Dynamical system analysis of a three fluid cosmological model: an invariant manifold approach

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Abstract The present paper considers a three-fluid cosmological model consisting of noninteracting dark matter, dark energy and baryonic matter in the background of the Friedman–Robertson–Walker–Lemaître flat spacetime. It has been assumed that the dark matter takes the form of dust whereas the dark energy is a quintessence (real) scalar field with exponential potential. It has been further assumed that the baryonic matter is a perfect fluid with barotropic equation of states. The field equations for this model takes the form of an autonomous dynamical system after some suitable changes of variables. Then a complete stability analysis is done considering all possible parameter (the adiabatic index of the baryonic matter and the parameter arising from the dark energy potential) values and for both the cases of hyperbolic and non-hyperbolic critical points. For non-hyperbolic critical points, the invariant manifold theory (center manifold approach) is applied. Finally various topologically different phase planes and vector field diagrams are produced and the cosmological interpretation of this model is presented.

1 Introduction

Recently, a large number of observational data from various sources such as Type Ia Supernova [1,2], CMB anisotropies [4,7], Large Scale Structures [3,5] and Baryon Acoustic Oscillations [6] suggests that we are in a spatially flat universe which after the big bang has undergone two accelerated expansion phases, one occurred before the radiation dominated era and the other one started not too long ago. We are presently in this accelerated expansion phase.

This present time accelerated expansion has been attributed to a unseen and unknown matter with very large negative pressure called the dark energy. Analysis of differ-

ent cosmological data suggests that our universe is composed of around 70% dark energy, about 25% dark matter and the rest accounts for baryonic matter and radiation.

Very few properties of the dark energy is known [9,12,24]. As far as the mathematical modelling of this exotic matter is concerned, the simplest choice for it has been the Cosmological Constant Λ [8,11,16,19]. Although these models are capable of explaining most of the observational data, all of them fails to explain the coincidence problem (why expansion is happening now and why is it accelerated) and the fine-tuning problem (why some of the parameters take exorbitantly high values where others are not). To resolve these issues, lots of dynamical dark energy models has been prescribed. There the dark energy has been modelled as a scalar field. Some scalar field models like quintessence [17,18], K -essence [21–23] and Tachyonic models have attracted lots of attention.

In this paper we consider a cosmological model consisting of non-interacting dark matter, dark energy and baryonic matter in the background of the Friedman–Robertson–Walker–Lemaître flat spacetime [25,26]. The dark matter has been assumed to take the form of dust whereas the dark energy is assumed to be a scalar field with exponential potential. It has been further assumed that the baryonic matter is a perfect fluid with barotropic equation of states. All these three fluids are assumed to be non-interacting and minimally coupled to gravity.

In order to study this model qualitatively, we derive the Einstein's field equations and the Klein–Gordon equation for the scalar field. After some suitable changes of variables, these equations take the form of an autonomous dynamical system. Then we find the critical points and analyze the stability of each critical point. We note that in [10] the model like ours have been considered. But the authors did not consider the cases of non-hyperbolic critical points. Our results differ from them in various ways. Firstly, we apply some of the very rich theories for the dynamical systems, namely the Invari-

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ant Manifold and the Center Manifold Theory [13–15,20] to compute the center manifolds for all the non-hyperbolic critical points and then continue to do the stability analysis for them. The other one is that we have considered all theoretically possible values of the parameters to do a complete analysis here, whereas in their article, they chose only a few suitable values. Lastly, we have presented all possible topologically different phase plane diagrams here where they included only some of them in their article.

The motivation to do stability analysis is that after considering all cosmological and observational constraints of data, the stable critical points in our model may depict our present universe as a global attractor. If they do fit with the data of present percentages of dark energy, dark matter and baryonic matter together with radiation in the universe then our model would successfully describe the universe.

The organization of this article is as follows: The Sect. 2 describes the Einstein field equations, Klein–Gordon scalar field equation and energy conservation relations for our model. Section 3 describes the formation of an autonomous system. The Sect. 4 is where we present our work on complete stability analysis. At the end of this section, we produce the phase plane diagrams for different topological cases. Finally Sect. 5 presents the cosmological interpretations of our results and concludes our work.

2 Equations

The homogeneous and isotropic flat Friedman–Robertson–Walker–Lemaître spacetime is the background of our model. This universe is assumed to be filled up by non-interacting dark matter, dark energy and baryon. Dark matter is assumed to be dust with energy density ρ_m and the dark energy is assumed to be a scalar field ϕ with the potential as $V(\phi)$. The density ρ_d and the pressure p_d of the scalar field follows the following equations:

$$\rho_d = \frac{1}{2}\dot{\phi}^2 + V(\phi) \quad (1)$$

and

$$p_d = \frac{1}{2}\dot{\phi}^2 - V(\phi). \quad (2)$$

Here $\dot{}$ denotes differentiation with respect to cosmic time t .

