



On-shell renormalization scheme for $\mathcal{N} = 1$ SQED and the NSVZ relation

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Abstract In this paper we investigate the renormalization of $\mathcal{N} = 1$ supersymmetric quantum electrodynamics, regularized by higher derivatives, in the on-shell scheme. It is demonstrated that in this scheme the exact Novikov, Shifman, Vainshtein, and Zakharov (NSVZ) equation relating the β -function to the anomalous dimension of the matter superfields is valid in all orders of the perturbation theory. This implies that the on-shell scheme enters the recently constructed continuous set of NSVZ subtraction schemes. To verify this statement, we compare the anomalous dimension of the matter superfields in the two-loop approximation and the β -function in the three-loop approximation, which are explicitly calculated in this scheme. The finite renormalizations relating the on-shell scheme to some other NSVZ subtraction schemes formulated previously are obtained.

1 Introduction

Among various renormalization schemes that can be used in quantum electrodynamics the subtraction on the mass shell is one of the most important (for a review, see, e.g., [1]). The reason for this is that in this scheme renormalized quantities such as masses, charges, and anomalous magnetic moments can be subjected to direct experimental measurements. This distinguishes it from the (modified) minimal subtraction or momentum subtraction schemes.

The $\mathcal{N} = 1$ supersymmetric generalization of quantum electrodynamics, besides the electron and photon, contains their superpartners, namely, a pair of complex scalar fields and a Majorana spinor. It is most convenient to describe this

theory using the superfield formalism with the gauge fixing term respecting supersymmetry. In this case $\mathcal{N} = 1$ supersymmetry appears to be a manifest symmetry of the theory, so that the perturbative calculations can be done in an $\mathcal{N} = 1$ supersymmetric way. This is to be contrasted with the approach, when the gauge superfield is put into the Wess–Zumino gauge, in which only its physical components survive. Although in this case the quantization is performed in terms of physical fields only, the manifest supersymmetry is lost.

An important feature of $\mathcal{N} = 1$ supersymmetric gauge theories is the existence of the relation between the β -function and the anomalous dimensions of the matter superfields [2–9]. For $\mathcal{N} = 1$ supersymmetric electrodynamics (SQED) considered in this paper it can be written as [10, 11]

$$\beta(\alpha) = \frac{N_f \alpha^2}{\pi} (1 - \gamma(\alpha)). \quad (1)$$

It is known that the NSVZ relation does not in general hold for an arbitrary renormalization prescription¹ and is valid only in certain subtraction schemes called the NSVZ schemes. Recently, it was discovered that all NSVZ subtraction schemes in $\mathcal{N} = 1$ SQED form a class that can be parameterized by a single function and a single constant [14]. Various schemes of this class are related by finite renormalizations satisfying a certain condition, which form a subgroup of the general renormalization group transformations [15–18]. This class in particular includes the so-called HD+MSL renormalization prescription (short for Higher Derivatives plus Minimal Subtraction of Logarithms) [19, 20]. In this case a theory is regularized by the higher covariant derivative method [21, 22] (see also Refs. [23–25] for its various $\mathcal{N} = 1$ supersymmetric versions) and only powers of $\ln \Lambda/\mu$

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¹ The general equations describing how the NSVZ relation changes under finite renormalizations can be found in [12, 13].

are included into the renormalization constants. Here Λ is the dimensionful parameter of the regularized theory, and μ is the subtraction point. Note the HD+MSL prescription gives the NSVZ- and NSVZ-like schemes for various theories, e.g., for the photino mass in the electrodynamics with softly broken supersymmetry [26]² or for the Adler D -function in $\mathcal{N} = 1$ SQCD [30].³ There are indications that the NSVZ scheme in the non-Abelian case is also given by this prescription in all orders [36]. This conjecture has been confirmed by explicit three-loop calculations in [37,38].

The $\overline{\text{DR}}$ scheme, most frequently used for practical calculations, does not enter the class of the NSVZ schemes, as was explicitly demonstrated in the three- [39] and four-loop [40] approximations. Nevertheless, a finite renormalization of the coupling constant, specially tuned in each order of the perturbation theory, allows constructing the NSVZ scheme with dimensional reduction [39,41,42]. The difference between calculations with the higher derivative regularization and with dimensional reduction for $\mathcal{N} = 1$ SQED in the three-loop approximation has been analyzed in Ref. [43].

Because the subtraction on the mass shell occupies a special place in electrodynamics, it would be interesting to find out whether a relation (1) is satisfied in this scheme and, therefore, whether it falls into the class of NSVZ schemes. Using the results of Ref. [44], the guess was made that the NSVZ relation in $\mathcal{N} = 1$ SQED is valid in the on-shell scheme [45]. Note that the explicit calculations in Ref. [44] were done only in the approximation, where the scheme dependence is not essential. In this paper we demonstrate that the NSVZ equation relating the β -function to the mass anomalous dimension is valid in the on-shell scheme in all orders. This statement is verified by the explicit calculation. Namely, the three-loop β -function is compared with the two-loop mass anomalous dimension in the on-shell scheme. This allows to check that Eq. (1) really holds in this case.

2 $\mathcal{N} = 1$ SQED: action and the higher derivative regularization

In the superfield language $\mathcal{N} = 1$ SQED with N_f flavors of massive Dirac fermions and their superpartners is described by the action

$$S = \frac{1}{4e_0^2} \text{Re} \int d^4x d^2\theta W^a W_a$$

² The NSVZ-like equation describing the renormalization of the gaugino mass has been proposed in [27–29].

³ This follows from the fact that the renormalization group functions defined in terms of the bare couplings satisfy the NSVZ and NSVZ-like equation with the higher derivative regularization. At present, it has been rigorously proved in all orders for $\mathcal{N} = 1$ SQED in [31,32], for the renormalization of the photino mass in softly broken SQED in [33], and for the Adler D -function in $\mathcal{N} = 1$ SQCD in [34,35].

$$+ \frac{1}{4} \sum_{i=1}^{N_f} \int d^4x d^4\theta \left(\phi_i^* e^{2V} \phi_i + \tilde{\phi}_i^* e^{-2V} \tilde{\phi}_i \right) + \frac{1}{2} \sum_{i=1}^{N_f} \left(\int d^4x d^2\theta m_0 \tilde{\phi}_i \phi_i + \text{c.c.} \right), \tag{2}$$

where m_0 is the bare mass of the chiral matter superfields. For simplicity, and in order not to deal with multiple thresholds, we assume the masses for different flavors to be equal.

