Multipole analysis for linearized $f(R, \mathcal{G})$ gravity with irreducible Cartesian tensors

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Abstract The field equations of $f(R, \mathcal{G})$ gravity are rewritten in the form of obvious wave equations with the stress–energy pseudotensor of the matter fields and the gravitational field as its source under the de Donder condition. The linearized field equations of $f(R, \mathcal{G})$ gravity are the same as those of linearized $f(R)$ gravity, and thus, their multipole expansions under the de Donder condition are also the same. It is also shown that the Gauss–Bonnet curvature scalar $\mathcal{G}$ does not contribute to the effective stress–energy tensor of gravitational waves in linearized $f(R, \mathcal{G})$ gravity, though $\mathcal{G}$ plays an important role in the nonlinear effects in general. Further, by applying the $1/r$ expansion in the distance to the source to the linearized $f(R, \mathcal{G})$ gravity, the energy, momentum, and angular momentum carried by gravitational waves in linearized $f(R, \mathcal{G})$ gravity are provided, which shows that $\mathcal{G}$, unlike the nonlinear term R^2 in the gravitational Lagrangian, does not contribute to them either.

1 Introduction

The detection of gravitational waves (GWs) by the LIGO and Virgo collaborations [1–5] is a milestone in GWs, and promotes the study of general relativity (GR) and astrophysics [6–8]. Their observations of GWs are very consistent with GR's prediction, which further confirms that GR is a great and successful theory of gravity. In despite of this, GR has to face many challenges, e.g., interpreting many data observed at infrared scales [9–12]. Introducing the modified gravity theories (MGTs) [13] is another approach to deal with these difficulties.

$f(R)$ gravity [14–19] is one of the simplest MGTs, and it replaces the Einstein–Hilbert action by the quantity $f(R)$ in the gravitational Lagrangian, where f is a general func-

tion of the Ricci scalar R . $f(R)$ gravity has many important implications in cosmology [20,21], and the typical one is that some models of $f(R)$ gravity can explain the inflation in early universe successfully [18]. Recently, a generalized modified Gauss–Bonnet (GB) gravity, whose gravitational Lagrangian is a general function of R and $\mathcal{G}$ as $f(R, \mathcal{G})$, has attracted considerable attention [22–27], where $\mathcal{G}$ is another interesting curvature scalar, and such models can give reasonable gravitational alternative for dark energy [28] and avoid ghost contributions [29].

The multipole analysis is the most useful way to describe the external field for the sources localized in a finite region of space [30]. The symmetric and trace-free (STF) formalism in terms of the irreducible Cartesian tensors, developed by Thorne [31] and Blanchet and Damour [32,33], is one of the important methods with respect to the multipole analysis, and it can render transparent the algebraic structure of vectorial and tensorial harmonics expansions. The relevant STF technique is summarized in Ref. [30]. The multipole analysis in terms of the STF formalism can be used to explore the nonlinear effects of a gravitational theory [31]. In this paper, we will apply the STF formalism to $f(R, \mathcal{G})$ gravity to investigate its related nonlinear effects.

In this paper, we assume that $f(R, \mathcal{G})$ is analytic at $R = R_0$ and $\mathcal{G} = \mathcal{G}_0$, and such models are physically interesting and allow to recover the results of GR and the correct boundary and asymptotic conditions in general. Then, $f(R, \mathcal{G})$ can be expressed as a power series:

$$\begin{aligned} f(R, \mathcal{G}) &= f(R_0, \mathcal{G}_0) + \frac{\partial f}{\partial R}(R_0, \mathcal{G}_0)(R - R_0) \\ &\quad + \frac{\partial f}{\partial \mathcal{G}}(R_0, \mathcal{G}_0)(\mathcal{G} - \mathcal{G}_0) + \cdots \\ &= a_0 + a_1 R + a_2 \mathcal{G} + a_{11} R^2 + a_{12} R \mathcal{G} + a_{22} \mathcal{G}^2 \\ &\quad + a_{111} R^3 + a_{112} R^2 \mathcal{G} + a_{122} R \mathcal{G}^2 \\ &\quad + a_{222} \mathcal{G}^3 + \cdots \end{aligned} \quad (1.1)$$

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Under the post-Minkowskian method, the leading order of the GB scalar is $\mathcal{G}^{(2)}$, so as the $f(R)$ gravity indicated in Ref. [34], the vacuum solution of 0-order field equations of $f(R, \mathcal{G})$ gravity shows that $a_0 = 0$. Moreover, to link the Lagrangian (1.1) to GR, $a_1 = 1$ is needed.

Because the integral of $\mathcal{G}$ over a 4-dimensional manifold is a topological invariant, the term $a_2\mathcal{G}$ in the above Lagrangian does not contribute to the field equation of $f(R, \mathcal{G})$ gravity. Under the post-Minkowskian method, the coupling constants a_{11} and a_{12} will appear in the second-order field equation of $f(R, \mathcal{G})$ gravity. This implies that $\mathcal{G}$ will play an important role in the nonlinear effects, and therefore, it is worthwhile to investigate the nonlinear effects contributed by $\mathcal{G}$.

In our preceding papers [35,36], referred to as I and II hereafter, respectively, the multipole analysis for linearized $f(R)$ gravity with irreducible Cartesian tensors has been presented, and the relevant results include:

- The field equations of $f(R)$ gravity are rewritten in the form of obvious wave equations with the stress–energy pseudotensor of the matter fields and the gravitational field as the sources under de Donder condition. This process is similar to that of GR [31,37];
- The field equations and the effective stress–energy tensor of GWs for linearized $f(R)$ gravity are presented, and the latter is a typical second-order nonlinear result;
- The STF formalism is applied to the linearized $f(R)$ gravity, and its explicit expression of multipole expansion is derived;
- The $1/r$ expansion in the distance to the source is applied to the linearized $f(R)$ gravity, and its multipole expansion in the radiation field with irreducible Cartesian tensors is presented;
- As the further second-order nonlinear results, the energy, momentum, and angular momentum carried by GWs in linearized $f(R)$ gravity are provided.

In this paper, we will follow the above process to discuss the multipole analysis for linearized $f(R, \mathcal{G})$ gravity with irreducible Cartesian tensors.

For $f(R, \mathcal{G})$ gravity, similarly to $f(R)$ gravity [35], one has to introduce the gravitational field amplitude

$$h^{\mu\nu} := \sqrt{-g}g^{\mu\nu} - \eta^{\mu\nu} \quad (1.2)$$

and the effective gravitational field amplitude

$$\tilde{h}^{\mu\nu} := f_R \sqrt{-g}g^{\mu\nu} - \eta^{\mu\nu}, \quad (1.3)$$

where $g^{\mu\nu}$ denotes the contravariant metric, $\eta^{\mu\nu}$ represents an auxiliary Minkowskian metric, g is the determinant of metric $g_{\mu\nu}$, and $f_R = \partial_R f$. With the help of $h^{\mu\nu}$ and $\tilde{h}^{\mu\nu}$,

by using the same method in Refs. [35,38], the field equations of $f(R, \mathcal{G})$ gravity can be rewritten in the form of obvious wave equations under the de Donder condition, and the source term, composed of all the nonlinear terms under the post-Minkowskian method, is the stress–energy pseudotensor of the matter fields and the gravitational field. This is the first purpose of the present paper. If $\tilde{h}^{\mu\nu}$ is a perturbation, the resulting field equations of linearized $f(R, \mathcal{G})$ gravity are the same as those of linearized $f(R)$ gravity [35], as expected. Furthermore, under this condition, the effective stress–energy tensor of GWs in linearized $f(R, \mathcal{G})$ gravity, as a typical second-order nonlinear result, is shown to be the same as that of linearized $f(R)$ gravity, which implies that the GB scalar $\mathcal{G}$ does not contribute to the effective stress–energy tensor of GWs in linearized $f(R, \mathcal{G})$ gravity, though $\mathcal{G}$ usually plays an important role in the nonlinear effects as mentioned before. This is the second purpose of the present paper.

As the further second-order nonlinear results, the energy, momentum, and angular momentum carried by GWs in linearized $f(R, \mathcal{G})$ gravity need to be discussed. Similarly to GR [31] and $f(R)$ gravity [36], the energy and the momentum can be derived directly by use of the effective stress–energy tensor of GWs, but the angular momentum cannot. So, a new way, not depending on the the effective stress–energy tensor of GWs, need to be found to deal with the angular momentum. In this paper, by following Refs. [31,36,39], the energy, momentum, and angular momentum carried by GWs in linearized $f(R, \mathcal{G})$ gravity will be dealt with in a unified way.

This unified way requires that the multipole expansion of the linearized $f(R, \mathcal{G})$ gravity is discussed firstly. By definitions (1.2) and (1.3), the true gravitational field amplitude $h^{\mu\nu}$ in $f(R, \mathcal{G})$ gravity can be read out from $\tilde{h}^{\mu\nu}$. For the linearized $f(R, \mathcal{G})$ gravity, the relation between $h^{\mu\nu}$ and $\tilde{h}^{\mu\nu}$ becomes linear and simple, namely,

$$h^{\mu\nu} = \tilde{h}^{\mu\nu} - 2a_{11}R^{(1)}\eta^{\mu\nu}, \quad (1.4)$$

where $R^{(1)}$ is the linear part of Ricci scalar R . This relation is the same as that in $f(R)$ gravity. As mentioned above, the linearized $f(R, \mathcal{G})$ gravity and the linearized $f(R)$ gravity have the same field equations, so the multipole expansions of $\tilde{h}^{\mu\nu}$ under the de Donder condition in these two gravitational models are also the same. Moreover, for these two models, $R^{(1)}$ satisfies the same Klein–Gordon (KG) equation with an external source and has the same multipole expansion. Therefore, Eq. (1.4) implies that these two gravitational models have the same multipole expansion, and in particular, they have the same multipole expansion in the radiation field. This is the third purpose of the present paper.

With the help of the relevant STF technique, the $1/r$ expansion in the distance to the source can be applied to the linearized $f(R, \mathcal{G})$ gravity. Based on this, the energy,

momentum, and angular momentum carried by GWs in linearized $f(R, \mathcal{G})$ gravity can be dealt with in the above unified way. Thus, we present the explicit expressions of the total power and rates of momentum and angular momentum carried by GWs in linearized $f(R, \mathcal{G})$ gravity, and prove that the GB curvature scalar $\mathcal{G}$ does not contribute to them either. This is the fourth purpose of the present paper.

This paper is organized as follows. In Sect. 2, the notation, the relevant formulas of STF formalism, and the review of the metric $f(R, \mathcal{G})$ gravity are described. In Sect. 3, by using the same method in Refs. [35, 38], we show that the field equations of $f(R, \mathcal{G})$ gravity can be rewritten in the form of obvious wave equations with the stress–energy pseudotensor of the matter fields and the gravitational field as the source under de Donder condition. Upon this, we derive the field equations and the effective stress–energy tensor of GWs for linearized $f(R, \mathcal{G})$ gravity. In Sect. 4, we show that the linearized $f(R, \mathcal{G})$ gravity and the linearized $f(R)$ gravity have the same multipole expansion. Then, we present its expression, and in particular, the one in the radiation field. Further, by using the $1/r$ expansion in the distance to the source, we evaluate the energy, momentum, and angular momentum carried by GWs in linearized $f(R, \mathcal{G})$ gravity systematically. In Sect. 5, we present the conclusions and make some discussions.

2 Preliminary

2.1 Notation

The notation in this paper is the same as that in I and II [35, 36]. The international system of units is used throughout the paper. The signature of the metric $g^{\mu\nu}$ is $(-, +, +, +)$. ϵ_{ijk} is the three-dimensional Levi-Civita symbol with $\epsilon_{123} = 1$. The greek indices run from 0 to 3 and the latin indices run from 1 to 3, where repeated indices indicate to be summed. In the linearized gravitational theory, $(x^\mu) = (ct, x_i)$ behave as though they were Minkowskian coordinates. The spherical coordinate system (ct, r, θ, φ) is defined by

$$x_1 = r \sin \theta \cos \varphi, \quad x_2 = r \sin \theta \sin \varphi, \quad x_3 = r \cos \theta. \quad (2.1)$$

As in the flat space, $\mathbf{x}$ is used to denote the radial vector, and x_i in Eq. (2.1) are its components. $\mathbf{n} = \mathbf{x}/r = (n_i)$ denotes the unit radial vector, where r is the length of $\mathbf{x}$, and $n_i = x_i/r$ are the components of $\mathbf{n}$. Obviously, by Eq. (2.1),

$$\partial_r = \frac{\partial}{\partial r} = n_1 \frac{\partial}{\partial x_1} + n_2 \frac{\partial}{\partial x_2} + n_3 \frac{\partial}{\partial x_3} = n_i \partial_i. \quad (2.2)$$

The symbol

$$B_{I_l} \equiv B_{i_1 i_2 \dots i_l} \quad (2.3)$$

is used to denote the Cartesian tensor with l indices [31], and especially, the tensor products of l radial and unit radial vectors are abbreviated by

$$X_{I_l} = X_{i_1 i_2 \dots i_l} := x_{i_1} x_{i_2} \dots x_{i_l}, \quad (2.4)$$

$$N_{I_l} = N_{i_1 i_2 \dots i_l} := n_{i_1} n_{i_2} \dots n_{i_l} \quad (2.5)$$

with

$$X_{I_l} = r^l N_{I_l}. \quad (2.6)$$

2.2 Relevant formulas in STF formalism

In this subsection, the relevant formulas in the STF formalism are listed [30–32, 35, 36]. The symmetric part and the STF part of a Cartesian tensor B_{I_l} are expressed by

$$B_{(I_l)} = B_{(i_1 i_2 \dots i_l)} := \frac{1}{l!} \sum_{\sigma} B_{i_{\sigma(1)} i_{\sigma(2)} \dots i_{\sigma(l)}}, \quad (2.7)$$

$$\begin{aligned} \hat{B}_{I_l} &\equiv B_{<I_l>} \equiv B_{<i_1 i_2 \dots i_l>} \\ &:= \sum_{k=0}^{[\frac{l}{2}]} b_k \delta_{(i_1 i_2} \dots \delta_{i_{2k-1} i_{2k}} S_{i_{2k+1} \dots i_l) a_1 a_1 \dots a_k a_k}, \end{aligned} \quad (2.8)$$

respectively, where σ runs over all permutations of $(12 \dots l)$,

$$S_{I_l} = B_{(I_l)}, \quad (2.9)$$

$$b_k = (-1)^k \frac{(2l - 2k - 1)!!}{(2l - 1)!!} \frac{l!}{(2k)!!(l - 2k)!}. \quad (2.10)$$

Moreover, there are

$$\hat{N}_{I_l} = \sum_{k=0}^{[\frac{l}{2}]} b_k \delta_{(i_1 i_2} \dots \delta_{i_{2k-1} i_{2k}} N_{i_{2k+1} \dots i_l)}, \quad (2.11)$$

$$\hat{\partial}_{I_l} = \sum_{k=0}^{[\frac{l}{2}]} b_k \delta_{(i_1 i_2} \dots \delta_{i_{2k-1} i_{2k}} \partial_{i_{2k+1} \dots i_l)} \Delta^k, \quad (2.12)$$

$$\partial_i n_j = \frac{1}{r} (\delta_{ij} - n_i n_j), \quad (2.13)$$

$$\partial_i \left(F \left(t - \frac{\epsilon r}{c} \right) \right) = -\frac{\epsilon n_i}{c} \partial_t F \left(t - \frac{\epsilon r}{c} \right), \quad (\epsilon^2 = 1), \quad (2.14)$$

$$\begin{aligned} & \hat{\partial}_{I_l} \left(\frac{F(t - \epsilon r/c)}{r} \right) \\ &= (-\epsilon)^l \hat{N}_{I_l} \sum_{k=0}^l \frac{(l+k)!}{(2\epsilon)^k k! (l-k)!} \frac{\partial_t^{l-k} F(t - \epsilon r/c)}{c^{l-k} r^{k+1}}, \quad (\epsilon^2 = 1), \end{aligned} \quad (2.15)$$

where $\Delta = \partial_a \partial_a$, $\partial_{I_l} \equiv \partial_{i_1 i_2 \dots i_l} := \partial_{i_1} \partial_{i_2} \dots \partial_{i_l}$ and ∂_t^{l-k} is the $(l-k)$ th derivative with respect to t .

The integral formulas over a unit sphere are

$$\int N_{I_{2l+1}} d\Omega = 0, \quad (2.16)$$

$$\int N_{I_{2l}} d\Omega = \frac{4\pi}{2l+1} \delta_{(i_1 i_2 \dots i_{2l-1} i_{2l})}, \quad (2.17)$$

where l is the non-negative integer and $d\Omega$ is the element of the solid angle about the radial vector.

2.3 Review of the metric $f(R, \mathcal{G})$ gravity

We restrict our attention to the metric $f(R, \mathcal{G})$ gravity [25], and its action is

$$S = \frac{1}{2\kappa c} \int dx^4 \sqrt{-g} f(R, \mathcal{G}) + S_M(g^{\mu\nu}, \psi), \quad (2.18)$$

where f is an arbitrary function of the Ricci scalar R and the GB curvature scalar $\mathcal{G}$, $\kappa = 8\pi G/c^4$, and $S_M(g^{\mu\nu}, \psi)$ is the matter action. Varying this action with respect to the metric $g^{\mu\nu}$ gives the gravitational field equations and the corresponding trace equation

$$H_{\mu\nu} = \kappa T_{\mu\nu}, \quad H = \kappa T, \quad (2.19)$$

where

$$\begin{aligned} H_{\mu\nu} = & -\frac{g_{\mu\nu}}{2} f + R_{\mu\nu} f_R + g_{\mu\nu} \square f_R - \nabla_\mu \nabla_\nu f_R \\ & + \frac{g_{\mu\nu}}{2} f_{\mathcal{G}} \mathcal{G} + 2R g_{\mu\nu} \square f_{\mathcal{G}} - 2R \nabla_\mu \nabla_\nu f_{\mathcal{G}} \\ & + 4R_{\nu}^{\lambda} \nabla_{\lambda} \nabla_{\mu} f_{\mathcal{G}} + 4R_{\mu}^{\lambda} \nabla_{\lambda} \nabla_{\nu} f_{\mathcal{G}} \\ & - 4g_{\mu\nu} R^{\alpha\beta} \nabla_{\alpha} \nabla_{\beta} f_{\mathcal{G}} \\ & - 4R_{\mu\nu} \square f_{\mathcal{G}} + 4R_{\mu\rho\nu\sigma} \nabla^{\rho} \nabla^{\sigma} f_{\mathcal{G}}, \end{aligned} \quad (2.20)$$

$$\begin{aligned} H = & -2f + R f_R + 3\square f_R + 2f_{\mathcal{G}} \mathcal{G} \\ & + 2R \square f_{\mathcal{G}} - 4R^{\alpha\beta} \nabla_{\alpha} \nabla_{\beta} f_{\mathcal{G}} \end{aligned} \quad (2.21)$$

with $f_{\mathcal{G}} = \partial_{\mathcal{G}} f$, and $T_{\mu\nu}$ is the stress–energy tensor of matter fields. In the present paper, only the polynomial $f(R, \mathcal{G})$ models, namely Eq. (1.1), is considered.