The baryonic matter is assumed to be a perfect fluid with linear equation of state

$$p_b = (\nu - 1)\rho_b \quad (3)$$

where p_b and ρ_b are the density and the pressure of the fluid and ν is the adiabatic index of the fluid satisfying $\frac{2}{3} < \nu \leq 2$. In particular $\nu = 1$ and $\nu = \frac{4}{3}$ corresponds to the dust and radiation respectively. Here we also assume that $\nu \neq 1$. All three matters are non-interacting and minimally coupled to gravity.

The Einstein field equations for this model is

$$3H^2 = k(\rho_m + \rho_d + \rho_b), \quad (4)$$

where H is the Hubble parameter and $k = 8\pi G$, where G is the gravitational constant, the speed of light has been scaled to 1.

The Klein–Gordon equation of the scalar field is

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0. \quad (5)$$

The energy conservation relations take the following form

$$\dot{\rho}_m + 3H\rho_m = 0 \quad (6)$$

$$\dot{\rho}_d + 3H(\rho_d + p_d) = 0 \quad (7)$$

$$\dot{\rho}_b + 3H(\rho_b + p_b) = 0. \quad (8)$$

From (4)–(8) we derive

$$2\dot{H} = -k(\rho_m + \rho_b + p_b + \dot{\phi}^2). \quad (9)$$

The Eqs. (4), (5) and (9) are the evolution equations for this model. Next we find suitable coordinate changes such that these evolution equations form a system of autonomous dynamical system. This is done in the following section.

3 The autonomous system

We introduce the following coordinate transformations of variables:

$$x = \sqrt{\frac{k}{6}} \frac{\dot{\phi}}{H}, \quad (10)$$

$$y = \sqrt{\frac{k}{3}} \frac{\sqrt{V(\phi)}}{H} \quad (11)$$

and the density parameters

$$\Omega_m = \frac{k\rho_m}{3H^2}, \quad (12)$$

$$\Omega_b = \frac{k\rho_b}{3H^2}, \quad (13)$$

$$\Omega_d = \frac{k\rho_d}{3H^2}. \quad (14)$$

These coordinate changes transform the Friedmann equation (4) and the Eq. (9) respectively as the following:

$$\Omega_d = x^2 + y^2, \quad (15)$$

$$\Omega_m + \Omega_b + x^2 + y^2 = 1 \quad (16)$$

and

$$\dot{H} = -3H^2 \left(x^2 + \frac{\Omega_m}{2} + \nu \frac{\Omega_b}{2} \right). \quad (17)$$

As Ω_m and Ω_b , the density parameters are non-negative real quantities, so from (16), $0 \leq \Omega_m \leq 1$, $0 \leq \Omega_b \leq 1$ and x and y satisfies $x^2 + y^2 \leq 1$. The strict equality is only possible if the energy densities of the dark matter and the baryonic matter is zero.

Differentiating the Eqs. (10)–(12) with respect to N where $N = \ln a(t)$ ($a(t)$ is the scale factor of the universe) and using (16), (17), (5) and (6), (7) and (8) we derive the following autonomous dynamical system:

$$\frac{dx}{dN} = 3x \left[x^2 - 1 + \frac{\Omega_m}{2} + \frac{\nu}{2}(1 - \Omega_m - x^2 - y^2) \right] - \sqrt{\frac{3}{2k}} \frac{1}{V} \frac{dV}{d\phi} y^2 \quad (18)$$

$$\frac{dy}{dN} = y \left[3x^2 + 3\frac{\nu}{2}(1 - \Omega_m - x^2 - y^2) + 3\frac{\Omega_m}{2} + \sqrt{\frac{3}{2k}} \frac{1}{V} \frac{dV}{d\phi} x \right] \quad (19)$$

$$\frac{d\Omega_m}{dN} = -3\Omega_m [1 - \Omega_m - 2x^2 - \nu(1 - \Omega_m - x^2 - y^2)]. \quad (20)$$

We end this section by expressing the relevant cosmological parameters in terms of the above transformed variables as

$$\Omega_d = x^2 + y^2, \quad (21)$$

$$\omega_d = \frac{p_d}{\rho_d} = \frac{x^2 - y^2}{x^2 + y^2}, \quad (22)$$

$$\omega_{eff} = \frac{p_d + p_b}{\rho_m + \rho_d + \rho_b} = -1 + 2x^2 + \nu(1 - \Omega_m - x^2 - y^2) + \Omega_m \quad (23)$$

and

$$q = - \left(1 + \frac{\dot{H}}{H^2} \right) = - \left[\left(1 - 3x^2 - \frac{3\nu}{2} \right) - \frac{3\Omega_m}{2}(1 - \nu) + \frac{3\nu}{2}(x^2 + y^2) \right]. \quad (24)$$

We note that for accelerated expansion, $\omega_{eff} \leq -\frac{1}{3}$ and $q \leq 0$ are two necessary conditions.

