



D-dependence of the gap between the critical temperatures in the one-dimensional gauge theories

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Abstract We investigate the dimensional dependence (*D*-dependence) of the difference (gap) between the critical temperatures associated with the uniform/non-uniform and non-uniform/gapped transitions in the large-*N* bosonic gauge theories with *D* matrix scalar fields on a S^1 -circled space. We use the equations describing these critical temperatures given in the $1/D$ expansion Mandal et al. (JHEP 1002:034, 2010). These transitions are related with the Gregory-Laflamme instabilities in the gravities and the Rayleigh-Plateau instabilities in the fluid dynamics, and qualitative similarities between these are expected. We find that the tendency in the *D*-dependence of the gap is opposite from those in the gravity and fluid side. This is interesting as a counterexample to the gauge/gravity and gauge/fluid correspondences.

1 Introduction

The model we consider in this paper is the one-dimensional large-*N* gauge theories given by the BFSS matrix model [2] with general *D*. The BFSS matrix model has come up in the evolutions of the superstring theory. Let us overview it [3].

In the superstring theory there are five theories defined in the *D*=10 space-time. The low energy sectors of those are the five types in the *D*=10 supergravity. [4] has proposed that superstrings are the rolled up supermembranes in the S^1 -compactified *D*=11 space-time, where supermembranes [5] can be obtained as the classical solutions in the *D*=11 supergravity [6]. [7] has proposed the relation $R = g_s l_s$ for the S^1 -compactification, and identified the *D*=11 supergravity without the S^1 -compactification as the low energy sector of the strongly coupled type IIA superstring theory. This has been reached by looking at the mass spectra between BPS black hole solutions in the type IIA supergravity and the KK modes in the S^1 -compactified *D*=11 supergravity the-

ory. This comprehensive theory is referred to as M-theory [8].

The BPS black hole solutions having played important roles in the identification above are zero-dimensional ones, but there are also spatially *p*-dimensional BPS black holes (black *p*-branes) in the type IIA supergravity. It is then needed to get quantum understanding of those and how those correspond to supermembranes in the *D*=11 supergravity theory. [9] has discovered *Dp*-branes, which are BPS states as those break SUSY half and the quantum objects for the black *p*-branes. The low-energy dynamics of *N* *Dp*-branes is described with *D*=*p*+1 $U(N)$ SYM, and the Hamiltonian of supermembranes is given by *D*=1 $SU(N)$ SYM, where *N* is infinity [10, 11].

Based on the fact that dynamics of supermembranes and *D*0-branes are described with a same SYM (and charges on the *D*2-branes obtained from membranes not winding on the S^1 -compactified space), [12] has proposed that membranes are composed of a large number of *D*0-branes.

However *D*=1 $U(N)$ SYM describing *N* *D*0-brane's dynamics is no more than the low-energy effective theory. However [2] has proposed that it is originally valid at the whole energy scale but is the one just seen from the standpoint of the infinite momentum frame (IMF) in the eleven dimensional space-time. By this, we have reached the microscopic descriptions of the M-theory in the IMF based on *N* *D*0-brane's dynamics using *D*=1 $U(N)$ SYM (BFSS matrix model). *N* has to be taken to infinities in the IMF, however [13–15] have proposed that finite *N* is possible by changing the S^1 -compactified direction to the light-cone.

One of the important interpretations of the bosonic BFSS (bBFSS) matrix model is the low-energy dynamics of bosonic *D*0-branes on $\mathbb{R}^{D=9} \times S^1_{(L)}$.

According to [1, 16, 17], a way to reach this interpretation is to consider a two-dimensional SYM on $\mathbb{R}^{D-1} \times S^1_{(L)} \times S^1_{(\beta)}$ first. This corresponds to the low-energy *D*1-brane system at

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finite temperature $T = \beta^{-1}$, where D1-branes wind around a L -direction overlapping. We perform a T-duality to the L -direction. As a result, L exchanges to $L' = 2\pi\alpha'/L$, and D1-branes exchange to D0-branes. We also take the high temperature limit. As a result, the β -direction dependence disappears and the $\mathbb{R}^{D-1} \times S^1_{(\beta \rightarrow 0)}$ part becomes \mathbb{R}^D effectively (see Sec. 2.2 for more specifically). Fermions also decouple. By doing like this, we can reach the bBFSS matrix model above. The eigenvalues of Wilson line wrapping around the L' -direction represent the position of the D0-branes in the L' -direction.

As such, BFSS matrix model has originally come up from contexts of the M-theory, however that with general D (1D gauge theories) also plays the role of the effective microscopic description of the low energy dynamics of D0-branes. Exploiting this, we can try to obtain understanding for the D-brane systems and black objects.

We list the studies for those based on low dimensional gauge theories: i) dynamical generation of space-times in IIB matrix model [18–20], ii) critical phenomena in strongly coupled 1D large- N gauge theories using Gaussian expansion method [21, 22], iii) stabilities of fuzzy spaces in IIB matrix model [23–29], iv) phase structures of low dimensional gauge theories [16, 30–36], v) correspondence between superstring theory and IIB matrix model [37–39], vi) phase structures of low dimensional gauge theories by $1/D$ expansion; [1, 40, 41], vii) descriptions of black holes in real-time using BFSS matrix model [42–45], viii) linear responses in D0-branes [46], ix) covariant matrix theory for D-particles [47].

The $1/D$ expansion has been performed in a 1D bosonic gauge theory on a S^1 -circled space [1]¹. The $1/D$ expansion is very important because it is the method regardless of the coupling constants; it is not the expansion with regard to coupling constants but around large D . Actually, [1] has succeeded in obtaining the results for not only the critical temperatures but also the transition-orders in the model above. This is very wonderful. Since the $1/D$ expansion takes the similar fashion with usual perturbative expansions, the analysis of the transition-orders has been possible for the first time.

The phase transitions occurring in the 1D bosonic gauge theories are two: 1) the uniform/non-uniform transition and 2) uniform/gapped transition.

The critical temperatures obtained by the $1/D$ expansion agree with the results of Monte Carlo (MC) simulation very well, however the transition-orders are obtained differently among [1], [17] and [49]: As temperature is risen,

- 1). in [1], the second-order transition occurs first, then the third-order transition occurs next as of $D = 9$,
- 2). in [17], the third-order transition occurs first, then the second-order transition occurs next, at $D = 9$,
- 3). in [49], only the first-order transition occurs until $D = 20$, then the transition switches to the situation in [1] at some large D .

One thing we can say is that the conclusion in [17] is wrong. At present we cannot conclude whether [1] or [49] is right. For this purpose, we need to confirm the existence of the D where transition changes from the 1st to the 2nd+3rd in the MC simulation of [49]. If we could confirm it in future, we could conclude that [49] is right.

In these studies, the D -dependence of the difference (gap) between the critical temperatures has not been investigated. Since the following gauge/gravity correspondence

1D gauge theories \iff D0 black-brane solutions

is one of the well-known correspondences, whether it agrees or not with the gravity and fluid sides is interesting. We here turn to the critical phenomena in those sides.

The critical phenomena in the gravity and fluid sides are Gregory-Laflamme (GL) instabilities [50, 51] and Rayleigh-Plateau (RP) instabilities, respectively. GL and RP instabilities can be interpreted as uniform/non-uniform and non-uniform/gapped transitions [52]. [53–56] and [57–59] address issues of these correspondences from the gravity side and the fluid side, respectively.

Among those studies, we would like to focus on the results in [56] and [59] on how the transition-orders vary depending on the number of transverse space dimensions. ([60–66] are studies related with this issue.) According to [56],

- 1). one first-order transition occurs in $d = \dots, 9, 10, 11$,
- 2). a first-order transition, then a higher-order transition occur in $d = 12, 13$,
- 3). a second-order transition, then a higher-order transition occur in $d = 14, 15, \dots$

(d is the number of space dimensions in $D = d + 1$ S^1 -compactified spaces).

Regarding the results in [59], we would like to refer readers to Table 1 in [59]; as the point in [59], only one first-order transition occurs at not-large D , while second-order and some transitions occur separately in succession at large D .

As such, we would like to investigate the D -dependence of the gap in the large- N 1D bosonic gauge theories on a S^1 -circled space with D matrix scalar fields. We perform this based on the $1/D$ expansion of [1].

¹ The study to have formulated the $1/D$ expansion first is [48] in the IIB matrix model [18], and we can regard the work of [1] as its extension to a S^1 -circled space.

The main result we obtain in this study is that the gap does not narrow even if D becomes smaller, on the other hand the gap narrows as D becomes larger. These mean that the two transitions keep on occurring separately at small D , while the two transitions asymptote and occur as a single transition effectively at large D . These tendencies are the opposite of the gravity and fluid sides above.

Of course there is no guarantee that the correspondences with the gravities and fluids are held in every point exactly, however we could expect qualitative similarities at least. Therefore, our result is interesting as a specific counterexample to that.

There may be a question that the results in this study may be error for the $1/D$ expansion. We comment on this in Sec. 9.

As the organization of this paper, in Sec. 2, our model is given. In Sec. 3-6 are the review for the $1/D$ expansion, and we obtain the equations of the critical temperatures. In Sec. 7, we show the D -dependence of the gap, then based on that we argue that the gauge/gravity and gauge/fluid correspondences do not always hold. In Sec. 8, we argue this in the Z_m symmetric solutions.

2 The model in this study

2.1 Our model

We begin with the one-dimensional $SU(N)^2$ bosonic Yang–Mills gauge theory given by the bosonic BFSS type matrix model (1D model):

$$S = \frac{1}{g^2} \int_0^\beta dt \text{Tr} \left(\frac{1}{2} \sum_{I=1}^D (D_0 Y^I)^2 - \frac{1}{4} \sum_{I,J=1}^D [Y^I, Y^J]^2 \right),$$

where A_0 and Y^I are the $N \times N$ bosonic Hermitian matrices, and t is the Euclidean time which can be related with the temperature T as $\beta = T^{-1}$. We have $D_0 = \partial_0 - i[A_0, \cdot]$. A_0 and Y^I obey the boundary conditions $Y^I(t) = Y^I(t + \beta)$ and $A_0(t) = A_0(t + \beta)$. D is a parameter.

Performing the rescaling $Y^I \rightarrow g Y^I$, we rewrite the one above into

$$S = \int_0^\beta dt \text{Tr} \left(\frac{1}{2} (D_0 Y^I)^2 - \frac{g^2}{4} [Y^I, Y^J]^2 \right). \tag{1}$$

We omit to write the summations for I in what follows.

² In Sec. 1 we have written that the low energy dynamics of N Dp-branes is described with $D = p + 1$ U(N) SYM, but the gauge group in the model in which we perform analysis is $SU(N)$. [1] to which we refer in this study also considers $SU(N)$.

We take $g^2 N$ to be constant: $g^2 N \equiv \lambda$, while taking large N as the large- N limit.³ We can see $[\lambda] = M^3$. Hence we define a dimensionless parameter $\lambda_{\text{eff}} = \lambda \beta^3$.