The regularization is introduced by adding to the action the term with higher derivatives

$$S_\Lambda = \frac{1}{4e_0^2} \text{Re} \int d^4x d^2\theta W^a \left[R \left(\frac{\partial^2}{\Lambda^2} \right) - 1 \right] W_a, \tag{3}$$

where R is a function which rapidly increases at large values of the argument and satisfies the condition $R(0) = 1$. Moreover, to regularize divergences in the one-loop approximation, it is necessary to insert the Pauli–Villars determinants in the generating functional [46]. Following Refs. [47,48], let us introduce n sets of the chiral Pauli–Villars superfields Φ_{iI} with masses M_I , where $I = 1, \dots, n$, and include

$$S_{\text{PV}} = \sum_{I=1}^n \left(\frac{1}{4} \sum_{i=1}^{N_f} \int d^4x d^4\theta \left(\Phi_i^* e^{2V} \Phi_i + \tilde{\Phi}_i^* e^{-2V} \tilde{\Phi}_i \right) + \frac{1}{2} \sum_{i=1}^{N_f} \left(\int d^4x d^2\theta M \tilde{\Phi}_i \Phi_i + \text{c.c.} \right) \right)_I \tag{4}$$

into the total action. Then, to cancel one-loop divergences, their Grassmannian parities $(-1)^{P_I}$ and masses M_I should satisfy the relations

$$\sum_{I=1}^n (-1)^{P_I} + 1 = 0; \quad \sum_{I=1}^n (-1)^{P_I} M_I^2 + m_0^2 = 0. \tag{5}$$

In the massless case the masses of the Pauli–Villars superfields M_I should be chosen proportional to the parameter Λ in the higher derivative term. However, in the massive case it is convenient to present them in the form

$$M_I^2 = a_I^2 \Lambda^2 + b_I^2 m_0^2, \tag{6}$$

where the coefficients a_I and b_I , independent of the coupling constant, satisfy the equations

$$\sum_{I=1}^n (-1)^{P_I} a_I^2 = 0; \quad \sum_{I=1}^n (-1)^{P_I} b_I^2 + 1 = 0, \tag{7}$$

which follow from Eq. (5). It should be noted that the derivative of M_I/Λ with respect to $\ln \Lambda$ or $\ln m_0$ is of the order m_0^2/Λ^2 and, therefore, can be neglected in the limit $\Lambda \rightarrow \infty$.

To complete the quantization, the gauge-fixing term

$$S_{\text{gf}} = -\frac{1}{32e_0^2 \xi_0} \int d^4x d^4\theta D^2 V R \left(\frac{\partial^2}{\Lambda^2} \right) \bar{D}^2 V \tag{8}$$

is added to the action. Below we will use the Feynman gauge in which the renormalized gauge fixing parameter is fixed as $\xi = 1$.

3 The on-shell subtraction scheme

To construct the on-shell scheme for $\mathcal{N} = 1$ SQED, let us consider the part of the effective action quadratic in the matter superfields. It can be presented in the form

$$\begin{aligned} \Gamma_\phi^{(2)} = & \frac{1}{4} \sum_{i=1}^{N_f} \int \frac{d^4 p}{(2\pi)^4} d^4 \theta \left(\phi_i^*(-p, \theta) \phi_i(p, \theta) \right. \\ & \left. + \tilde{\phi}_i^*(-p, \theta) \tilde{\phi}_i(p, \theta) \right) G(p/\Lambda, m_0/\Lambda, \alpha_0) \\ & + \frac{1}{2} \sum_{i=1}^{N_f} \left(\int \frac{d^4 p}{(2\pi)^4} d^2 \theta m_0 \tilde{\phi}_i(-p, \theta) \phi_i(p, \theta) \right. \\ & \left. \times J(p/\Lambda, m_0/\Lambda, \alpha_0) + \text{c.c.} \right), \end{aligned} \tag{9}$$

where the functions G and J are normalized in such a way that in the tree approximation $G = 1$ and $J = 1$. From the expression (9) it is possible to construct the exact superfield propagators for the matter superfields, see Ref. [32] for details. In the coordinate representation they are written as

$$\begin{aligned} \left(\frac{\delta^2 \Gamma}{\delta \phi_{ix} \delta \phi_{jy}^*} \right)^{-1} &= \left(\frac{\delta^2 \Gamma}{\delta \tilde{\phi}_{ix} \delta \tilde{\phi}_{jy}^*} \right)^{-1} \\ &= -\frac{G D_y^2 \bar{D}_x^2}{4(\partial^2 G^2 + m_0^2 J^2)} \delta_{xy}^8 \delta_{ij}; \\ \left(\frac{\delta^2 \Gamma}{\delta \phi_{ix} \delta \tilde{\phi}_{jy}} \right)^{-1} &= -\frac{m_0 J \bar{D}^2}{\partial^2 G^2 + m_0^2 J^2} \delta_{xy}^8 \delta_{ij}; \\ \left(\frac{\delta^2 \Gamma}{\delta \phi_{ix}^* \delta \tilde{\phi}_{jy}^*} \right)^{-1} &= -\frac{m_0 J D^2}{\partial^2 G^2 + m_0^2 J^2} \delta_{xy}^8 \delta_{ij}. \end{aligned} \tag{10}$$

In the momentum representation all these propagators contain the denominator

$$p^2 G^2(p) - m_0^2 J^2(p), \tag{11}$$

where all arguments of the functions G and J except for the momentum p were omitted.

The renormalized mass in the on-shell scheme is defined as the pole of the propagators (10),

$$m = m_0 \left. \frac{J(p)}{G(p)} \right|_{p^2=m^2}. \tag{12}$$

It is convenient to introduce the mass renormalization constant $Z_m \equiv m_0/m$, which in the scheme under consideration is given by the expression

$$Z_m = \left. \frac{G(p)}{J(p)} \right|_{p^2=m^2}. \tag{13}$$

The matter superfield renormalization constant Z in the on-shell scheme is given by the residue at this pole. For all propagators (10) the result is the same,

$$Z^{-1} = G(p) \left(1 + 2m^2 \frac{\partial}{\partial p^2} \ln \frac{G(p)}{J(p)} \right) \Big|_{p^2=m^2}. \tag{14}$$

Note that, due to the superpotential non-renormalization in $\mathcal{N} = 1$ supersymmetric theories [49], it is usually assumed that $Z Z_m = 1$. However, in the one-shell scheme it is not so, because

$$Z^{-1} Z_m^{-1} = J(p) \left(1 + 2m^2 \frac{\partial}{\partial p^2} \ln \frac{G(p)}{J(p)} \right) \Big|_{p^2=m^2}. \tag{15}$$

Although this expression is not equal to 1, it is finite in the ultraviolet region due to the non-renormalization of the superpotential. This implies that the renormalization constants Z and Z_m^{-1} differ by a finite factor.

Note that in the component formulation of the theory the scalars and the spinors will have the same renormalization constants only if the theory is regularized and quantized in a manifestly supersymmetric way. In the case of using the Wess–Zumino gauge this equality will be lost. This can be seen already in the one-loop approximation, see Ref. [50]. On the other hand, since the relation between the bare and the pole mass must be gauge-independent, the equality between the fermion and the scalar masses must be preserved after renormalization in the on-shell scheme whichever of the two quantization methods is used [51].