4 Stability analysis

We start working on the stability analysis of the dynamical system (18) in this section. We assume that the potential of the scalar field representing the dark energy is exponential, ie $\frac{1}{V} \frac{dV}{d\phi} = \text{some constant}$. We choose $\sqrt{\frac{3}{2k}} \frac{1}{V} \frac{dV}{d\phi} = \alpha$. Then the autonomous system (18) transforms into:

$$\frac{dx}{dN} = 3x \left[x^2 - 1 + \frac{\Omega_m}{2} + \frac{\nu}{2}(1 - \Omega_m - x^2 - y^2) \right] - \alpha y^2 \quad (25)$$

$$\frac{dy}{dN} = y \left[3x^2 + 3\frac{\nu}{2}(1 - \Omega_m - x^2 - y^2) + 3\frac{\Omega_m}{2} + \alpha x \right] \quad (26)$$

$$\frac{d\Omega_m}{dN} = -3\Omega_m [1 - \Omega_m - 2x^2 - \nu(1 - \Omega_m - x^2 - y^2)]. \quad (27)$$

There are ten critical points of this autonomous system. They are listed in the Table 1.

The value of the relevant cosmological parameters for each of the critical points are given in the Table 2.

For every critical point and for each of their subcases we will do two successive change of variables to bring them to a form with which the calculation of center manifold and dynamics of the reduced system will be easy. For each of the cases and subcases the transformations are as follows:

If

$$X = \begin{pmatrix} x \\ y \\ \Omega_m \end{pmatrix}, \quad \bar{X} = \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{\Omega}_m \end{pmatrix} \quad \text{and} \quad \bar{\bar{X}} = \begin{pmatrix} \bar{\bar{x}} \\ \bar{\bar{y}} \\ \bar{\bar{\Omega}}_m \end{pmatrix},$$

then $\bar{X} = X - A$ and $\bar{\bar{X}} = P^{-1}\bar{X}$ where A and P is some 3×1 and 3×3 matrices respectively with P being non-singular. The exact form of A and P will vary from case to

Table 1 Critical points

Critical point name	Critical point
C_1	(0, 0, 0)
C_2	(1, 0, 0)
C_3	(0, 0, 1)
C_4	(-1, 0, 0)
C_5	$(-\frac{\alpha}{3}, \frac{1}{3}\sqrt{9-\alpha^2}, 0)$
C_6	$(-\frac{\alpha}{3}, -\frac{1}{3}\sqrt{9-\alpha^2}, 0)$
C_7	$(-\frac{3\nu}{2\alpha}, \frac{3}{2\alpha}\sqrt{2\nu-\nu^2}, 0)$
C_8	$(-\frac{3\nu}{2\alpha}, -\frac{3}{2\alpha}\sqrt{2\nu-\nu^2}, 0)$
C_9	$(-\frac{3}{2\alpha}, \frac{3}{2\alpha}, (1 - \frac{9}{2\alpha^2}))$
C_{10}	$(-\frac{3}{2\alpha}, -\frac{3}{2\alpha}, (1 - \frac{9}{2\alpha^2}))$

Table 2 Values of the different cosmological parameters at the critical points

Critical points	ω_d	ω_{eff}	Ω_m	Ω_d	Ω_b	q
C_1	Undefined	$\nu - 1$	0	0	1	$\frac{3\nu}{2} - 1$
C_2	1	1	0	1	0	2
C_3	Undefined	0	1	0	0	$\frac{1}{2}$
C_4	1	1	0	1	0	2
C_5	$\frac{2\alpha^2}{9} - 1$	$\frac{2\alpha^2}{9} - 1$	0	1	0	$\frac{\alpha^2}{3} - 1$
C_6	$\frac{2\alpha^2}{9} - 1$	$\frac{2\alpha^2}{9} - 1$	0	1	0	$\frac{\alpha^2}{3} - 1$
C_7	$\nu - 1$	$\nu - 1$	0	$\frac{9\nu}{2\alpha^2}$	$1 - \frac{9\nu}{2\alpha^2}$	$\frac{3\nu}{2} - 1$
C_8	$\nu - 1$	$\nu - 1$	0	$\frac{9\nu}{2\alpha^2}$	$1 - \frac{9\nu}{2\alpha^2}$	$\frac{3\nu}{2} - 1$
C_9	0	0	$1 - \frac{9}{2\alpha^2}$	$\frac{9}{2\alpha^2}$	0	$\frac{1}{2}$
C_{10}	0	0	$1 - \frac{9}{2\alpha^2}$	$\frac{9}{2\alpha^2}$	0	$\frac{1}{2}$

Table 3 C_1 (reduced system)

ν	Center manifold	Reduced system
0	$\bar{x} = O(\bar{\Omega}_m^3), \bar{y} = -\frac{\alpha}{3}\bar{\Omega}_m^2 + O(\bar{\Omega}_m^3)$	$\bar{\Omega}_m' = -\frac{\alpha^2}{3}\bar{\Omega}_m^3 + O(\bar{\Omega}_m^4)$
1	$\bar{y} = O(\bar{x}^3), \bar{\Omega}_m' = O(\bar{x}^3)$	$\bar{x}' = 0$
2	$\bar{x} = O(\bar{y}^3), \bar{\Omega}_m' = O(\bar{y}^3)$	$\bar{y}' = 0$