2.2 Possible λ and β for the description by our model

Our model (1) with $D = 9$ can be obtained from the high-temperature limit and the T-duality of the $SU(N)$ $\mathcal{N} = 8$ SYM on a circle with a period L at finite temperature $T_2 = \beta_2^{-1}$:

$$S = \frac{1}{g_2^2} \int_0^L dx \int_0^{\beta_2} dt \text{Tr} \left(\frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2} \sum_{I=1}^8 (D_\mu Y^I)^2 - \frac{1}{4} \sum_{I,J=1}^8 [Y^I, Y^J]^2 \right) + \text{fermions}, \tag{2}$$

where μ, ν take two values t, x , L is common to the L in the description of Sect. 1, and fermions are anti-periodic in the t -circle. We refer to (2) as 2D SYM in what follows.

The 2D SYM is characterized by the two dimensionless parameters

$$\lambda' = \lambda_2 L^2, \quad t' = L/\beta_2, \tag{3}$$

where $\lambda_2 \equiv g_2^2 N$ is the 't Hooft coupling in the 2D SYM.

The high-temperature limit is taken, which leads to decoupling of the t -dependence. As a result 8 changes to 9. Fermions also decouple. We also take the T-duality.⁴

We consider this to be the effective theory for the D0-branes in the S^1 -compactified $D = 9$ space-time at finite temperature, where the x -cycle plays the role of the finite temperature after the T-duality. We denote the period of the S^1 direction by L' . We have noted the relation between L' and L in Sect. 1. D0-branes are assumed to be distributed on the same S^1 -circle.

When λ' is large, the dynamics on both the x -cycle and the β_2 -cycle becomes effective. However, even if λ' is large, if β_2 is small, the final contributions of the dynamics from the t -cycle can be ignorable since the space itself is small. Likewise, even if λ' is large, if L' has some small values, the final contribution from the x -cycle can be ignorable. These features can be written in the qualitative manner as [16]:

- The t -dependence is ignorable for $\lambda'^{1/3} < t'$.

³ We can change the overall factor g^2 arbitrarily as $g^2 \rightarrow \kappa g^2$ by the rescalings: $(Y^I, A_0) \rightarrow (\kappa^{-1/3} Y^I, \kappa^{1/3} A_0)$ and $(t, \beta) \rightarrow \kappa^{1/3} (t, \beta)$ without changing physics as long as λ_{eff} is fixed.

⁴ One reason to perform the T-dual has to do with the regions other than $\lambda' \gg 1$. In such parameter regions the winding modes and the α' -corrections become effective, which conflicts with the fact that D1-branes are solutions at the supergravity level. However, we can keep those as a solution at the supergravity level by performing the T-dual [16].

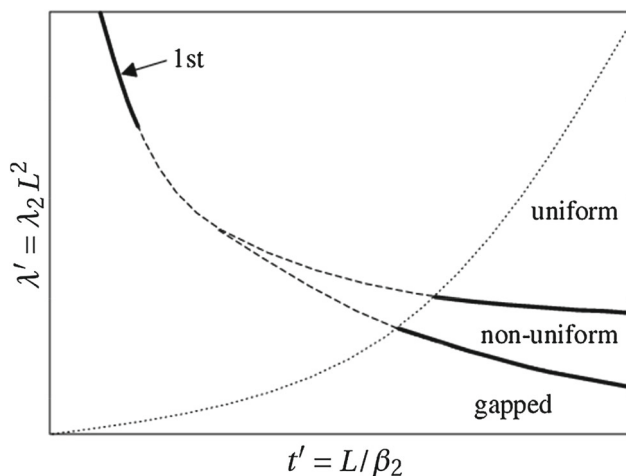


Fig. 1 Phase structure in 2D SYM (2). As going to the right side, it becomes a more higher-temperature region. On the other hand, as going up, it becomes a more strongly coupled region. The “1st” in the upper-left region can be known from the GL instability in the gravity side. The bottom-right region separated by the fine dotted line is the region effectively described by our 1D model (1); “uniform”, “non-uniform” and “gapped” represent the phases

- The x -dependence is ignorable for $1/\lambda' > t'$.

The boundary of $\lambda^{1/3} < t'$ is plotted in Fig. 1.

In particular, when we realize the following situation:

$$\lambda^{1/3} \ll t' \tag{4}$$

by taking the high-temperature limit, the 2D SYM reduces to our 1D model (1). This time, the parameters in the 2D SYM and our model (1) are linked as

$$g_2^2/\beta_2 = g^2, \quad L' = \beta. \tag{5}$$

Using these we can rewrite the condition (4) as

$$\lambda_{\text{eff}} \ll t'^4, \tag{6}$$

where λ_{eff} is given under (1). Therefore, when the condition (6) is valid, we can consider our 1D model (1) instead of the 2D SYM.

Let us mention the conclusion in this section. Since the high-temperature limit is taken, t' goes to ∞ . This time, we can assign any finite values to β and λ without breaking (6) by exploiting the rescaling in the footnote under (1). Therefore, practically we can always include the uniform/non-uniform and the non-uniform/gapped transitions in the parameter region where the description by our 1D model (1) is possible.

3 Preliminary for the analysis of the effective action

From this section to Sect. 5, we review how to obtain the effective action in Ref. [1], and in Sect. 6, we review how to obtain the equations of the critical behaviors in Ref. [1].

Writing Y^I as $Y^I(t) = \sum_{a=1}^{N^2-1} Y_a^I(t) t_a$, we can rewrite the potential term as

$$\begin{aligned} -\text{Tr}[Y^I, Y^J][Y^I, Y^J] &= (Y_a^I Y_b^J) M_{ab,cd} (Y_c^I Y_d^J), \\ M_{ab,cd} &= -\frac{1}{4} \text{Tr} \left([t_a, t_c][t_b, t_d] + (a \leftrightarrow b) \right. \\ &\quad \left. + (c \leftrightarrow d) + (a \leftrightarrow b, c \leftrightarrow d) \right), \end{aligned} \tag{7}$$

where t_a are the generators of $SU(N)$ with the orthogonal condition: $\text{tr}(t_a t_b) = \delta_{ab}$, and Y_a^I are coefficients.

Introducing a matrix B_{ab} satisfying $M_{ab,cd}^{-1} B_{cd} = i g^2 Y_a^I Y_b^I$, (1) can be written as

$$S = \int_0^\beta dt \left(\frac{1}{2} (D_0 Y_a^I)^2 - \frac{i}{2} B_{ab} Y_a^I Y_b^I + \frac{1}{4g^2} B_{ab} M_{ab,cd}^{-1} B_{cd} \right). \tag{8}$$

Here, when we introduce B_{ab} , some factor appears in the distribution function, but we ignore it as it is just a numerical factor [1]. We can see that B_{ab} plays the role of the squared masses for Y_a^I .

Integrating out Y^I , we can write the action as

$$S_{\text{eff}} = \frac{1}{g^2} \left(\frac{1}{4} \int_0^\beta dt B_{ab} M_{ab,cd}^{-1} B_{cd} + \frac{g^2 D}{2} \log \det (D_0^2 + iB) \right). \tag{9}$$

In the above, it is known that B_{ab} will get some value for the large D [1]. If we write it as $B_{ab} = i \Delta_0^2 \delta_{ab}$, Δ_0^2 will turn out to be real and play the role of squared mass, which guarantees that we are on a stable vacuum.

We consider B_{ab} with quantum fluctuations as

$$\begin{aligned} \bar{B}_{ab}(t) &= B_0 \delta_{ab} + g b_{ab}(t), \\ \text{where } B_0 &= i \Delta^2 \quad \text{and} \quad \int_0^\beta dt b_{aa}(t) = 0. \end{aligned} \tag{10}$$

Replacing B_{ab} in (8) with this \bar{B}_{ab} we can obtain

$$\begin{aligned} S &= -\frac{\beta N \Delta^4}{8g^2} + \int_0^\beta dt \left(\frac{1}{4} b_{ab} M_{ab,cd}^{-1} b_{cd} + \frac{1}{2} (D_0 Y_a^I)^2 \right. \\ &\quad \left. - \frac{i}{2} B_0 (Y_a^I)^2 - \frac{ig}{2} b_{ab} Y_a^I Y_b^I \right), \end{aligned} \tag{11}$$

where we have used $M_{ab,cd}^{-1} \delta_{cd} = \delta_{ab}/2N$ ($a, b = 1, \dots, N^2 - 1$) in [1]. The $SU(N)$ gauge symmetry exists in

our model at each $t \in [0, \beta]$. We can separate off the volume factor for the gauge transformation by inserting the unity (56) as

$$Z = \int \mathcal{D}\theta \cdot \int \mathcal{D}\alpha \mathcal{D}b \mathcal{D}Y \left\{ 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \times \left(- \int_0^\beta dt \frac{ig}{2} b_{ab} Y_a^I Y_b^I \right)^n \right\} \exp -(S + S_{FP}), \tag{12}$$

$$S + S_{FP} = DN^2 \left\{ - \frac{\beta \Delta^4}{8\tilde{\lambda}} + \frac{1}{D} \sum_{n=1}^{\infty} \frac{|u_n|^2}{n} + \frac{1}{DN^2} \int_0^\beta dt \left(\frac{1}{4} b_{ab} M_{ab,cd}^{-1} b_{cd} - \frac{1}{2} Y_a^I ((D_0)^2 + iB_0) Y_a^I \right) \right\}, \tag{13}$$

where $\tilde{\lambda} \equiv \lambda D$, and $u_n = \frac{1}{N} \text{Tr} P \exp i \int_0^\beta dt A_t = \frac{1}{N} \sum_{i=1}^N e^{in\alpha_i}$. Here, we are now taking the static diagonal gauge $(A_0)_{ij} = \alpha_i \delta_{ij} / \beta$, $(i, j = 1, \dots, N)$. u_n are the Wilson lines twining around the t -direction n times.

Let us look at the terms in (13). The second term will turn out to be indispensable, because it plays a critical role in the determination of the sign of the $|u_1|^2$ coefficient in the effective action (32). Thus let us include it. Therefore, we have to take into account the $1/D$ correction to $1/D$ order.

The term of the summation in (12) and the third term in (13) are the interaction term. We comment on the contribution from this term in Appendix B. The θ -integral gives just a gauge volume, which we disregard.

We perform the one-loop integral for Y without interaction terms in the next section. We quote the contribution from the interaction term from [1] (we explain how to derive a necessary equation in the analysis of the interaction term in Appendix B). It will start with $1/D$ and $1/N^2$ orders (see under (E.33) and (A.17) in Ref. [1]). We address only the $1/D$ corrections to $1/D$ order considering the large- N limit.

4 One-loop integral of Y^I

Taking SU(3) to make our calculation process concrete, we write down the expression for the part to become the one-loop integration of Y^I , explicitly. Then deducing the expression for arbitrary N , we perform the one-loop order path-integral.