3 The field equations and stress–energy pseudotensor of $f(R, \mathcal{G})$ gravity under de Donder condition

3.1 Obvious wave equation in $f(R, \mathcal{G})$ gravity

By following GR [37, 38] and $f(R)$ gravity [35], we begin to rewrite the field equations of $f(R, \mathcal{G})$ gravity in the form of obvious wave equations in a fictitious flat spacetime under the de Donder condition. Firstly, by Eq. (1.2), the gravitational field amplitude $h^{\mu\nu}$ is defined as

$$h^{\mu\nu} := \bar{g}^{\mu\nu} - \eta^{\mu\nu}, \quad (3.1)$$

where

$$\bar{g}^{\mu\nu} := \sqrt{-g} g^{\mu\nu} \quad (3.2)$$

is the densitized inverse metric. It should be noted that $h^{\mu\nu}$ in Eq. (3.1) is not necessarily a perturbation. According to the Ref. [38],

$$\begin{aligned} \Gamma^{\lambda\mu\nu} &:= g^{\mu\alpha} g^{\nu\beta} \Gamma_{\alpha\beta}^{\lambda} \\ &= \frac{1}{2g} (\bar{g}^{\mu\rho} \partial_{\rho} \bar{g}^{\nu\lambda} + \bar{g}^{\nu\rho} \partial_{\rho} \bar{g}^{\mu\lambda} - \bar{g}^{\lambda\rho} \partial_{\rho} \bar{g}^{\mu\nu}) \\ &\quad + \frac{1}{2} (y^{\mu} g^{\nu\lambda} + y^{\nu} g^{\mu\lambda} - y^{\lambda} g^{\mu\nu}), \end{aligned} \quad (3.3)$$

$$y^{\mu} := g^{\mu\rho} \partial_{\rho} \ln \sqrt{-g}. \quad (3.4)$$

In terms of $h^{\mu\nu}$, the Christoffel symbols read

$$\begin{aligned} \Gamma_{\sigma\nu}^{\mu} = & -\frac{1}{2\sqrt{-g}} (g_{\nu\alpha} \partial_{\sigma} h^{\alpha\mu} + g_{\sigma\alpha} \partial_{\nu} h^{\alpha\mu} - g_{\sigma\alpha} g_{\nu\beta} g^{\mu\rho} \partial_{\rho} h^{\alpha\beta}) \\ & + \frac{1}{4\sqrt{-g}} (\delta_{\nu}^{\mu} g_{\epsilon\pi} \partial_{\sigma} h^{\epsilon\pi} + \delta_{\sigma}^{\mu} g_{\epsilon\pi} \partial_{\nu} h^{\epsilon\pi} \\ & - g_{\sigma\nu} g_{\epsilon\pi} g^{\mu\rho} \partial_{\rho} h^{\epsilon\pi}), \end{aligned} \quad (3.5)$$

and then, the Riemann tensor, Ricci tensor, and Ricci scalar read, respectively,

$$\begin{aligned} R^{\mu\nu\rho\sigma} = & -\frac{1}{2\sqrt{-g}} \left(g^{\rho\epsilon} g^{\nu\pi} \partial_{\epsilon} \partial_{\pi} h^{\mu\sigma} \right. \\ & - g^{\sigma\epsilon} g^{\nu\pi} \partial_{\epsilon} \partial_{\pi} h^{\mu\rho} + g^{\sigma\epsilon} g^{\mu\pi} \partial_{\epsilon} \partial_{\pi} h^{\nu\rho} \\ & - g^{\rho\epsilon} g^{\mu\pi} \partial_{\epsilon} \partial_{\pi} h^{\nu\sigma} + \frac{1}{2} \left(g^{\mu\rho} g^{\sigma\epsilon} g^{\nu\pi} - g^{\mu\sigma} g^{\rho\epsilon} g^{\nu\pi} \right. \\ & \left. \left. + g^{\nu\sigma} g^{\rho\epsilon} g^{\mu\pi} - g^{\nu\rho} g^{\sigma\epsilon} g^{\mu\pi} \right) g_{\alpha\beta} \partial_{\epsilon} \partial_{\pi} h^{\alpha\beta} \right) \\ & - \frac{1}{2g} \left(\frac{1}{2} \left(g^{\rho\delta} \partial_{\delta} h^{\mu\lambda} \partial_{\lambda} h^{\nu\sigma} - g^{\sigma\delta} \partial_{\delta} h^{\mu\lambda} \partial_{\lambda} h^{\nu\rho} \right. \right. \\ & + g^{\sigma\delta} \partial_{\delta} h^{\nu\lambda} \partial_{\lambda} h^{\mu\rho} - g^{\rho\delta} \partial_{\delta} h^{\nu\lambda} \partial_{\lambda} h^{\mu\sigma} \\ & + g^{\mu\delta} \partial_{\delta} h^{\rho\lambda} \partial_{\lambda} h^{\nu\sigma} - g^{\mu\delta} \partial_{\delta} h^{\sigma\lambda} \partial_{\lambda} h^{\nu\rho} \\ & \left. \left. + g^{\nu\delta} \partial_{\delta} h^{\sigma\lambda} \partial_{\lambda} h^{\mu\rho} - g^{\nu\delta} \partial_{\delta} h^{\rho\lambda} \partial_{\lambda} h^{\mu\sigma} \right) \right) \end{aligned}$$

$$\begin{aligned}
& + g^{\lambda\tau} \partial_\lambda h^{\mu\sigma} \partial_\tau h^{\nu\rho} - g^{\lambda\tau} \partial_\lambda h^{\mu\rho} \partial_\tau h^{\nu\sigma} \Big) \\
& + \frac{1}{2} g_{\alpha\beta} \Big(g^{\rho\epsilon} g^{\sigma\pi} \partial_\epsilon h^{\nu\beta} \partial_\pi h^{\mu\alpha} - g^{\sigma\epsilon} g^{\rho\pi} \partial_\epsilon h^{\nu\beta} \partial_\pi h^{\mu\alpha} \\
& + g^{\mu\epsilon} g^{\nu\pi} \partial_\epsilon h^{\sigma\beta} \partial_\pi h^{\rho\alpha} - g^{\mu\epsilon} g^{\nu\pi} \partial_\epsilon h^{\rho\beta} \partial_\pi h^{\sigma\alpha} \\
& + g^{\rho\epsilon} g^{\nu\pi} \partial_\epsilon h^{\mu\alpha} \partial_\pi h^{\sigma\beta} - g^{\sigma\epsilon} g^{\nu\pi} \partial_\epsilon h^{\mu\alpha} \partial_\pi h^{\rho\beta} \\
& + g^{\sigma\epsilon} g^{\mu\pi} \partial_\epsilon h^{\nu\alpha} \partial_\pi h^{\rho\beta} - g^{\rho\epsilon} g^{\mu\pi} \partial_\epsilon h^{\nu\alpha} \partial_\pi h^{\sigma\beta} \\
& + 2g^{\rho\epsilon} g^{\nu\pi} \partial_\epsilon h^{\sigma\beta} \partial_\pi h^{\mu\alpha} \\
& - 2g^{\sigma\epsilon} g^{\nu\pi} \partial_\epsilon h^{\rho\beta} \partial_\pi h^{\mu\alpha} + 2g^{\sigma\epsilon} g^{\mu\pi} \partial_\epsilon h^{\rho\beta} \partial_\pi h^{\nu\alpha} \\
& - 2g^{\rho\epsilon} g^{\mu\pi} \partial_\epsilon h^{\sigma\beta} \partial_\pi h^{\nu\alpha} \Big) \\
& + \frac{1}{4} \Big(g^{\sigma\delta} g^{\rho\nu} \partial_\delta h^{\mu\lambda} - g^{\rho\delta} g^{\sigma\nu} \partial_\delta h^{\mu\lambda} \\
& + g^{\rho\delta} g^{\sigma\mu} \partial_\delta h^{\nu\lambda} - g^{\sigma\delta} g^{\rho\mu} \partial_\delta h^{\nu\lambda} + g^{\mu\delta} g^{\rho\nu} \partial_\delta h^{\sigma\lambda} \\
& - g^{\mu\delta} g^{\sigma\nu} \partial_\delta h^{\rho\lambda} + g^{\nu\delta} g^{\sigma\mu} \partial_\delta h^{\rho\lambda} \\
& - g^{\nu\delta} g^{\rho\mu} \partial_\delta h^{\sigma\lambda} \Big) g_{\alpha\beta} \partial_\lambda h^{\alpha\beta} \\
& + \frac{1}{4} \Big(g^{\sigma\nu} \partial_\lambda h^{\rho\mu} - g^{\rho\nu} \partial_\lambda h^{\sigma\mu} + g^{\rho\mu} \partial_\lambda h^{\sigma\nu} \\
& - g^{\sigma\mu} \partial_\lambda h^{\rho\nu} \Big) g^{\lambda\tau} g_{\alpha\beta} \partial_\tau h^{\alpha\beta} \\
& + \frac{1}{8} g_{\alpha\beta} g_{\lambda\tau} \Big(g^{\rho\epsilon} g^{\mu\pi} g^{\sigma\nu} - g^{\sigma\epsilon} g^{\mu\pi} g^{\rho\nu} \\
& + g^{\sigma\epsilon} g^{\nu\pi} g^{\rho\mu} - g^{\rho\epsilon} g^{\nu\pi} g^{\sigma\mu} \Big) \Big(4\partial_\epsilon h^{\beta\tau} \partial_\pi h^{\alpha\lambda} \\
& + \partial_\epsilon h^{\lambda\tau} \partial_\pi h^{\alpha\beta} \Big) \\
& + \frac{1}{8} \Big(g^{\mu\sigma} g^{\rho\nu} - g^{\mu\rho} g^{\sigma\nu} \Big) g^{\lambda\tau} g_{\alpha\beta} g_{\epsilon\pi} \partial_\lambda h^{\alpha\beta} \partial_\tau h^{\epsilon\pi} \Big),
\end{aligned} \tag{3.6}$$

$$\begin{aligned}
R^{\mu\nu} &= \frac{1}{2\sqrt{-g}} g^{\alpha\beta} \partial_\alpha \partial_\beta h^{\mu\nu} - \frac{1}{2g} g^{\mu\alpha} g_{\beta\tau} \partial_\lambda h^{\nu\tau} \partial_\alpha h^{\beta\lambda} \\
&+ \frac{1}{4g} g^{\mu\alpha} g^{\nu\beta} g_{\lambda\tau} g_{\epsilon\pi} \partial_\alpha h^{\lambda\pi} \partial_\beta h^{\tau\epsilon} \\
&+ \frac{1}{2g} g_{\alpha\beta} g^{\lambda\tau} \partial_\lambda h^{\mu\alpha} \partial_\tau h^{\nu\beta} \\
&+ \frac{1}{2g} \partial_\alpha h^{\mu\beta} \partial_\beta h^{\nu\alpha} - \frac{1}{2g} g^{\nu\alpha} g_{\beta\tau} \partial_\lambda h^{\mu\tau} \partial_\alpha h^{\beta\lambda} \\
&- \frac{1}{8g} g^{\mu\alpha} g^{\nu\beta} g_{\tau\epsilon} g_{\lambda\pi} \partial_\alpha h^{\lambda\pi} \partial_\beta h^{\tau\epsilon} \\
&- \frac{1}{4g} g^{\mu\nu} g_{\rho\tau} g_{\epsilon\sigma} g^{\alpha\beta} \partial_\alpha h^{\rho\sigma} \partial_\beta h^{\tau\epsilon} \\
&- \frac{1}{4\sqrt{-g}} g^{\mu\nu} g^{\alpha\beta} g_{\rho\sigma} \partial_\alpha \partial_\beta h^{\rho\sigma} - \frac{1}{2\sqrt{-g}} g^{\mu\alpha} \partial_\alpha \partial_\lambda h^{\nu\lambda} \\
&- \frac{1}{2\sqrt{-g}} g^{\nu\alpha} \partial_\alpha \partial_\lambda h^{\mu\lambda} + \frac{1}{4g} g^{\mu\nu} g_{\rho\sigma} \partial_\alpha h^{\rho\sigma} \partial_\lambda h^{\alpha\lambda} \\
&- \frac{1}{2g} \partial_\alpha h^{\mu\nu} \partial_\lambda h^{\alpha\lambda},
\end{aligned} \tag{3.7}$$

$$R = -\frac{1}{4g} g^{\alpha\beta} g_{\rho\tau} g_{\epsilon\sigma} \partial_\alpha h^{\rho\sigma} \partial_\beta h^{\tau\epsilon}$$

$$\begin{aligned}
& - \frac{1}{2\sqrt{-g}} g_{\rho\sigma} g^{\alpha\beta} \partial_\alpha \partial_\beta h^{\rho\sigma} \\
& - \frac{1}{\sqrt{-g}} \partial_\alpha \partial_\beta h^{\alpha\beta} + \frac{1}{2g} g_{\rho\sigma} \partial_\alpha h^{\rho\sigma} \partial_\lambda h^{\alpha\lambda} \\
& - \frac{1}{2g} g_{\mu\nu} \partial_\lambda h^{\mu\tau} \partial_\tau h^{\nu\lambda} - \frac{1}{8g} g^{\mu\nu} g_{\tau\epsilon} g_{\lambda\pi} \partial_\mu h^{\lambda\pi} \partial_\nu h^{\tau\epsilon}.
\end{aligned} \tag{3.8}$$

In order to apply the de Donder condition, we need to define the effective gravitational field amplitude $\tilde{h}^{\mu\nu}$ by Eq. (1.3), namely,

$$\tilde{h}^{\mu\nu} := \tilde{g}^{\mu\nu} - \eta^{\mu\nu}, \tag{3.9}$$

$$\tilde{g}^{\mu\nu} := f_R \sqrt{-g} g^{\mu\nu}. \tag{3.10}$$

Obviously, besides the information of metric, it also contains the information of the function f_R , which is, from (1.1),

$$f_R = 1 + 2a_{11}R + a_{12}\mathcal{G} + 3a_{111}R^2 + 2a_{112}R\mathcal{G} + a_{122}\mathcal{G}^2 + \dots \tag{3.11}$$

For GR, the de Donder condition is the condition for the harmonic coordinates [37, 38]:

$$\square x^\mu = 0 \Leftrightarrow \partial_\mu \tilde{g}^{\mu\nu} = \partial_\mu h^{\mu\nu} = 0. \tag{3.12}$$

For $f(R)$ gravity [35], the de Donder condition in GR has been generalized to

$$\partial_\mu \tilde{g}^{\mu\nu} = \partial_\mu \tilde{h}^{\mu\nu} = 0 \Leftrightarrow \square x^\mu = -g^{\mu\nu} \partial_\nu \ln f_R, \tag{3.13}$$

where the definitions of $\tilde{h}^{\mu\nu}$ and $\tilde{g}^{\mu\nu}$ are the same as Eqs. (3.9) and (3.10), respectively. In this paper, we also adopt the de Donder condition (3.13) for $f(R, \mathcal{G})$ gravity.

Now, we will express the Riemann tensor, the Ricci tensor, and the Ricci scalar in terms of $\tilde{h}^{\mu\nu}$ with the help of the de Donder condition (3.13). By Eqs. (3.1), (3.2), (3.9), and (3.10), there is

$$h^{\mu\nu} = \frac{1}{f_R} \tilde{h}^{\mu\nu} + \left(\frac{1}{f_R} - 1 \right) \eta^{\mu\nu}. \tag{3.14}$$

It immediately results in

$$\partial_\lambda h^{\mu\nu} = \frac{1}{f_R} \partial_\lambda \tilde{h}^{\mu\nu} - \tilde{g}^{\mu\nu} \partial_\lambda \ln f_R. \tag{3.15}$$

With the help of the de Donder condition (3.13), the substitution of (3.14) and (3.15) in Eqs. (3.6), (3.7) and (3.8) gives rise to the expressions of $R^{\mu\nu\rho\sigma}$, $R^{\mu\nu}$ and R in terms of $\tilde{h}^{\mu\nu}$:

$$\begin{aligned}
R^{\mu\nu\rho\sigma} &= -\frac{1}{2f_R\sqrt{-g}} \Big(g^{\rho\epsilon} g^{\nu\pi} \partial_\epsilon \partial_\pi \tilde{h}^{\mu\sigma} - g^{\sigma\epsilon} g^{\nu\pi} \partial_\epsilon \partial_\pi \tilde{h}^{\mu\rho} \\
&+ g^{\sigma\epsilon} g^{\mu\pi} \partial_\epsilon \partial_\pi \tilde{h}^{\nu\rho} - g^{\rho\epsilon} g^{\mu\pi} \partial_\epsilon \partial_\pi \tilde{h}^{\nu\sigma}
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2} \left(g^{\mu\rho} g^{\sigma\epsilon} g^{v\pi} - g^{\mu\sigma} g^{\rho\epsilon} g^{v\pi} \right. \\
& + g^{v\sigma} g^{\rho\epsilon} g^{\mu\pi} - g^{v\rho} g^{\sigma\epsilon} g^{\mu\pi} \left. \right) g_{\alpha\beta} \partial_\epsilon \partial_\pi \tilde{h}^{\alpha\beta} \Big) \\
& - \frac{1}{2g f_R^2} \left(\frac{1}{2} \left(g^{\rho\delta} \partial_\delta \tilde{h}^{\mu\lambda} \partial_\lambda \tilde{h}^{v\sigma} - g^{\sigma\delta} \partial_\delta \tilde{h}^{\mu\lambda} \partial_\lambda \tilde{h}^{v\rho} \right. \right. \\
& + g^{\sigma\delta} \partial_\delta \tilde{h}^{v\lambda} \partial_\lambda \tilde{h}^{\mu\rho} - g^{\rho\delta} \partial_\delta \tilde{h}^{v\lambda} \partial_\lambda \tilde{h}^{\mu\sigma} \\
& + g^{\mu\delta} \partial_\delta \tilde{h}^{\rho\lambda} \partial_\lambda \tilde{h}^{v\sigma} - g^{\mu\delta} \partial_\delta \tilde{h}^{\sigma\lambda} \partial_\lambda \tilde{h}^{v\rho} \\
& + g^{v\delta} \partial_\delta \tilde{h}^{\sigma\lambda} \partial_\lambda \tilde{h}^{\mu\rho} - g^{v\delta} \partial_\delta \tilde{h}^{\rho\lambda} \partial_\lambda \tilde{h}^{\mu\sigma} \\
& + g^{\lambda\tau} \partial_\lambda \tilde{h}^{\mu\sigma} \partial_\tau \tilde{h}^{v\rho} \\
& - g^{\lambda\tau} \partial_\lambda \tilde{h}^{\mu\rho} \partial_\tau \tilde{h}^{v\sigma} \Big) + \frac{1}{2} g_{\alpha\beta} \left(g^{\rho\epsilon} g^{\sigma\pi} \partial_\epsilon \tilde{h}^{v\beta} \partial_\pi \tilde{h}^{\mu\alpha} \right. \\
& - g^{\sigma\epsilon} g^{\rho\pi} \partial_\epsilon \tilde{h}^{v\beta} \partial_\pi \tilde{h}^{\mu\alpha} + g^{\mu\epsilon} g^{v\pi} \partial_\epsilon \tilde{h}^{\sigma\beta} \partial_\pi \tilde{h}^{\rho\alpha} \\
& - g^{\mu\epsilon} g^{v\pi} \partial_\epsilon \tilde{h}^{\rho\beta} \partial_\pi \tilde{h}^{\sigma\alpha} \\
& + g^{\rho\epsilon} g^{v\pi} \partial_\epsilon \tilde{h}^{\mu\alpha} \partial_\pi \tilde{h}^{\sigma\beta} - g^{\sigma\epsilon} g^{v\pi} \partial_\epsilon \tilde{h}^{\mu\alpha} \partial_\pi \tilde{h}^{\rho\beta} \\
& + g^{\sigma\epsilon} g^{\mu\pi} \partial_\epsilon \tilde{h}^{v\alpha} \partial_\pi \tilde{h}^{\rho\beta} - g^{\rho\epsilon} g^{\mu\pi} \partial_\epsilon \tilde{h}^{v\alpha} \partial_\pi \tilde{h}^{\sigma\beta} \\
& + 2g^{\rho\epsilon} g^{v\pi} \partial_\epsilon \tilde{h}^{\sigma\beta} \partial_\pi \tilde{h}^{\mu\alpha} \\
& - 2g^{\sigma\epsilon} g^{v\pi} \partial_\epsilon \tilde{h}^{\rho\beta} \partial_\pi \tilde{h}^{\mu\alpha} + 2g^{\sigma\epsilon} g^{\mu\pi} \partial_\epsilon \tilde{h}^{\rho\beta} \partial_\pi \tilde{h}^{v\alpha} \\
& \left. - 2g^{\rho\epsilon} g^{\mu\pi} \partial_\epsilon \tilde{h}^{\sigma\beta} \partial_\pi \tilde{h}^{v\alpha} \right) \\
& + \frac{1}{4} \left(g^{\sigma\delta} g^{\rho\nu} \partial_\delta \tilde{h}^{\mu\lambda} - g^{\rho\delta} g^{\sigma\nu} \partial_\delta \tilde{h}^{\mu\lambda} \right. \\
& + g^{\rho\delta} g^{\sigma\mu} \partial_\delta \tilde{h}^{v\lambda} - g^{\sigma\delta} g^{\rho\mu} \partial_\delta \tilde{h}^{v\lambda} \\
& + g^{\mu\delta} g^{\rho\nu} \partial_\delta \tilde{h}^{\sigma\lambda} - g^{\mu\delta} g^{\sigma\nu} \partial_\delta \tilde{h}^{\rho\lambda} \\
& + g^{v\delta} g^{\sigma\mu} \partial_\delta \tilde{h}^{\rho\lambda} - g^{v\delta} g^{\rho\mu} \partial_\delta \tilde{h}^{\sigma\lambda} \Big) g_{\alpha\beta} \partial_\lambda \tilde{h}^{\alpha\beta} \\
& + \frac{1}{4} \left(g^{\sigma\nu} \partial_\lambda \tilde{h}^{\rho\mu} - g^{\rho\nu} \partial_\lambda \tilde{h}^{\sigma\mu} + g^{\rho\mu} \partial_\lambda \tilde{h}^{\sigma\nu} \right. \\
& \left. - g^{\sigma\mu} \partial_\lambda \tilde{h}^{\rho\nu} \right) g^{\lambda\tau} g_{\alpha\beta} \partial_\tau \tilde{h}^{\alpha\beta} \\
& + \frac{1}{8} g_{\alpha\beta} g_{\lambda\tau} \left(g^{\rho\epsilon} g^{\mu\pi} g^{\sigma\nu} - g^{\sigma\epsilon} g^{\mu\pi} g^{\rho\nu} \right. \\
& + g^{\sigma\epsilon} g^{v\pi} g^{\rho\mu} - g^{\rho\epsilon} g^{v\pi} g^{\sigma\mu} \Big) (4\partial_\epsilon \tilde{h}^{\beta\tau} \partial_\pi \tilde{h}^{\alpha\lambda} \\
& + \partial_\epsilon \tilde{h}^{\lambda\tau} \partial_\pi \tilde{h}^{\alpha\beta} \Big) + \frac{1}{8} \left(g^{\mu\sigma} g^{\rho\nu} \right. \\
& \left. - g^{\mu\rho} g^{\sigma\nu} \right) g^{\lambda\tau} g_{\alpha\beta} g_{\epsilon\pi} \partial_\lambda \tilde{h}^{\alpha\beta} \partial_\tau \tilde{h}^{\epsilon\pi} \Big) \\
& + \frac{1}{4} \left(g^{\mu\sigma} g^{v\rho} - g^{\mu\rho} g^{v\sigma} \right) g^{\lambda\tau} \partial_\lambda \ln f_R \partial_\tau \ln f_R \\
& + \frac{1}{4} \left(g^{\rho\epsilon} g^{\mu\pi} g^{\sigma\nu} - g^{\sigma\epsilon} g^{\mu\pi} g^{\rho\nu} + g^{\sigma\epsilon} g^{v\pi} g^{\rho\mu} \right. \\
& \left. - g^{\rho\epsilon} g^{v\pi} g^{\sigma\mu} \right) \left(2\partial_\epsilon \partial_\pi \ln f_R \right. \\
& + \partial_\epsilon \ln f_R \partial_\pi \ln f_R \Big) - \frac{1}{4f_R \sqrt{-g}} \left(\left(g^{\sigma\delta} g^{\rho\nu} \partial_\delta \tilde{h}^{\mu\lambda} \right. \right. \\
& - g^{\rho\delta} g^{\sigma\nu} \partial_\delta \tilde{h}^{\mu\lambda} + g^{\rho\delta} g^{\sigma\mu} \partial_\delta \tilde{h}^{v\lambda} \\
& - g^{\sigma\delta} g^{\rho\mu} \partial_\delta \tilde{h}^{v\lambda} + g^{\mu\delta} g^{\rho\nu} \partial_\delta \tilde{h}^{\sigma\lambda} - g^{\mu\delta} g^{\sigma\nu} \partial_\delta \tilde{h}^{\rho\lambda} \\
& + g^{v\delta} g^{\sigma\mu} \partial_\delta \tilde{h}^{\rho\lambda} - g^{v\delta} g^{\rho\mu} \partial_\delta \tilde{h}^{\sigma\lambda} \Big) \partial_\lambda \ln f_R \\
& + \left(g^{\sigma\nu} \partial_\lambda \tilde{h}^{\rho\mu} - g^{\rho\nu} \partial_\lambda \tilde{h}^{\sigma\mu} + g^{\rho\mu} \partial_\lambda \tilde{h}^{\sigma\nu} - g^{\sigma\mu} \partial_\lambda \tilde{h}^{\rho\nu} \right) \\
& \times g^{\lambda\tau} \partial_\tau \ln f_R + \frac{1}{2} g_{\alpha\beta} \left(g^{\rho\epsilon} g^{\mu\pi} g^{\sigma\nu} - g^{\sigma\epsilon} g^{\mu\pi} g^{\rho\nu} \right. \\
& + g^{\sigma\epsilon} g^{v\pi} g^{\rho\mu} - g^{\rho\epsilon} g^{v\pi} g^{\sigma\mu} \Big) \left(\partial_\pi \tilde{h}^{\alpha\beta} \partial_\epsilon \ln f_R \right. \\
& + \partial_\epsilon \tilde{h}^{\alpha\beta} \partial_\pi \ln f_R \Big) \\
& + \left(g^{\mu\sigma} g^{\rho\nu} - g^{\mu\rho} g^{\sigma\nu} \right) g^{\lambda\tau} g_{\epsilon\pi} \partial_\tau \tilde{h}^{\epsilon\pi} \partial_\lambda \ln f_R \Big), \quad (3.16)
\end{aligned}$$

$$\begin{aligned}
R^{\mu\nu} = & -\frac{1}{2f_R \sqrt{-g}} g^{\alpha\beta} \partial_\alpha \tilde{h}^{\mu\nu} \partial_\beta \ln f_R \\
& + \frac{1}{2f_R \sqrt{-g}} g^{\alpha\beta} \partial_\alpha \partial_\beta \tilde{h}^{\mu\nu} - \frac{1}{2} g^{\mu\nu} g^{\alpha\beta} \partial_\alpha \ln f_R \partial_\beta \ln f_R \\
& + \frac{1}{2} g^{\mu\alpha} g^{v\beta} \partial_\alpha \ln f_R \partial_\beta \ln f_R \\
& + \frac{1}{2} g^{\mu\nu} g^{\alpha\beta} \partial_\alpha \partial_\beta \ln f_R + g^{\mu\alpha} g^{v\beta} \partial_\alpha \partial_\beta \ln f_R \\
& - \frac{1}{8f_R^2} g_{\beta\tau} g^{\alpha(\mu} \partial_\lambda \tilde{h}^{v)\tau} \partial_\alpha \tilde{h}^{\beta\lambda} \\
& + \frac{1}{4g f_R^2} g_{\lambda\tau} g_{\epsilon\pi} g^{\mu\alpha} g^{v\beta} \partial_\alpha \tilde{h}^{\lambda\pi} \partial_\beta \tilde{h}^{\tau\epsilon} \\
& - \frac{1}{2f_R \sqrt{-g}} g_{\rho\sigma} g^{\mu(\alpha} g^{\beta)\nu} \partial_\alpha \tilde{h}^{\rho\sigma} \partial_\beta \ln f_R \\
& + \frac{1}{2g f_R^2} g_{\alpha\beta} g^{\lambda\tau} \partial_\lambda \tilde{h}^{\mu\alpha} \partial_\tau \tilde{h}^{v\beta} \\
& + \frac{1}{2g f_R^2} \partial_\alpha \tilde{h}^{\mu\beta} \partial_\beta \tilde{h}^{v\alpha} \\
& + \frac{1}{f_R \sqrt{-g}} g^{\alpha(\mu} \partial_\alpha \tilde{h}^{v)\beta} \partial_\beta \ln f_R \\
& - \frac{1}{8g f_R^2} g_{\tau\epsilon} g_{\lambda\pi} g^{\mu\alpha} g^{v\beta} \partial_\alpha \tilde{h}^{\lambda\pi} \partial_\beta \tilde{h}^{\tau\epsilon} \\
& - \frac{1}{4g f_R^2} g_{\rho\tau} g_{\epsilon\sigma} g^{\mu\nu} g^{\alpha\beta} \partial_\alpha \tilde{h}^{\rho\sigma} \partial_\beta \tilde{h}^{\tau\epsilon} \\
& - \frac{1}{4f_R \sqrt{-g}} g^{\mu\nu} g^{\alpha\beta} g_{\rho\sigma} \partial_\alpha \partial_\beta \tilde{h}^{\rho\sigma} \\
& + \frac{1}{4f_R \sqrt{-g}} g^{\mu\nu} g^{\alpha\beta} g_{\rho\sigma} \partial_\alpha \tilde{h}^{\rho\sigma} \partial_\beta \ln f_R, \quad (3.17)
\end{aligned}$$

$$\begin{aligned}
R = & -\frac{1}{4gf_R^2}g_{\rho\tau}g_{\epsilon\sigma}g^{\alpha\beta}\partial_\alpha\tilde{h}^{\rho\sigma}\partial_\beta\tilde{h}^{\tau\epsilon} \\
& -\frac{3}{2}g^{\alpha\beta}\partial_\alpha\ln f_R\partial_\beta\ln f_R \\
& -\frac{1}{2f_R\sqrt{-g}}g_{\rho\sigma}g^{\alpha\beta}\partial_\alpha\partial_\beta\tilde{h}^{\rho\sigma}+3g^{\alpha\beta}\partial_\alpha\partial_\beta\ln f_R \\
& -\frac{1}{2gf_R^2}g_{\alpha\beta}\partial_\lambda\tilde{h}^{\alpha\tau}\partial_\tau\tilde{h}^{\beta\lambda} \\
& -\frac{1}{8gf_R^2}g^{\alpha\beta}g_{\tau\epsilon}g_{\lambda\pi}\partial_\alpha\tilde{h}^{\lambda\pi}\partial_\beta\tilde{h}^{\tau\epsilon}. \quad (3.18)
\end{aligned}$$

Further, the GB curvature scalar is

$$\begin{aligned}
\mathcal{G} = & \frac{1}{2gf_R^2}\left(2g_{\lambda\alpha}g_{\tau\beta}-g_{\lambda\tau}g_{\alpha\beta}\right)\left(g^{\rho\nu}g^{\epsilon\pi}\right. \\
& \left.-g^{\rho\epsilon}g^{\nu\pi}\right)\partial_\rho\partial_\nu\tilde{h}^{\lambda\tau}\partial_\epsilon\partial_\pi\tilde{h}^{\alpha\beta} \\
& +\frac{2}{gf_R^2}g_{\sigma\tau}g^{\rho\epsilon}\partial_\rho\partial_\nu\tilde{h}^{\pi\tau}\partial_\epsilon\partial_\pi\tilde{h}^{\nu\sigma} \\
& -\frac{1}{gf_R^2}\partial_\rho\partial_\nu\tilde{h}^{\epsilon\pi}\partial_\epsilon\partial_\pi\tilde{h}^{\rho\nu} \\
& +\frac{1}{gf_R^2}g_{\lambda\tau}g^{\rho\nu}\partial_\rho\partial_\nu\tilde{h}^{\epsilon\pi}\partial_\epsilon\partial_\pi\tilde{h}^{\lambda\tau} \\
& -\frac{2}{f_R\sqrt{-g}}g^{\rho\nu}\partial_\rho\partial_\nu\tilde{h}^{\epsilon\pi}\partial_\epsilon\partial_\pi\ln f_R \\
& +2\left(g^{\rho\nu}g^{\epsilon\pi}-g^{\rho\epsilon}g^{\nu\pi}\right)\partial_\rho\partial_\nu\ln f_R\partial_\epsilon\partial_\pi\ln f_R \\
& +\text{higher-order terms.} \quad (3.19)
\end{aligned}$$

We begin to consider the field equations (2.19) for $f(R, \mathcal{G})$ gravity. $H^{\mu\nu}$ in Eq. (2.20) can be split into three parts,

$$H^{\mu\nu} := H_1^{\mu\nu} + H_2^{\mu\nu} + H_3^{\mu\nu}, \quad (3.20)$$

where

$$H_1^{\mu\nu} := -\frac{1}{2}g^{\mu\nu}f + R^{\mu\nu}f_R + \frac{g^{\mu\nu}}{2}f_{\mathcal{G}}\mathcal{G}, \quad (3.21)$$

$$H_2^{\mu\nu} := (g^{\mu\nu}\square - \nabla^\mu\nabla^\nu)f_R, \quad (3.22)$$

$$\begin{aligned}
H_3^{\mu\nu} := & 2g^{\mu\nu}R\square f_{\mathcal{G}} - 2R\nabla^\mu\nabla^\nu f_{\mathcal{G}} \\
& + 4R^\nu{}_\lambda\nabla^\lambda\nabla^\mu f_{\mathcal{G}} + 4R^\mu{}_\lambda\nabla^\lambda\nabla^\nu f_{\mathcal{G}} \\
& - 4g^{\mu\nu}R^{\alpha\beta}\nabla_\alpha\nabla_\beta f_{\mathcal{G}} - 4R^{\mu\nu}\square f_{\mathcal{G}} \\
& + 4R^{\mu\rho\nu\sigma}\nabla_\rho\nabla_\sigma f_{\mathcal{G}} \quad (3.23)
\end{aligned}$$

with

$$f_{\mathcal{G}} = a_2 + a_{12}R + 2a_{22}\mathcal{G} + a_{112}R^2 + 2a_{122}R\mathcal{G} + 3a_{222}\mathcal{G}^2 + \dots \quad (3.24)$$

By the expressions of f_R and $f_{\mathcal{G}}$, Eq. (3.21) reads

$$H_1^{\mu\nu} = G^{\mu\nu} + 2a_{11}R^{\mu\nu}R - \frac{a_{11}}{2}g^{\mu\nu}R^2 + \dots, \quad (3.25)$$

where $G^{\mu\nu}$ is the Einstein tensor, and the omitted terms will produce the third-order or higher-order terms under the post-Minkowskian method. By Eqs. (3.17) and (3.18), the expression of $G^{\mu\nu}$ in terms of $\tilde{h}^{\mu\nu}$ can be obtained [35],

$$\begin{aligned}
G^{\mu\nu} = & -\frac{1}{2f_R\sqrt{-g}}g^{\alpha\beta}\partial_\alpha\tilde{h}^{\mu\nu}\partial_\beta\ln f_R \\
& +\frac{1}{2f_R\sqrt{-g}}g^{\alpha\beta}\partial_\alpha\partial_\beta\tilde{h}^{\mu\nu} \\
& +\frac{1}{4}g^{\mu\nu}g^{\alpha\beta}\partial_\alpha\ln f_R\partial_\beta\ln f_R \\
& +\frac{1}{2}g^{\mu\alpha}g^{\nu\beta}\partial_\alpha\ln f_R\partial_\beta\ln f_R \\
& -g^{\mu\nu}g^{\alpha\beta}\partial_\alpha\partial_\beta\ln f_R + g^{\mu\alpha}g^{\nu\beta}\partial_\alpha\partial_\beta\ln f_R \\
& -\frac{1}{gf_R^2}g_{\beta\tau}g^{\alpha(\mu}\partial_\lambda\tilde{h}^{\nu)\tau}\partial_\alpha\tilde{h}^{\beta\lambda} \\
& -\frac{1}{2f_R\sqrt{-g}}g_{\rho\sigma}g^{\mu(\alpha}g^{\beta)\nu}\partial_\alpha\tilde{h}^{\rho\sigma}\partial_\beta\ln f_R \\
& +\frac{1}{2gf_R^2}g_{\alpha\beta}g^{\lambda\tau}\partial_\lambda\tilde{h}^{\mu\alpha}\partial_\tau\tilde{h}^{\nu\beta} + \frac{1}{2gf_R^2}\partial_\alpha\tilde{h}^{\mu\beta}\partial_\beta\tilde{h}^{\nu\alpha} \\
& +\frac{1}{f_R\sqrt{-g}}g^{\alpha(\mu}\partial_\alpha\tilde{h}^{\nu)\beta}\partial_\beta\ln f_R \\
& +\frac{1}{4gf_R^2}g^{\mu\nu}g_{\alpha\beta}\partial_\lambda\tilde{h}^{\alpha\tau}\partial_\tau\tilde{h}^{\beta\lambda} \\
& +\frac{1}{4f_R\sqrt{-g}}g^{\mu\nu}g^{\alpha\beta}g_{\rho\sigma}\partial_\alpha\tilde{h}^{\rho\sigma}\partial_\beta\ln f_R \\
& +\frac{1}{16gf_R^2}(2g^{\mu\alpha}g^{\nu\beta}-g^{\mu\nu}g^{\alpha\beta})(2g_{\lambda\tau}g_{\epsilon\pi} \\
& -g_{\epsilon\tau}g_{\lambda\pi})\partial_\alpha\tilde{h}^{\lambda\pi}\partial_\beta\tilde{h}^{\tau\epsilon}. \quad (3.26)
\end{aligned}$$

For scalar f_R ,

$$\nabla^\mu\nabla^\nu f_R = g^{\mu\alpha}g^{\nu\beta}\partial_\alpha\partial_\beta f_R - \Gamma^{\lambda\mu\nu}\partial_\lambda f_R,$$

$$\square f_R = g^{\alpha\beta}\partial_\alpha\partial_\beta f_R - \Gamma^\lambda{}_\lambda\partial_\lambda f_R,$$

where the expression of $\Gamma^{\lambda\mu\nu}$ is in Eq. (3.3), and $\Gamma^\lambda{}_\lambda := g^{\mu\nu}\Gamma^\lambda{}_{\mu\nu}$. By use of (3.1), (3.2), (3.4), (3.13), (3.14), and (3.15), Eq. (3.3) can be expressed as

$$\begin{aligned}
\Gamma^{\lambda\mu\nu} = & -\frac{1}{f_R\sqrt{-g}}g^{\rho(\mu}\partial_\rho\tilde{h}^{\nu)\lambda} + \frac{1}{2f_R\sqrt{-g}}g^{\lambda\rho}\partial_\rho\tilde{h}^{\mu\nu} \\
& +\frac{1}{2f_R\sqrt{-g}}g_{\alpha\beta}g^{\rho(\mu}g^{\nu)\lambda}\partial_\rho\tilde{h}^{\alpha\beta} \\
& -\frac{1}{4f_R\sqrt{-g}}g_{\alpha\beta}g^{\mu\nu}g^{\lambda\rho}\partial_\rho\tilde{h}^{\alpha\beta}
\end{aligned}$$

$$-g^{\rho(\mu}g^{\nu)\lambda}\partial_\rho\ln f_R + \frac{1}{2}g^{\mu\nu}g^{\lambda\rho}\partial_\rho\ln f_R. \quad (3.27)$$

Therefore,

$$\begin{aligned} H_2^{\mu\nu} = & f_R g^{\mu\nu} g^{\alpha\beta} \partial_\alpha \partial_\beta \ln f_R - f_R g^{\mu\alpha} g^{\nu\beta} \partial_\alpha \partial_\beta \ln f_R \\ & - f_R g^{\mu\alpha} g^{\nu\beta} \partial_\alpha \ln f_R \partial_\beta \ln f_R \\ & - \frac{1}{\sqrt{-g}} g^{\rho(\mu} \partial_\rho \tilde{h}^{\nu)\lambda} \partial_\lambda \ln f_R \\ & + \frac{1}{2\sqrt{-g}} g^{\lambda\rho} \partial_\rho \tilde{h}^{\mu\nu} \partial_\lambda \ln f_R \\ & + \frac{1}{2\sqrt{-g}} g_{\alpha\beta} g^{\rho(\mu} g^{\nu)\lambda} \partial_\rho \tilde{h}^{\alpha\beta} \partial_\lambda \ln f_R \\ & - \frac{1}{4\sqrt{-g}} g_{\alpha\beta} g^{\mu\nu} g^{\lambda\rho} \partial_\rho \tilde{h}^{\alpha\beta} \partial_\lambda \ln f_R \\ & - f_R g^{\rho(\mu} g^{\nu)\lambda} \partial_\rho \ln f_R \partial_\lambda \ln f_R \\ & + \frac{1}{2} f_R g^{\mu\nu} g^{\lambda\rho} \partial_\rho \ln f_R \partial_\lambda \ln f_R. \end{aligned} \quad (3.28)$$

For scalar f_G , similarly to f_R , there are

$$\nabla^\mu \nabla^\nu f_G = g^{\mu\alpha} g^{\nu\beta} \partial_\alpha \partial_\beta f_G - \Gamma^{\lambda\mu\nu} \partial_\lambda f_G, \quad (3.29)$$

$$\nabla_\mu \nabla_\nu f_G = \partial_\mu \partial_\nu f_G - \Gamma_{\mu\nu}^\lambda \partial_\lambda f_G, \quad (3.30)$$

$$\square f_G = g^{\alpha\beta} \partial_\alpha \partial_\beta f_G - \Gamma^\lambda \partial_\lambda f_G. \quad (3.31)$$

By Eqs. (3.16)–(3.19), (3.27) and the expressions of f_R and f_G , namely Eqs. (3.11) and (3.24), Eq. (3.23) reads

$$\begin{aligned} H_3^{\mu\nu} = & 2R g^{\mu\nu} g^{\alpha\beta} \partial_\alpha \partial_\beta f_G - 2R g^{\mu\alpha} g^{\nu\beta} \partial_\alpha \partial_\beta f_G \\ & + 4R^{\nu\alpha} g^{\mu\beta} \partial_\alpha \partial_\beta f_G \\ & + 4R^{\mu\alpha} g^{\nu\beta} \partial_\alpha \partial_\beta f_G - 4g^{\mu\nu} R^{\alpha\beta} \partial_\alpha \partial_\beta f_G \\ & - 4R^{\mu\nu} g^{\alpha\beta} \partial_\alpha \partial_\beta f_G + 4R^{\mu\rho\nu\sigma} \partial_\rho \partial_\sigma f_G + \dots, \end{aligned} \quad (3.32)$$

where similarly to Eq. (3.25), the omitted terms will also produce the third-order or higher-order terms under the post-Minkowskian method.