Table 4 Summary for the critical point C_1

Case	ν	Stability (RS)	Stability (DS)
a	$\nu < 0$	NA	Stable
b	$\nu = 0$	Stable	Stable
c	$0 < \nu < 1$	NA	Saddle
d	$\nu = 1$	Center	Center-saddle
e	$1 < \nu < 2$	NA	Saddle
f	$\nu = 2$	Center	Center-unstable
g	$2 < \nu$	NA	Unstable

case and we will mention them while studying each of the cases and subcases.

4.1 Critical point C_1

For critical point C_1 , $A = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and the Jacobian matrix of the system (25)–(27) at this critical point has the characteristic polynomial

$$\lambda^3 + (6 - 6\nu)\lambda^2 + \left[\frac{3\nu(3\nu - 3)}{2} + \frac{(3\nu - 6)(9\nu - 6)}{4} \right] \lambda - \frac{3\nu(3\nu - 3)(3\nu - 6)}{4} = 0, \quad (28)$$

which has eigen values as $3\nu - 3$, $\frac{3\nu}{2} - 3$ and $\frac{3\nu}{2}$. The stability analysis when ν is not 0, 1 or 2 (hyperbolic cases) is an easy application of the stability analysis of the linear system and the Hartman–Gröbman Theorem.

So now we will present the Table 3 containing the non-hyperbolic subcases and the result of their stability analysis.

In the Table 3, represents derivative with respect to N .

$$\text{For all the subcases in the Table 3, } P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

We summarize our results for C_1 in the Table 4 below to end this subsection.

In the Table 4, RS stands for the Reduced System, DS stands for the whole system and NA stands for Not Applicable (Hyperbolic Cases).

4.2 Critical point C_2

For critical point C_2 , $A = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and the Jacobian matrix of the system (25)–(27) at this critical point has the characteristic polynomial

Table 5 C_2 (reduced system)

ν	α	Center manifold	Reduced system
2	-3	$\bar{\Omega}_m = O(\ (\bar{x}, \bar{y})\ ^3)$	$\bar{x}' = -3\bar{x}\bar{y}^2, \bar{y}' = -3\bar{x}\bar{y}$
2	$\neq -3$	$\bar{y} = O(\bar{x}^3), \bar{\Omega}_m = O(\bar{x}^3)$	$\bar{x}' = 0$
$\neq 2$	-3	$\bar{y} = O(\bar{x}^3), \bar{\Omega}_m = -\frac{1}{2}\bar{x}^2 + O(\bar{x}^3)$	$\bar{x}' = -\frac{3}{2}\bar{x}^3$

Table 6 A summary for the critical point C_2

Case	ν	α	Stability (RS)	Stability (DS)
a	< 2	< -3	NA	Saddle
b	< 2	-3	Stable	Saddle
c	< 2	> -3	NA	Unstable
d	2	> -3	Center	Center-unstable
e	2	-3	Stable	Saddle
f	2	< -3	Center	Center-saddle
g	> 2	< -3	NA	Saddle
h	> 2	-3	Stable	Saddle
i	> 2	> -3	NA	Saddle

$$\lambda^3 + (3\nu - \alpha - 12)\lambda^2 + [18 - (3\nu - 9)(\alpha + 3) - 9\nu]\lambda - (9\nu - 18)(\alpha + 3) = 0, \quad (29)$$

which has eigen values as $3, 6 - 3\nu$ and $\alpha + 3$. The stability analysis when ν is not 2 and α is not -3 (hyperbolic cases) is again an easy application of the stability analysis of the linear system and the Hartman–Gröbman Theorem.

So we will present the Table 5 containing all the non-hyperbolic subcases and the result of their stability analysis. Later we will write the stability results for all possible subcases for C_2 .

In the Table 5, When $\nu = 2$ and $\alpha = -3$ the P is $\begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

$$\text{For } \nu = 2 \text{ and } \alpha \neq -3, P = \begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

$$\text{When } \nu \neq 2 \text{ and } \alpha = -3 \text{ then } P = \begin{pmatrix} 0 & -\frac{1}{2} & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

We end this subsection by summarizing our results for C_2 in the Table 6.

4.3 Critical point C_3

$$\text{For } C_3 \text{ the } A = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

The Jacobian Matrix at C_3 has the characteristic polynomial

$$\lambda^3 + (3\nu - 3)\lambda^2 - \frac{9}{4}\lambda + \left[\frac{27}{4} - \frac{27\nu}{4}\right] = 0 \quad (30)$$

which has the eigenvalues $-\frac{3}{2}, \frac{3}{2}$ and $3 - 3\nu$.