We start with

$$Y = \sum_{a=1}^8 Y_a t_a = \frac{1}{2} \begin{pmatrix} Y_3 + Y_8/\sqrt{3} & Y_1 - iY_2 & Y_4 - iY_5 \\ Y_1 + iY_2 & -Y_3 + Y_8/\sqrt{3} & Y_6 - iY_7 \\ Y^4 + iY_5 & Y_6 + iY_7 & -2Y_8/\sqrt{3} \end{pmatrix} \equiv \begin{pmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{21} & Y_{22} & Y_{23} \\ Y_{31} & Y_{32} & Y_{33} \end{pmatrix}, \tag{14}$$

$$A_0^{\theta_0} = \sum_{a=1}^8 A_a^{\theta_0} t_a = 2 \text{diag}(\lambda_1, \lambda_2, -(\lambda_1 + \lambda_2)) \equiv \text{diag}(\alpha_1, \alpha_2, -(\alpha_1 + \alpha_2)), \tag{15}$$

where t^a are 1/2 of the Gell-Mann matrices, and $A_a^{\theta_0}$ and Y_a are some constants as the components of the vector. Since we take the time-independent diagonal gauge, we can take the components $A_a^{\theta_0}$ freely as long as this gauge is kept with the traceless condition. Therefore, we have taken $A_a^{\theta_0}$ in the (15), $A_3^{\theta_0} = 4\lambda_1 - A_8^{\theta_0}/\sqrt{3}$, $A_8^{\theta_0} = 2\sqrt{3}(\lambda_1 + \lambda_2)$ and $A_a^{\theta_0} = 0$ for $a = 1, 2, 4, 5, 6, 7$. We have omitted the index I in Y^I .

We show $D_0 Y(t) = \partial_0 Y(t) - i[A_0^{\theta_0}, Y(t)]$ and $D_0^2 Y(t)$ concretely:

$$D_0 Y(t) = \begin{pmatrix} \partial_0 Y_{11} & (\partial_0 - i\alpha_{12}) Y_{12} & (\partial_0 - i\alpha_{13}) Y_{13} \\ (\partial_0 - i\alpha_{21}) Y_{21} & \partial_0 Y_{22} & (\partial_0 - i\alpha_{23}) Y_{23} \\ (\partial_0 - i\alpha_{31}) Y_{31} & (\partial_0 - i\alpha_{32}) Y_{32} & \partial_0 Y_{33} \end{pmatrix}, \tag{16}$$

$$D_0^2 Y(t) = \begin{pmatrix} \partial_0 (D_0 Y)_{11} & (\partial_0 - i\alpha_{12}) (D_0 Y)_{12} & (\partial_0 - i\alpha_{13}) (D_0 Y)_{13} \\ (\partial_0 - i\alpha_{21}) (D_0 Y)_{21} & \partial_0 (D_0 Y)_{22} & (\partial_0 - i\alpha_{23}) (D_0 Y)_{23} \\ (\partial_0 - i\alpha_{31}) (D_0 Y)_{31} & (\partial_0 - i\alpha_{32}) (D_0 Y)_{32} & \partial_0 (D_0 Y)_{33} \end{pmatrix} = \begin{pmatrix} \partial_0^2 Y_{11} & (\partial_0 - i\alpha_{12})^2 Y_{12} & (\partial_0 - i\alpha_{13})^2 Y_{13} \\ (\partial_0 - i\alpha_{21})^2 Y_{21} & \partial_0^2 Y_{22} & (\partial_0 - i\alpha_{23})^2 Y_{23} \\ (\partial_0 - i\alpha_{31})^2 Y_{31} & (\partial_0 - i\alpha_{32})^2 Y_{32} & \partial_0^2 Y_{33} \end{pmatrix}, \tag{17}$$

where $\alpha_{ij} \equiv \alpha_i - \alpha_j$.

We proceed our calculation by performing the plane-wave expansion:

$$Y_{ij} = \frac{1}{\sqrt{\beta}} \sum_{n=-\infty}^{\infty} Y_{ij}^n e^{ik_n t}, \quad k_n \equiv \frac{2\pi n}{\beta}. \tag{18}$$

4.1 Expression of action

We can write our action as

$$S = -\frac{1}{2} \text{tr} \int_0^\beta dt (Y D_0^2 Y + iB_0 Y Y), \tag{19}$$

where $Z = \int DY \exp(-S)$. We now compute the expressions of the kinetic and mass terms.

We obtain the expression of the kinetic term, $-\int_0^\beta dt \text{tr} (Y^i D_0^2 Y^i)$. From (16),

$$-\int_0^\beta dt \text{tr} (Y D_0^2 Y) = \frac{1}{\beta} \sum_{n=-\infty}^{\infty} e^{ik_m+n t} \text{tr} \begin{pmatrix} Y_{11}^m & Y_{12}^m & Y_{13}^m \\ Y_{21}^m & Y_{22}^m & Y_{23}^m \\ Y_{31}^m & Y_{32}^m & Y_{33}^m \end{pmatrix} \times \begin{pmatrix} k_n^2 Y_{11}^n & (k_n - \alpha_{12})^2 Y_{12}^n & (k_n - \alpha_{13})^2 Y_{13}^n \\ (k_n - \alpha_{21})^2 Y_{21}^n & k_n^2 Y_{22}^n & (k_n - \alpha_{23})^2 Y_{23}^n \\ (k_n - \alpha_{31})^2 Y_{31}^n & (k_n - \alpha_{32})^2 Y_{32}^n & k_n^2 Y_{33}^n \end{pmatrix}$$

$$\begin{aligned}
 &= \sum_{n=-\infty}^{\infty} \sum_{i=1}^3 \text{tr} \\
 &\left(\begin{array}{ccc}
 (k_n - \alpha_{i1})^2 Y_{1i}^{-n} Y_{i1}^n & - & - \\
 - & (k_n - \alpha_{i2})^2 Y_{2i}^{-n} Y_{i2}^n & - \\
 - & - & (k_n - \alpha_{i3})^2 Y_{3i}^{-n} Y_{i3}^n
 \end{array} \right). \tag{20}
 \end{aligned}$$

In the above, we have used the Kronecker delta-function, $\frac{1}{\beta} \int_0^\beta dt e^{i\frac{2\pi(m-n)t}{\beta}} = \delta_{mn}$,⁵ and $k_{-n} = -k_n$ and $\alpha_{ij} = -\alpha_{ji}$. We have written the components relevant to the trace at the last. We can obtain the expression of $\int_0^\beta dt \text{tr} (D_0 Y^i)^2$ from (16) in the same way, which agrees with (16).

We next obtain the expression of the mass term, which is written as⁶

$$\begin{aligned}
 &\int_0^\beta dt \text{tr} (BYY) \\
 &= i \Delta^2 \int_0^\beta dt Y_a Y_b \delta_{ab} \text{tr}(t_a t_b) \\
 &= i \Delta^2 \int_0^\beta dt \text{tr} \left(\begin{array}{ccc}
 \frac{1}{4} \sum_{a=1,2,3,4,5} Y_a^2 + \frac{Y_8^2}{12} & 0 & 0 \\
 0 & \frac{1}{4} \sum_{a=2,3,6,7} Y_a^2 + \frac{Y_8^2}{12} & 0 \\
 0 & 0 & \frac{1}{4} \sum_{a=4,5,6,7} Y_a^2 + \frac{Y_8^2}{3}
 \end{array} \right) \\
 &= i \Delta^2 \sum_{i=1}^3 \int_0^\beta dt \text{tr} \left(\begin{array}{ccc}
 Y_{1i} Y_{i1} & 0 & 0 \\
 0 & Y_{2i} Y_{i2} & 0 \\
 0 & 0 & Y_{3i} Y_{i3}
 \end{array} \right) \\
 &= i \Delta^2 \sum_{i=1}^3 \sum_{n=-\infty}^{\infty} \text{tr} \left(\begin{array}{ccc}
 Y_{1i}^{-n} Y_{i1}^n & 0 & 0 \\
 0 & Y_{2i}^{-n} Y_{i2}^n & 0 \\
 0 & 0 & Y_{3i}^{-n} Y_{i3}^n
 \end{array} \right), \tag{21}
 \end{aligned}$$

where the Y_{ij} in the forth line are given in (14). The third line appears to depend on N , but in the fourth and fifth lines, we can deduce the expression at arbitrary N .

From (20) and (21), we can now write the action as

$$\begin{aligned}
 S &= \frac{1}{2} \sum_{n=-\infty}^{\infty} \sum_{i=1}^3 \text{tr} \left(\begin{array}{ccc}
 ((k_n - \alpha_{i1})^2 + \Delta^2) Y_{1i}^{-n} Y_{i1}^n & - & - \\
 - & ((k_n - \alpha_{i2})^2 + \Delta^2) Y_{2i}^{-n} Y_{i2}^n & - \\
 - & - & ((k_n - \alpha_{i3})^2 + \Delta^2) Y_{3i}^{-n} Y_{i3}^n
 \end{array} \right) \\
 &= \frac{1}{2} \sum_{n=-\infty}^{\infty} \sum_{i,j} ((k_n - \alpha_{ji})^2 + \Delta^2) Y_{ij}^{-n} Y_{ji}^n. \tag{22}
 \end{aligned}$$

We omitting the index I in Y^I in the description above.

⁵ The Kronecker-delta function in the non-compactified space is $\frac{1}{2\pi} \int_0^{2\pi} dx e^{i(m-n)x} = \delta_{mn}$.

⁶ In the calculation (21), the transitions from the second to the third lines and from the third to the fourth lines may be difficult to understand instantly, so we have written those explicitly.

4.2 Degree of freedom to be integrated

We confirm the degree of freedom with regard to Y to be integrated. To this purpose, let us write the plane-wave expanded scalar matrix field Y and its Hermitian conjugate in a qualitative manner as

$$\begin{aligned}
 Y &\sim \begin{pmatrix} a_{-n} + ib_{-n} & c_{-n} + id_{-n} \\ f_{-n} + ig_{-n} & h_{-n} + ij_{-n} \end{pmatrix} e^{-int} + \begin{pmatrix} a_0 + ib_0 & c_0 + id_0 \\ f_0 + ig_0 & h_0 + ij_0 \end{pmatrix} \\
 &+ \begin{pmatrix} a_n + ib_n & c_n + id_n \\ f_n + ig_n & h_n + ij_n \end{pmatrix} e^{int}, \tag{23}
 \end{aligned}$$

$$\begin{aligned}
 Y^\dagger &\sim \begin{pmatrix} a_{-n} - ib_{-n} & f_{-n} - ig_{-n} \\ c_{-n} - id_{-n} & h_{-n} - ij_{-n} \end{pmatrix} e^{int} + \begin{pmatrix} a_0 - ib_0 & f_0 - ig_0 \\ c_0 - id_0 & h_0 - ij_0 \end{pmatrix} \\
 &+ \begin{pmatrix} a_n - ib_n & f_n - ig_n \\ c_n - id_n & h_n - ij_n \end{pmatrix} e^{-int}, \tag{24}
 \end{aligned}$$

where the characters used above, a, b, \dots, j , are the ones used only in this subsection.