The combination of Eqs. (3.20), (3.24)–(3.26), (3.28), and (3.32) brings about the expression of $H^{\mu\nu}$:

$$H^{\mu\nu} = -\frac{1}{2gf_R^2} (\square_\eta \tilde{h}^{\mu\nu} - \Lambda^{\mu\nu}) \quad (3.33)$$

with $\square_\eta := \eta^{\mu\nu} \partial_\mu \partial_\nu$ and

$$\begin{aligned} \Lambda^{\mu\nu} = & \Lambda_{GR}^{\mu\nu}(\tilde{h}^{\alpha\beta}) - \frac{1}{2}(1+2f_R)\tilde{g}^{\mu\nu}\tilde{g}^{\alpha\beta}\partial_\alpha\ln f_R\partial_\beta\ln f_R \\ & - (1-4f_R)\tilde{g}^{\mu\alpha}\tilde{g}^{\nu\beta}\partial_\alpha\ln f_R\partial_\beta\ln f_R \\ & - 2(f_R-1)\tilde{g}^{\mu\nu}\tilde{g}^{\alpha\beta}\partial_\alpha\partial_\beta\ln f_R \\ & + 2(f_R-1)\tilde{g}^{\mu\alpha}\tilde{g}^{\nu\beta}\partial_\alpha\partial_\beta\ln f_R \end{aligned}$$

$$\begin{aligned} & - 2(1-f_R)\tilde{g}^{\alpha(\mu}\partial_\alpha\tilde{h}^{\nu)\beta}\partial_\beta\ln f_R \\ & - \frac{1}{2}(1-f_R)\tilde{g}_{\rho\sigma}\tilde{g}^{\mu\nu}\tilde{g}^{\alpha\beta}\partial_\alpha\tilde{h}^{\rho\sigma}\partial_\beta\ln f_R \\ & - (f_R-1)\tilde{g}^{\alpha\beta}\partial_\alpha\tilde{h}^{\mu\nu}\partial_\beta\ln f_R \\ & - (f_R-1)\tilde{g}_{\rho\sigma}\tilde{g}^{\mu(\alpha}\tilde{g}^{\beta)\nu}\partial_\alpha\tilde{h}^{\rho\sigma}\partial_\beta\ln f_R \\ & + a_{11}\sqrt{-g}f_R\tilde{g}^{\mu\nu}R^2 + 4a_{11}gf_R^2R^{\mu\nu}R \\ & - 4a_{12}\left(\tilde{g}^{\mu\nu}\tilde{g}^{\alpha\beta} - \tilde{g}^{\mu\alpha}\tilde{g}^{\nu\beta}\right)R\partial_\alpha\partial_\beta R \\ & - 16a_{12}\sqrt{-g}f_R\tilde{g}^{\beta(\mu}R^{\nu)\alpha}\partial_\alpha\partial_\beta R \\ & + 8a_{12}\sqrt{-g}f_R\tilde{g}^{\mu\nu}R^{\alpha\beta}\partial_\alpha\partial_\beta R \\ & + 8a_{12}\sqrt{-g}f_R\tilde{g}^{\alpha\beta}R^{\mu\nu}\partial_\alpha\partial_\beta R \\ & + 8a_{12}gf_R^2R^{\mu\rho\nu\sigma}\partial_\rho\partial_\sigma R + \dots, \end{aligned} \quad (3.34)$$

where

$$\begin{aligned} \Lambda_{GR}^{\mu\nu}(\tilde{h}^{\alpha\beta}) = & -\tilde{h}^{\alpha\beta}\partial_\alpha\partial_\beta\tilde{h}^{\mu\nu} + \partial_\alpha\tilde{h}^{\mu\beta}\partial_\beta\tilde{h}^{\nu\alpha} \\ & + \frac{1}{2}\tilde{g}^{\mu\nu}\tilde{g}_{\alpha\beta}\partial_\lambda\tilde{h}^{\alpha\tau}\partial_\tau\tilde{h}^{\beta\lambda} - \tilde{g}^{\mu\alpha}\tilde{g}_{\beta\tau}\partial_\lambda\tilde{h}^{\nu\tau}\partial_\alpha\tilde{h}^{\beta\lambda} \\ & - \tilde{g}^{\nu\alpha}\tilde{g}_{\beta\tau}\partial_\lambda\tilde{h}^{\mu\tau}\partial_\alpha\tilde{h}^{\beta\lambda} \\ & + \tilde{g}_{\alpha\beta}\tilde{g}^{\lambda\tau}\partial_\lambda\tilde{h}^{\mu\alpha}\partial_\tau\tilde{h}^{\nu\beta} \\ & + \frac{1}{8}(2\tilde{g}^{\mu\alpha}\tilde{g}^{\nu\beta} - \tilde{g}^{\mu\nu}\tilde{g}^{\alpha\beta})(2\tilde{g}_{\lambda\tau}\tilde{g}^{\epsilon\pi} \\ & - \tilde{g}_{\tau\epsilon}\tilde{g}_{\lambda\pi})\partial_\alpha\tilde{h}^{\lambda\pi}\partial_\beta\tilde{h}^{\tau\epsilon}, \end{aligned} \quad (3.35)$$

and

$$\tilde{g}_{\mu\nu} := \frac{1}{\sqrt{-g}f_R}g_{\mu\nu} \quad (3.36)$$

satisfies

$$\tilde{g}_{\mu\lambda}\tilde{g}^{\lambda\nu} = \delta_\mu^\nu. \quad (3.37)$$

In Eqs. (3.34) and (3.35), the quantity $\Lambda_{GR}^{\mu\nu}(h^{\alpha\beta})$ is the corresponding term in GR with the replacement of the variables $h^{\alpha\beta}$ by $\tilde{h}^{\alpha\beta}$. It is easy to check that $\Lambda^{\mu\nu}$ is made of, at least, quadratic in the effective gravitational field amplitude $\tilde{h}^{\mu\nu}$ and its first and second derivatives. Moreover, $\Lambda^{\mu\nu}$ can reduce to the expression of $\Lambda_{GR}^{\mu\nu}(h^{\alpha\beta})$ when $f(R, \mathcal{G})$ gravity reduces to GR by Eqs. (3.11) and (3.14). Equations (3.33) and (3.34) show that the coupling constant a_{12} appears in the expression of $H^{\mu\nu}$, which, from the Lagrangian (1.1), implies that the GB curvature scalar $\mathcal{G}$ should have nontrivial effect in the theory. According to the Eq. (3.33), the field equations of $f(R, \mathcal{G})$ gravity are transformed into the form of an obvious wave equation

$$\square_\eta \tilde{h}^{\mu\nu} = 2\kappa\tau^{\mu\nu} \quad (3.38)$$

under the de Donder condition, where the source term

$$\tau^{\mu\nu} = |g|f_R^2 T^{\mu\nu} + \frac{1}{2\kappa} \Lambda^{\mu\nu} \quad (3.39)$$

is the stress–energy pseudotensor of the matter fields and the gravitational field. Obviously, Eqs. (3.13) and (3.38) imply the conservation of the stress–energy pseudotensor in $f(R, \mathcal{G})$ gravity, namely,

$$\partial_\mu \tau^{\mu\nu} = 0. \quad (3.40)$$

3.2 Linearized $f(R, \mathcal{G})$ gravity

If $\tilde{h}^{\mu\nu}$ is a perturbation, namely,

$$|\tilde{h}^{\mu\nu}| \ll 1, \quad (3.41)$$

the linearized gravitational field equations are

$$\square_\eta \tilde{h}^{\mu\nu} = 2\kappa T^{\mu\nu} \quad (3.42)$$

by (3.38) and (3.39). Eq. (3.42) is the basis of multipole expansion of linearized $f(R, \mathcal{G})$ gravity with irreducible Cartesian tensors. It is easy to check that the field equations (3.42) of linearized $f(R, \mathcal{G})$ gravity are the same as those in linearized $f(R)$ gravity [35]. This is a trivial result because the leading term of $\mathcal{G}$ in the Lagrangian (1.1) gives a topological invariant, and its next leading term only relates to the higher-order effects.

Above field equations (3.42) are the linear results, but the effective stress–energy tensor of GWs for the linearized $f(R, \mathcal{G})$ gravity are second-order nonlinear result, so in order to obtain its expression, we need some formulas to derive the quadratic terms of the stress–energy pseudotensor of the gravitational field. Firstly, by Eqs. (3.9) and (3.37), we have

$$\tilde{g}_{\mu\nu} = \eta_{\mu\nu} - \tilde{h}_{\mu\nu}. \quad (3.43)$$

Next, by Eq. (3.11),

$$f_R = 1 + 2a_{11}R^{(1)} + o(\tilde{h}^{\mu\nu}), \quad (3.44)$$

where $o(\tilde{h}^{\mu\nu})$ is the higher-order terms of $\tilde{h}^{\mu\nu}$, and then inserting f_R into Eq. (3.14) gives rise to

$$h^{\mu\nu} = \tilde{h}^{\mu\nu} - 2a_{11}R^{(1)}\eta^{\mu\nu}, \quad (3.45)$$

$$h = \tilde{h} - 8a_{11}R^{(1)}, \quad (3.46)$$

where $h = \eta_{\mu\nu}h^{\mu\nu}$ and $\tilde{h} = \eta_{\mu\nu}\tilde{h}^{\mu\nu}$. Equations (3.45) and (3.46) show that $h^{\mu\nu}$ and h are also perturbations, and thus, with the help of Eqs. (3.1) and (3.2),

$$g = |g_{\mu\nu}| = |\tilde{g}^{\mu\nu}| = -1 - h + o(h^{\mu\nu}) = -1 - \tilde{h}$$

$$+ 8a_{11}R^{(1)} + o(\tilde{h}^{\mu\nu}), \quad (3.47)$$

so by Eq. (3.44), there are

$$\sqrt{-g}f_R = 1 + \frac{\tilde{h}}{2} - 2a_{11}R^{(1)} + o(\tilde{h}^{\mu\nu}), \quad (3.48)$$

$$-gf_R^2 = 1 + \tilde{h} - 4a_{11}R^{(1)} + o(\tilde{h}^{\mu\nu}). \quad (3.49)$$

The substitution of Eqs. (3.9), (3.16)–(3.18), (3.43), (3.44), (3.48), and (3.49) in Eq. (3.34) gives rise to the quadratic term of $\Lambda^{\mu\nu}$:

$$\begin{aligned} \Lambda^{\mu\nu(2)}_{f(R)} &= \Lambda^{\mu\nu(2)}_{f(R)} - 4a_{12}\eta^{\mu\nu}R^{(1)}\square_\eta R^{(1)} \\ &\quad + 4a_{12}R^{(1)}\partial^\mu\partial^\nu R^{(1)} - 8a_{12}R^{\alpha\nu(1)}\partial^\mu\partial_\alpha R^{(1)} \\ &\quad - 8a_{12}R^{\alpha\mu(1)}\partial^\nu\partial_\alpha R^{(1)} \\ &\quad + 8a_{12}\eta^{\mu\nu}R^{\alpha\beta(1)}\partial_\alpha\partial_\beta R^{(1)} + 8a_{12}R^{\mu\nu(1)}\square_\eta R^{(1)} \\ &\quad - 8a_{12}R^{\mu\rho\nu\sigma(1)}\partial_\rho\partial_\sigma R^{(1)}, \end{aligned} \quad (3.50)$$

where $\partial^\mu = \eta^{\mu\lambda}\partial_\lambda$. In Eq. (3.50),

$$\begin{aligned} \Lambda^{\mu\nu(2)}_{f(R)} &= \Lambda^{\mu\nu(2)}_{GR}(\tilde{h}^{\alpha\beta}) + a_{11}\eta^{\mu\nu}R^{(1)2} - 4a_{11}R^{\mu\nu(1)}R^{(1)} \\ &\quad - 6a_{11}^2\eta^{\mu\nu}\partial^\alpha R^{(1)}\partial_\alpha R^{(1)} + 12a_{11}^2\partial^\mu R^{(1)}\partial^\nu R^{(1)} \\ &\quad - 8a_{11}^2\eta^{\mu\nu}R^{(1)}\square_\eta R^{(1)} + 8a_{11}^2R^{(1)}\partial^\mu\partial^\nu R^{(1)} \end{aligned} \quad (3.51)$$

is the corresponding term in $f(R)$ gravity, and

$$\begin{aligned} \Lambda^{\mu\nu(2)}_{GR}(\tilde{h}^{\alpha\beta}) &= -\tilde{h}^{\alpha\beta}\partial_\alpha\partial_\beta\tilde{h}^{\mu\nu} + \frac{1}{2}\partial^\mu\tilde{h}_{\alpha\beta}\partial^\nu\tilde{h}^{\alpha\beta} - \frac{1}{4}\partial^\mu\tilde{h}\partial^\nu\tilde{h} \\ &\quad + \partial_\beta\tilde{h}^{\mu\alpha}(\partial^\beta\tilde{h}^\nu_\alpha + \partial_\alpha\tilde{h}^{\nu\beta}) - \partial^\mu\tilde{h}_{\alpha\beta}\partial^\alpha\tilde{h}^{\nu\beta} \\ &\quad - \partial^\nu\tilde{h}_{\alpha\beta}\partial^\alpha\tilde{h}^{\mu\beta} + \eta^{\mu\nu}\left(-\frac{1}{4}\partial_\lambda\tilde{h}_{\alpha\beta}\partial^\lambda\tilde{h}^{\alpha\beta}\right. \\ &\quad \left. + \frac{1}{8}\partial_\lambda\tilde{h}\partial^\lambda\tilde{h} + \frac{1}{2}\partial_\alpha\tilde{h}_{\beta\lambda}\partial^\beta\tilde{h}^{\alpha\lambda}\right) \end{aligned} \quad (3.52)$$

is the quadratic term of $\Lambda^{\mu\nu}_{GR}(\tilde{h}^{\alpha\beta})$. The cubic term and the higher-power terms of $\Lambda^{\mu\nu}$ are also useful, because they can be used to explain the additional degree of freedom, namely, the ghost in $f(R, \mathcal{G})$ gravity [40]. But, as is mentioned above, the effective stress–energy tensor of GWs for the linearized $f(R, \mathcal{G})$ gravity are second-order nonlinear result, so the quadratic term of $\Lambda^{\mu\nu}$ is enough for this paper.

We need to simplify $\Lambda^{\mu\nu(2)}$. By use of Eqs. (3.9), (3.10), (3.13), (3.36), (3.43), (3.44), (3.48), and (3.49), Eqs. (3.16), (3.17), and (3.18) reduce to

$$\begin{aligned} R^{\mu\nu\rho\sigma(1)} &= \frac{1}{2}\left(\partial^\sigma\partial^\nu\tilde{h}^{\mu\rho} - \partial^\rho\partial^\nu\tilde{h}^{\mu\sigma} + \partial^\rho\partial^\mu\tilde{h}^{\nu\sigma} - \partial^\sigma\partial^\mu\tilde{h}^{\nu\rho}\right) \\ &\quad - \frac{1}{4}\left(\eta^{\mu\rho}\partial^\sigma\partial^\nu\tilde{h} - \eta^{\mu\sigma}\partial^\rho\partial^\nu\tilde{h}\right) \end{aligned}$$

$$\begin{aligned}
& + \eta^{\nu\sigma} \partial^\rho \partial^\mu \tilde{h} - \eta^{\nu\rho} \partial^\sigma \partial^\mu \tilde{h} \Big) \\
& + a_{11} \Big(\eta^{\mu\rho} \partial^\sigma \partial^\nu R^{(1)} - \eta^{\mu\sigma} \partial^\rho \partial^\nu R^{(1)} \\
& + \eta^{\nu\sigma} \partial^\rho \partial^\mu R^{(1)} - \eta^{\nu\rho} \partial^\sigma \partial^\mu R^{(1)} \Big), \quad (3.53)
\end{aligned}$$

$$\begin{aligned}
R^{\mu\nu(1)} &= \frac{1}{2} \square_\eta \tilde{h}^{\mu\nu} - \frac{1}{4} \eta^{\mu\nu} \square_\eta \tilde{h} + a_{11} \eta^{\mu\nu} \square_\eta R^{(1)} \\
&+ 2a_{11} \partial^\mu \partial^\nu R^{(1)}, \quad (3.54)
\end{aligned}$$

$$R^{(1)} = -\frac{1}{2} \square_\eta \tilde{h} + 6a_{11} \square_\eta R^{(1)} \quad (3.55)$$

under the de Donder condition, and they satisfy

$$\begin{aligned}
\partial_\rho \partial_\sigma R^{\mu\rho\nu\sigma(1)} &= \square_\eta R^{\mu\nu(1)} - \frac{1}{2} \partial^\mu \partial^\nu R^{(1)}, \\
\partial_\mu R^{\mu\nu(1)} &= \frac{1}{2} \partial^\nu R^{(1)},
\end{aligned}$$

which can also be derived by the Bianchi identity, in fact. Furthermore, Eqs. (2.19) and (3.33) lead to

$$H^{\mu\nu(1)} = \frac{1}{2} \square_\eta \tilde{h}^{\mu\nu} = \kappa T^{\mu\nu(1)}, \quad (3.56)$$

$$H^{(1)} = \frac{1}{2} \square_\eta \tilde{h} = \kappa T^{(1)}, \quad (3.57)$$

so outside the sources, there are

$$R^{(1)} = 6a_{11} \square_\eta R^{(1)}, \quad (3.58)$$

$$R^{\mu\nu(1)} = \frac{1}{6} \eta^{\mu\nu} R^{(1)} + 2a_{11} \partial^\mu \partial^\nu R^{(1)} \quad (3.59)$$

by Eqs. (3.55) and (3.54), respectively. Moreover, Eqs. (3.55) and (3.57) imply

$$\square_\eta R^{(1)} - m^2 R^{(1)} = m^2 \kappa T^{(1)}, \quad (3.60)$$

where

$$m^2 := \frac{1}{6a_{11}}, \quad (3.61)$$

which shows that $R^{(1)}$ satisfies a massive KG equation with an external source, and it is the same as that in linearized $f(R)$ gravity.