Hence the stability analysis is almost trivial if $\nu \neq 1$. Because then it is an immediate application of the stability analysis of the linear cases and the Hartman–Gröbman theorem.

Therefore we assume that $\nu = 1$ and proceed with our stability analysis in Table 7.

$$\text{For the lone subcase in the Table 7, } P \text{ is } \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

At the end, the summary of the results for C_3 is listed in the Table 8.

4.4 Critical point C_4

For critical point C_4 , $A = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$ and the Jacobian matrix of the system (25) at this critical point has the characteristic polynomial

$$\lambda^3 + (3\nu + \alpha - 12)\lambda^2 + [(3\nu - 6)(\alpha - 6) - 3\alpha + 9]\lambda - (3\nu - 6)(3\alpha - 9) = 0, \quad (31)$$

which has eigen values as $3, 6 - 3\nu$ and $3 - \alpha$. The stability analysis when ν is not 2 and α is not 3 (hyperbolic cases) is easy again as said before.

Now we will present the Table 9 containing all the non-hyperbolic subcases and the result of their stability analysis.

Table 7 C_3 (reduced system)

ν	Center manifold	Reduced system
1	$\bar{y} = O(\bar{x}^3), \bar{\Omega}_m = O(\bar{x}^3)$	$\bar{x}' = 0$

Table 8 The critical point C_3 , a summary

Case	ν	Stability(RS)	Stability(DS)
a	< 1	NA	Saddle
b	1	Center	Center-saddle
c	> 1	NA	Saddle

Table 9 C_4 (reduced system)

ν	α	Center manifold	Reduced System
2	3	$\bar{\Omega}_m = O(\ (\bar{x}, \bar{y})\ ^3)$	$\bar{x}' = -3\bar{x}\bar{y}^2, \bar{y}' = 3\bar{x}\bar{y}$
2	$\neq 3$	$\bar{y} = O(\bar{x}^3), \bar{\Omega}_m = O(\bar{x}^3)$	$\bar{x}' = 0$
$\neq 2$	3	$\bar{y} = O(\bar{x}^3), \bar{\Omega}_m = \frac{1}{2}\bar{x}^2 + O(\bar{x}^3)$	$\bar{x}' = -\frac{3}{2}\bar{x}^3$

Later we will write the stability results for all possible subcases for C_4 .

When $\nu = 2$ and $\alpha = 3$ the P is $\begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

For $\nu = 2$ and $\alpha \neq 3$, $P = \begin{pmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

When $\nu \neq 2$ and $\alpha = 3$ then $P = \begin{pmatrix} 0 & \frac{1}{2} & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

We end this subsection by summarizing our results for C_4 in a Table 10.

4.5 Critical point C_5

Here it is necessary that $|\alpha| \leq 3$.

$$A = \begin{pmatrix} \frac{-\alpha}{3} \\ \frac{1}{3}\sqrt{9-\alpha^2} \\ 0 \end{pmatrix}$$

for this case.

The Jacobian matrix of the system (25)–(27) at this critical point has the characteristic polynomial

$$\begin{aligned} &\lambda^3 + \left(3\nu - \frac{5}{3}\alpha^2 + 6\right)\lambda^2 \\ &+ \left[18\nu - 3\nu\alpha^2 - 7\alpha^2 + \frac{8}{9}\alpha^4 + 9\right]\lambda \\ &- \frac{(9\nu - 2\alpha^2)(2\alpha^4 - 27\alpha^2 + 81)}{27} = 0. \end{aligned} \quad (32)$$

This characteristic polynomial has eigenvalues as $\frac{2\alpha^3}{3} - 3\nu$, $\frac{\alpha^2}{3} - 3$ and $\frac{2\alpha^3}{3} - 3$. The stability analysis when $\nu \neq \frac{2\alpha^2}{9}$

and $\alpha \neq \pm 3, \pm \frac{3}{\sqrt{2}}$ (hyperbolic cases) is easy as said in the above subsections.

Now we will present the Table 11 containing all the non-hyperbolic subcases and the result of their stability analysis. Since for C_5 , the total number of all possible subcases is too many, we would not present the results for the complete case in a table at the end as we did in the previous subsections. We would rather present a diagram in $\alpha - \nu$ plane to show the stability analysis for all possible values of α 's and ν 's.

In the Table 11, “id” means “identical” and $r = \frac{\sqrt{2}(3\nu-4)}{4(2\nu-3)}$, $s = \frac{\sqrt{2}(1-\nu)}{4(2\nu-3)}$, $t = \frac{3\sqrt{9-\alpha^2}(9\alpha^2-4\alpha^4)}{4\alpha^6-72\alpha^4+405\alpha^2-729}$ and $u = -\frac{4\alpha^2\sqrt{9-\alpha^2}}{2\alpha^2-9}$. We also note that in the subcase before the last one in the Table 11, the ordered pair (ν, α) is not permitted to take the values $(2, 3)$, $(2, -3)$, $(1, \frac{3}{\sqrt{2}})$ and $(1, -\frac{3}{\sqrt{2}})$.