Quantum corrections to the two-point Green function of the gauge superfield are encoded in the function $d(k/\Lambda, m_0/\Lambda, \alpha_0)$, which enters the effective action as

$$\begin{aligned} \Gamma_V^{(2)} - S_{\text{gf}} = & -\frac{1}{16\pi} \int \frac{d^4 k}{(2\pi)^4} d^4 \theta \\ & \times V(-k, \theta) \partial^2 \Pi_{1/2} V(k, \theta) \\ & \times d^{-1}(k/\Lambda, m_0/\Lambda, \alpha_0), \end{aligned} \tag{16}$$

where $\partial^2 \Pi_{1/2} \equiv -D^a \bar{D}^2 D_a / 8$, and the normalization constant is chosen in such a way that in the tree approximation $d^{-1} = \alpha_0^{-1}$. The function $d(k/\Lambda, m_0/\Lambda, \alpha_0)$ is the invariant charge [18] of the supersymmetric quantum electrodynamics. In the limit $k \rightarrow 0$ it gives the value of the fine-structure constant as a function of m_0/Λ and α_0 in the supersymmetric case. In the on-shell subtraction scheme this value plays the role of the renormalized coupling constant α . The β -function in this scheme is defined as

$$\beta(\alpha) = \left. \frac{d\alpha}{d \ln m} \right|_{\alpha_0, \Lambda = \text{const}}, \tag{17}$$

where m is the pole mass defined earlier.

4 The three-loop β -function in the on-shell scheme

An important feature of using the higher covariant derivative regularization in supersymmetric theories is the factorization of the loop integrals contributing to the function $d^{-1}(k/\Lambda, m_0/\Lambda, \alpha_0)$ in the limit $k \rightarrow 0$ into integrals of double total derivatives with respect to the momenta. This was first discovered in explicit calculations for $\mathcal{N} = 1$ SQED in [52] (total derivatives) and [44] (double total derivatives). The rigorous all-order proof for the Abelian case has been done in Refs. [31,32]. (The factorization into double total derivatives seems to be a general feature of supersymmetric theories and theories with softly broken supersymmetry regularized by higher covariant derivatives, see, e.g., the calculations of Refs. [33,37,38,53–55].)

In $\mathcal{N} = 1$ SQED the double total derivatives are taken with respect to the momenta of the matter loops to which the external lines of the gauge superfield are attached. If a double total derivative acts on a massless propagator, it produces a delta-function singularity which gives rise to a non-vanishing contribution. However, if a double total derivative acts only on massive propagators, the integral of this total derivative vanishes. This implies that in massive $\mathcal{N} = 1$ SQED the only nonvanishing contribution to the function $d^{-1}(k/\Lambda, m_0/\Lambda, \alpha_0)$ at $k = 0$ comes from the one-loop approximation. In the case of using the higher derivative regularization it is possible to write the one-loop contribution to this function in the form⁴

$$d^{-1}(K/\Lambda = 0, m_0/\Lambda, \alpha_0) = \alpha_0^{-1} + 2\pi N_f \int \frac{d^4 Q}{(2\pi)^4} \frac{\partial}{\partial Q^\mu} \frac{\partial}{\partial Q_\mu} \times \left(-\frac{1}{Q^2} \ln(Q^2 + m_0^2) + \sum_{I=1}^n c_I \frac{1}{Q^2} \ln(Q^2 + M_I^2) \right), \tag{18}$$

where $c_I = (-1)^{P_I+1}$, see Refs. [52,56]. It is important that this expression is exact. All higher order contributions in the massive case vanish as integrals of total derivatives acting on non-singular functions [31,32]. Note that the singularities are absent, because all propagators are massive.

The integral in Eq. (18) can easily be calculated, see, e.g., [25]. Taking into account that $d(0, m_0/\Lambda, \alpha_0) = \alpha$ is the renormalized charge in the on-shell scheme and omitting terms suppressed by powers of m_0/Λ , we obtain

$$\alpha^{-1} - \alpha_0^{-1} = \frac{N_f}{\pi} \left(\ln \frac{\Lambda}{m_0} + \sum_{I=1}^n c_I \ln a_I \right). \tag{19}$$

⁴ In our notation capital letters denote Euclidean momenta.

Next, following [44], the right-hand side is expressed in terms of the renormalized mass,

$$\alpha^{-1} - \alpha_0^{-1} = \frac{N_f}{\pi} \left(\ln \frac{\Lambda}{m} - \ln Z_m + \sum_{I=1}^n c_I \ln a_I \right). \tag{20}$$

Then differentiating with respect to $\ln m$ gives the NSVZ relation

$$\beta(\alpha) = \frac{N_f \alpha^2}{\pi} \left(1 + \gamma_m(\alpha) \right) \tag{21}$$

written in terms of the mass anomalous dimension

$$\gamma_m(\alpha) = \left. \frac{d \ln Z_m}{d \ln m} \right|_{\alpha_0, \Lambda = \text{const}}. \tag{22}$$

Thus, the NSVZ equation similar to Eq. (1) is indeed valid in the on-shell scheme. It relates the β -function in a given order to the mass anomalous dimension in the previous order. Note that in the on-shell scheme the mass anomalous dimension differs from the anomalous dimension of the matter superfields taken with the opposite sign.⁵

Using Eq. (21) it is possible to construct the three-loop β -function in the on-shell scheme by calculating the mass anomalous dimension in the two-loop approximation. This is done in this paper for the theory regularized by higher derivatives. Methods of evaluating Feynman integrals with the help of this regularization are not described in the literature in enough detail, while there is some interest in investigating various $D = 4$ techniques for calculating quantum corrections (see the review [58]). That is why in Appendix A we describe in detail how the renormalization constant Z_m is obtained in the two-loop approximation. The result is given by the expression

$$\begin{aligned} \ln Z_m = & -\frac{\alpha_0}{\pi} \left(\ln \frac{\Lambda}{m} + \frac{1}{2} - \frac{A}{2} \right) + \frac{\alpha_0^2}{\pi^2} \left(\frac{N_f}{2} \ln^2 \frac{\Lambda}{m} \right. \\ & \left. + \ln \frac{\Lambda}{m} \left(\frac{3N_f}{2} + \frac{1}{2} + N_f \sum_{I=1}^n c_I \ln a_I \right) + O(1) \right) \\ & + O(\alpha_0^3). \end{aligned} \tag{23}$$

In this equation the symbol $O(1)$ denotes finite terms that do not vanish in the limit $\Lambda \rightarrow \infty$, and the constant A is defined by

$$A \equiv 2 \int_0^\infty dK \ln \frac{K}{\Lambda} \frac{d}{dK} \left(\frac{1}{R_K} \right), \tag{24}$$

⁵ Exactly as in the case of (non-supersymmetric) QED [57], in the lowest-order approximation the corresponding renormalization constants differ by an ultraviolet finite but infrared divergent term, $Z^{-1} Z_m^{-1} = 1 + \alpha(1 - \ln m/\kappa)/\pi + O(\alpha^2)$, where κ is a small photon mass.