Now, we use the above formulas to simplify $\Lambda^{\mu\nu(2)}$. Firstly, applying Eqs. (3.53)–(3.55) to Eqs. (3.50) and (3.51) gives rise to

$$\begin{aligned}
\Lambda^{\mu\nu(2)} &= \Lambda_{f(R)}^{\mu\nu(2)} + 8a_{11}a_{12} \eta^{\mu\nu} \partial^\alpha \partial^\beta R^{(1)} \partial_\alpha \partial_\beta R^{(1)} \\
&- \frac{2a_{12}}{9a_{11}} \eta^{\mu\nu} \left(R^{(1)} \right)^2 \\
&- 16a_{11}a_{12} \partial^\mu \partial_\alpha R^{(1)} \partial^\nu \partial^\alpha R^{(1)}
\end{aligned}$$

$$\begin{aligned}
& + \frac{8a_{12}}{3} R^{(1)} \partial^\mu \partial^\nu R^{(1)} - 4a_{12} \partial_\rho \partial_\sigma R^{(1)} \left(\left(\partial^\mu \partial^\nu \tilde{h}^{\rho\sigma} \right. \right. \\
& + \partial^\rho \partial^\sigma \tilde{h}^{\mu\nu} - \partial^\rho \partial^\nu \tilde{h}^{\mu\sigma} - \partial^\mu \partial^\sigma \tilde{h}^{\rho\nu} \Big) \\
& - \frac{1}{2} \left(\eta^{\rho\sigma} \partial^\mu \partial^\nu \tilde{h} + \eta^{\mu\nu} \partial^\rho \partial^\sigma \tilde{h} - \eta^{\mu\sigma} \partial^\rho \partial^\nu \tilde{h} \right. \\
& \left. \left. - \eta^{\rho\nu} \partial^\mu \partial^\sigma \tilde{h} \right) \right), \quad (3.62)
\end{aligned}$$

where

$$\begin{aligned}
\Lambda_{f(R)}^{\mu\nu(2)} &= \Lambda_{GR}^{\mu\nu(2)}(\tilde{h}^{\alpha\beta}) + 12a_{11}^2 \partial^\mu R^{(1)} \partial^\nu R^{(1)} \\
&- \eta^{\mu\nu} \left(a_{11} \left(R^{(1)} \right)^2 + 6a_{11}^2 \partial^\alpha R^{(1)} \partial_\alpha R^{(1)} \right). \quad (3.63)
\end{aligned}$$

It is easy to check that $\Lambda^{\mu\nu}$ in (3.39) can reduce to the expression of $\Lambda_{GR}^{\mu\nu}$, when $f(R, \mathcal{G})$ gravity reduces to GR by Eq. (3.35). For GR, if $h^{\mu\nu}$ is the perturbation, the effective stress–energy tensor of GWs are [35]

$$t_{GR}^{\mu\nu} = \frac{1}{2\kappa} \left\langle \Lambda_{GR}^{\mu\nu(2)} \right\rangle,$$

where the average $\langle \dots \rangle$ has been taken over a small spatial volume (several wavelengths) surrounding each point. By averaging, the curvature in a macroscopic region about a given point in spacetime is probed, which yields a gauge invariant of the gravitational field strength. The averaging can be thought of as smoothing out the rapidly varying ripples of the radiation, leaving only the coarsegrained component that acts as a source for the background curvature [34]. Similarly, the effective stress–energy tensor of GWs in linearized $f(R, \mathcal{G})$ gravity should be defined as

$$t^{\mu\nu} := \frac{1}{2\kappa} \left\langle \Lambda^{\mu\nu(2)} \right\rangle,$$

and the relevant rules for the average are [34]

$$\left\langle \partial_\mu X \right\rangle = 0, \quad (3.64)$$

$$\left\langle A(\partial_\mu B) \right\rangle = -\left\langle (\partial_\mu A)B \right\rangle, \quad (3.65)$$

where X, A, B are three arbitrary quantities. By use of Eqs. (3.13), (3.56)–(3.58), (3.64), and (3.65), the average of Eq. (3.62) outside the sources reduces to

$$\begin{aligned}
\left\langle \Lambda^{\mu\nu(2)} \right\rangle &= \left\langle \Lambda_{f(R)}^{\mu\nu(2)} \right\rangle = \left\langle \Lambda_{GR}^{\mu\nu(2)}(\tilde{h}^{\alpha\beta}) \right\rangle \\
&+ 12a_{11}^2 \left\langle \partial^\mu R^{(1)} \partial^\nu R^{(1)} \right\rangle. \quad (3.66)
\end{aligned}$$

That is, after the average over a small spatial volume, the terms associated with the coupling constant a_{12} in Eq. (3.62) are cancelled each other, and the dependence on a_{12} vanishes. It gives the effective stress–energy tensor of GWs in linearized $f(R, \mathcal{G})$ gravity:

$$t^{\mu\nu} = \frac{1}{2\kappa} \left\langle \Lambda_{f(R)}^{\mu\nu(2)} \right\rangle = t_{f(R)}^{\mu\nu} = t_{GR}^{\mu\nu}(\tilde{h}^{\alpha\beta}) + \frac{6a_{11}^2}{\kappa} \left\langle \partial^\mu R^{(1)} \partial^\nu R^{(1)} \right\rangle, \quad (3.67)$$

where

$$t_{GR}^{\mu\nu}(\tilde{h}^{\alpha\beta}) = \frac{1}{4\kappa} \left\langle \partial^\mu \tilde{h}_{\alpha\beta} \partial^\nu \tilde{h}^{\alpha\beta} - \frac{1}{2} \partial^\mu \tilde{h} \partial^\nu \tilde{h} \right\rangle \quad (3.68)$$

is the effective stress–energy tensor of GWs under the de Donder condition in linearized GR with the replacement of the variables $h^{\alpha\beta}$ by $\tilde{h}^{\alpha\beta}$ [41].

It should be noted that the above result only depends on the coupling constant a_{11} in the Lagrangian (1.1), and has nothing to do with the GB curvature scalar $\mathcal{G}$. In other words, the effective stress–energy tensor of GWs in linearized $f(R, \mathcal{G})$ gravity is the same as that in linearized $f(R)$ gravity [34, 35]. As mentioned before, for $f(R, \mathcal{G})$ gravity, the GB curvature scalar $\mathcal{G}$ plays an important role in the nonlinear effects in general. However, although the effective stress–energy tensor of GWs in $f(R, \mathcal{G})$ gravity is the typical second-order nonlinear result, the above result shows that $\mathcal{G}$ does not contribute to it.

4 Energy, momentum, and angular momentum carried by GWs in linearized $f(R, \mathcal{G})$ gravity

4.1 Multipole expansion of linearized $f(R, \mathcal{G})$ gravity

As the further second-order nonlinear results, the energy, momentum, and angular momentum carried by GWs in linearized $f(R, \mathcal{G})$ gravity need to be discussed. For GR [31] and $f(R)$ gravity [36], the fluxes of the energy and the momentum can be evaluated by using the above effective stress–energy tensor of GWs, but the flux of the angular momentum cannot [31]. This problem also exists in the $f(R, \mathcal{G})$ gravity. Therefore, a way, not depending on the effective stress–energy tensor of GWs, to deal with the angular momentum is needed. In this section, by following Refs. [31, 36, 39], we will calculate the energy, momentum, and angular momentum carried by GWs in linearized $f(R, \mathcal{G})$ gravity in a unified way. This unified way requires that the multipole expansion of the linearized $f(R, \mathcal{G})$ gravity is discussed firstly.

The definitions of the effective gravitational field amplitude $\tilde{h}^{\mu\nu}$ and the gravitational amplitude $h^{\mu\nu}$, namely Eqs. (3.9) and (3.1), respectively, show that the former contains not only the information of metric but also the information of function f_R , and the latter carries all pure information of metric, which implies that $h^{\mu\nu}$ is the true gravitational amplitude, and that we need to find out the multipole expansion of $h^{\mu\nu}$. For the linearized $f(R, \mathcal{G})$ gravity, $h^{\mu\nu}$ is expressed by Eq. (3.45), which shows that we should deal with the multipole expansions of tensor part $\tilde{h}^{\mu\nu}$ and of the scalar part associated with $R^{(1)}$, respectively.

Firstly, we discuss the multipole expansion of $\tilde{h}^{\mu\nu}$. As mentioned in the preceding section, the linearized field equations (3.42) of linearized $f(R, \mathcal{G})$ gravity are the same as those in linearized $f(R)$ gravity [35]. So, under the same gauge condition, namely the de Donder condition (3.13), the multipole expansions of $\tilde{h}^{\mu\nu}$ in these two gravitational models are also the same, and their expression is [36]

$$\left\{ \begin{aligned} \tilde{h}^{00}(t, \mathbf{x}) &= -\frac{4G}{c^2} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \partial_{l_l} \left(\frac{\hat{M}_{l_l}(u)}{r} \right), \\ \tilde{h}^{0i}(t, \mathbf{x}) &= \frac{4G}{c^3} \sum_{l=1}^{\infty} \frac{(-1)^l}{l!} \partial_{l_{l-1}} \left(\frac{\partial_t \hat{M}_{i l_{l-1}}(u)}{r} \right) \\ &\quad + \frac{4G}{c^3} \sum_{l=1}^{\infty} \frac{(-1)^l}{(l+1)!} \epsilon_{iab} \partial_{a l_{l-1}} \left(\frac{\hat{S}_{b l_{l-1}}(u)}{r} \right), \\ \tilde{h}^{ij}(t, \mathbf{x}) &= -\frac{4G}{c^4} \sum_{l=2}^{\infty} \frac{(-1)^l}{l!} \partial_{l_{l-2}} \left(\frac{\partial_t^2 \hat{M}_{ij l_{l-2}}(u)}{r} \right) \\ &\quad - \frac{8G}{c^4} \sum_{l=2}^{\infty} \frac{(-1)^l}{(l+1)!} \partial_{a l_{l-2}} \left(\frac{\epsilon_{ab(i} \partial_{|t|} \hat{S}_{j) b l_{l-2}}(u)}{r} \right), \end{aligned} \right. \quad (4.1)$$

where

$$\left\{ \begin{aligned} \hat{M}_{l_l}(u) &= \frac{1}{c^2} \int d^3 x' \left(\hat{X}'_{l_l} \left(\bar{T}_l^{00}(u, \mathbf{x}') + \bar{T}_l^{aa}(u, \mathbf{x}') \right) \right. \\ &\quad \left. - \frac{4(2l+1)}{c(l+1)(2l+3)} \hat{X}'_{a l_l} \partial_t \bar{T}_{l+1}^{0a}(u, \mathbf{x}') \right. \\ &\quad \left. + \frac{2(2l+1)}{c^2(l+1)(l+2)(2l+5)} \hat{X}'_{ab l_l} \partial_t^2 \bar{T}_{l+2}^{ab}(u, \mathbf{x}') \right), \\ \hat{S}_{l_l}(u) &= \frac{1}{c} \int d^3 x' \left(\epsilon_{ab < i_1} \hat{X}'_{|a| i_2 \dots i_l} \bar{T}_l^{0b}(u, \mathbf{x}') \right. \\ &\quad \left. - \frac{2l+1}{c(l+2)(2l+3)} \epsilon_{ab < i_1} \hat{X}'_{|ac| i_2 \dots i_l} \partial_t \bar{T}_{l+1}^{cb}(u, \mathbf{x}') \right), \quad l \geq 1 \end{aligned} \right. \quad (4.2)$$

are referred to as the mass-type and current-type source multipole moments, respectively [37]; the subscript $i l_{l-1} = i i_1 \dots i_{l-1}$, and the other subscripts like $ij l_{l-2}$, $a l_l$ are also similar, the symbol $(i|t|j)$ represents that t is not the symmetric indices, the symbol $< i_1 | a | i_2 \dots i_l >$ and $< i_1 | ac | i_2 \dots i_l >$ represent that a and ac are not STF indices, respectively; and [30]

$$\bar{T}_l^{\mu\nu}(u, \mathbf{x}') := \frac{(2l+1)!!}{2^{l+1}l!} \int_{-1}^1 (1-z^2)^l T^{\mu\nu}\left(u + \frac{z\mathbf{r}'}{c}, \mathbf{x}'\right) dz. \quad (4.3)$$

The meaning of other notations related to Eqs. (4.1) and (4.2) can be found in Sect. 2.

The Eq. (4.1) can be simplified by adopting the further gauge. Under an infinitesimal coordinate transformation, $x'^\mu \rightarrow x^\mu + \varepsilon^\mu$,

$$h'^{\mu\nu} = h^{\mu\nu} + \partial^\mu \varepsilon^\nu + \partial^\nu \varepsilon^\mu - \eta^{\mu\nu} \partial_\alpha \varepsilon^\alpha. \quad (4.4)$$

Moreover, $R^{(1)}$ is an invariant under the coordinate transformation, so from Eq. (3.45),

$$\tilde{h}'^{\mu\nu} = \tilde{h}^{\mu\nu} + \partial^\mu \varepsilon^\nu + \partial^\nu \varepsilon^\mu - \eta^{\mu\nu} \partial_\alpha \varepsilon^\alpha, \quad (4.5)$$

which shows that the transverse and traceless (TT) gauge can be applied to the effective gravitational field amplitude $\tilde{h}^{\mu\nu}$ like that in GR [42]. Therefore, Eq. (4.1) can be written as

$$\begin{cases} \tilde{h}_{TT}^{00}(t, \mathbf{x}) = 0, \\ \tilde{h}_{TT}^{0i}(t, \mathbf{x}) = 0, \\ \tilde{h}_{TT}^{ij}(t, \mathbf{x}) = -\frac{4G}{c^4} \left(\sum_{l=2}^{\infty} \frac{(-1)^l}{l!} \partial_{l-2} \left(\frac{\partial_t^2 \hat{M}_{ijl-2}(u)}{r} \right) \right)^{TT} \\ \quad - \frac{8G}{c^4} \left(\sum_{l=2}^{\infty} \frac{(-1)^l}{(l+1)!} \partial_{al-2} \left(\frac{\epsilon_{ab(i} \partial_{|l|} \hat{S}_{j)b l-2}(u)}{r} \right) \right)^{TT} \end{cases} \quad (4.6)$$

under the TT gauge, where the symbol TT represents the TT projection operator which is defined as

$$(X_{ij})^{TT} := P_{ik} P_{jl} X_{kl} - \frac{1}{2} P_{ij} P_{kl} X_{kl}, \quad (4.7)$$

where X_{ij} is an arbitrary symmetric spatial tensor, and

$$P_{ij} = \delta_{ij} - n_i n_j. \quad (4.8)$$

According to Refs. [41, 43], $t_{GR}^{\mu\nu}(\tilde{h}^{\alpha\beta})$ in Eq. (3.68) can also be simplified by imposing the TT gauge outside the source, namely,

$$t_{GR}^{\mu\nu}(\tilde{h}^{\alpha\beta}) = \frac{1}{4\kappa} \left\langle \partial^\mu \tilde{h}_{pq}^{TT} \partial^\nu \tilde{h}_{pq}^{TT} \right\rangle. \quad (4.9)$$

$R^{(1)}$ in linearized $f(R, \mathcal{G})$ gravity satisfies the massive KG equation (3.60) with an external source, which is the same as that in linearized $f(R)$ gravity [35], so in these two gravitational models, $R^{(1)}$ has the same multipole expansion. Thus, the multipole expansion of $R^{(1)}$ under the condition $r'/r \ll 1$ is [36],

$$R^{(1)}(t, \mathbf{x}) = -\frac{m^2 \kappa}{4\pi} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \int d^3 x' \frac{c}{r'} \hat{X}'_l(\theta', \varphi') \hat{T}_l(t, \mathbf{x}, \mathbf{x}'), \quad (4.10)$$

where $r = |\mathbf{x}|$, $r' = |\mathbf{x}'|$,

$$X'_l(\theta', \varphi') = X'_{i_1 i_2 \dots i_l}(\theta', \varphi') := x'_{i_1} x'_{i_2} \dots x'_{i_l} = r'^l N_l(\theta', \varphi'), \quad (4.11)$$

$$\begin{aligned} \hat{T}_l(t, \mathbf{x}, \mathbf{x}') &= \frac{(2l+1)!!}{2^{l+1}l!} \left(\int_{u-\frac{r'}{c}}^{u+\frac{r'}{c}} dt' \left[\hat{\partial}_l \left(\frac{1}{r} \left(1 - \frac{c^2}{r'^2} (t' - u)^2 \right) \right) \right] T^{(1)}(t', \mathbf{x}') \right. \\ &\quad \left. - \int_{-\infty}^{u-\frac{r'}{c}} dt' \left[\hat{A}(t, r; t', r') \hat{\partial}_l \left(\frac{1}{r} \left(1 - \frac{c^2}{r'^2} (t' - u)^2 \right) \right) \right] T^{(1)}(t', \mathbf{x}') \right) \end{aligned} \quad (4.12)$$

with

$$\begin{aligned} \hat{A}(t, r; t', r') &:= \exp \left(\frac{m^2 r r'}{2} (\text{int } t')_- \right) \\ &\quad - \exp \left(\frac{m^2 r r'}{2} (\text{int } t')_+ \right), \end{aligned} \quad (4.13)$$

$$(\text{int } t')_{\pm} \tilde{f}(t') := -\frac{c}{r'} \int_{t'}^{u \pm \frac{r'}{c}} \tilde{f}(\tau) \left(1 + \frac{c}{r} (u - \tau) \right) d\tau. \quad (4.14)$$

Upon obtaining Eqs. (4.1) and (4.10), we can write down the multipole expansion of $h^{\mu\nu}$ under the de Donder condition by Eq. (3.45). Under the condition $r'/r \ll 1$, it is

$$\begin{cases} h^{00}(t, \mathbf{x}) = -\frac{4G}{c^2} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \partial_{l-2} \left(\frac{\hat{M}_l(u)}{r} \right) \\ \quad - \frac{2G}{3c^4} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \int d^3 x' \frac{c}{r'} \hat{X}'_l(\theta', \varphi') \hat{T}_l(t, \mathbf{x}, \mathbf{x}'), \\ h^{0i}(t, \mathbf{x}) = \frac{4G}{c^3} \sum_{l=1}^{\infty} \frac{(-1)^l}{l!} \partial_{l-1} \left(\frac{\partial_t \hat{M}_{il-1}(u)}{r} \right) \\ \quad + \frac{4G}{c^3} \sum_{l=1}^{\infty} \frac{(-1)^l}{(l+1)!} \epsilon_{iab} \partial_{al-1} \left(\frac{\hat{S}_{bil-1}(u)}{r} \right), \\ h^{ij}(t, \mathbf{x}) = -\frac{4G}{c^4} \sum_{l=2}^{\infty} \frac{(-1)^l}{l!} \partial_{l-2} \left(\frac{\partial_t^2 \hat{M}_{ijl-2}(u)}{r} \right) \\ \quad - \frac{8G}{c^4} \sum_{l=2}^{\infty} \frac{(-1)^l}{(l+1)!} \partial_{al-2} \left(\frac{\epsilon_{ab(i} \partial_{|l|} \hat{S}_{j)b l-2}(u)}{r} \right) \\ \quad + \frac{2G}{3c^4} \delta^{ij} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \int d^3 x' \frac{c}{r'} \hat{X}'_l(\theta', \varphi') \hat{T}_l(t, \mathbf{x}, \mathbf{x}'), \end{cases} \quad (4.15)$$

which is the same as that in the linearized $f(R)$ gravity [35], as expected.