When $\nu = 1$ and $\alpha = \frac{3}{\sqrt{2}}$ the P is $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

For $\nu \neq 1$ and $\alpha = \frac{3}{\sqrt{2}}$, $P = \begin{pmatrix} 1 & \frac{1}{\sqrt{2}} & \frac{2-\nu}{\nu-1} \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

For $\nu = 1$ and $\alpha = -\frac{3}{\sqrt{2}}$ the P is $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

and when $\nu \neq 1$ and $\alpha = -\frac{3}{\sqrt{2}}$, $P = \begin{pmatrix} -1 & -\frac{1}{\sqrt{2}} & -\frac{2-\nu}{\nu-1} \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

Also in the final two subcases in the Table 11

$$P = \begin{pmatrix} -\frac{2\alpha(9-\alpha^2)^{3/2}}{2\alpha^4-27\alpha^2+81} & \frac{\sqrt{9-\alpha^2}}{\alpha} & \frac{3}{2\alpha} \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Table 10 C_4 , a stability analysis

Case	ν	α	Stability (RS)	Stability (DS)
a	< 2	< 3	NA	Unstable
b	< 2	3	Stable	Saddle
c	< 2	> 3	NA	Saddle
d	2	> 3	Center	Center-saddle
e	2	3	Stable	Saddle
f	2	< 3	Center	Center-unstable
g	> 2	< 3	NA	Saddle
h	> 2	3	Stable	Saddle
i	> 2	> 3	NA	Saddle

Table 11 C_5 (reduced System)

ν	α	Center manifold	Reduced system
2	3	id to subcase (e) of C_4	id to subcase (e) of C_4
$\neq 2$	3	id to subcase (b) and (h) of C_4	id to subcase (b) and (h) of C_4
2	-3	id to subcase (e) of C_2	id to subcase (e) of C_2
$\neq 2$	-3	id to subcase (b) and (h) of C_2	id to subcase (b) and (h) of C_2
1	$\frac{3}{\sqrt{2}}$	$\bar{\Omega}_m = \frac{1}{\sqrt{2}}\bar{x}^2 + O(\ (\bar{x}, \bar{y})\ ^3)$	$\bar{x}' = -3\sqrt{2}\bar{x}^2, \bar{y}' = -3\sqrt{2}\bar{x}\bar{y}$
$\neq 1$	$\frac{3}{\sqrt{2}}$	$\bar{x} = r\bar{y}^2 + O(\bar{y}^3), \bar{\Omega}_m = s\bar{y}^2 + O(\bar{y}^3)$	$\bar{y}' = -3\bar{y}^2$
1	$-\frac{3}{\sqrt{2}}$	$\bar{\Omega}_m = \frac{1}{\sqrt{2}}\bar{x}^2 + O(\ (\bar{x}, \bar{y})\ ^3)$	$\bar{x}' = 3\sqrt{2}\bar{x}^2, \bar{y}' = 3\sqrt{2}\bar{x}\bar{y}$
$\neq 1$	$-\frac{3}{\sqrt{2}}$	$\bar{x} = r\bar{y}^2 + O(\bar{y}^3), \bar{\Omega}_m = s\bar{y}^2 + O(\bar{y}^3)$	$\bar{y}' = -3\bar{y}^2$
$\frac{2\alpha^2}{9}$	α	$\bar{y} = t\bar{x}^2 + O(\bar{x}^3), \bar{\Omega}_m = O(\bar{x}^3)$	$\bar{x}' = -u\bar{x}^2 + O(\bar{x}^3)$
0	0	$\bar{y} = O(\bar{x}^3), \bar{\Omega}_m = O(\bar{x}^3)$	$\bar{x}' = O(\bar{x}^3)$

and $P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 1 \\ 0 & 1 & 0 \end{pmatrix}$

respectively.

Since we will find that the stability diagram for C_5 and C_6 is identical, it will be presented at the end of the next subsection. We now proceed with the next subsection.

4.6 Critical point C_6

Here it is also necessary that $|\alpha| \leq 3$.

$$A = \begin{pmatrix} -\frac{\alpha}{3} \\ -\frac{1}{3}\sqrt{9-\alpha^2} \\ 0 \end{pmatrix} \text{ in this case.}$$

The Jacobian matrix of the system (25)–(27) at this critical point has the characteristic polynomial

$$\lambda^3 + \left(3\nu - \frac{5}{3}\alpha^2 + 6\right)\lambda^2 + \left[18\nu - 3\nu\alpha^2 - 7\alpha^2 + \frac{8}{9}\alpha^4 + 9\right]\lambda + \frac{(9\nu - 2\alpha^2)(2\alpha^4 - 27\alpha^2 + 81)}{27} = 0. \quad (33)$$

This characteristic polynomial has eigenvalues as $\frac{2\alpha^3}{3} - 3\nu$, $\frac{\alpha^2}{3} - 3$ and $\frac{2\alpha^3}{3} - 3$. The stability analysis when $\nu \neq \frac{2\alpha^2}{9}$ and $\alpha \neq \pm 3, \pm \frac{3}{\sqrt{2}}$ (hyperbolic cases) is again easy.