where $R_K \equiv R(K^2/\Lambda^2)$. (For the regulator $R(K^2/\Lambda^2) = 1 + (K^2/\Lambda^2)^n$ this integral vanishes, so that $A = 0$.) Differentiating (23) with respect to $\ln m$ and expressing the result in terms of α using (20) we obtain

$$\gamma_m(\alpha) = \frac{d \ln Z_m}{d \ln m} = \frac{\alpha}{\pi} - \frac{\alpha^2(3N_f + 1)}{2\pi^2} + O(\alpha^3). \tag{25}$$

As expected in the on-shell scheme, any dependence on the regularization details has disappeared. After substituting the mass anomalous dimension (25) into Eq. (21) the three-loop result for the β -function takes the form

$$\beta(\alpha) = \frac{N_f \alpha^2}{\pi} \left(1 + \frac{\alpha}{\pi} - \frac{\alpha^2(3N_f + 1)}{2\pi^2} + O(\alpha^3) \right). \tag{26}$$

Comparing it with the corresponding result in the $\overline{\text{DR}}$ -scheme (i.e., in the case of using dimensional reduction supplemented by modified minimal subtractions) [39]

$$\begin{aligned} \beta_{\overline{\text{DR}}}(\alpha_{\overline{\text{DR}}}) &= \frac{\alpha_{\overline{\text{DR}}}^2 N_f}{\pi} \\ &\times \left(1 + \frac{\alpha_{\overline{\text{DR}}}}{\pi} - \frac{\alpha_{\overline{\text{DR}}}^2}{4\pi^2} (3N_f + 2) + O(\alpha_{\overline{\text{DR}}}^3) \right), \end{aligned} \tag{27}$$

we see that the terms linear in N_f coincide. This follows from the scheme-independence of these terms proved in [20] in all orders, which is related to the so-called conformal symmetry limit of perturbative quenched quantum electrodynamics [59].

5 Relations between the on-shell scheme and other NSVZ schemes

In the previous section it was demonstrated that the NSVZ relation (21) is valid in the on-shell scheme in all orders. Therefore, this scheme belongs to the class of NSVZ schemes described in Ref. [14], which also includes the all-order HD+MSL prescription and the NSVZ scheme constructed with dimensional reduction in the three-loop approximation in Refs. [39,43]. According to Ref. [14] any two NSVZ subtraction schemes can be related by a finite renormalization

$$\begin{aligned} \alpha'(\alpha_0, \Lambda/\mu) &= \alpha'(\alpha(\alpha_0, \Lambda/\mu)); \\ Z'(\alpha'(\alpha), \Lambda/\mu) &= z(\alpha) Z(\alpha, \Lambda/\mu), \end{aligned} \tag{28}$$

which is subjected to the constraint

$$\frac{1}{\alpha'(\alpha)} - \frac{1}{\alpha} = \frac{N_f}{\pi} \ln z(\alpha) + B, \tag{29}$$

where B is a constant.

First, let us find the finite renormalization relating the on-shell scheme to the HD+MSL scheme. According to the HD+MSL prescription, the calculations are to be carried out with the higher derivative regularization and only powers of

$\ln \Lambda/\mu$ are included into renormalization constants, so that in this scheme

$$\begin{aligned} \frac{1}{\alpha_{\text{HD+MSL}}} &= \frac{1}{\alpha_0} + \frac{N_f}{\pi} \ln \frac{\Lambda}{\mu} + \frac{\alpha_0 N_f}{\pi^2} \ln \frac{\Lambda}{\mu} + O(\alpha_0^2); \\ \ln Z_{\text{HD+MSL}} &= \frac{\alpha_0}{\pi} \ln \frac{\Lambda}{\mu} + O(\alpha_0^2). \end{aligned} \tag{30}$$

Comparing these equations with Eqs. (20) and (23) we derive the required finite renormalization,

$$\begin{aligned} \alpha_{\text{HD+MSL}}^{-1} \Big|_{\mu=m} &= \alpha_{\text{OS}}^{-1} - \frac{N_f}{\pi} \sum_{I=1}^n c_I \ln a_I \\ &+ \frac{N_f \alpha_{\text{OS}}}{\pi^2} \left(-\frac{1}{2} + \frac{A}{2} \right) + O(\alpha_{\text{OS}}^2); \\ \ln Z_{\text{HD+MSL}} \Big|_{\mu=m} + \ln Z_m &= \frac{\alpha_{\text{OS}}}{\pi} \left(-\frac{1}{2} + \frac{A}{2} \right) + O(\alpha_{\text{OS}}^2). \end{aligned} \tag{31}$$

Evidently, in this case the condition (29) is satisfied with $B = -(N_f/\pi) \sum_{I=1}^n c_I \ln a_I$.

Also it is possible to find the finite renormalization which relates the on-shell scheme to the NSVZ scheme constructed with the dimensional reduction. (For short, we will call this scheme ‘‘DR+NSVZ’’.) In the case of using the DR+NSVZ scheme the renormalization group functions (RGFs) can be found in [43] and have the form

$$\begin{aligned} \frac{\beta_{\text{DR+NSVZ}}(\alpha_{\text{DR+NSVZ}})}{\alpha_{\text{DR+NSVZ}}^2} &= \frac{N_f}{\pi} \left(1 + \frac{\alpha_{\text{DR+NSVZ}}}{\pi} - \frac{\alpha_{\text{DR+NSVZ}}^2}{2\pi^2} (1 + N_f) + O(\alpha_{\text{DR+NSVZ}}^3) \right); \\ \gamma_{\text{DR+NSVZ}}(\alpha_{\text{DR+NSVZ}}) &= -\frac{\alpha_{\text{DR+NSVZ}}}{\pi} + \frac{\alpha_{\text{DR+NSVZ}}^2}{2\pi^2} (1 + N_f) \\ &+ O(\alpha_{\text{DR+NSVZ}}^3). \end{aligned} \tag{32}$$

These expressions should be compared with Eqs. (25) and (26). With the help of the standard equations describing how RGFs transform under finite renormalizations [60] we obtain the finite renormalization after which RGFs in the on-shell scheme are converted into RGFs in the DR+NSVZ scheme,

$$\begin{aligned} \alpha_{\text{DR+NSVZ}}^{-1} \Big|_{\mu=m} &= \alpha_{\text{OS}}^{-1} + \frac{N_f}{\pi} z_1 + \frac{N_f \alpha_{\text{OS}}}{\pi^2} (-1 + z_1) + O(\alpha_{\text{OS}}^2); \\ \ln Z_{\text{DR+NSVZ}}(\alpha_{\text{DR+NSVZ}}) \Big|_{\mu=m} + \ln Z_{m, \text{OS+DRED}}(\alpha_{\text{OS}}) &= \frac{\alpha_{\text{OS}}}{\pi} (-1 + z_1) + O(\alpha_{\text{OS}}^2). \end{aligned} \tag{33}$$