From the condition $Y = Y^\dagger$, we can obtain the following condition:

$$\begin{aligned}
 c_{-n} &= f_{+n}, \quad d_{-n} = -g_{+n}, \quad f_{-n} = c_{+n}, \quad g_{-n} = -d_{+n}, \\
 &\text{for the non-diagonal elements,} \\
 a_{-n} &= a_{+n}, \quad b_{-n} = -b_{+n}, \quad h_{-n} = h_{+n}, \quad j_{-n} = -j_{+n}, \\
 &\text{for the diagonal elements.}
 \end{aligned}$$

Plugging these into the Y in (23), it can be written as

$$\begin{aligned}
 Y &\sim \dots + \begin{pmatrix} a_{+n} - ib_{+n} & c_{-n} + id_{-n} \\ c_n - id_n & h_n - ij_n \end{pmatrix} e^{-int} \\
 &+ \dots + \begin{pmatrix} a_0 & c_0 + id_0 \\ c_0 - id_0 & h_0 \end{pmatrix} \\
 &+ \dots + \begin{pmatrix} a_n + ib_n & c_n + id_n \\ c_{-n} - id_{-n} & h_n + ij_n \end{pmatrix} e^{int} + \dots. \tag{25}
 \end{aligned}$$

We can see that the degrees of freedom to be integrated are the parts corresponding to the following ones:

- For all the diagonal elements:
 - Real part:
 - $a_n (n = 0, 1, 2, \dots),$
 - $h_n (n = 0, 1, 2, \dots),$
 - Imaginary part:
 - $b_n (n = 1, 2, \dots),$
 - $j_n (n = 1, 2, \dots).$
- For one-side of the non-diagonal elements:
 - Real part: $c_n (n = -2, -1, 0, 1, 2, \dots),$
 - Imaginary part: $d_n (n = -2, -1, 0, 1, 2, \dots).$

Therefore the integral measure except for the factors is given as

$$\mathcal{D}Y \propto \prod_{i=1}^N \left(\prod_{n=0}^{\infty} d(\text{Re}Y_{ii}^n) \prod_{n=1}^{\infty} d(\text{Im}Y_{ii}^n) \right) \cdot \prod_{i>j}^N \left(\prod_{n=-\infty}^{\infty} d(\text{Re}Y_{ij}^n) d(\text{Im}Y_{ij}^n) \right). \tag{26}$$

4.3 Path-integral

We can see from (23) that we have the relation $Y_{ij}^n = Y_{ji}^{-n*}$. Exploiting this, we can decompose the description of the action (22) into components as

$$\begin{aligned} & \int \mathcal{D}Y \exp \frac{1}{2} \int_0^\beta dt Y ((D_0)^2 + iB_0) Y \\ &= \int \mathcal{D}Y \exp -\frac{1}{2} \sum_{n=-\infty}^{\infty} \left[\sum_{i=1}^N (k_n^2 + \Delta^2) Y_{ii}^{n*} Y_{ii}^n \right. \\ & \quad \left. + 2 \sum_{i<j} \left((k_n - \alpha_{ji})^2 + \Delta^2 \right) Y_{ji}^{n*} Y_{ji}^n \right] \\ &= \int \mathcal{D}Y \exp -\left[\sum_{i=1}^N \left\{ \frac{\Delta^2}{2} (Y_{ii}^0)^2 \right. \right. \\ & \quad \left. \left. + \sum_{n=1}^{\infty} (k_n^2 + \Delta^2) \left((\text{Re}Y_{ii}^n)^2 + (\text{Im}Y_{ii}^n)^2 \right) \right\} \right. \\ & \quad \left. + \sum_{n=-\infty}^{\infty} \sum_{i<j} \left((k_n - \alpha_{ji})^2 + \Delta^2 \right) \left((\text{Re}Y_{ji}^n)^2 + (\text{Im}Y_{ji}^n)^2 \right) \right]. \tag{27} \end{aligned}$$

In the above, we have written the expression at general N based on (22) (and omitted parentheses as $\exp[\dots]$).

We perform the path-integrals of Y in (27). We show its calculation process in Appendix C. As a result we get the following result:

$$\begin{aligned} (27) &= 2^{\frac{DN}{2}} \left(\frac{1}{\beta} \prod_{n=1}^{\infty} k_n^2 \right)^{-DN^2} \exp \\ & \quad - DN^2 \left(\frac{\beta\Delta}{2} - \sum_{n=1}^{\infty} \frac{e^{-n\beta\Delta}}{n} |u_n|^2 \right). \tag{28} \end{aligned}$$

Adding the FP term obtained in Appendix A.3 and the corrections arising from the interaction term to $1/D$ order (we quote from (4.21) in Ref. [1]),

$$Z = \int \mathcal{D}\alpha e^{-(S_{1\text{-loop}} + S_{\text{int}} + S_{\text{FP}})}, \tag{29}$$

where

$$S_{1\text{-loop}} + S_{\text{int}} + S_{\text{FP}} = DN^2 (c_0 + c_2 |u_1|^2 + c_4 |u_1|^4 + \dots),$$

$$\begin{aligned} c_0 &= -\frac{\beta\Delta^4}{8\tilde{\lambda}} + \frac{\beta\Delta}{2} + \frac{\beta\Delta}{D} \left((1+y)^{1/2} - 1 - y - \frac{y^2}{4} \right), \\ c_2 &= \frac{1}{D} - x + \frac{\beta\Delta}{D} xy \left((1+y)^{-1/2} + (1+y)^{-1} - 4 - 3y \right), \\ c_4 &= -\frac{\beta\Delta}{2D} x^2 y^2 \left\{ \frac{1}{2} (2 + (1+y)^{-3/2}) \right. \\ & \quad \left. + (2 + \beta\Delta) (2 + (1+y)^{-2}) \right\}, \\ x &\equiv e^{-\beta\Delta} \quad \text{and} \quad y \equiv \frac{\tilde{\lambda}}{4\Delta^3}. \tag{30} \end{aligned}$$

S_{int} represents the corrections from the interaction term and “...” represents negligible corrections. All the $1/D$ order terms except for $1/D$ in c_2 are the terms from S_{int} . $1/N$ corrections from S_{int} do not appear in our analysis, because it starts from $1/N^2$ in S_{int} as written under (E.33) and (A.17) in Ref. [1].

5 Evaluation of Δ at the saddle-point

We fix Δ to the saddle-point by taking its variation in (30) instead of performing the path-integral. Note that this is the saddle-point method, so it can work at large- N .

It turns out that we cannot obtain the Δ exactly. However we can solve in terms of the power series of $|u_1|^2$. We finally obtain the approximated solution to order $|u_1|^2$ in the $1/D$ expansion as

$$\Delta = \tilde{\lambda}^{1/3} \left\{ 1 + \frac{2}{3} e^{-\beta\Delta} |u_1|^2 + \left(\frac{7\sqrt{5}}{30} - \frac{9}{32} \right) \frac{1}{D} + \mathcal{O}(D^{-2}) \right\} + \dots \tag{31}$$

We can see that the $1/D$ part is consistent with (4.25) in Ref. [1]. “...” represents corrections which will be ignorable when $|u_1|$ is small.

Plugging (31) into the effective action (30), we can obtain the following Ginzburg–Landau (GL) type effective action:⁷

$$\begin{aligned} S_{\text{GL}}|_{\Delta \text{ s.p.}} &= DN^2(c'_0 + c'_2|u_1|^2 + c'_4|u_1|^4 + \dots), \\ c'_0 &= \left\{ \frac{3}{8} + \frac{1}{2} \left(\sqrt{5} - \frac{81}{32} \right) \frac{1}{D} + \mathcal{O}(D^{-2}) \right\} \beta \tilde{\lambda}^{1/3}, \\ c'_2 &= \left\{ -e^{-\beta \tilde{\lambda}^{1/3}} + \left(1 - \frac{\beta \tilde{\lambda}^{1/3}}{e^{\beta \tilde{\lambda}^{1/3}}} \left(\frac{203}{160} - \frac{\sqrt{5}}{3} \right) \right) \frac{1}{D} + \mathcal{O}(D^{-2}) \right\}, \\ c'_4 &= \frac{1}{7200} \frac{\beta \tilde{\lambda}^{1/3}}{e^{2\beta \tilde{\lambda}^{1/3}}} \left\{ 2400 + \left(9543 - 1564\sqrt{5} + 8\beta \tilde{\lambda}^{1/3} (687 - 200\sqrt{5}) \right) \frac{1}{D} + \mathcal{O}(D^{-2}) \right\}. \end{aligned} \tag{32}$$

The above is consistent with (4.26) in Ref. [1].

6 Equations of the critical temperatures

Let us obtain the equations of the critical phenomena based on GL action (32). We can see that the coefficient of $|u_1|^2$ is positive for $\beta \gg 1$, which means that $|u_1| = 0$ and the confinement (uniform) phase is realized. However, when the temperature is increased, the sign of the coefficient of $|u_1|^2$ will flip to negative at some temperature. As a result, $|u_1|$ gets some finite value and the phase switches to the deconfinement phase (non-uniform). We can get the critical temperature $T_1 = \beta_1^{-1}$ for this from the condition $c'_2|_{\beta=\beta_1} = 0$. In the actual calculation, we obtain

$$0 = \frac{(480\alpha_D + 160\sqrt{5} - 609) \ln D}{480D^2} + \mathcal{O}(D^{-3}),$$

where we have put β_1 as $\frac{\ln D}{\tilde{\lambda}^{1/3}} (1 + \frac{\alpha_D}{D})$ and obtained the result with regard to α_D . We find $\alpha_D = \frac{203}{160} - \frac{\sqrt{5}}{3}$. Finally, T_1 is obtained:

⁷ The term $1/D$ in c'_2 in (32) comes from the gauge fixing. Saying roughly, the other terms come from the integrals for Y and b . We can see that the uniform/non-uniform transition in our model is determined by which one is larger.

$$T_1 = \frac{\tilde{\lambda}^{1/3}}{\ln D} \left\{ 1 - \left(\frac{203}{160} - \frac{\sqrt{5}}{3} \right) \frac{1}{D} \right\} + \mathcal{O}(D^{-2}). \tag{33}$$

The above is consistent with (4.30) in Ref. [1].