As to the multipole expansion of $h^{\mu\nu}$ in the radiation field, we still need to deal with the tensor part $\tilde{h}^{\mu\nu}$ and the scalar part associated with $R^{(1)}$, separately. By Eq. (2.15), we have

$$\hat{\partial}_{l_l} \left(\frac{F(u)}{r} \right) = \hat{N}_{l_l}(\theta, \varphi) \frac{(-1)^l}{c^l} \frac{\partial_u^l F(u)}{r} + O\left(\frac{1}{r^2}\right), \quad (4.16)$$

where $O(1/r^2)$ contains the terms whose orders are equal to or higher than $1/r^2$. The $1/r$ expansion in the distance to the source requires formula (4.16) to be applied to Eqs. (4.6) and (4.10), and then the multipole expansions of $\tilde{h}^{\mu\nu}$ and $R^{(1)}$ are obtained, respectively,

$$\begin{cases} \tilde{h}_{TT}^{00}(t, \mathbf{x}) = 0, \\ \tilde{h}_{TT}^{0i}(t, \mathbf{x}) = 0, \\ \tilde{h}_{TT}^{ij}(t, \mathbf{x}) = -\frac{4G}{c^2 r} \sum_{l=2}^{\infty} \left(\frac{1}{c^l l!} \left(\partial_u^l \hat{M}_{ij l_{l-2}}(u) N_{l_{l-2}}(\theta, \varphi) \right)^{TT} \right. \\ \quad \left. - \frac{2l}{c^{l+1}(l+1)!} \left(\epsilon_{ab(i} \partial_{|u|}^l \hat{S}_{j)b l_{l-2}}(u) N_{a l_{l-2}}(\theta, \varphi) \right)^{TT} \right) \\ \quad + O\left(\frac{1}{r^2}\right) \end{cases} \quad (4.17)$$

and

$$R^{(1)}(t, \mathbf{x}) = -\frac{m^2 \kappa}{4\pi r} \sum_{l=0}^{\infty} \frac{1}{c^l l!} \hat{K}_{l_l}(t, r) N_{l_l}(\theta, \varphi) + O\left(\frac{1}{r^2}\right), \quad (4.18)$$

where

$$\begin{aligned} \hat{K}_{l_l}(t, r) &= \frac{(2l+1)!!}{2^{l+1}l!} \int d^3x' \frac{c}{r'} \hat{X}'_{l_l}(\theta', \varphi') \\ &\quad \left(\int_{u-\frac{r'}{c}}^{u+\frac{r'}{c}} dt' \left[\frac{d^l}{du^l} \left(1 - \frac{c^2}{r'^2} (t' - u)^2 \right)^l \right] T^{(1)}(t', \mathbf{x}') \right. \\ &\quad \left. - \int_{-\infty}^{u-\frac{r'}{c}} dt' \left[\hat{A}(t, r; t', r') \frac{d^l}{du^l} \left(1 - \frac{c^2}{r'^2} (t' - u)^2 \right)^l \right] T^{(1)}(t', \mathbf{x}') \right) \end{aligned} \quad (4.19)$$

is the effective l -pole radiative moment of $R^{(1)}$. Eqs. (4.17) and (4.18), of course, are the same as those in the linearized $f(R)$ gravity, respectively [36].

According to the above conclusions, Eq. (3.45) implies that $h^{\mu\nu}$ in linearized $f(R, \mathcal{G})$ gravity and in linearized $f(R)$ gravity have the same multipole expansion under the TT gauge in the radiation field, and it is [36]

$$\begin{cases} h^{00}(t, \mathbf{x}) = -\frac{2G}{3c^4 r} \sum_{l=0}^{\infty} \frac{1}{c^l l!} \hat{K}_{l_l}(t, r) N_{l_l}(\theta, \varphi) + O\left(\frac{1}{r^2}\right), \\ h^{0i}(t, \mathbf{x}) = 0, \\ h^{ij}(t, \mathbf{x}) = -\frac{4G}{c^2 r} \sum_{l=2}^{\infty} \left(\frac{1}{c^l l!} \left(\partial_u^l \hat{M}_{ij l_{l-2}}(u) N_{l_{l-2}}(\theta, \varphi) \right)^{TT} \right. \\ \quad \left. - \frac{2l}{c^{l+1}(l+1)!} \left(\epsilon_{ab(i} \partial_{|u|}^l \hat{S}_{j)b l_{l-2}}(u) N_{a l_{l-2}}(\theta, \varphi) \right)^{TT} \right) \\ \quad + \frac{2G}{3c^4 r} \delta^{ij} \sum_{l=0}^{\infty} \frac{1}{c^l l!} \hat{K}_{l_l}(t, r) N_{l_l}(\theta, \varphi) + O\left(\frac{1}{r^2}\right). \end{cases} \quad (4.20)$$

4.2 Energy, momentum, and angular momentum carried by GWs in linearized $f(R, \mathcal{G})$ gravity

For $f(R, \mathcal{G})$ gravity, the energy, momentum, and angular momentum carried by the gravitational field inside a volume V enclosed by a large sphere S can be defined in the unified way with the help of stress-energy pseudotensor of the matter fields and the gravitational field [31, 36, 39]:

$$E = \int_V d^3x \tau^{00}, \quad (4.21)$$

$$P_a = \frac{1}{c} \int_V d^3x \tau^{0a}, \quad (4.22)$$

$$J_a = \frac{1}{c} \int_V d^3x \epsilon_{abc} x_b \tau^{0c}, \quad (4.23)$$

respectively, and then, by Eq. (3.40) and the Gauss theorem, there are

$$\frac{dE}{dt} = c \int_V d^3x \partial_0 \tau^{00} = -c \int_V d^3x \partial_i \tau^{i0} = -c \int d\Omega r^2 n_i \tau^{i0}, \quad (4.24)$$

$$\frac{dP_a}{dt} = \int_V d^3x \partial_0 \tau^{0a} = - \int_V d^3x \partial_i \tau^{ia} = - \int d\Omega r^2 n_i \tau^{ia}, \quad (4.25)$$

$$\begin{aligned} \frac{dJ_a}{dt} &= \int_V d^3x \partial_0 (\epsilon_{abc} x_b \tau^{0c}) = \int_V d^3x \epsilon_{abc} x_b \partial_0 \tau^{0c} \\ &= - \int_V d^3x \epsilon_{abc} x_b \partial_i \tau^{ic} \\ &= - \int_V d^3x \partial_i (\epsilon_{abc} x_b \tau^{ic}) = - \int d\Omega r^3 \epsilon_{abc} n_i n_b \tau^{ic}, \end{aligned} \quad (4.26)$$

where n_i is also the unit normal vector of the large sphere S , and $d\Omega$ is the element of solid angle whose integral domain is 4π . Thus, as indicated in Refs. [31, 39], for the linearized $f(R, \mathcal{G})$ gravity, the fluxes of energy, momentum, and angular momentum carried by the outward-propagating GW should be

$$\frac{dE}{d\Omega dt} = +cr^2 \left\langle n_i (\tau_{\text{rad}}^{i0})^{(2)} \right\rangle, \quad (4.27)$$

$$\frac{dP_a}{d\Omega dt} = +r^2 \left\langle n_i (\tau_{\text{rad}}^{ia})^{(2)} \right\rangle, \quad (4.28)$$

$$\frac{dJ_a}{d\Omega dt} = +r^3 \left\langle \epsilon_{abc} n_i n_b (\tau_{\text{rad}}^{ic})^{(2)} \right\rangle, \quad (4.29)$$

respectively, where

$$(\tau_{\text{rad}}^{\mu\nu})^{(2)} = \frac{1}{2\kappa} \left(\Lambda^{\mu\nu(2)} \right)$$

is the quadratic term of the stress–energy pseudotensor under the TT gauge in the radiation field.

Next, the leading terms of the fluxes of energy, momentum, and angular momentum carried by GWs in linearized $f(R, \mathcal{G})$ gravity are expressed by

$$\left(\frac{dE}{d\Omega dt} \right)^{[0]} = cr^2 \left\langle n_i (\tau_{\text{rad}}^{i0})^{(2)} \right\rangle^{[2]}, \quad (4.30)$$

$$\left(\frac{dP_a}{d\Omega dt} \right)^{[0]} = r^2 \left\langle n_i (\tau_{\text{rad}}^{ia})^{(2)} \right\rangle^{[2]}, \quad (4.31)$$

$$\left(\frac{dJ_a}{d\Omega dt} \right)^{[0]} = r^3 \left\langle \epsilon_{abc} n_i n_b (\tau_{\text{rad}}^{ic})^{(2)} \right\rangle^{[3]}, \quad (4.32)$$

respectively, where the superscript $[m]$ ($m = 0, 1, 2, 3 \dots$) represents that the term of the $1/r^m$ order for the corresponding quantity is taken, and then, there are

$$\frac{dE}{d\Omega dt} = \left(\frac{dE}{d\Omega dt} \right)^{[0]} + O\left(\frac{1}{r}\right), \quad (4.33)$$

$$\frac{dP_a}{d\Omega dt} = \left(\frac{dP_a}{d\Omega dt} \right)^{[0]} + O\left(\frac{1}{r}\right), \quad (4.34)$$

$$\frac{dJ_a}{d\Omega dt} = \left(\frac{dJ_a}{d\Omega dt} \right)^{[0]} + O\left(\frac{1}{r}\right). \quad (4.35)$$

In the following, the fluxes of the energy, momentum, and the angular momentum are used to denote their leading terms (4.30)–(4.32), respectively. Moreover, we also need to prove

$$\left\langle \epsilon_{abc} n_i n_b (\tau_{\text{rad}}^{ic})^{(2)} \right\rangle^{[2]} = 0. \quad (4.36)$$

Because if it does not hold, Eq. (4.32) shows that the flux of angular momentum carried by the outward-propagating GW will diverge, and this is impossible.

By Eqs. (3.50)–(3.52), $(\tau_{\text{rad}}^{i0})^{(2)}$ and $(\tau_{\text{rad}}^{ij})^{(2)}$ under the TT gauge are, respectively,

$$(\tau_{\text{rad}}^{i0})^{(2)} = (\tau_{\text{rad}}^{i0})_{f(R)}^{(2)} + \frac{1}{\kappa} \left(-\frac{4a_{12}}{3} R^{(1)} \partial_0 \partial_i R^{(1)} + 8a_{11}a_{12} \partial_i \partial_\alpha R^{(1)} \partial_0 \partial^\alpha R^{(1)} \right.$$

$$\left. + 2a_{12} \partial_p \partial_q R^{(1)} \partial_0 \partial_i \tilde{h}_{pq}^{TT} - 2a_{12} \partial_\rho \partial_q R^{(1)} \partial^\rho \partial_0 \tilde{h}_{iq}^{TT} \right), \quad (4.37)$$

$$(\tau_{\text{rad}}^{ij})^{(2)} = (\tau_{\text{rad}}^{ij})_{f(R)}^{(2)} + \frac{1}{\kappa} \left(-\frac{a_{12}}{9a_{11}} \delta_{ij} (R^{(1)})^2 + 4a_{11}a_{12} \delta_{ij} \partial^\alpha \partial^\beta R^{(1)} \partial_\alpha \partial_\beta R^{(1)} + \frac{4a_{12}}{3} R^{(1)} \partial_i \partial_j R^{(1)} - 8a_{11}a_{12} \partial_i \partial_\alpha R^{(1)} \partial_j \partial^\alpha R^{(1)} - 2a_{12} \partial_p \partial_q R^{(1)} \partial_i \partial_j \tilde{h}_{pq}^{TT} - 2a_{12} \partial_\rho \partial_\sigma R^{(1)} \partial^\rho \partial^\sigma \tilde{h}_{ij}^{TT} + 2a_{12} \partial_\rho \partial_q R^{(1)} \partial^\rho \partial_j \tilde{h}_{iq}^{TT} + 2a_{12} \partial_\sigma \partial_q R^{(1)} \partial^\sigma \partial_i \tilde{h}_{jq}^{TT} \right), \quad (4.38)$$

where

$$(\tau_{\text{rad}}^{i0})_{f(R)}^{(2)} = \frac{1}{2\kappa} \left(-\frac{1}{2} \partial_0 \tilde{h}_{pq}^{TT} \partial_i \tilde{h}_{pq}^{TT} + \partial_0 \tilde{h}_{pq}^{TT} \partial_p \tilde{h}_{iq}^{TT} - 12a_{11}^2 \partial_0 R^{(1)} \partial_i R^{(1)} \right), \quad (4.39)$$

$$(\tau_{\text{rad}}^{ij})_{f(R)}^{(2)} = \frac{1}{2\kappa} \left(-\tilde{h}_{pq}^{TT} \partial_p \partial_q \tilde{h}_{ij}^{TT} + \frac{1}{2} \partial_i \tilde{h}_{pq}^{TT} \partial_j \tilde{h}_{pq}^{TT} - \partial_0 \tilde{h}_{ip}^{TT} \partial_0 \tilde{h}_{jp}^{TT} + \partial_q \tilde{h}_{ip}^{TT} \partial_q \tilde{h}_{jp}^{TT} + \partial_p \tilde{h}_{iq}^{TT} \partial_q \tilde{h}_{jp}^{TT} - \partial_i \tilde{h}_{pq}^{TT} \partial_p \tilde{h}_{jq}^{TT} - \partial_j \tilde{h}_{pq}^{TT} \partial_p \tilde{h}_{iq}^{TT} + \delta_{ij} \left(\frac{1}{4} \partial_0 \tilde{h}_{pq}^{TT} \partial_0 \tilde{h}_{pq}^{TT} - \frac{1}{4} \partial_m \tilde{h}_{pq}^{TT} \partial_m \tilde{h}_{pq}^{TT} + \frac{1}{2} \partial_p \tilde{h}_{qm}^{TT} \partial_q \tilde{h}_{pm}^{TT} \right) + 12a_{11}^2 \partial_i R^{(1)} \partial_j R^{(1)} - \delta_{ij} \left(a_{11} (R^{(1)})^2 + 6a_{11}^2 \partial^\alpha R^{(1)} \partial_\alpha R^{(1)} \right) \right), \quad (4.40)$$

and then, we can use them to calculate the average parts in Eqs. (4.30)–(4.32), and (4.36). Firstly, the average parts in Eqs. (4.30) and (4.31) are, respectively,

$$\left\langle n_i (\tau_{\text{rad}}^{i0})^{(2)} \right\rangle^{[2]} = \left\langle n_i (\tau_{\text{rad}}^{i0})_{f(R)}^{(2)} \right\rangle^{[2]} + \frac{1}{\kappa} \left(-\frac{4a_{12}}{3} \left\langle n_i R^{(1)} \partial_0 \partial_i R^{(1)} \right\rangle^{[2]} + 8a_{11}a_{12} \left\langle n_i \partial_i \partial_\alpha R^{(1)} \partial_0 \partial^\alpha R^{(1)} \right\rangle^{[2]} + 2a_{12} \left\langle n_i \partial_p \partial_q R^{(1)} \partial_0 \partial_i \tilde{h}_{pq}^{TT} \right\rangle^{[2]} - 2a_{12} \left\langle n_i \partial_\rho \partial_q R^{(1)} \partial^\rho \partial_0 \tilde{h}_{iq}^{TT} \right\rangle^{[2]} \right), \quad (4.41)$$

$$\begin{aligned}
\left\langle n_i(\tau_{\text{rad}}^{ia})^{(2)} \right\rangle^{[2]} &= \left\langle n_i(\tau_{\text{rad}}^{ia})_{f(R)}^{(2)} \right\rangle^{[2]} + \frac{1}{\kappa} \left(-\frac{a_{12}}{9a_{11}} \left\langle n_a(R^{(1)})^2 \right\rangle^{[2]} \right. \\
&\quad + 4a_{11}a_{12} \left\langle n_a \partial^\alpha \partial^\beta R^{(1)} \partial_\alpha \partial_\beta R^{(1)} \right\rangle^{[2]} \\
&\quad + \frac{4a_{12}}{3} \left\langle n_i R^{(1)} \partial_i \partial_a R^{(1)} \right\rangle^{[2]} \\
&\quad - 8a_{11}a_{12} \left\langle n_i \partial_i \partial_\alpha R^{(1)} \partial_a \partial^\alpha R^{(1)} \right\rangle^{[2]} \\
&\quad - 2a_{12} \left\langle n_i \partial_p \partial_q R^{(1)} \partial_i \partial_a \tilde{h}_{pq}^{TT} \right\rangle^{[2]} \\
&\quad - 2a_{12} \left\langle n_i \partial_\rho \partial_\sigma R^{(1)} \partial^\rho \partial^\sigma \tilde{h}_{ia}^{TT} \right\rangle^{[2]} \\
&\quad + 2a_{12} \left\langle n_i \partial_\rho \partial_q R^{(1)} \partial^\rho \partial_a \tilde{h}_{iq}^{TT} \right\rangle^{[2]} \\
&\quad \left. + 2a_{12} \left\langle n_i \partial_\sigma \partial_q R^{(1)} \partial^\sigma \partial_i \tilde{h}_{aq}^{TT} \right\rangle^{[2]} \right). \quad (4.42)
\end{aligned}$$

By using of Eqs. (2.13), (3.64), (3.65), (4.17), and (4.18),

$$\begin{aligned}
\left\langle n_i(\tau_{\text{rad}}^{i0})^{(2)} \right\rangle^{[2]} &= \left\langle n_i(\tau_{\text{rad}}^{i0})_{f(R)}^{(2)} \right\rangle^{[2]} \\
&\quad + \frac{1}{\kappa} \left(-\frac{4a_{12}}{3} \left\langle n_i R^{(1)} \partial_0 \partial_i R^{(1)} \right\rangle^{[2]} \right. \\
&\quad + 8a_{11}a_{12} \left\langle n_i R^{(1)} \partial_0 \partial_i \square_\eta R^{(1)} \right\rangle^{[2]} \\
&\quad - 2a_{12} \left\langle n_i \partial_p R^{(1)} \partial_0 \partial_i \partial_q \tilde{h}_{pq}^{TT} \right\rangle^{[2]} \\
&\quad \left. + 2a_{12} \left\langle n_i \partial_\rho R^{(1)} \partial^\rho \partial_0 \partial_q \tilde{h}_{iq}^{TT} \right\rangle^{[2]} \right), \quad (4.43)
\end{aligned}$$

$$\begin{aligned}
\left\langle n_i(\tau_{\text{rad}}^{ia})^{(2)} \right\rangle^{[2]} &= \left\langle n_i(\tau_{\text{rad}}^{ia})_{f(R)}^{(2)} \right\rangle^{[2]} \\
&\quad + \frac{1}{\kappa} \left(-\frac{a_{12}}{9a_{11}} \left\langle n_a(R^{(1)})^2 \right\rangle^{[2]} \right. \\
&\quad + 4a_{11}a_{12} \left\langle n_a R^{(1)} \square_\eta^2 R^{(1)} \right\rangle^{[2]} \\
&\quad + \frac{4a_{12}}{3} \left\langle n_i R^{(1)} \partial_i \partial_a R^{(1)} \right\rangle^{[2]} \\
&\quad - 8a_{11}a_{12} \left\langle n_i R^{(1)} \partial_i \partial_a \square_\eta R^{(1)} \right\rangle^{[2]} \\
&\quad + 2a_{12} \left\langle n_i \partial_p R^{(1)} \partial_i \partial_a \partial_q \tilde{h}_{pq}^{TT} \right\rangle^{[2]} \\
&\quad \left. + 2a_{12} \left\langle n_i \partial_\sigma R^{(1)} \partial^\sigma \square_\eta \tilde{h}_{ia}^{TT} \right\rangle^{[2]} \right)
\end{aligned}$$

From Eq. (3.13), the de Donder condition under the TT gauge is

$$\partial_p \tilde{h}_{pq}^{TT} = 0, \quad (4.45)$$

and then, with the help of the Eq. (3.56) in the radiation field and Eq. (3.58), Eqs. (4.43) and (4.44) reduce to

$$\left\langle n_i(\tau_{\text{rad}}^{i0})^{(2)} \right\rangle^{[2]} = \left\langle n_i(\tau_{\text{rad}}^{i0})_{f(R)}^{(2)} \right\rangle^{[2]}, \quad (4.46)$$

$$\left\langle n_i(\tau_{\text{rad}}^{ia})^{(2)} \right\rangle^{[2]} = \left\langle n_i(\tau_{\text{rad}}^{ia})_{f(R)}^{(2)} \right\rangle^{[2]}. \quad (4.47)$$