However, these equations contain an undefined constant z_1 , which reflects the arbitrariness of choosing a renormalization point in the DR+NSVZ scheme. This constant can be found by comparing the one-loop expressions for the renormalized function d^{-1} in the limit $k \rightarrow 0$,

$$\begin{aligned} \alpha_{\text{OS}}^{-1} &= d_{\text{OS}}^{-1} \Big|_{k \rightarrow 0} = d_{\text{DR+NSVZ}}^{-1} \Big|_{k \rightarrow 0} = \alpha_{\text{DR+NSVZ}}^{-1} \\ &+ \frac{N_f}{\pi} \ln \frac{\mu}{m} + O(\alpha_{\text{DR+NSVZ}}), \end{aligned} \tag{34}$$

so that $z_1 = 0$. This is analogous to the case of (non-supersymmetric) QED in which a similar coefficient also vanishes, $\alpha_{\overline{MS}}^{-1}|_{\mu=m} = \alpha_{OS}^{-1} + O(\alpha_{OS})$, see Ref. [61].

In the case $\mu \neq m$ the considered finite renormalization takes the form

$$\begin{aligned} \alpha_{DR+NSVZ}^{-1} &= \alpha_{OS}^{-1} - \frac{N_f}{\pi} \ln \frac{\mu}{m} - \frac{N_f \alpha_{OS}}{\pi^2} \left(1 + \ln \frac{\mu}{m}\right) \\ &+ O\left(\alpha_{OS}^2\right); \\ \ln Z_{DR+NSVZ}(\alpha_{DR+NSVZ}) + \ln Z_{m, OS+DRED}(\alpha_{OS}) \\ &= -\frac{\alpha_{OS}}{\pi} \left(1 + \ln \frac{\mu}{m}\right) + O\left(\alpha_{OS}^2\right). \end{aligned} \tag{35}$$

One can easily verify that the constraint (29) is also satisfied for the functions (35) with $B = -N_f \ln(\mu/m)/\pi$.

6 Conclusion

We have explicitly demonstrated that the NSVZ equation in $\mathcal{N} = 1$ SQED is valid in the on-shell scheme in all orders. In this case it relates the β -function to the mass anomalous dimension. The NSVZ relation appears in the on-shell scheme due to the fact that quantum corrections to the photon polarization operator in the limit of zero momentum are given by integrals of double total derivatives with the higher derivative regularization. In the massive case these total derivatives act on nonsingular expressions in all orders beyond the one-loop approximation. The remaining one-loop contribution produces the NSVZ relation between the β -function and the mass anomalous dimension. This implies that the β -function in a given order can be found by calculating the mass anomalous dimension in the previous order. In this paper, having calculated the latter to the two-loop order, we obtained the β -function in the on-shell scheme to the three-loop order.

It was also investigated how the on-shell scheme in $\mathcal{N} = 1$ SQED is related to other known NSVZ schemes, namely HD+MSL and the NSVZ scheme based on dimensional reduction. Finite renormalizations relating the on-shell scheme to these two schemes have been constructed. They were shown to satisfy the constraint (29) derived in [14].

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A The renormalization constant Z_m in the two-loop order

This appendix is devoted to the calculation of the two-loop renormalization constant Z_m in the on-shell scheme. In particular, we describe the technique of evaluating the $D = 4$ loop integrals appearing with the higher derivative regularization.

A.1 Z_m as a sum of loop integrals

In the considered approximation the logarithm of the renormalization constant Z_m is written as

$$\begin{aligned} \ln Z_m &= \ln G(m) - \ln J(m) = \Delta G(m) - \Delta J(m) \\ &- \frac{(\Delta G(m))^2}{2} + \frac{(\Delta J(m))^2}{2} + \dots, \end{aligned} \tag{36}$$

where $\Delta G \equiv G - 1$ and $\Delta J \equiv J - 1$. Using the results of Refs. [47, 48],⁶ after the Wick rotation it is possible to present this expression as a sum of Euclidean loop integrals

$$\begin{aligned} \ln Z_m &= \alpha_0 I_{\text{one-loop}} + \alpha_0^2 \left(I_1 + I_2 + I_3 + I_4 + I_5 \right. \\ &\left. - \frac{1}{2} (I_{\text{one-loop}})^2 \right) + O\left(\alpha_0^3\right). \end{aligned} \tag{37}$$

In explicit expressions for these integrals (presented below), Euclidean momenta will be denoted by capital letters. Due to the higher derivative regularization, denominators of the integrands contain the function R_K . In the simplest case it can be chosen as $R_K = 1 + K^{2n}/\Lambda^{2n}$. However, in the general case considered here it is sufficient to require that $R_K(0) = 1$ and (due to the presence of higher powers of the momentum) $R_K \rightarrow \infty$ in the limit $K \rightarrow \infty$. In Eq. (37)

$$\begin{aligned} &\alpha_0 I_{\text{one-loop}} \\ &= -8\pi\alpha_0 \int \frac{d^4 K}{(2\pi)^4} \frac{1}{K^2 R_K ((P + K)^2 + m^2)} \Bigg|_{P^2 = -m^2} \end{aligned} \tag{38}$$

is the one-loop contribution, while the remaining integrals

⁶ With the higher derivative regularization the superdiagrams contributing to the function G have first been calculated in Refs. [62, 63].

$$I_1 \equiv \frac{m_0^2 - m^2}{\alpha_0} \int \frac{d^4 K}{(2\pi)^4} \times \frac{8\pi}{K^2 R_K ((P + K)^2 + m^2)^2} \Big|_{p^2 = -m^2}; \tag{39}$$

$$I_2 \equiv -128\pi^2 \int \frac{d^4 K}{(2\pi)^4} \frac{d^4 L}{(2\pi)^4} \times \frac{P_\mu (K + L)^\mu - m^2}{K^2 R_K L^2 R_L ((P + K)^2 + m^2)((P + L)^2 + m^2)} \times \frac{1}{((P + K + L)^2 + m^2)} \Big|_{p^2 = -m^2}; \tag{40}$$

$$I_3 \equiv 128\pi^2 \int \frac{d^4 K}{(2\pi)^4} \frac{d^4 L}{(2\pi)^4} \frac{1}{K^2 R_K L^2 R_L} \times \frac{m^2}{((P + K)^2 + m^2)^2 ((P + K + L)^2 + m^2)} \Big|_{p^2 = -m^2}; \tag{41}$$

$$I_4 \equiv 64\pi^2 \int \frac{d^4 K}{(2\pi)^4} \frac{d^4 L}{(2\pi)^4} \frac{1}{K^2 R_K L^2 R_L} \times \frac{1}{((P + K)^2 + m^2)((P + K + L)^2 + m^2)} \Big|_{p^2 = -m^2}; \tag{42}$$

$$I_5 \equiv 64\pi^2 N_f \int \frac{d^4 K}{(2\pi)^4} \frac{d^4 L}{(2\pi)^4} \frac{1}{K^2 R_K^2 ((P + K)^2 + m^2)} \times \left(\frac{1}{(L^2 + m^2)((L + K)^2 + m^2)} + \sum_{l=1}^n (-1)^{P_l} \frac{1}{(L^2 + M_l^2)((L + K)^2 + M_l^2)} \right) \Big|_{p^2 = -m^2} \tag{43}$$

correspond to the two-loop approximation. Note that in these integrals terms proportional to $\alpha_0^2 (P^2 + m^2)$ were omitted, because they evidently vanish due to the condition $P^2 = -m^2$. Also all these integrals were expressed in terms of the renormalized mass m . Therefore, the one-loop superdiagrams give both the integral $I_{\text{one-loop}}$ and the integral I_1 . (The latter one is produced by the one-loop superdiagrams containing an insertion of the one-loop mass counterterm.)