Using this result, we can see how $|u_1|$ arises at $T = T_1$ as

$$\begin{aligned} (u_1|_{T=T_1+\delta T})^2 &= \mathcal{O}(D^{-1}) + \frac{\ln D}{2\tilde{\lambda}^{1/3}} \left(3D + \frac{-9543 + 1564\sqrt{5} + 594 \ln D}{800} + \mathcal{O}(D^{-1}) \right) \delta T \\ &\quad - \frac{3(\ln D)^3}{4\tilde{\lambda}^{2/3}} \left(3D + \frac{-3051 + 382\sqrt{5} + 297 \ln D}{400} + \mathcal{O}(D^{-1}) \right) \delta T^2 + \mathcal{O}(\delta T^3). \end{aligned} \tag{34}$$

We have computed the above according to $(u_1)^2 = -c'_2/(2c'_4) \geq 0$. We can confirm that $c'_2|_{T=T_1+\delta T} \sim D^{-3} + D^{-1}\delta T + \dots$ and $c'_4|_{T=T_1+\delta T} \sim D^{-2} + D^{-2}\delta T + \dots$. Since $u_1|_{T=T_1+\delta T}$ should vanish at $\delta T = 0$, we disregard the part $\mathcal{O}(D^{-1})$ in what follows.

From (34),

$$\begin{aligned} |u_1|_{T=T_1+\delta T} &= \sqrt{\frac{D \ln D}{\tilde{\lambda}^{1/3}} \left(\sqrt{\frac{3}{2}} + \frac{-9543 + 1564\sqrt{5} + 594 \ln D}{1600\sqrt{6}D} + \mathcal{O}(D^{-2}) \right) \delta T^{1/2}} \\ &\quad - \frac{3}{4} \sqrt{\frac{3D(\ln D)^5}{2\tilde{\lambda}}} \left(1 - \frac{887 + 12\sqrt{5} - 198 \ln D}{1600D} + \mathcal{O}(D^{-2}) \right) \delta T^{3/2} + \mathcal{O}(\delta T^{5/2}). \end{aligned} \tag{35}$$

The one above does not agree with (4.14) in [1] concerning $(\ln D)^{5/2}(\delta T^3/\tilde{\lambda})^{1/2}$ or $(\ln D)^{3/2}(\delta T/\tilde{\lambda}^{1/3})^{5/2}$. I have confirmed that the one above is right⁸.

It is well known from Ref. [52] that the eigenvalue density function is given as $\rho(\alpha) = \frac{\beta}{2\pi} (1 + 2|u_1| \cos(\beta\alpha))$. Therefore, the region where there is no eigenvalues arises in the eigenvalue distributions when $|u_1|$ reaches $1/2$. According to [52], the third-order phase transition occurs at that time.

⁸ I have confirmed this by actually inquiring of T. Morita in [1].

We obtain the critical temperature for this by solving with regard to $\delta\beta$ in

$$\left. \frac{\delta S_{GL}|_{\Delta \text{ at s.p.}}}{\delta |u_1|} \right|_{\beta=\beta_1+\delta\beta \text{ and } |u_1|=1/2} = 0,$$

where $S_{GL}|_{\Delta \text{ at s.p.}}$ is given in (32), and β_1 is given above (33). $\frac{\delta S_{GL}|_{\Delta \text{ at s.p.}}}{\delta |u_1|}$ leads to $2c'_2 + c'_4$. Expanding the one above regarding $\delta\beta$ to the first-order, then putting $\delta\beta$ as $\delta\beta_1/D + \delta\beta_2/D^2$, we solve $\delta\beta_{1,2}$ order by order. As a result we finally obtain

$$\delta\beta = \frac{\ln D}{D\tilde{\lambda}^{1/3}} \left\{ -\frac{1}{6} + \frac{1}{D} \left(\frac{85051}{76800} - \frac{1127\sqrt{5}}{1800} + \left(-\frac{499073}{460800} + \frac{203\sqrt{5}}{480} \right) \ln D \right) \right\} + \mathcal{O}(D^{-3}). \quad (36)$$

We can see that the one above agrees with (4.31) in [1].

Denoting the critical temperature for this case as T_2 , its result is

$$T_2 = \frac{\tilde{\lambda}^{1/3}}{\ln D} \left\{ 1 + \frac{1}{6D} \left(1 - 6 \left(\frac{203}{160} - \frac{\sqrt{5}}{3} \right) \right) \right\} + \mathcal{O}(D^{-2}), \quad (37)$$

where $T_2 = \frac{1}{\beta_1+\delta\beta} = \frac{1}{\beta_1} \left(1 - \frac{\delta\beta}{\beta_1} \right) + \mathcal{O}(\delta\beta^2)$, we have expanded with regard to $1/D$.

Finally, we can check the transition-order of the uniform/non-uniform case. However, since it is not important for the issues we treat in this study, we perform it in Appendix D.

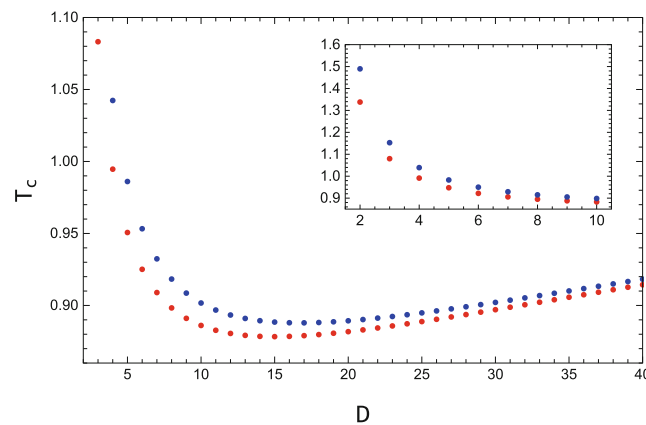


Fig. 2 D -dependence of the gap between $T_{1,2}$ against D ; the red and blue points represent T_1 and T_2 respectively, which are results evaluated with (33) and (37). We can see that the gap does not narrow even

7 D -dependence of the gap between $T_{1,2}$

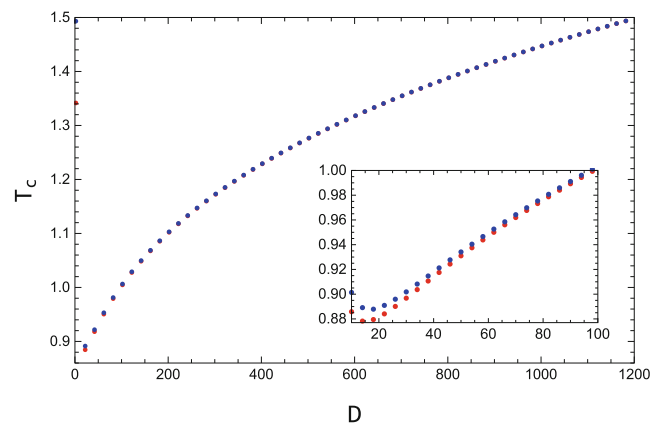
In this section, we check the D -dependence of the gap between the critical temperatures associated with the uniform/non-uniform and non-uniform/gapped transitions.

In Fig. 2, we represent $T_{1,2}$ in (33) and (37) against D , where we treat $\tilde{\lambda}$ as $D\lambda$ in those expressions as in (13), and plug unit in λ .

We can see that even if D becomes smaller, the gap between $T_{1,2}$ does not close, while as D grows, the gap narrows. These mean that the two transitions do not asymptote at small D , while asymptote and become a single transition effectively at large D .

Since higher-order corrections of the $1/D$ expansion become effective when D is small, what we mentioned above concerning small D may be an error of that. However, we can see in the table in the last of Sect. 4 in Ref. [1] that the results of the $1/D$ expansion are not incorrect so much from the numerical results of the Monte Carlo simulation (MC simulation) at $D = 2$, and as can be seen there the numerical difference between $T_{1,2}$ is $1.3 - 1.12 = 0.18$. This numerical value is not be as small as ignorable and can be considered as the sign of the existence of the gap. Therefore, the gap keeps appearing at small D even in the MC simulation. Therefore, we can consider that the tendency we have found above is right even at small D .

These tendencies are completely opposite from the tendency of GL and RP instabilities, where we have summarized those tendencies in Sect. 1. From these results, we can conclude that the gauge/gravity and gauge/fluid correspondences do not always hold in every point.



for small D , while gets smaller as D gets larger. These are an opposite tendency from GL instabilities in gravities and RP instabilities in fluid dynamics

8 D-dependence of the gap in the Z_m symmetric solutions

In this section, we generalize the D -dependence of the gap between the uniform/non-uniform and non-uniform/gapped transitions into the critical temperatures of the Z_m symmetric solutions.

First, let us define the Z_m symmetric solutions. Since we are now taking the static diagonal gauge, we can write the gauge field as

$$(A_0)_{ij} = \alpha_i \delta_{ij} / \beta, \quad \text{where } i, j = 1, \dots, N. \tag{38}$$

Then considering a set $\{N_1, N_2, \dots, N_m\}$, where $N_k \in \mathbb{Z}$ with $\sum_{k=1}^m N_k = N$, let us consider the following gauge field configuration:

$$\alpha_i = 2\pi l/m + \alpha_j^{(l)}, \quad \text{where } \sum_{k=1}^{l-1} N_k < i \leq \sum_{k=1}^l N_k, \\ j = i - \sum_{k=1}^{l-1} N_k. \tag{39}$$

We can consider $2\pi l/m$ as the mean position of α_i belonging in N_l .

We can see that this configuration is Z_m symmetric if $\alpha_j^{(l)}$ are expanding evenly around $2\pi l/m$. Equation (39) is the definition for the Z_m symmetric solutions. We will refer to the Z_m symmetric solutions as “ Z_m -solution” in what follows. We can understand that (39) can be the solutions in what follows.

What we have treated so far can be considered as the case with $m = 1$, and what we will perform in this section is the generalization of the D -dependence of the gap between the uniform/non-uniform and non-uniform/gapped transitions in Sect. 7 into the framework of Z_m symmetric solutions.

Here, if $\alpha_j^{(l)}$ belonging to $2\pi l/m$ for some l are completely separated from $\alpha_j^{(l')}$ belonging to $2\pi l'/m$ for any l' except for l and, forming a mob, we refer to those configurations as “multi-cut Z_m -solution”.

In what follows we assume $N_l \sim O(N)$ (which leads to $m \ll N$) and $N_1 = N_2 = \dots = N_m$. In addition, normally $\alpha_j^{(l)} \ll 1$ may be assumed, however, since in this section we consider the transitions between the uniform phase and the Z_m -solutions, we assume that $\alpha_j^{(l)}$ are expanding widely in such a way that $\alpha_j^{(l)}$ and $\alpha_j^{(l')}$ belonging to the mobs next to each other merge and form a uniform state, or are at the moment to start to separate and form the Z_m -solutions. We will not consider the situations with $\alpha_j^{(l)} \ll 1$.