Inserting them into Eqs. (4.30) and (4.31), respectively, gives rise to

$$\left(\frac{dE}{d\Omega dt} \right)^{[0]} = cr^2 \left\langle n_i(\tau_{\text{rad}}^{i0})_{f(R)}^{(2)} \right\rangle^{[2]} = \left(\frac{dE}{d\Omega dt} \right)_{f(R)}^{[0]}, \quad (4.48)$$

$$\left(\frac{dP_a}{d\Omega dt} \right)^{[0]} = r^2 \left\langle n_i(\tau_{\text{rad}}^{ia})_{f(R)}^{(2)} \right\rangle^{[2]} = \left(\frac{dP_a}{d\Omega dt} \right)_{f(R)}^{[0]}, \quad (4.49)$$

where $\left(\frac{dE}{d\Omega dt} \right)_{f(R)}^{[0]}$ and $\left(\frac{dP_a}{d\Omega dt} \right)_{f(R)}^{[0]}$ are the fluxes of energy and momentum carried by the outward-propagating GW in linearized $f(R)$ gravity, respectively, and thus, according to the Ref. II [36],

$$\begin{aligned}
\left(\frac{dE}{d\Omega dt} \right)^{[0]} &= \frac{cr^2}{4\kappa} \left\langle \partial_0 \tilde{h}_{pq}^{TT} \partial_0 \tilde{h}_{pq}^{TT} \right\rangle^{[2]} \\
&\quad - \frac{6cr^2 a_{11}^2}{\kappa} \left\langle \partial_r R^{(1)} \partial_0 R^{(1)} \right\rangle^{[2]}, \quad (4.50)
\end{aligned}$$

$$\begin{aligned}
\left(\frac{dP_a}{d\Omega dt} \right)^{[0]} &= -\frac{r^2}{4\kappa} \left\langle \partial_0 \tilde{h}_{pq}^{TT} \partial_a \tilde{h}_{pq}^{TT} \right\rangle^{[2]} \\
&\quad + \frac{6r^2 a_{11}^2}{\kappa} \left\langle \partial_r R^{(1)} \partial_a R^{(1)} \right\rangle^{[2]}. \quad (4.51)
\end{aligned}$$

The calculation of the average part in Eq. (4.36) is similar to the above process.

$$\begin{aligned}
\left\langle \epsilon_{abc} n_i n_b(\tau_{\text{rad}}^{ic})^{(2)} \right\rangle^{[2]} &= \left\langle \epsilon_{abc} n_i n_b(\tau_{\text{rad}}^{ic})_{f(R)}^{(2)} \right\rangle^{[2]} \\
&\quad + \frac{1}{\kappa} \left(\frac{4a_{12}}{3} \left\langle \epsilon_{abc} n_i n_b R^{(1)} \partial_i \partial_c R^{(1)} \right\rangle^{[2]} \right)
\end{aligned}$$

$$\begin{aligned}
& -8a_{11}a_{12}\left\langle\epsilon_{abc}n_in_b\partial_i\partial_\alpha R^{(1)}\partial_c\partial^\alpha R^{(1)}\right\rangle^{[2]} \\
& -2a_{12}\left\langle\epsilon_{abc}n_in_b\partial_p\partial_q R^{(1)}\partial_i\partial_c\tilde{h}_{pq}^{TT}\right\rangle^{[2]} \\
& -2a_{12}\left\langle\epsilon_{abc}n_in_b\partial_\rho\partial_\sigma R^{(1)}\partial^\rho\partial^\sigma\tilde{h}_{ic}^{TT}\right\rangle^{[2]} \\
& +2a_{12}\left\langle\epsilon_{abc}n_in_b\partial_\rho\partial_q R^{(1)}\partial^\rho\partial_c\tilde{h}_{iq}^{TT}\right\rangle^{[2]} \\
& +2a_{12}\left\langle\epsilon_{abc}n_in_b\partial_\sigma\partial_q R^{(1)}\partial^\sigma\partial_i\tilde{h}_{cq}^{TT}\right\rangle^{[2]}\Bigg) \\
& =\left\langle\epsilon_{abc}n_in_b(\tau_{\text{rad}}^{ic})_{f(R)}^{(2)}\right\rangle^{[2]} \\
& +\frac{1}{\kappa}\left(\frac{4a_{12}}{3}\left\langle\epsilon_{abc}n_in_b R^{(1)}\partial_i\partial_c R^{(1)}\right\rangle^{[2]} \right. \\
& -8a_{11}a_{12}\left\langle\epsilon_{abc}n_in_b R^{(1)}\partial_i\partial_c\Box_\eta R^{(1)}\right\rangle^{[2]} \\
& +2a_{12}\left\langle\epsilon_{abc}n_in_b\partial_p R^{(1)}\partial_i\partial_c\partial_q\tilde{h}_{pq}^{TT}\right\rangle^{[2]} \\
& +2a_{12}\left\langle\epsilon_{abc}n_in_b\partial_\sigma R^{(1)}\partial^\sigma\Box_\eta\tilde{h}_{ic}^{TT}\right\rangle^{[2]} \\
& -2a_{12}\left\langle\epsilon_{abc}n_in_b\partial_\rho R^{(1)}\partial^\rho\partial_c\partial_q\tilde{h}_{iq}^{TT}\right\rangle^{[2]} \\
& \left.-2a_{12}\left\langle\epsilon_{abc}n_in_b\partial_\sigma R^{(1)}\partial^\sigma\partial_i\partial_q\tilde{h}_{cq}^{TT}\right\rangle^{[2]}\right) \\
& =\left\langle\epsilon_{abc}n_in_b(\tau_{\text{rad}}^{ic})_{f(R)}^{(2)}\right\rangle^{[2]}, \quad (4.52)
\end{aligned}$$

and then, by the result in Ref. II [36], Eq. (4.36) holds, which ensures that the flux of angular momentum carried by the outward-propagating GW does not diverge.

Now, we begin to calculate the average part in Eq. (4.32). Eq. (4.38) results in

$$\begin{aligned}
& \left\langle\epsilon_{abc}n_in_b(\tau_{\text{rad}}^{ic})_{f(R)}^{(2)}\right\rangle^{[3]} \\
& =\left\langle\epsilon_{abc}n_in_b(\tau_{\text{rad}}^{ic})_{f(R)}^{(2)}\right\rangle^{[3]} \\
& +\frac{1}{\kappa}\left(\frac{4a_{12}}{3}\left\langle\epsilon_{abc}n_in_b R^{(1)}\partial_i\partial_c R^{(1)}\right\rangle^{[3]} \right. \\
& -8a_{11}a_{12}\left\langle\epsilon_{abc}n_in_b\partial_i\partial_\alpha R^{(1)}\partial_c\partial^\alpha R^{(1)}\right\rangle^{[3]} \\
& -2a_{12}\left\langle\epsilon_{abc}n_in_b\partial_p\partial_q R^{(1)}\partial_i\partial_c\tilde{h}_{pq}^{TT}\right\rangle^{[3]} \\
& \left.-2a_{12}\left\langle\epsilon_{abc}n_in_b\partial_\rho\partial_\sigma R^{(1)}\partial^\rho\partial^\sigma\tilde{h}_{ic}^{TT}\right\rangle^{[3]}\right)
\end{aligned}$$

$$\begin{aligned}
& +2a_{12}\left\langle\epsilon_{abc}n_in_b\partial_\rho\partial_q R^{(1)}\partial^\rho\partial_c\tilde{h}_{iq}^{TT}\right\rangle^{[3]} \\
& +2a_{12}\left\langle\epsilon_{abc}n_in_b\partial_\sigma\partial_q R^{(1)}\partial^\sigma\partial_i\tilde{h}_{cq}^{TT}\right\rangle^{[3]}\Bigg). \quad (4.53)
\end{aligned}$$

By Eqs. (3.64) and (3.65), the average part of the third term in Eq. (4.53) is

$$\begin{aligned}
& \left\langle\epsilon_{abc}n_in_b\partial_i\partial_\alpha R^{(1)}\partial_c\partial^\alpha R^{(1)}\right\rangle^{[3]} \\
& =-\left\langle\epsilon_{abc}n_in_b\partial_\alpha R^{(1)}\partial_i\partial_c\partial^\alpha R^{(1)}\right\rangle^{[3]} \\
& -\left\langle\epsilon_{abc}(\partial_in_i)n_b\partial_\alpha R^{(1)}\partial_c\partial^\alpha R^{(1)}\right\rangle^{[3]} \\
& -\left\langle\epsilon_{abc}n_i(\partial_in_b)\partial_\alpha R^{(1)}\partial_c\partial^\alpha R^{(1)}\right\rangle^{[3]} \\
& =\left\langle\epsilon_{abc}n_in_b R^{(1)}\partial_i\partial_c\Box_\eta R^{(1)}\right\rangle^{[3]} \\
& +\left\langle\epsilon_{abc}(\partial_p n_i)n_b R^{(1)}\partial_i\partial_c\partial_p R^{(1)}\right\rangle^{[3]} \\
& +\left\langle\epsilon_{abc}n_i(\partial_p n_b) R^{(1)}\partial_i\partial_c\partial_p R^{(1)}\right\rangle^{[3]} \\
& +\frac{2}{r}\left\langle\epsilon_{abc}n_b R^{(1)}\partial_c\Box_\eta R^{(1)}\right\rangle^{[2]} \\
& =\frac{1}{6a_{11}}\left\langle\epsilon_{abc}n_in_b R^{(1)}\partial_i\partial_c R^{(1)}\right\rangle^{[3]} \\
& +\frac{1}{r}\left\langle\epsilon_{abc}n_b R^{(1)}\partial_c\Delta R^{(1)}\right\rangle^{[2]} \\
& -\frac{2}{r}\left\langle\epsilon_{abc}n_p n_i n_b R^{(1)}\partial_i\partial_c\partial_p R^{(1)}\right\rangle^{[2]} \\
& +\frac{1}{3a_{11}r}\left\langle\epsilon_{abc}n_b R^{(1)}\partial_c R^{(1)}\right\rangle^{[2]}, \quad (4.54)
\end{aligned}$$

where $\Delta = \partial_i \partial_i$, Eq. (2.13) has been used in the second step, and Eqs. (3.58) and $\epsilon_{abc}\partial_b\partial_c = 0$ have been used in the third step. Equations (2.13) and (4.18) bring about

$$\left(\partial_k R^{(1)}\right)^{[1]} = \left(\partial_k (R^{(1)})^{[1]}\right)^{[1]} = n_k \left(\partial_r (R^{(1)})^{[1]}\right)^{[1]}, \quad (4.55)$$

$$\begin{aligned}
\left(\partial_j \partial_k R^{(1)}\right)^{[1]} & = \left(\partial_j (\partial_k (R^{(1)})^{[1]})\right)^{[1]} = n_k \left(\partial_j \left(\partial_r (R^{(1)})^{[1]}\right)^{[1]}\right)^{[1]} \\
& = n_j n_k \left(\partial_r \left(\partial_r (R^{(1)})^{[1]}\right)^{[1]}\right)^{[1]}, \quad (4.56)
\end{aligned}$$

$$\left(\partial_i \partial_j \partial_k R^{(1)}\right)^{[1]} = \left(\partial_i (\partial_j \partial_k (R^{(1)})^{[1]})\right)^{[1]}$$

$$\begin{aligned}
&= n_j n_k \left(\partial_i \left(\partial_r \left(\partial_r (R^{(1)})^{[1]} \right)^{[1]} \right)^{[1]} \right)^{[1]} \\
&= n_i n_j n_k \left(\partial_r \left(\partial_r \left(\partial_r (R^{(1)})^{[1]} \right)^{[1]} \right)^{[1]} \right)^{[1]}, \quad (4.57)
\end{aligned}$$

and then, there are

$$\begin{aligned}
&\left\langle \epsilon_{abc} n_b R^{(1)} \partial_c \Delta R^{(1)} \right\rangle^{[2]} \\
&= \left\langle \epsilon_{abc} n_b (R^{(1)})^{[1]} (\partial_c \Delta R^{(1)})^{[1]} \right\rangle^{[2]} = 0, \quad (4.58)
\end{aligned}$$

$$\begin{aligned}
&\left\langle \epsilon_{abc} n_p n_i n_b R^{(1)} \partial_i \partial_c \partial_p R^{(1)} \right\rangle^{[2]} \\
&= \left\langle \epsilon_{abc} n_p n_i n_b (R^{(1)})^{[1]} (\partial_i \partial_c \partial_p R^{(1)})^{[1]} \right\rangle^{[2]} = 0, \quad (4.59)
\end{aligned}$$

$$\begin{aligned}
&\left\langle \epsilon_{abc} n_b R^{(1)} \partial_c R^{(1)} \right\rangle^{[2]} \\
&= \left\langle \epsilon_{abc} n_b (R^{(1)})^{[1]} (\partial_c R^{(1)})^{[1]} \right\rangle^{[2]} = 0, \quad (4.60)
\end{aligned}$$

where $\epsilon_{abc} n_b n_c = 0$ have been used, so by Eq. (4.54), the average part of the third term in Eq. (4.53) is

$$\begin{aligned}
&\left\langle \epsilon_{abc} n_i n_b \partial_i \partial_\alpha R^{(1)} \partial_c \partial^\alpha R^{(1)} \right\rangle^{[3]} \\
&= \frac{1}{6a_{11}} \left\langle \epsilon_{abc} n_i n_b R^{(1)} \partial_i \partial_c R^{(1)} \right\rangle^{[3]}. \quad (4.61)
\end{aligned}$$

By using Eqs. (3.64) and (3.65) again, the average part of the fourth term in Eq. (4.53) is

$$\begin{aligned}
&\left\langle \epsilon_{abc} n_i n_b \partial_p \partial_q R^{(1)} \partial_i \partial_c \tilde{h}_{pq}^{TT} \right\rangle^{[3]} \\
&= - \left\langle \epsilon_{abc} (\partial_p n_i) n_b \partial_q R^{(1)} \partial_i \partial_c \tilde{h}_{pq}^{TT} \right\rangle^{[3]} \\
&\quad - \left\langle \epsilon_{abc} n_i (\partial_p n_b) \partial_q R^{(1)} \partial_i \partial_c \tilde{h}_{pq}^{TT} \right\rangle^{[3]} \\
&= \frac{2}{r} \left\langle \epsilon_{abc} n_p n_i n_b \partial_q R^{(1)} \partial_i \partial_c \tilde{h}_{pq}^{TT} \right\rangle^{[2]} \\
&\quad - \frac{1}{r} \left\langle \epsilon_{abc} n_i \partial_q R^{(1)} \partial_i \partial_c \tilde{h}_{bq}^{TT} \right\rangle^{[2]} \\
&= - \frac{2}{r} \left\langle \epsilon_{abc} n_p n_i n_b R^{(1)} \partial_i \partial_c \partial_q \tilde{h}_{pq}^{TT} \right\rangle^{[2]} \\
&\quad + \frac{1}{r} \left\langle \epsilon_{abc} n_i R^{(1)} \partial_i \partial_c \partial_q \tilde{h}_{bq}^{TT} \right\rangle^{[2]} = 0, \quad (4.62)
\end{aligned}$$

where Eqs. (2.13) and (4.45) have been used. Similar reason can be used to calculate average parts of the fifth, the sixth, and the seventh terms in Eq. (4.53), namely,

$$\begin{aligned}
&\left\langle \epsilon_{abc} n_i n_b \partial_\rho \partial_\sigma R^{(1)} \partial^\rho \partial^\sigma \tilde{h}_{ic}^{TT} \right\rangle^{[3]} \\
&= - \left\langle \epsilon_{abc} (\partial_\rho n_i) n_b \partial_\sigma R^{(1)} \partial^\rho \partial^\sigma \tilde{h}_{ic}^{TT} \right\rangle^{[3]} \\
&\quad - \left\langle \epsilon_{abc} n_i (\partial_\rho n_b) \partial_\sigma R^{(1)} \partial^\rho \partial^\sigma \tilde{h}_{ic}^{TT} \right\rangle^{[3]} \\
&= - \left\langle \epsilon_{abc} (\partial_q n_i) n_b \partial_\sigma R^{(1)} \partial_q \partial^\sigma \tilde{h}_{ic}^{TT} \right\rangle^{[3]} \\
&\quad - \left\langle \epsilon_{abc} n_i (\partial_q n_b) \partial_\sigma R^{(1)} \partial_q \partial^\sigma \tilde{h}_{ic}^{TT} \right\rangle^{[3]} \\
&= \frac{2}{r} \left\langle \epsilon_{abc} n_q n_i n_b \partial_\sigma R^{(1)} \partial_q \partial^\sigma \tilde{h}_{ic}^{TT} \right\rangle^{[2]} \\
&\quad - \frac{1}{r} \left\langle \epsilon_{abc} n_i \partial_\sigma R^{(1)} \partial_b \partial^\sigma \tilde{h}_{ic}^{TT} \right\rangle^{[2]} \\
&= - \frac{2}{r} \left\langle \epsilon_{abc} n_q n_i n_b R^{(1)} \partial_q \square_\eta \tilde{h}_{ic}^{TT} \right\rangle^{[2]} \\
&\quad + \frac{1}{r} \left\langle \epsilon_{abc} n_i R^{(1)} \partial_b \square_\eta \tilde{h}_{ic}^{TT} \right\rangle^{[2]} = 0, \quad (4.63)
\end{aligned}$$

$$\begin{aligned}
&\left\langle \epsilon_{abc} n_i n_b \partial_\rho \partial_q R^{(1)} \partial^\rho \partial_c \tilde{h}_{iq}^{TT} \right\rangle^{[3]} \\
&= - \left\langle \epsilon_{abc} (\partial_q n_i) n_b \partial_\rho R^{(1)} \partial^\rho \partial_c \tilde{h}_{iq}^{TT} \right\rangle^{[3]} \\
&\quad - \left\langle \epsilon_{abc} n_i (\partial_q n_b) \partial_\rho R^{(1)} \partial^\rho \partial_c \tilde{h}_{iq}^{TT} \right\rangle^{[3]} \\
&= \frac{2}{r} \left\langle \epsilon_{abc} n_q n_i n_b \partial_\rho R^{(1)} \partial^\rho \partial_c \tilde{h}_{iq}^{TT} \right\rangle^{[2]} \\
&\quad - \frac{1}{r} \left\langle \epsilon_{abc} n_i \partial_\rho R^{(1)} \partial^\rho \partial_c \tilde{h}_{ib}^{TT} \right\rangle^{[2]} \\
&= - \frac{2}{r} \left\langle \epsilon_{abc} n_q n_i n_b R^{(1)} \partial_c \square_\eta \tilde{h}_{iq}^{TT} \right\rangle^{[2]} \\
&\quad + \frac{1}{r} \left\langle \epsilon_{abc} n_i R^{(1)} \partial_c \square_\eta \tilde{h}_{ib}^{TT} \right\rangle^{[2]} = 0, \quad (4.64)
\end{aligned}$$

$$\begin{aligned}
&\left\langle \epsilon_{abc} n_i n_b \partial_\sigma \partial_q R^{(1)} \partial^\sigma \partial_i \tilde{h}_{cq}^{TT} \right\rangle^{[3]} \\
&= - \left\langle \epsilon_{abc} (\partial_q n_i) n_b \partial_\sigma R^{(1)} \partial^\sigma \partial_i \tilde{h}_{cq}^{TT} \right\rangle^{[3]} \\
&\quad - \left\langle \epsilon_{abc} n_i (\partial_q n_b) \partial_\sigma R^{(1)} \partial^\sigma \partial_i \tilde{h}_{cq}^{TT} \right\rangle^{[3]}
\end{aligned}$$

$$\begin{aligned}
&= \frac{2}{r} \left\langle \epsilon_{abc} n_q n_i n_b \partial_\sigma R^{(1)} \partial^\sigma \partial_i \tilde{h}_{cq}^{TT} \right\rangle^{[2]} \\
&= -\frac{2}{r} \left\langle \epsilon_{abc} n_q n_i n_b R^{(1)} \partial_i \square_\eta \tilde{h}_{cq}^{TT} \right\rangle^{[2]} = 0, \quad (4.65)
\end{aligned}$$

where Eq. (3.56) in the radiation field has been used in the above derivations, and $\tilde{h}_{ii}^{TT} = 0$ and $\epsilon_{abc} \tilde{h}_{bc}^{TT} = 0$ have been used in the derivations of Eqs. (4.64) and (4.65), respectively. Therefore, Eq. (4.53) reduces to

$$\left\langle \epsilon_{abc} n_i n_b (\tau_{\text{rad}}^{ic})^{(2)} \right\rangle^{[3]} = \left\langle \epsilon_{abc} n_i n_b (\tau_{\text{rad}}^{ic})^{(2)}_{f(R)} \right\rangle^{[3]}. \quad (4.66)$$