A.2 One-loop contribution

The one-loop contribution is given by the expression

$$\ln Z_m = \ln G(m) - \ln J(m) = - \int \frac{d^4 K}{(2\pi)^4} \frac{8\pi\alpha_0}{K^2 R_K ((P + K)^2 + m^2)} \Big|_{p^2 = -m^2} + O(\alpha_0^2), \tag{44}$$

where the bare mass m_0 was replaced by the renormalized mass m , because their difference $\Delta m^2 \equiv m^2 - m_0^2$ is pro-

portional to α_0 and is essential in the next order. The integral in Eq. (44) can be calculated in four-dimensional spherical coordinates using the method of Refs. [62,63]. Introducing the variable $x \equiv \cos \theta_3$, where θ_3 is the angle between the vectors K^μ and P^μ , it can be rewritten in the form

$$\ln Z_m = -\alpha_0 \int_0^\infty dK \frac{1}{\pi^2 R_K} \oint_C dx \frac{\sqrt{1-x^2}}{K + 2imx} + O(\alpha_0^2). \tag{45}$$

The contour C is presented in Fig. 1. To calculate this integral it is necessary to find the residues at the points $x = \infty$ and $x = iK/2m$. The result written as an integral over $z \equiv K^2/\Lambda^2$ has the form

$$\ln Z_m = \frac{\alpha_0}{\pi} \int_0^\infty dz \frac{\Lambda^2}{4m^2 R(z)} \left(1 - \sqrt{1 + \frac{4m^2}{z\Lambda^2}} \right) + O(\alpha_0^2). \tag{46}$$

It is convenient to introduce the new variable

$$\eta \equiv \frac{a}{4} \exp \left\{ -\frac{2z}{a} + \frac{2z}{a} \sqrt{1 + \frac{a}{z}} - 1 + \ln \left(1 + \frac{2z}{a} + 2\sqrt{\frac{z}{a} + \frac{z^2}{a^2}} \right) \right\}, \tag{47}$$

where $a \equiv 4m^2/\Lambda^2$, such that

$$\frac{d\eta}{2\eta} = \frac{dz}{a} \left(-1 + \sqrt{1 + \frac{a}{z}} \right). \tag{48}$$

Note that $z = 0$ and $z \rightarrow \infty$ correspond to $\eta = \exp(-1)a/4$ and $\eta \rightarrow \infty$, respectively. Therefore the integral under consideration can be rewritten as

$$\ln Z_m = -\frac{\alpha_0}{2\pi} \int_{\exp(-1)a/4}^\infty \frac{d\eta}{\eta R(z(\eta))} + O(\alpha_0^2). \tag{49}$$

This integral diverges in the limit $a \rightarrow 0$, when $z(\eta) = \eta + O(a)$. However, if the function $R^{-1}(z)$ is expanded in powers of a , then only the leading term will produce a divergent integral, while the other terms are given by convergent integrals, which vanish in the limit $a \rightarrow 0$. Therefore, omitting the terms suppressed by powers of m^2/Λ^2 we obtain

$$\ln Z_m = -\frac{\alpha_0}{2\pi} \int_{\exp(-1)m^2/\Lambda^2}^\infty \frac{d\eta}{\eta R(\eta)} + O(\alpha_0^2). \tag{50}$$

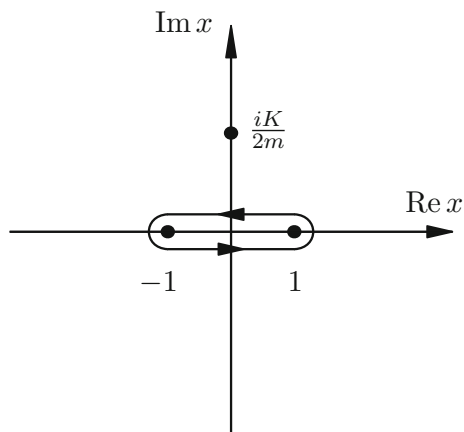


Fig. 1 The contour C in the integrals (45) and (55)

Integrating by parts, it is possible to extract the divergent part of the remaining integral,

$$\int_{\exp(-1)m^2/\Lambda^2}^{\infty} d\eta \frac{1}{\eta R(\eta)} = \frac{1}{R(\eta)} \ln \eta \Big|_{\exp(-1)m^2/\Lambda^2}^{\infty} - \int_{\exp(-1)m^2/\Lambda^2}^{\infty} d\eta \ln \eta \frac{d}{d\eta} \left(\frac{1}{R(\eta)} \right) = 2 \ln \frac{\Lambda}{m} + 1 - A + O(m^2/\Lambda^2), \tag{51}$$

where the constant A is given by the equation

$$A \equiv \int_0^{\infty} d\eta \ln \eta \frac{d}{d\eta} \left(\frac{1}{R(\eta)} \right), \tag{52}$$

which is equivalent to Eq. (24). Thus, in the one-loop approximation

$$\ln Z_m = -\frac{\alpha_0}{\pi} \left(\ln \frac{\Lambda}{m} + \frac{1}{2} - \frac{A}{2} \right) + O(\alpha_0^2). \tag{53}$$

A.3 Two-loop contribution

To find the two-loop contribution to the renormalization constant Z_m , it is necessary to calculate the integrals $I_1 - I_5$ in Eq. (37).

The integral I_1 is convergent in the ultraviolet region, but diverges in the infrared one. That is why it is necessary to regularize it by introducing a small photon mass κ ,

$$I_1 = \frac{m_0^2 - m^2}{\alpha_0} \int \frac{d^4 K}{(2\pi)^4} \times \frac{8\pi}{(K^2 + \kappa^2)R_K((P + K)^2 + m^2)^2} \Big|_{P^2 = -m^2}. \tag{54}$$

This integral is convergent, so that it is possible to take the limit $\Lambda \rightarrow \infty$ omitting terms suppressed by powers of Λ^{-1} .