Now we will present the Table 12 containing all the non-hyperbolic subcases and the result of their stability analysis. For C_6 too the total number of all possible subcases is many, so we would not present the results for the complete case in a table. We would rather present a diagram in $\alpha - \nu$ plane to show the stability analysis for all possible values of α 's and ν 's as we did for C_5 .

In the Table 12 r, s, t and u take usual values as defined in the subsection of C_5 . As like in the previous subsection here is also in the subcase before the last one in the Table 12, the ordered pair (ν, α) is not permitted to take the values $(2, 3), (2, -3), (1, \frac{3}{\sqrt{2}})$ and $(1, -\frac{3}{\sqrt{2}})$.

$$\text{When } \nu = 1 \text{ and } \alpha = \frac{3}{\sqrt{2}}, P = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

$$\text{For } \nu \neq 1 \text{ and } \alpha = \frac{3}{\sqrt{2}}, P = \begin{pmatrix} -1 & \frac{1}{\sqrt{2}} & -\frac{2-\nu}{\nu-1} \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

$$\text{For } \nu = 1 \text{ and } \alpha = -\frac{3}{\sqrt{2}}, P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Table 12 C_6 (reduced system)

ν	α	Center manifold	Reduced system
2	3	id to subcase (e) of C_4	id to subcase (e) of C_4
$\neq 2$	3	id to subcase (b) and (h) of C_4	id to subcase (b) and (h) of C_4
2	-3	id to subcase (e) of C_2	id to subcase (e) of C_2
$\neq 2$	-3	id to subcase (b) and (h) of C_2	id to subcase (b) and (h) of C_2
1	$\frac{3}{\sqrt{2}}$	$\bar{\Omega}_m = -\frac{1}{\sqrt{2}}\bar{x}^2 + O(\ (\bar{x}, \bar{y})\ ^3)$	$\bar{x}' = -3\sqrt{2}\bar{x}^2, \bar{y}' = -3\sqrt{2}\bar{x}\bar{y}$
$\neq 1$	$\frac{3}{\sqrt{2}}$	$\bar{x} = -r\bar{y}^2 + O(\bar{y}^3), \bar{\Omega}_m = -s\bar{y}^2 + O(\bar{y}^3)$	$\bar{y}' = -3\bar{y}^2$
1	$-\frac{3}{\sqrt{2}}$	$\bar{\Omega}_m = -\frac{1}{\sqrt{2}}\bar{x}^2 + O(\ (\bar{x}, \bar{y})\ ^3)$	$\bar{x}' = 3\sqrt{2}\bar{x}^2, \bar{y}' = 3\sqrt{2}\bar{x}\bar{y}$
$\neq 1$	$-\frac{3}{\sqrt{2}}$	$\bar{x} = -r\bar{y}^2 + O(\bar{y}^3), \bar{\Omega}_m = -s\bar{y}^2 + O(\bar{y}^3)$	$\bar{y}' = -3\bar{y}^2$
$\frac{2\alpha^2}{9}$	α	$\bar{y} = -t\bar{x}^2 + O(\bar{x}^3), \bar{\Omega}_m = O(\bar{x}^3)$	$\bar{x}' = u\bar{x}^2 + O(\bar{x}^3)$
0	0	$\bar{y} = O(\bar{x}^3), \bar{\Omega}_m = O(\bar{x}^3)$	$\bar{x}' = O(\bar{x}^3)$

Fig. 1 Stability diagram (C_5, C_6)

and when $\nu \neq 1$ and $\alpha = -\frac{3}{\sqrt{2}}$, $P = \begin{pmatrix} 1 - \frac{1}{\sqrt{2}} & \frac{2-\nu}{\nu-1} \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

Also in the final two subcases

$$P = \begin{pmatrix} \frac{2\alpha(9-\alpha^2)^{3/2}}{2\alpha^4-27\alpha^2+81} - \frac{\sqrt{9-\alpha^2}}{\alpha} & \frac{3}{2\alpha} \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and $P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 1 \\ 0 & 1 & 0 \end{pmatrix}$ respectively.

We finish this subsection by producing the stability diagram for both C_5 and C_6 in $\alpha - \nu$ plane in the Fig. 1.

Here the cyan shaded area is the area bounded by $\alpha = \frac{3}{\sqrt{2}}$, $\alpha = -\frac{3}{\sqrt{2}}$ and $\nu = \frac{2\alpha^2}{9}$ represents the $\alpha - \nu$ pairs for which (25)–(27) is stable. The yellow shaded region represents parameter values for which (25)–(27) is saddle. Lastly the origin is center-stable here.

4.7 Critical point C_7

In this case, it is necessary that $0 \leq \nu \leq 2$ and $\alpha \neq 0$.