For the Z_m -solutions, we can see

$$u_n = \frac{1}{N} \sum_{k=1}^N e^{in\alpha_k} = 0 \quad \text{if } n \neq km, \quad k = 1, 2, \dots \tag{40}$$

Therefore, in the situation with a Z_m -solution, we can write the effective action (30) in the following form:

$$S_{\text{eff.}}^{(m)} = -\frac{DN^2}{m} \left\{ -\frac{m\beta\Delta^4}{8\tilde{\lambda}} + \frac{m\beta\Delta}{2} + \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{1}{D} - e^{-km\beta\Delta} \right) |u_{km}|^2 + \dots \right\}, \tag{41}$$

where $S_{\text{eff.}}^{(m)}$ means the effective action for a Z_m -solution. In the above, there is no $1/D$ correction as long as we consider up to $1/D$ order. This is because it turns out that all the $1/D$ order corrections are below $1/D^2$ order for the Z_m -solutions with $m \geq 2$,

Let us explain the above more precisely. Considering x given in (29), the x^p ($p \geq 2$) always accompany to the terms concerning $|u_p|^2$ (we find this in the appendix in [1]), and we can see that in the higher-temperature regions where the Z_m -solutions with $m \geq 2$ begin to appear as a saddle-point solution (we below mention the reason of this), x behaves as $x \sim \frac{1}{D} \exp \frac{1}{1+\tilde{\lambda}^{-1/3} \ln D \Delta T} \sim 1/D$ for $D \gg 1$ and $\Delta T \gg 1$.

We have obtained this “ $x \sim 1/D$ ” by writing the higher temperatures $T_{\text{high temp.}}$ and Δ as $T_{\text{high temp.}} = T_1 + \Delta T \sim \tilde{\lambda}^{1/3} / \ln D + \Delta T$ and $\Delta \sim \tilde{\lambda}^{1/3}$, where we have taken the leading terms of these.

Considering (30) removing the $1/D$ corrections arising from S_{int} and ignoring the overall factor $1/m$, we can see that (41) can match with such a (30) only by identifying $\beta \rightarrow m\beta$ and $|u_1| \rightarrow |u_{km}|$. Therefore we can write the effective action for a Z_m -solution by referring (32) in the case of the Z_1 solution to

$$\frac{S_{\text{eff.}}^{(m)}}{DN^2} = \frac{3\beta}{8} \tilde{\lambda}^{1/3} + \frac{c_2^{(m)}}{m} |u_{km}|^2 + \frac{c_4^{(m)}}{m} |u_{km}|^4 + \dots, \quad \text{where } k = 1, \\ c_2^{(n)} = -e^{-n\beta\tilde{\lambda}^{1/3}} + \frac{1}{D}, \quad c_4^{(n)} = \frac{n\beta\tilde{\lambda}^{1/3}}{3e^{2n\beta\tilde{\lambda}^{1/3}}}. \tag{42}$$

Note that the contributions with $k = 1$ in (41) are dominant in the above corresponding to the fact that the contribution with $n = 1$ in (28) is dominant in (32).

Since the effective action (42) is (32) in which just the temperature is exchanged, $\beta \rightarrow m\beta$, we can get the critical temperatures $T_1^{(m)}$ and $T_2^{(m)}$:

$$T_1^{(m)} / m = T_1, \quad T_2^{(m)} / m = T_2, \tag{43}$$

where $T_1^{(m)}$ and $T_2^{(m)}$ mean the critical temperatures for the uniform/ Z_m -solution and the Z_m -solution/ Z_m multi-cut

solution transitions, respectively. We can represent $T_{1,2}$ in (33) and (37) as $T_{1,2}^{(1)}$.

As the conclusion in this section, since the critical temperatures (43) are given just by constant multiples of $T_{1,2}$, we can see that the D -dependence of the gap between the uniform/ Z_m -solution and the Z_m -solution/ Z_m multi-cut solution transitions has the same tendency as the gap between the critical temperatures of the uniform/non-uniform and non-uniform/gapped transitions we have pointed out in Sect. 7, and we can plot the qualitatively identical figure to Fig. 2.

9 Conclusion and comment

Let us summarize the result in this study, which is the totally opposite tendency in the D -dependence of the gap between the two critical temperatures toward the gaps in GL and RP instabilities in the gravity and fluid sides. We have plotted it in Fig. 2.

Gauge/gravity and gauge/fluid correspondences are widely believed to hold (at least qualitatively), and the following correspondence

1D gauge theories \iff D0 black-brane solutions

is known well and the one having been studied very much until now. Our result means that the gauge/gravity and gauge/fluid correspondences concerning 1D gauge theories do not hold in the point of the D -dependence of the gap between the two critical temperatures. This is a specific counterexample to the gauge/gravity and gauge/fluid correspondences concerning 1D gauge theories.

Our analysis has based on the $1/D$ expansion of [1]. Therefore, there may be a question that the results in this study may be error for the $1/D$ expansion. We have mentioned the case when D is small in Sec. 7, so we mention the case when D is large.

Saying from my experience of MC simulation in [49], the two critical temperatures obtained from the $1/D$ expansion can match with the results of MC simulation well. Further, those can match better as D gets larger in the MC simulation until $D = 20$. Therefore, as long as saying concerning the two critical temperatures, the $1/D$ expansion would keep on capturing the two critical temperatures rightly even at large D , and it seems that the behavior of the gap at large D we have obtained in this study is not wrong. If we performed MC simulation with large D (but not so large that transitions disappear in effect) and grow it little by little, we could observe that the gap narrows gradually.

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Data Availability Statement This manuscript has associated data in a data repository [Authors' comment: All data included in this manuscript are available upon request by contacting with the corresponding author.]

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A Our Faddeev–Popov (FP) term

Let us begin with a general formula of the delta-function. We consider some function $f(x)$ ($f(x_0) = 0$) expanded around $x = x_0$ in a delta-function as

$$\delta(f(x)) = \delta\left(f(x_0) + f'(x)|_{x=x_0}(x - x_0) + \mathcal{O}\left((x - x_0)^2\right)\right). \tag{44}$$

At this time, the following formula holds:

$$\left|f'(x)|_{x=x_0}\right| \int dx \delta(f(x)) = 1. \tag{45}$$

$\left|f'(x)|_{x=x_0}\right|$ corresponds to the FP determinant.

Here, let us mention that we represent the unitary matrices as $U = \exp(i g \epsilon)$, where $\epsilon \equiv \sum_{a=1}^{N^2-1} \theta^a t^a$ (t^a are the generators of the $SU(N)$ Lie algebra and θ^a are these coefficients) in what follows.

From now on, we consider the one-dimensional system with $SU(N)$ gauge freedom such as in our model. Gauge transformations act on gauge fields $A_0(t)$ as $A_0^\theta(t) = \frac{i}{g} U(t) \partial_0 U^\dagger(t) + U^\dagger(t) A_0(t) U(t) = A_0(t) + D_0 \epsilon(t) + \mathcal{O}(\theta^2)$ in general, where $U(t) = \exp(i \epsilon(t))$, $D_0 = \partial_0 - i g [A_0(t), \cdot]$, and the θ in the shoulder of $A_0(t)$ means that $A_0(t)$ got a gauge transformation for θ from the configuration of $A_0(t)$.

We pick up the time-independent configuration:

$$\partial_0 A_0^\theta \Big|_{\theta=\theta_0} = 0 \tag{46}$$

in the path-integral for $SU(N)$ transformation.

Even if we remove the t -dependence from the gauge matrix field A_0 , there still remains the t -independent $SU(N)$ gauge freedom in the A_0 . We fix it by the diagonalization gauge: $A_0^{\eta_0} = \text{diag}(\alpha_1, \dots, \alpha_N)$.

In what follows, we first obtain the FP terms arisen from the time-independent gauge and the diagonal gauge individually. Then we obtain the FP term as a whole by summing the FP terms in each gauge fixing.

A.1 FP term from the gauge fixing, $\partial_0 A_0 = 0$

In order to compose the unity ((45) in this case) for the time-independent gauge (46), we consider the deviations arising by the gauge transformation from the configuration satisfying (46) as

$$\delta \left(\partial_0 (A_0^\theta)_{ij} \Big|_{\theta=\theta_0} \right) = \partial_0 D_0 \epsilon_{ij}(t) = \begin{cases} \partial_0^2 \epsilon_{ii}(t), \\ \partial_0 (D_0)_{ij} \epsilon_{ij}(t) \text{ for } i \neq j, \end{cases} \tag{47}$$

where the δ in the l.h.s. means the gauge transformation, and $(D_0)_{ij} = \partial_t \mathbb{I}_{ij} + i\alpha_{ij}$ ($\alpha_{ij} \equiv \alpha_i - \alpha_j$ and $i, j = 1, \dots, N$). Here we note that the analysis in what follows will be performed in the situation that the configuration of the gauge matrix field on which the gauge transformations act is the time-independent and diagonal one.

Therefore, the unity for (46) can be written as

$$\begin{aligned} 1 &= \prod_{t=0}^{\beta} \left[\int d\theta \prod_{i=1}^N \left(\frac{\delta \Theta_{ii}^\theta}{\delta(\epsilon_{ii})} \Big|_{\theta=\theta_0} \delta(\Theta_{ii}^\theta) \right) \prod_{i>j} \left(\frac{\delta(\Theta_{ij}^\theta)}{\delta(\text{Re } \epsilon_{ij})} \Big|_{\theta=\theta_0} \delta(\text{Re } \Theta_{ij}^\theta) \right) \left(\frac{\delta(\Theta_{ij}^\theta)}{\delta(\text{Im } \epsilon_{ij})} \Big|_{\theta=\theta_0} \delta(\text{Im } \Theta_{ij}^\theta) \right) \right] \\ &= \int d\theta \prod_{i=1}^N (\partial_0^2 \delta \cdot (\Theta_{ii}^\theta)) \cdot \prod_{i>j} (\partial_0 D_0^\theta)_{ij}^2 \delta(\text{Re } \Theta_{ij}^\theta) \delta(\text{Im } \Theta_{ij}^\theta), \end{aligned} \tag{48}$$

where $\Theta_{ij}^\theta \equiv \partial_0 (A_0^\theta)_{ij}$, and the integral is for the $SU(N)$ gauge transformation space. ∂_0^2 and $\partial_0 D_0^\theta$ are just formal expressions for present use only. We omit writing $\prod_{t=0}^{\beta}$ in the second line and from now on.

We here evaluate the FP determinant part in (48). We have

$$\begin{aligned} \prod_{i=1}^N (\partial_0)^2 \cdot \prod_{i>j} (\partial_0 D_0^\theta)_{ij}^2 &= \prod_{n \neq 0} \prod_{i \geq j} \left(\frac{2\pi i n}{\beta} \right)^2 \left(\frac{2\pi i n}{\beta} - i\alpha_{ij} \right)^2 \\ &= \prod_{i \geq j} \left(\prod_{n \neq 0} \left(\frac{2\pi n}{\beta} \right)^4 \cdot \left(\frac{\sin \frac{\beta}{2} \alpha_{ij}}{\frac{\beta}{2} \alpha_{ij}} \right)^2 \right), \end{aligned} \tag{49}$$

where we have performed the plane-wave expansion without the zero-mode [30]. Note α_i in α_{ij} in (49) are the elements in the diagonalized gauge matrix field.