Therefore, the function R_K in the considered expression can be replaced by 1. The resulting integral can be calculated in the four-dimensional spherical coordinates. After the substitution $x = \cos \theta_3$ it takes the form

$$I_1 = \frac{m_0^2 - m^2}{\alpha_0} \int_0^{\infty} dK \frac{K}{\pi^2(K^2 + \kappa^2)} \oint_C dx \frac{\sqrt{1-x^2}}{(K + 2imx)^2}, \tag{55}$$

where the contour C is presented in Fig. 1. The integral over x can be found by calculating the residues at the points $x = \infty$ and $x = iK/2m$,

$$I_1 = \frac{m_0^2 - m^2}{2m^2\alpha_0} \int_0^{\infty} dK \frac{K}{\pi(K^2 + \kappa^2)} \left(1 - \frac{1}{\sqrt{1 + 4m^2/K^2}} \right) = \frac{m_0^2 - m^2}{2m^2\pi\alpha_0} \int_0^{\infty} dK \left(\frac{K}{K^2 + \kappa^2} - \frac{1}{\sqrt{K^2 + 4m^2}} + \frac{\kappa^2}{(K^2 + \kappa^2)\sqrt{K^2 + 4m^2}} \right). \tag{56}$$

Taking into account that $m_0 = Z_m m$ and omitting the last term in the round brackets (which gives a convergent integral proportional to $\kappa \rightarrow 0$) this expression can be written as

$$I_1 = \frac{Z_m^2 - 1}{4\pi\alpha_0} \left(\ln(1 + K^2/\kappa^2) - 2 \operatorname{arcsinh}(K/2m) \right) \Big|_0^{\infty} = \frac{Z_m^2 - 1}{2\pi\alpha_0} \ln \frac{m}{\kappa}. \tag{57}$$

After substituting the one-loop result for Z_m from Eq. (53) with the considered accuracy the integral I_1 takes the form

$$I_1 = -\frac{1}{\pi^2} \left(\ln \frac{\Lambda}{m} + \frac{1}{2} - \frac{A}{2} \right) \ln \frac{m}{\kappa}. \tag{58}$$

The integral I_2 is convergent and does not contain infrared divergences. This implies that (due to the condition $P^2 = -m^2$) this integral is equal to a finite number and does not contribute to γ_m in the considered approximation. Below we will omit such terms.

The integral I_3 diverges in the infrared region and should be regularized by introducing the small photon mass κ ,

$$I_3 \equiv 128\pi^2 \int \frac{d^4 K}{(2\pi)^4} \frac{d^4 L}{(2\pi)^4} \frac{1}{(K^2 + \kappa^2)R_K(L^2 + \kappa^2)R_L} \times \frac{m^2}{((P + K)^2 + m^2)((P + K + L)^2 + m^2)} \Big|_{P^2 = -m^2}. \tag{59}$$

This expression can be equivalently rewritten as

$$I_3 = 128\pi^2 \int \frac{d^4K}{(2\pi)^4} \frac{d^4L}{(2\pi)^4} \times \frac{1}{(K^2 + \kappa^2)R_K} \times \frac{m^2}{(L^2 + \kappa^2)R_L((P + K)^2 + m^2)((P + L)^2 + m^2)} \times \left(1 + \frac{(P + L)^2 - (P + K + L)^2}{(P + K + L)^2 + m^2}\right) \Big|_{p^2=-m^2} \equiv I'_3 + I''_3. \tag{60}$$

The integral I'_3 (corresponding to 1 in the round brackets) can be presented as a product of two integrals which have already been calculated above,

$$I'_3 = 128\pi^2 \int \frac{d^4K}{(2\pi)^4} \frac{d^4L}{(2\pi)^4} \frac{m^2}{(K^2 + \kappa^2)R_K((P + K)^2 + m^2)^2} \times \int \frac{d^4L}{(2\pi)^4} \frac{1}{(L^2 + \kappa^2)R_L((P + L)^2 + m^2)} = \frac{1}{2\pi^2} \ln \frac{m}{\kappa} \left(2 \ln \frac{\Lambda}{m} + 1 - A\right). \tag{61}$$

The expression for I''_3 (which is obtained from the second term in the round brackets in Eq. (60)) is not divergent in the infrared region, so that it is possible to set κ to 0,

$$I''_3 \Big|_{\kappa=0} = -128\pi^2 \int \frac{d^4K}{(2\pi)^4} \frac{d^4L}{(2\pi)^4} \frac{1}{K^2 R_K L^2 R_L} \times \frac{m^2 K^\mu (K + 2P + 2L)_\mu}{((P + K)^2 + m^2)^2 ((P + L)^2 + m^2)} \times \frac{1}{((P + K + L)^2 + m^2)} \Big|_{p^2=-m^2} \tag{62}$$

It is also not divergent in the ultraviolet region. Therefore, it is a finite number, which does not contribute to the two-loop mass anomalous dimension. This implies that

$$I_3 = \frac{1}{2\pi^2} \ln \frac{m}{\kappa} \left(2 \ln \frac{\Lambda}{m} + 1 - A\right) + \text{a finite constant.} \tag{63}$$

After some transformations the expression $I_4 - (I_{\text{one-loop}})^2/2$ can be rewritten as

$$I_4 - \frac{1}{2}(I_{\text{one-loop}})^2 = -64\pi^2 \int \frac{d^4K}{(2\pi)^4} \frac{d^4L}{(2\pi)^4} \frac{1}{K^2 R_K L^2 R_L ((P + K)^2 + m^2)} \times \frac{L_\mu K^\mu}{((P + L)^2 + m^2)((P + K + L)^2 + m^2)} \Big|_{p^2=-m^2}. \tag{64}$$

This integral is convergent in the infrared region. Therefore, it depends on Λ/m and can be presented as

$$I_4 - \frac{1}{2}(I_{\text{one-loop}})^2 = f_2 \ln^2 \frac{\Lambda}{m} + f_1 \ln \frac{\Lambda}{m} + f_0 + \text{terms proportional to } m/\Lambda. \tag{65}$$

Let us calculate the derivative of the integral (64) with respect to $\ln \Lambda$ in the case $m = 0$,

$$\frac{d}{d \ln \Lambda} \left(I_4 - \frac{1}{2}(I_{\text{one-loop}})^2 \right) \Big|_{m \rightarrow 0} = -64\pi^2 \frac{d}{d \ln \Lambda} \int \frac{d^4K}{(2\pi)^4} \frac{d^4L}{(2\pi)^4} \frac{L_\mu K^\mu}{K^4 R_K L^4 R_L (K + L)^2}. \tag{66}$$

Following Ref. [52], this integral can be presented as

$$\frac{1}{2\pi^2} \left[\int_0^\infty \frac{dK}{K^3} \int_0^K dL L \frac{d}{d \ln \Lambda} \left(\frac{1}{R_K R_L} \right) + \int_0^\infty dK K \int_K^\infty \frac{dL}{L^3} \frac{d}{d \ln \Lambda} \left(\frac{1}{R_K R_L} \right) \right] = \frac{1}{\pi^2} \int_0^\infty \frac{dK}{K^3} \int_0^K dL L \frac{d}{d \ln \Lambda} \left(\frac{1}{R_K R_L} \right). \tag{67}$$