Inserting it into Eq. (4.32) gives rise to

$$\left(\frac{dJ_a}{d\Omega dt} \right)^{[0]} = r^3 \left\langle \epsilon_{abc} n_i n_b (\tau_{\text{rad}}^{ic})^{(2)}_{f(R)} \right\rangle^{[3]} = \left(\frac{dJ_a}{d\Omega dt} \right)^{[0]}_{f(R)}, \quad (4.67)$$

where $\left(\frac{dJ_a}{d\Omega dt} \right)^{[0]}_{f(R)}$ is the flux of angular momentum carried by the outward-propagating GW in linearized $f(R)$ gravity, and thus, according to the Ref. II [36],

$$\begin{aligned}
\left(\frac{dJ_a}{d\Omega dt} \right)^{[0]} &= \frac{r^2}{2\kappa} \left\langle \epsilon_{abc} \left(\tilde{h}_{bi}^{TT} \partial_0 \tilde{h}_{ic}^{TT} - \frac{1}{2} x_b \partial_c \tilde{h}_{pq}^{TT} \partial_0 \tilde{h}_{pq}^{TT} \right) \right\rangle^{[2]} \\
&\quad + \frac{6r^3 a_{11}^2}{\kappa} \left\langle \epsilon_{abc} n_b \partial_r R^{(1)} \partial_c R^{(1)} \right\rangle^{[3]}. \quad (4.68)
\end{aligned}$$

The coupling constant a_{12} in the Lagrangian (1.1) appears in the expressions of $(\tau_{\text{rad}}^{i0})^{(2)}$ and $(\tau_{\text{rad}}^{ij})^{(2)}$, which shows that the fluxes of the energy, momentum, and angular momentum carried by GWs in linearized $f(R, \mathcal{G})$ gravity depend on the GB curvature scalar $\mathcal{G}$. However, the above derivations of Eqs. (4.48), (4.49), and (4.67) show that, similarly to that of the effective stress–energy tensor of GWs, the terms associated with the coupling constant a_{12} in the fluxes of the energy, momentum, and angular momentum are cancelled each other after the average over a small spatial volume, and the dependence on a_{12} also vanishes. Therefore, the fluxes of the energy, momentum, and angular momentum carried by GWs in linearized $f(R, \mathcal{G})$ gravity are the same as those in linearized $f(R)$ gravity. Moreover, the fluxes of energy and momentum carried by GW in linearized $f(R, \mathcal{G})$ gravity could also be directly expressed in terms of the effective stress–energy tensor $t^{\mu\nu}$, namely,

$$\frac{dE}{d\Omega dt} = +cr^2 n_i t^{i0}, \quad \frac{dP_a}{d\Omega dt} = +r^2 n_i t^{ia}. \quad (4.69)$$

Equation (3.67) shows that the effective stress–energy tensor of GWs in linearized $f(R, \mathcal{G})$ gravity is the same as that in linearized $f(R)$ gravity, so there are

$$\frac{dE}{d\Omega dt} = +cr^2 n_i t_{f(R)}^{i0}, \quad \frac{dP_a}{d\Omega dt} = +r^2 n_i t_{f(R)}^{ia}, \quad (4.70)$$

and then, according to the Ref. II [36], both of the leading terms in Eq. (4.69) can recover Eqs. (4.50) and (4.51), respectively.

Integrating the fluxes of the energy, momentum, and angular momentum carried by GWs in linearized $f(R, \mathcal{G})$ gravity over a sphere can give rise to the total power and rates of momentum and angular momentum, respectively, and then, according to the Ref. II [36] again, they are

$$\begin{aligned}
\left(\frac{dE}{dt} \right)^{[0]} &= \left(\frac{dE}{dt} \right)^{[0]}_{GR} - \sum_{l=0}^{\infty} \frac{G}{3c^{2l+4}} \frac{1}{l!(2l+1)!!} \\
&\quad \times \left\langle \partial_t \hat{K}_{l_l}(t, r) \partial_r \hat{K}_{l_l}(t, r) \right\rangle, \quad (4.71)
\end{aligned}$$

$$\begin{aligned}
\left(\frac{dP_a}{dt} \right)^{[0]} &= \left(\frac{dP_a}{dt} \right)^{[0]}_{GR} + \sum_{l=0}^{\infty} \frac{2G}{3c^{2l+5}} \frac{1}{l!(2l+3)!!} \\
&\quad \times \left\langle \partial_r \hat{K}_{l_l}(t, r) \partial_r \hat{K}_{a l_l}(t, r) \right\rangle, \quad (4.72)
\end{aligned}$$

$$\begin{aligned}
\left(\frac{dJ_a}{dt} \right)^{[0]} &= \left(\frac{dJ_a}{dt} \right)^{[0]}_{GR} - \sum_{l=1}^{\infty} \frac{G}{3c^{2l+4}} \frac{1}{(l-1)!(2l+1)!!} \\
&\quad \times \left\langle \epsilon_{abc} \hat{K}_{b l_{l-1}}(t, r) \partial_r \hat{K}_{c l_{l-1}}(t, r) \right\rangle, \quad (4.73)
\end{aligned}$$

where

$$\begin{aligned}
\left(\frac{dE}{dt} \right)^{[0]}_{GR} &= \sum_{l=2}^{\infty} \frac{G}{c^{2l+1}} \left(\frac{(l+1)(l+2)}{l(l-1)l!(2l+1)!!} \right. \\
&\quad \times \left\langle \partial_u^{l+1} \hat{M}_{l_l}(u) \partial_u^{l+1} \hat{M}_{l_l}(u) \right\rangle \\
&\quad + \frac{4l(l+2)}{c^2(l-1)(l+1)!(2l+1)!!} \\
&\quad \times \left. \left\langle \partial_u^{l+1} \hat{S}_{l_l}(u) \partial_u^{l+1} \hat{S}_{l_l}(u) \right\rangle \right), \quad (4.74)
\end{aligned}$$

$$\begin{aligned}
\left(\frac{dP_a}{dt} \right)^{[0]}_{GR} &= \sum_{l=2}^{\infty} \frac{2G}{c^{2l+3}} \left(\frac{(l+2)(l+3)}{l(l+1)l!(2l+3)!!} \right. \\
&\quad \times \left\langle \partial_u^{l+1} \hat{M}_{l_l}(u) \partial_u^{l+2} \hat{M}_{a l_l}(u) \right\rangle \\
&\quad + \frac{4(l+2)}{(l-1)(l+1)!(2l+1)!!} \\
&\quad \times \left. \left\langle \epsilon_{abc} \partial_u^{l+1} \hat{M}_{b l_{l-1}}(u) \partial_u^{l+1} \hat{S}_{c l_{l-1}}(u) \right\rangle \right. \\
&\quad + \frac{4(l+3)}{c^2(l+1)!(2l+3)!!} \\
&\quad \times \left. \left\langle \partial_u^{l+1} \hat{S}_{l_l}(u) \partial_u^{l+2} \hat{S}_{a l_l}(u) \right\rangle \right), \quad (4.75)
\end{aligned}$$

$$\begin{aligned}
\left(\frac{dJ_a}{dt}\right)_{GR}^{[0]} &= \sum_{l=2}^{\infty} \frac{G}{c^{2l+1}} \left(\frac{(l+1)(l+2)}{(l-1)!(2l+1)!!} \right. \\
&\quad \times \left\langle \epsilon_{abc} \partial_u^l \hat{M}_{bI_{l-1}}(u) \partial_u^{l+1} \hat{M}_{cI_{l-1}}(u) \right\rangle \\
&\quad + \frac{4l^2(l+2)}{c^2(l-1)(l+1)!(2l+1)!!} \\
&\quad \times \left\langle \epsilon_{abc} \partial_u^l \hat{S}_{bI_{l-1}}(u) \partial_u^{l+1} \hat{S}_{cI_{l-1}}(u) \right\rangle \Bigg) \quad (4.76)
\end{aligned}$$

are the corresponding results in GR with the replacement of the variables h_{pq}^{TT} by $\tilde{h}_{pq}^{TT}$.

Obviously, these expressions include two parts, which are associated with the tensor part $\tilde{h}^{\mu\nu}$ and the scalar part $R^{(1)}$ in the multipole expansion of linearized $f(R, \mathcal{G})$ gravity, respectively. The former are the GR-like part, and the latter, contributed by the R^2 term in the Lagrangian (1.1), are the corrections to the results in GR, which implies that the GB curvature scalar $\mathcal{G}$ does not contribute to the energy, momentum, and angular momentum carried by GWs in linearized $f(R, \mathcal{G})$ gravity.

For GR, with the stress–energy pseudotensor of the gravitational field, after obtaining the energy, momentum, and angular momentum carried by GWs, any higher order nonlinear effect has been analyzed by the so-called multipolar post-Minkowskian solution, which forms the basis of “post-Newtonian gravitational wave generation formalism of GR” [37]. With the help of this formalism, GWs emitted by inspiralling compact binaries, namely binary star systems composed of neutron stars and/or black holes in their late stage of evolution, can be accurately modelled. Seeing that $f(R, \mathcal{G})$ gravity have many very important applications in cosmology, we try to construct its post-Newtonian gravitational wave generation by following the above thought for GR. Then, the most primary task for us is to analyzing the 2-order (the lowest order) nonlinear effects of $f(R, \mathcal{G})$ gravity by using its stress–energy pseudotensor of the gravitational field, which is exactly the subject of the present paper. Moreover, as indicated in Ref. [21], the GB scalar in $f(R, \mathcal{G})$ gravity, given by the squares of the curvatures, could make a small correction to the propagation of GWs in $f(R)$ gravity, which shows that it is very meaningful to study the contribution of the GB scalar in the GWs of $f(R, \mathcal{G})$ gravity. The results in our paper imply that in terms of the energy, momentum, and angular momentum carried by GWs, the linearized $f(R, \mathcal{G})$ gravity behaves as the linearized $f(R)$ gravity.

5 Conclusions and discussions

In this paper, similarly to GR [31, 37] and $f(R)$ gravity [35], the field equations of $f(R, \mathcal{G})$ gravity have been rewritten in

the form of obvious wave equations in a fictitious flat space-time under the de Donder condition, even though the effective gravitational field amplitude $\tilde{h}^{\mu\nu}$ is not a perturbation. The corresponding source term of this wave equation, composed of all the nonlinear terms under the post-Minkowskian method, is obtained, and it is the stress–energy pseudotensor of the matter fields and the gravitational field. For the linearized $f(R, \mathcal{G})$ gravity, the corresponding field equations, as the linear results, are the same as those of the linearized $f(R)$ gravity, as is expected. Under the post-Minkowskian method, the coupling constant a_{12} in the Lagrangian (1.1) appears in the second-order field equations of $f(R, \mathcal{G})$ gravity, which implies that the GB curvature scalar $\mathcal{G}$ usually plays an important role in the nonlinear effects. Thus, as a typical second-order nonlinear result, the effective stress–energy tensor of GWs in linearized $f(R, \mathcal{G})$ gravity should include the contribution from $\mathcal{G}$. However, it is shown that $\mathcal{G}$ has nothing to do with the effective stress–energy tensor of GWs in linearized $f(R, \mathcal{G})$ gravity, and the effective stress–energy tensor of GWs in linearized $f(R, \mathcal{G})$ gravity is the same as that in linearized $f(R)$ gravity.

Besides the effective stress–energy tensor of GWs, the energy, momentum, and angular momentum carried by GWs in linearized $f(R, \mathcal{G})$ gravity, as the further second-order nonlinear results, need to be discussed. As in GR [31] and $f(R)$ gravity [36], the above effective stress–energy tensor of GWs can be used to evaluate the fluxes of the energy and the momentum but not to evaluate the flux of the angular momentum [31]. Therefore, we need to find a new way, not depending on the effective stress–energy tensor of GWs, to deal with the angular momentum. According to the Refs. [31, 36, 39], we can deal with the energy, momentum, and angular momentum carried by GWs in linearized $f(R, \mathcal{G})$ gravity in a unified way. What this way requires is the stress–energy pseudotensor of $f(R, \mathcal{G})$ gravity in the radiation field, instead of the effective stress–energy tensor of GWs. By the above unified way, the multipole expansion of the linearized $f(R, \mathcal{G})$ gravity need to be discussed firstly. The definitions of the gravitational field amplitude $h^{\mu\nu}$ and the effective gravitational field amplitude $\tilde{h}^{\mu\nu}$ show that they satisfy the same relation, and especially, the linearized one (1.4) as that in $f(R)$ gravity. Further, for the linearized $f(R, \mathcal{G})$ gravity and the linearized $f(R)$ gravity, $\tilde{h}^{\mu\nu}$ and $R^{(1)}$ satisfy the same field equations, respectively, and therefore, these two gravitational models have the same multipole expansion, and in particular, they have the same multipole expansion in the radiation field.

With the help of the relevant STF technique, the $1/r$ expansion in the distance to the source can be applied to the linearized $f(R, \mathcal{G})$ gravity. Based on this, the energy, momentum, and angular momentum carried by GWs in linearized $f(R, \mathcal{G})$ gravity can be dealt with. In the expressions of the fluxes of the energy, momentum, and angular momen-

tum carried by GWs in linearized $f(R, \mathcal{G})$ gravity, the existence of the coupling constant a_{12} in the Lagrangian (1.1) implies the GB curvature scalar $\mathcal{G}$ should play an important role. However, because of the average over a small spatial volume, the terms associated with the coupling constant a_{12} in these expressions are cancelled each other, and the dependence on a_{12} vanishes. Therefore, $\mathcal{G}$ does not contribute to the energy, momentum, and angular momentum carried by GWs in linearized $f(R, \mathcal{G})$ gravity, which implies that these fluxes are the same as those in linearized $f(R)$ gravity. Thus, by integrating these fluxes over a sphere, we present the general expressions of the total power and rates of momentum and angular momentum in the GWs for linearized $f(R, \mathcal{G})$ gravity. Same as those in $f(R)$ gravity, these expressions include the GR-like parts and the corrections, contributed by the R^2 term in the Lagrangian (1.1), to the corresponding results in GR.

As mentioned above, in terms of the energy, momentum, and angular momentum carried by GWs, the linearized $f(R, \mathcal{G})$ gravity behave as the linearized $f(R)$ gravity, but for $f(R, \mathcal{G})$ gravity, where is the contribution of the GB curvature scalar $\mathcal{G}$? One obvious answer is that $\mathcal{G}$ perhaps plays an important role in the higher-order nonlinear effects. The similar situation also appears for the $f(R, \mathcal{G})$ gravity under the weak-field and slow-motion method [25]. Under this case, the GB curvature $\mathcal{G}$ has nothing to do with the effects at $(v/c)^2$ and $(v/c)^3$ orders, and only contributes to the effects at $(v/c)^4$ order or the higher-order, where v/c is the perturbation parameter of the weak-field and slow-motion method. In the present gravitational Lagrangian (1.1), the GB curvature scalar $\mathcal{G}$ works only by the term $a_{12}R\mathcal{G}$, so $f(R, \mathcal{G})$ in Eq. (1.1) is equivalent to $f(R) + a_{12}R\mathcal{G}$. Further, $f(R)$ gravity in the metric formalism can be cast in the form of Brans-Dicke (BD) theory with a potential for the effective scalar-field degree of freedom (scalon) [44], and thus, the theory with Lagrangian $f(R) + a_{12}R\mathcal{G}$ can be cast as a large class of GB gravity whose Lagrangian is equivalent to above BD theory coupled by the GB curvature scalar $\mathcal{G}$. Therefore, the conclusion in this paper also holds for such GB gravity.

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References

1. B.P. Abbott et al., Phys. Rev. Lett. **116**, 061102 (2016)
2. B.P. Abbott et al., Phys. Rev. Lett. **116**, 241103 (2016)
3. B.P. Abbott et al., Phys. Rev. Lett. **118**, 221101 (2017)
4. B.P. Abbott et al., Phys. Rev. Lett. **119**, 141101 (2017)
5. B.P. Abbott et al., Phys. Rev. Lett. **119**, 161101 (2017)
6. B.P. Abbott et al., Phys. Rev. Lett. **116**, 221101 (2016)
7. B.P. Abbott et al., Astrophys. J. **818**, L22 (2016)
8. B.P. Abbott et al., Astrophys. J. **848**, L12 (2017)
9. P. Astier et al., SNLS Collaboration. Astron. Astrophys. **447**, 31 (2006)
10. D.J. Eisenstein et al., SDSS Collaboration. Astrophys. J. **633**, 560 (2005)
11. A.G. Riess et al., Supernova Search Team Collaboration. Astrophys. J. **607**, 665 (2004)
12. D.N. Spergel et al., WMAP Collaboration. Astrophys. J. Suppl. **170**, 377 (2007)
13. T.P. Sotiriou, Rev. Mod. Phys. **82**, 451 (2010)
14. H.A. Buchdahl, Mon. Not. R. Astron. Soc. **150**, 1 (1970)
15. A.A. Starobinsky, Phys. Lett. **91B**, 99 (1980)
16. J. York, W. James, Phys. Rev. Lett. **28**, 1082 (1972)
17. G.W. Gibbons, S.W. Hawking, Phys. Rev. D **15**, 2752 (1977)
18. S. Nojiri, S.D. Odintsov, Phys. Rep. **505**, 59 (2011)
19. S. Nojiri, S.D. Odintsov, V.K. Oikonomou, Phys. Rep. **692**, 1 (2017)
20. S. Capozziello, M. De Laurentis, S. Nojiri, S.D. Odintsov, Phys. Rev. D **95**, 083524 (2017)
21. S. Nojiri, S.D. Odintsov, Phys. Lett. B **779**, 425 (2018)
22. G. Cognola, E. Elizalde, S. Nojiri, S.D. Odintsov, S. Zerbini, Phys. Rev. D **73**, 084007 (2006)
23. M. Alimohammadi, A. Ghalee, Phys. Rev. D **79**, 063006 (2009)
24. K. Bamba, S.D. Odintsov, L. Sebastiani, S. Zerbini, Eur. Phys. J. C **67**, 295 (2010)
25. B. Wu, B.-Q. Ma, Phys. Rev. D **92**, 044012 (2015)
26. M. Farasat Shamir, A. Komal, Int. J. Geom. Methods Mod. Phys. **14**, 12 (2017)
27. S.D. Odintsov, V.K. Oikonomou, S. Banerjee, Nucl. Phys. B **938**, 935 (2019)
28. S. Nojiri, S.D. Odintsov, Phys. Lett. B **631**, 1 (2005)
29. S. Nojiri, S.D. Odintsov, V.K. Oikonomou, Phys. Rev. D **99**, 044050 (2019)
30. T. Damour, B.R. Iyer, Phys. Rev. D **43**, 3259 (1991)
31. K.S. Thorne, Rev. Mod. Phys. **52**, 299 (1980)
32. L. Blanchet, T. Damour, Philos. Trans. R. Soc. A **320**, 379 (1986)
33. L. Blanchet, T. Damour, Ann. Inst. Henri Poincaré **50**, 377 (1989)
34. C.P.L. Berry, J.R. Gair, Phys. Rev. D **83**, 104022 (2011)
35. B. Wu, C.-G. Huang, Phys. Rev. D **96**, 104052 (2017)
36. B. Wu, C.-G. Huang, Phys. Rev. D **97**, 084027 (2018)
37. L. Blanchet, Living Rev. Relativ. **17**, 2 (2014)
38. V. Fock, *The Theory of Space, Time and Gravitation* (Pergamon Press, London, 1959)
39. P.C. Peters, Phys. Rev. **136**, B1224 (1964)
40. A. De Felice, T. Tanaka, Progr. Theor. Phys. **124**, 503 (2010)
41. S. Carroll, *Spacetime and Geometry: An Introduction to General Relativity* (Addison-Wesley, San Francisco, 2004)
42. J. Näf, P. Jetzer, Phys. Rev. D **84**, 024027 (2011)
43. M. Maggiore, *Gravitational Waves* (Oxford University Press, Oxford, 2008)
44. A. De Felice, S. Tsujikawa, Living Rev. Relativ. **13**, 3 (2010)