We then calculate a part of (49),

$$\prod_{i \geq j} \left(\frac{\sin \frac{\beta}{2} \alpha_{ij}}{\frac{\beta}{2} \alpha_{ij}} \right)^2 = \prod_{i > j} \left(\frac{\sin \frac{\beta}{2} \alpha_{ij}}{\frac{\beta}{2} \alpha_{ij}} \right)^2. \tag{50}$$

Here,

$$\begin{aligned} \prod_{i>j} \sin^2 \left(\frac{\beta}{2} \alpha_{ij} \right) &= \exp \sum_{i \neq j} \left(\log \frac{1}{2i} + \frac{i\beta}{2} \alpha_{ij} - \sum_{n=1}^{\infty} \frac{e^{-in\beta\alpha_{ij}}}{n} \right) \\ &= \frac{1}{2^{N(N-1)}} \exp \left(-N^2 \sum_{n=1}^{\infty} \frac{|u_n|^2}{n} \right), \end{aligned} \tag{51}$$

where we have assumed that N is an even number. Therefore,

$$\begin{aligned} (48) &= \frac{1}{\beta^{N(N-1)}} \prod_{n \neq 0} \left(\frac{2\pi n}{\beta} \right)^{2N(N+1)} \cdot \prod_{i>j} \frac{1}{(\alpha_{ij})^2} \\ &\quad \cdot \exp \left(-N^2 \sum_{n=1}^{\infty} \frac{|u_n|^2}{n} \right) \\ &\quad \times \int d\theta \prod_{i=1}^N \delta(\Theta_{ii}^\theta) \cdot \prod_{i>j} \delta(\text{Re } \Theta_{ij}^\theta) \delta(\text{Im } \Theta_{ij}^\theta). \end{aligned} \tag{52}$$

A.2 FP term from gauge fixing, $A_{0ij} = \alpha_i \delta_{ij}$

We fix the remaining t -independent $SU(N)$ gauge freedom by the diagonal gauge: $A_0^{\eta_0} = \text{diag}(\alpha_1, \dots, \alpha_N)$.

Since gauge transformations from $A_0^{\eta_0}$ can be written as

$$\begin{aligned} \delta A_0^{\eta_0} &= (1 - i\epsilon) A_0^{\eta_0} (1 + i\epsilon) - A_0^{\eta_0} + \mathcal{O}(\theta^2) \\ &= i[A_0^{\eta_0}, \epsilon] + \mathcal{O}(\theta^2), \end{aligned} \tag{53}$$

the configuration deviating from the diagonalized one simultaneously can be written as

$$(\delta A_0^{\eta_0})_{ij} = i\alpha_{ij} \epsilon_{ij}. \tag{54}$$

This can be seen from the case of $SU(3)$, (16). The unity for the diagonalized constraint is therefore given as

$$\begin{aligned}
 1 &= \prod_{i>j} \left| \frac{(\delta A_0^{\eta_0})_{ij}}{\delta \epsilon_{ij}} \right|^2 \int d\theta \delta(\operatorname{Re} A_{0ij}^\theta) \delta(\operatorname{Im} A_{0ij}^\theta) \\
 &= \int d\theta \prod_{i>j} (\alpha_{ij})^2 \delta(\operatorname{Re} A_{0ij}^\theta) \delta(\operatorname{Im} A_{0ij}^\theta). \tag{55}
 \end{aligned}$$

A.3 Total FP term

We can now obtain the unity when we impose the t -independent diagonalized constraints by combining (48) and (55) as

$$\begin{aligned}
 1 &= \frac{1}{\beta^{N(N-1)}} \prod_{n \neq 0} \left(\frac{2\pi n}{\beta} \right)^{2N(N+1)} \cdot \int d\theta \exp \left(-N^2 \sum_{n=1}^{\infty} \frac{|u_n|^2}{n} \right) \\
 &\quad \times \prod_i \delta(\Theta_{ii}^\theta) \cdot \prod_{i>j} \delta(\operatorname{Re} \Theta_{ij}^\theta) \delta(\operatorname{Im} \Theta_{ij}^\theta) \delta(\operatorname{Re} A_{0ij}^\theta) \delta(\operatorname{Im} A_{0ij}^\theta). \tag{56}
 \end{aligned}$$

Note that $\prod_{i=0}^\beta$ attaching to the whole is omitted in the expression above.

B Derivation of (E.8) in Ref. [1]

From (12), we write the contribution arisen from the interaction term as

$$- \sum_{n=1}^{\infty} \frac{1}{(2n)!} \left(\frac{-ig}{2} \right)^{2n} \left\langle \prod_{\alpha=1}^{2n} \int dt_\alpha b_{a_\alpha b_\alpha} Y_{a_\alpha}^{I_\alpha}(t_\alpha) Y_{b_\alpha}^{I_\alpha}(t_\alpha) \right\rangle, \tag{57}$$

where $\langle \mathcal{A} \rangle \equiv \int \mathcal{D}b \mathcal{D}Y \mathcal{A} \exp -\frac{1}{DN^2} \int_0^\beta dt (bM^{-1}b - \frac{1}{2}Y(D_0^2 + iB)Y)$. Equation (57) is a summation of the $(n + 1)$ -loop diagrams $(n = 1, 2, \dots, \infty)$ in Fig. 3.

We can see $\frac{(2n-1)!!(n-1)!2^{n-1}2^{2n}}{(2n)!2^{2n}}$ emerges from Wick contractions. We mention the origin of each factor:

(a) $(2n)!2^{2n}$ in the denominator comes from the denominator in (57).

- (b) $(2n - 1)!!$ is the number of pairs made by combining the interaction terms bYY two by two (each pair forms a 1PI diagram made of one loop of Y with two lines of b).
- (c) $(n - 1)!$ is the number of patterns to combine those 1PI diagrams to form a single big loop by the n loops of Fig. 3.
- (d) 2^{n-1} comes from the option to put each 1PI diagram upward or downward.
- (e) 2^{2n} comes from the two patterns in combining Y 's in the two interaction terms bYY each to form one 1PI diagram.

Since there is no explanation between (E.1) and (E.8) in [1], let us consider how to derive (E.8). Upon evaluating (57), we consider the two-loop contribution as the most simple example.

Focusing on a part with $n = 1$ in (57),

$$\begin{aligned}
 & - \frac{1}{2} \left(\frac{-ig}{2} \right)^2 \int dt_2 dt_1 \\
 & \quad \left\langle b_{a_1 b_1} Y_{a_1}^{I_1}(t_2) Y_{b_1}^{I_1}(t_2) \cdot b_{a_2 b_2} Y_{a_2}^{I_2}(t_1) Y_{b_2}^{I_2}(t_1) \right\rangle \\
 & = \frac{2^4 g^2}{8} \langle b_{a_1 b_1} b_{a_2 b_2} t_{ij}^{a_1} t_{kl}^{b_1} t_{qp}^{a_2} t_{nm}^{b_2} \int dt_2 dt_1 \\
 & \quad \left\langle Y_{ji}^{I_1}(t_2) Y_{lk}^{I_1}(t_2) \cdot Y_{pq}^{I_2}(t_1) Y_{mn}^{I_2}(t_1) \right\rangle \\
 & = 2g^2 M_{a_1 b_1, a_2 b_2} t_{ij}^{a_1} t_{kl}^{b_1} t_{qp}^{a_2} t_{nm}^{b_2} \int dt_2 dt_1 \\
 & \quad \left(\left\langle Y_{ji}^{I_1}(t_2) Y_{pq}^{I_2}(t_1) \right\rangle \left\langle Y_{lk}^{I_1}(t_2) Y_{mn}^{I_2}(t_1) \right\rangle \right. \\
 & \quad \left. + \left\langle Y_{ji}^{I_1}(t_2) Y_{mn}^{I_2}(t_1) \right\rangle \left\langle Y_{lk}^{I_1}(t_2) Y_{pq}^{I_2}(t_1) \right\rangle \right), \tag{58}
 \end{aligned}$$

where $\langle b_{a_1 b_1} b_{a_2 b_2} \rangle = M_{a_1 b_1, a_2 b_2}$, and we have used $Y_a^I = 2 \operatorname{tr}(t^a Y^I)$ with $Y^I = \sum_{a=1}^{N^2} Y_a^I t^a$. As we have the invariance $M_{a_1 b_1, a_2 b_2} = M_{a_1 b_1, b_2 a_2}$, (58) can be written as

$$\begin{aligned}
 (58) & = 4g^2 M_{a_1 b_1, a_2 b_2} t_{ij}^{a_1} t_{kl}^{b_1} t_{qp}^{a_2} t_{nm}^{b_2} \int dt_2 dt_1 \\
 & \quad \left\langle Y_{ji}^{I_1}(t_2) Y_{pq}^{I_2}(t_1) \right\rangle \left\langle Y_{lk}^{I_1}(t_2) Y_{mn}^{I_2}(t_1) \right\rangle. \tag{59}
 \end{aligned}$$

We take the leading contribution in the large- N (this would be the point). In this case, the contribution in the case that

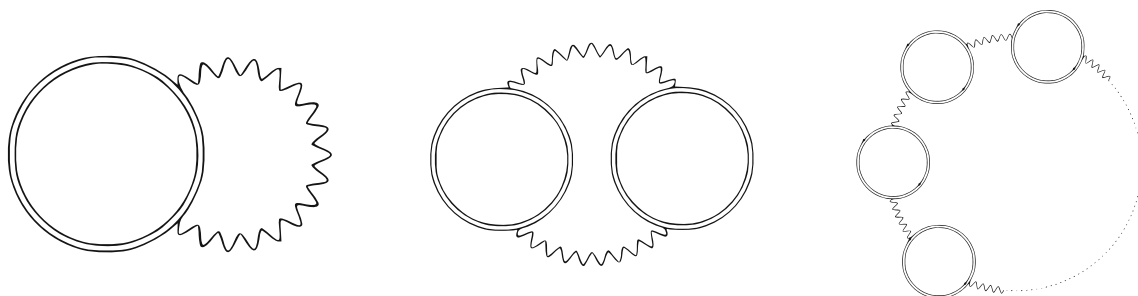


Fig. 3 Two loops (left), three loops (center) and n loops (right); double lines mean the propagators of Y , and wavy lines mean the propagators of b

each of the indices for the inner and outer lines in the Y 's loop becomes, respectively, the same term as will be picked up. Therefore,

$$(59)|_{\text{large}-N} = 4g^2 M_{a_1 b_1, a_2 b_2} t_{ij}^{a_1} t_{ij}^{b_1} t_{ji}^{a_2} t_{ji}^{b_2} \int dt_2 dt_1 \left\langle Y_{ji}^{I_1}(t_2) Y_{ij}^{I_2}(t_1) \right\rangle \left\langle Y_{ji}^{I_1}(t_2) Y_{ij}^{I_2}(t_1) \right\rangle. \tag{60}$$

