanish at $r \rightarrow \infty$. Therefore the power series becomes a polynomial of degree n by imposing the following two conditions [106–113]

$$c_{n+1} = 0, \quad \theta_0 = 2n, \tag{70}$$

where $n = 1, 2, 3 \dots$. Hence, there are two conditions that must be satisfied in order that the series terminates. Note that $n = 1, 2, 3 \dots$ is the quantum number associated with the radial modes. By analysing the condition $\theta_0 = 2n$, we obtain

$$B^2 + \frac{\tilde{\lambda}}{\beta} - 2D - 4AD = 2n, \quad n = 1, 2 \dots \tag{71}$$

Substituting various term into the above equation, we get the following energy eigenvalues E_n as

$$\begin{aligned} E^2 - \bar{m}^2 - \frac{5}{2}\alpha_0 E s - k^2 + \frac{9}{16}\alpha_0^2 &= (2n+3)\alpha_0 E \\ \Rightarrow E_n &= \frac{1}{2} \left[\alpha_0 \left(2n+3 + \frac{5}{2}s\right) \right. \\ &\quad \left. \pm \sqrt{\alpha_0^2 \left(2n+3 + \frac{5}{2}s\right)^2 + 4 \left(\bar{m}^2 + k^2 - \frac{9}{16}\alpha_0^2\right)} \right] \end{aligned}$$

$$= \Omega \left(2n + 3 + \frac{5}{2}s \right) \pm \sqrt{\Omega^2 \left(2n + 3 + \frac{5}{2}s \right)^2 + (m + 2\xi \Omega^2)^2 + k^2 - \frac{9}{4} \Omega^2}. \quad (72)$$

Hence, Eq. (72) corresponds to the allowed energies of Dirac particle in a flat Gödel-type space-time in curved space. The corresponding eigen functions are given by

$$\Phi_n(r) = r e^{-\frac{|l|}{\sqrt{2\Omega E_n}} r} e^{-\frac{r^2}{2}} H(r). \quad (73)$$

For $m = 0 = k$, the energy eigenvalue reduces to

$$E_n = \Omega \left[\left(2n + 3 + \frac{5}{2}s \right) \pm \sqrt{\left(2n + 3 + \frac{5}{2}s \right)^2 + 4 \left(\xi^2 \Omega^2 - \frac{9}{16} \right)} \right], \quad n = 1, 2, \dots \quad (74)$$

We have seen in Eq. (74) that the presence of term $4 \left(\xi^2 \Omega^2 - \frac{9}{16} \right)$ causes an asymmetry in the energy levels that are no longer Landau-type energy levels, which differs from the results obtained in Refs. [60, 76].

3 Conclusions

In Ref. [76], the behavior of a Dirac particle in a class of Gödel-type space-time backgrounds with a cosmic string in the Einstein–Cartan theory, were investigated. They obtained the corresponding Dirac equations with a torsion in all three classes of Gödel-type space-time backgrounds, namely, flat or linear case which is also called the Som–Raychaudhuri class of space-time with torsion that contain a cosmic string passing through z -axis, spherical symmetric Gödel-type space-time, and in hyperbolic coordinates and solved them analytically. In the case of Som–Raychaudhuri space-time with a cosmic string, they obtained the allowed energies for this relativistic quantum system and shown an analogy between the relativistic energy levels and the Landau levels, where the rotation plays the role of a uniform magnetic field on the z -direction. In torsion free case, the allowed energies are analogous to the relativistic Landau levels for a Dirac particle in the cosmic string space-time [106], where the parameter associated with rotation plays the role of the cyclotron frequency. The presence of the topological defect breaks the degeneracy of the relativistic energy levels. In addition, they have solved analytically the Dirac equation in presence of topological defects in spherical and hyperbolic Gödel-type space-time with a torsion.

In this article, we have studied the Dirac equation and Dirac oscillator in curved space considering the line element in a class of Gödel-type geometries called flat Gödel-type space-time. We have obtained the allowed energies for this relativistic quantum system in the presence of non-minimal coupling of the gravitational field with the background curvature. In Sect. 2.1, we have studied the Dirac equation in flat space and seen that the allowed energies (43) depend on the vorticity parameter characterizing the space-time, and the non-minimal coupling constant (ξ). The allowed energies of the system are not same to the relativistic Landau levels of a Dirac particle with cosmic string space-time [106]. In the absence of non-minimal coupling constant and $m = 0 = k$, the energy levels are $E_n = 2 \Omega (2n + 3 + s)$, where $n = 1, 2, 3$. In the limit of zero vorticity parameter, $\Omega \rightarrow 0$, the energy levels Eq. (51) are not the same with the relativistic energy levels of bound states for the Dirac oscillator in the Minkowski space-time in cylindrical symmetry system with a cosmic string [77]. In Sect. 2.2, we have solved analytically the Dirac equation in curved space in the presence of non-minimal coupling of the gravitational field with the background curvature. We have obtained the energy eigenvalues of the system by Eq. (72) and seen that the energy levels depend on the vorticity parameter (Ω), and the non-minimal coupling constant (ξ). The obtained energy levels are no longer Landau-type energy levels, differs from the results obtained in Ref. [60] and to the result with cosmic string obtained in Ref. [76].

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