After the substitution $L = \rho K$ in the last integral, this expression can be rewritten in the form⁷

$$\frac{1}{\pi^2} \int_0^\infty \frac{dK}{K} \int_0^1 d\rho \rho \frac{d}{d \ln \Lambda} \left(\frac{1}{R_K R_{\rho K}} \right) = -\frac{1}{\pi^2} \int_0^1 d\rho \rho \int_0^\infty d \ln K \frac{d}{d \ln K} \left(\frac{1}{R_K R_{\rho K}} \right) = \frac{1}{2\pi^2}. \tag{68}$$

This implies that in Eq. (65) $f_2 = 0$ and $f_1 = 1/2\pi^2$. Therefore, omitting terms proportional to m/Λ , we obtain

$$I_4 - \frac{1}{2}(I_{\text{one-loop}})^2 = \frac{1}{2\pi^2} \ln \frac{\Lambda}{m} + \text{a finite constant.} \tag{69}$$

The remaining integral I_5 can be calculated using the equation

$$\int \frac{d^4L}{(2\pi)^4} \frac{1}{(L^2 + m^2)((K + L)^2 + m^2)} + \sum_{I=1}^n (-1)^{P_I} \int \frac{d^4L}{(2\pi)^4} \frac{1}{(L^2 + M_I^2)((K + L)^2 + M_I^2)} = J(K/m) + \sum_{I=1}^n (-1)^{P_I} J(K/M_I), \tag{70}$$

where

$$J(K/M) \equiv -\frac{1}{8\pi^2} \left(\ln \frac{M}{K} + \sqrt{1 + \frac{4M^2}{K^2}} \operatorname{arctanh} \sqrt{\frac{K^2}{K^2 + 4M^2}} \right)$$

⁷ This result agrees with the calculation of Ref. [52] carried out for the particular case $R(x) = 1 + x^n$.

$$\approx \begin{cases} \frac{1}{4\pi^2} \frac{M^2}{K^2} \ln \frac{M}{K} & \text{if } K \rightarrow \infty \\ -\frac{1}{8\pi^2} \left(\ln \frac{M}{K} + 1 \right) & \text{if } K \rightarrow 0 \end{cases}, \tag{71}$$

see, e.g., Ref. [52]. This implies that the integral

$$64\pi^2 N_f \int \frac{d^4 K}{(2\pi)^4} \frac{1}{K^2 R_K^2 ((P + K)^2 + m^2)} \times J(K/m) \Big|_{p^2=-m^2} \tag{72}$$

is convergent in both ultraviolet and infrared regions. Therefore, it is equal to a finite constant, and only the terms with the Pauli–Villars masses M_I nontrivially contribute to the divergent part of the integral I_5 . To calculate them, let us consider the expression

$$64\pi^2 N_f \frac{d}{d \ln \Lambda} \int \frac{d^4 K}{(2\pi)^4} \frac{1}{K^2 R_K^2 ((P + K)^2 + m^2)} \times J(K/m) \Big|_{p^2=-m^2} \equiv J_1 + J_2 + J_3, \tag{73}$$

where $M = a\Lambda$ with a being a finite constant and

$$J_1 \equiv -8N_f \int \frac{d^4 K}{(2\pi)^4} \frac{1}{K^2 R_K^2 ((P + K)^2 + m^2)} \Big|_{p^2=-m^2} \tag{74}$$

$$J_2 \equiv -8N_f \int \frac{d^4 K}{(2\pi)^4} \frac{1}{K^2 ((P + K)^2 + m^2)} \times \frac{d}{d \ln \Lambda} \left(\frac{1}{R_K^2} \right) \ln \frac{\Lambda}{K} \Big|_{p^2=-m^2}; \tag{75}$$

$$J_3 \equiv 64\pi^2 N_f \int \frac{d^4 K}{(2\pi)^4} \frac{1}{K^2 ((P + K)^2 + m^2)} \frac{d}{d \ln \Lambda} \times \left[\frac{1}{R_K^2} \left(J(K/M) + \frac{1}{8\pi^2} \ln \frac{\Lambda}{K} \right) \right] \Big|_{p^2=-m^2}. \tag{76}$$

Repeating the calculation of Appendix A.2 we obtain

$$J_1 = -\frac{N_f}{\pi^2} \left(\ln \frac{\Lambda}{m} + \frac{1}{2} \right) - J_2 + O\left(\frac{m^2}{\Lambda^2}\right). \tag{77}$$

To find the integral J_3 , we note that the derivative of the function in the square brackets with respect to $\ln \Lambda$ is equal to the one with respect to $\ln K$ multiplied by (-1) . Therefore,

$$J_3 \Big|_{m \rightarrow 0} = -8N_f \int_0^\infty \frac{dK}{K} \frac{d}{d \ln K} \times \left[\frac{1}{R_K^2} \left(J(K/M) + \frac{1}{8\pi^2} \ln \frac{\Lambda}{K} \right) \right]$$

$$= -8N_f \left[\frac{1}{R_K^2} \left(J(K/M) + \frac{1}{8\pi^2} \ln \frac{\Lambda}{K} \right) \right] \Big|_0^\infty = -\frac{N_f}{\pi^2} (1 + \ln a). \tag{78}$$

This implies that the expression (72) can be written as

$$\frac{N_f}{\pi^2} \left[-\frac{1}{2} \ln^2 \frac{\Lambda}{m} - \ln \frac{\Lambda}{m} \left(\ln a + \frac{3}{2} \right) \right] + O(1). \tag{79}$$

Consequently, the integral I_5 takes the form

$$I_5 = \frac{N_f}{\pi^2} \left[\frac{1}{2} \ln^2 \frac{\Lambda}{m} + \ln \frac{\Lambda}{m} \left(\sum_{I=1}^n c_I \ln a_I + \frac{3}{2} \right) + \text{a finite constant} \right], \tag{80}$$

where $c_I = (-1)^{P_I+1}$ and terms vanishing in the limit $\Lambda \rightarrow \infty$ were omitted.

Collecting the results (53), (58), (63), (69), and (80) we obtain that the two-loop mass renormalization constant is given by the expression (23),

$$\ln Z_m = -\frac{\alpha_0}{\pi} \left(\ln \frac{\Lambda}{m} + \frac{1}{2} - \frac{A}{2} \right) + \frac{\alpha_0^2}{\pi^2} \left[\frac{N_f}{2} \ln^2 \frac{\Lambda}{m} + \ln \frac{\Lambda}{m} \left(N_f \sum_{I=1}^n c_I \ln a_I + \frac{3N_f + 1}{2} \right) + \text{a finite constant} \right] + O\left(\alpha_0^3\right), \tag{81}$$

and does not contain infrared divergences.

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