Using the composite propagator given in (E.1) of [1],

$$\sum_{j,p} \sum_{I,J} \left\langle Y_{ij}^I(t) Y_{pq}^J(t') \right\rangle \left\langle Y_{jk}^I(t) Y_{lp}^J(t') \right\rangle \equiv DN \sum_n G_{n,ik} e^{i \frac{2\pi n}{\beta} (t-t')} \delta_{iq} \delta_{kl}, \tag{61}$$

(60) can be written as

$$(60) = 4g^2 \beta g^2 DN M_{a_1 b_1, a_2 b_2} t_{ij}^{a_1} t_{ij}^{b_1} t_{ji}^{a_2} t_{ji}^{b_2} \sum_n \sum_{j,i} G_{n,ji}. \tag{62}$$

It would be difficult to evaluate this. However, based on the following two points:

1. The result should become (E.8) when $D = 0$ including coefficients except for β ,
2. standing up behavior of Wilson line $|u_1|$ just above T_1 , (35),

we would be able to analogize how (60) will be written finally as

$$(60) = -\beta d_1 \frac{g^2 DN}{2} \sum_n \sum_{q,m} G_{n,qm}. \tag{63}$$

Performing the above by increasing n in $(n + 1)$ loops, we can reduce the contributions at $(n + 1)$ loops as

$$(57) = -d_n \frac{(-)^n}{2n} (\beta g^2 DN)^n \sum_{m=-\infty}^{\infty} \sum_{i,j=1}^N (G_{m,ij})^n, \tag{64}$$

where $d_1 = -1$, $d_2 = 3$, and $d_n = 1$ for $n = 3, 4, \dots$

C Calculation process from (27) to (28)

We show the calculation process from (27) to (28).

$$\begin{aligned} (27) &= \left(\prod_{i=1}^N \sqrt{\frac{\Delta^2}{2}} \cdot \prod_{i=1}^N \prod_{n=1}^{\infty} (k_n^2 + \Delta^2) \right. \\ &\quad \left. \cdot \prod_{j>i} \prod_{n=-\infty}^{\infty} ((k_n - \alpha_{ji})^2 + \Delta^2) \right)^{-D} \\ &= 2^{\frac{DN}{2}} \left(\prod_{i=1}^N \Delta \cdot \prod_{i=1}^N \prod_{n=1}^{\infty} (k_n^2 + \Delta^2) \right. \\ &\quad \left. \cdot \prod_{i \neq j} \prod_{n=1}^{\infty} ((k_n - \alpha_{ji})^2 + \Delta^2) \cdot \prod_{i \neq j} \sqrt{\alpha_{ji}^2 + \Delta^2} \right)^{-D} \\ &= 2^{\frac{DN}{2}} \prod_{i,j} \left(\sqrt{\alpha_{ji}^2 + \Delta^2} \cdot \prod_{n=1}^{\infty} ((k_n - \alpha_{ji})^2 + \Delta^2) \right)^{-D} \\ &= 2^{\frac{DN}{2}} \left(\prod_{i,j} \sqrt{\alpha_{ji} + \Delta^2} \cdot \prod_{n=1}^{\infty} k_n^2 \right)^{-D} \\ &\quad \left(\prod_{i,j} \prod_{n=1}^{\infty} \left(1 - \frac{\beta(\alpha_{ji} - i\Delta)}{2\pi n} \right) \left(1 - \frac{\beta(\alpha_{ji} + i\Delta)}{2\pi n} \right) \right)^{-D} \\ &= 2^{\frac{DN}{2}} \prod_{i,j} \left(\frac{\sqrt{2}}{\beta} \prod_{n=1}^{\infty} k_n^2 \cdot \sqrt{-\cos \beta \alpha_{ji} + \cosh \beta \Delta} \right)^{-D} \\ &= 2^{\frac{DN}{2}} \left(\frac{1}{\beta} \prod_{n=1}^{\infty} k_n^2 \right)^{-DN^2} \\ &\quad \exp -\frac{D}{2} \left[N^2 \beta \Delta + 2N \ln(1 - e^{-\beta \Delta}) \right. \\ &\quad \left. + \sum_{i \neq j} \left\{ \ln(1 - e^{-\beta(\Delta - i\alpha_{ji})}) + \ln(1 - e^{-\beta(\Delta + i\alpha_{ji})}) \right\} \right] \\ &= 2^{\frac{DN}{2}} \left(\frac{1}{\beta} \prod_{n=1}^{\infty} k_n^2 \right)^{-DN^2} \exp -DN^2 \left(\frac{\beta \Delta}{2} - \sum_{n=1}^{\infty} \frac{e^{-n\beta \Delta}}{n} |u_n|^2 \right). \end{aligned}$$

In the above, we have used the following relations:

- $\prod_{i,j} \left(1 - \frac{\alpha_{ji} - i\Delta}{k_n} \right) \left(1 - \frac{\alpha_{ji} + i\Delta}{k_n} \right) = \prod_{i,j} \sqrt{\left(1 - \frac{\alpha_{ji} - i\Delta}{k_n} \right) \left(1 - \frac{\alpha_{ji} + i\Delta}{k_n} \right) \left(1 - \frac{\alpha_{ij} - i\Delta}{k_n} \right) \left(1 - \frac{\alpha_{ij} + i\Delta}{k_n} \right)}$, (65)
- $\prod_{n=1}^{\infty} \left(1 - \left(\frac{\alpha_{ji} - i\Delta}{k_n} \right)^2 \right) \left(1 - \left(\frac{\alpha_{ji} + i\Delta}{k_n} \right)^2 \right) = \frac{2(-\cos(\beta \alpha_{ji}) + \cosh(\beta \Delta))}{\beta^2(\alpha_{ji}^2 + \Delta^2)}$, (66)
- $\sum_{i \neq j} \log(1 - e^{-\beta(\Delta - i\alpha_{ji})}) = \sum_{i \neq j} \log(1 - e^{-\beta(\Delta + i\alpha_{ji})})$
- $= -N^2 \sum_{n=1}^{\infty} \frac{e^{-n\beta \Delta}}{n} |u_n|^2 - N \ln(1 - e^{-\beta \Delta})$. (67)

D Transition-order of the uniform/non-uniform transition

In this appendix we check the transition-order associated with the uniform/non-uniform transition at.

By substituting $\beta = 1/(T_1 + \delta T)$ into c'_2 in (32), we can obtain as⁹

$$\begin{aligned}
 c'_2|_{T=T_1+\delta T} &= \mathcal{O}(D^{-3}) + \frac{(\ln D)^2}{\tilde{\lambda}^{1/3} D} \left\{ -1 + \frac{-609 + 160\sqrt{5}}{480D} \right. \\
 &\quad \left. + \mathcal{O}(D^{-3}) \right\} \delta T \\
 &\quad + \frac{(-2 + \ln D)(\ln D)^3}{2\tilde{\lambda}^{2/3} D} \left\{ -1 + \frac{-609 + 160\sqrt{5}}{240D} \right. \\
 &\quad \left. + \mathcal{O}(D^{-3}) \right\} (\delta T)^2 + \mathcal{O}(\delta T^3). \tag{69}
 \end{aligned}$$

Note that the term $1/D$ appearing in c'_2 in (32) does not appear in (69). This is because it is canceled with that appearing as $-e^{\beta\tilde{\lambda}^{1/3}}|_{T=T_1+\delta T} \sim -1/D + \dots$. Since our analysis is supposed to $1/D$ order, we disregard the part $\mathcal{O}(D^{-3})$.

Multiplying by (34), and performing the expansion regarding δT , then performing the expansion regarding $1/D$,¹⁰

$$c'_2 |u_1|^2|_{T=T_1+\delta T}$$

⁹ We have also calculated c'_0 at $T = T_1 + \delta T$. However, it is not important in the analysis for the transition-order associated with the uniform/non-uniform transition that we are now performing. Therefore we show its result here:

$$\begin{aligned}
 c'_0|_{T=T_1+\delta T} &= \frac{3 \ln D}{8} \left(1 + \frac{(160\sqrt{5} - 337)}{160D} + \mathcal{O}(D^{-2}) \right) \tag{68} \\
 &\quad - \frac{(\ln D)^2}{8\tilde{\lambda}^{1/3}} \left(3 + \frac{(160\sqrt{5} - 201)}{80D} + \mathcal{O}(D^{-2}) \right) \delta T \\
 &\quad + \frac{(\ln D)^3}{8\tilde{\lambda}^{2/3}} \left(3 + \frac{(207 + 160\sqrt{5})}{160D} + \mathcal{O}(D^{-2}) \right) (\delta T)^2 \\
 &\quad - \frac{(\ln D)^4}{8\tilde{\lambda}} \left(3 + \frac{51}{10D} + \mathcal{O}(D^{-2}) \right) (\delta T)^3 + \dots \\
 &\sim \frac{\ln D}{8} (3 + 1/D + \mathcal{O}(D^{-2})) \sum_{n=0}^{\infty} \left(-\frac{\ln D \cdot \delta T}{\tilde{\lambda}^{1/3}} \right)^n,
 \end{aligned}$$

¹⁰ L.h.s. in (70) is written as $c'_2|_{T=T_1+\delta T} \cdot |u_1|^2|_{T=T_1+\delta T}$ if writing exactly along the actual manipulation.

$$\begin{aligned}
 &= \frac{(\ln D)^3}{2\tilde{\lambda}^{2/3}} \left(3 + \frac{(-3249 + 382\sqrt{5} + 297 \ln D)}{400D} \right. \\
 &\quad \left. + \mathcal{O}(D^{-2}) \right) (\delta T)^2 + \mathcal{O}(\delta T^3). \tag{70}
 \end{aligned}$$

Using the relation $c'_2|u_1|^2 = -c'^2_2/2c'_4$ and $c'_4|u_1|^4 = c'^2_2/4c'_4$, we can calculate $c'_4|u_1|^4|_{T=T_1+\delta T}$ as

$$c'_4|u_1|^4|_{T=T_1+\delta T} = -\frac{1}{2}c'_2|u_1|^2|_{T=T_1+\delta T}. \tag{71}$$

Therefore, from (70) and (71), writing the D - and δT -dependences only,

$$\bullet \frac{S_{GL}}{DN^2}|_{T=T_1-\delta T} \sim c'_0|_{T=T_1-\delta T}, \tag{72}$$

$$\begin{aligned}
 \bullet \frac{S_{GL}}{DN^2}|_{T=T_1+\delta T} &\sim c'_0|_{T=T_1+\delta T} \\
 &+ \left(1 + \mathcal{O}(D^{-2}) \right) (\delta T)^2 + \mathcal{O}(\delta T^3). \tag{73}
 \end{aligned}$$

Note that $|u_1| = 0$ for $T < T_1$. This leads to the conclusion that the transition-orders for the uniform/non-uniform transition in the large- N 1D bosonic models is second.

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