A takes the form $\begin{pmatrix} -\frac{3\nu}{2\alpha} \\ \frac{3\sqrt{\nu(2-\nu)}}{2\alpha} \\ 0 \end{pmatrix}$.

The Jacobian matrix of the system (25)–(27) at C_7 has the following characteristic polynomial:

$$\lambda^3 + \left(6 - \frac{9\nu}{2}\right)\lambda^2 + \left[\frac{9(\nu-2)(9\nu^2-2\alpha^2)}{4\alpha^2}\right]\lambda - \frac{27\nu(9\nu-2\alpha^2)(\nu^2-3\nu+2)}{4\alpha^2} = 0. \quad (34)$$

It has eigenvalues as $3(\nu-1)$, $\frac{3}{4\alpha}(\nu\alpha-2\alpha+\sqrt{(2-\nu)(36\nu^2-9\nu\alpha^2+2\alpha^2)})$ and $\frac{3}{4\alpha}(\nu\alpha-2\alpha-\sqrt{(2-\nu)(36\nu^2-9\nu\alpha^2+2\alpha^2)})$. The stability analysis when $\nu \neq 1$, $\nu \neq \frac{2\alpha^2}{9}$ and $\nu \neq 2$, $\nu \neq 0$ (hyperbolic cases) are easy by application of linear stability analysis and reduction of non-linear case to linear case under some specific conditions.

So we present the Table 13 containing all the non-hyperbolic subcases. For C_7 also, we will present a diagram in $\alpha - \nu$ plane to show the stability analysis for all possible values of α 's and ν 's at the end.

In the Table 13, $\nu = \frac{4\alpha^2-36}{4\alpha}$.

For the last two subcases in the Table 13, P is $\begin{pmatrix} 0 & w & -w \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

$$\text{and } \begin{pmatrix} \frac{3}{2\alpha} & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

respectively with $w = \frac{\alpha\sqrt{36-\alpha^2}-\alpha^2+9}{2\alpha^2-9}$.

Again we will find that the stability diagram for C_7 is exactly same as C_8 . Hence it will be presented at the end of the next subsection.

Table 13 C_7 (reduced system)

ν	α	Center manifold	Reduced system
1	$\frac{3}{\sqrt{2}}$	id to subcase (e) of C_5	id to subcase (e) of C_5
1	$-\frac{3}{\sqrt{2}}$	id to subcase (g) of C_6	id to subcase (g) of C_6
$\neq 1$	$\nu = \frac{2\alpha^2}{9}, \alpha > 0$	id to subcase (i) of C_5	id to subcase (i) of C_5
$\neq 1$	$\nu = \frac{2\alpha^2}{9}, \alpha < 0$	id to subcase (i) of C_6	id to subcase (i) of C_6
1	$\nu \neq \frac{2\alpha^2}{9}$	$\bar{y} = O(\bar{x}^3), \bar{\Omega}_m = O(\bar{x}^3)$	$\bar{x}' = 0$
2	α	$\bar{x} = O(\ \bar{y}, \bar{\Omega}_m\ ^3)$	$\bar{y}' = \alpha \bar{y} \bar{\Omega}_m, \bar{\Omega}_m' = -\nu \bar{y}^2$
0	α	id to subcase (a) of C_1	id to subcase (a) of C_1

4.8 Critical point C_8

In this case also, it is necessary that $0 \leq \nu \leq 2$ and $\alpha \neq 0$ and

$$A \text{ takes the form } \begin{pmatrix} -\frac{3\nu}{2\alpha} \\ -\frac{3\sqrt{\nu(2-\nu)}}{2\alpha} \\ 0 \end{pmatrix}.$$

The Jacobian matrix of the system (25)–(27) at C_8 has the characteristic polynomial:

$$\lambda^3 + \left(6 - \frac{9\nu}{2}\right)\lambda^2 + \left[\frac{9(\nu-2)(9\nu^2-2\alpha^2)}{4\alpha^2}\right]\lambda - \frac{27\nu(9\nu-2\alpha^2)(\nu^2-3\nu+2)}{4\alpha^2} = 0. \quad (35)$$

This polynomial has eigenvalues as $3(\nu-1)$, $\frac{3}{4\alpha}(\nu\alpha - 2\alpha + \sqrt{(2-\nu)(36\nu^2-9\nu\alpha^2+2\alpha^2)})$ and $\frac{3}{4\alpha}(\nu\alpha - 2\alpha - \sqrt{(2-\nu)(36\nu^2-9\nu\alpha^2+2\alpha^2)})$. The stability analysis when $\nu \neq 1$, $\nu \neq \frac{2\alpha^2}{9}$ and $\nu \neq 2$, $\nu \neq 0$ (hyperbolic cases) are easy again.

So we present the Table 14 containing all the non-hyperbolic subcases for C_8 . For this case also, at the end we will present a diagram in $\alpha - \nu$ plane to show the stability analysis for all possible values of α 's and ν 's.

Where “ ν ” takes its value as defined for C_7 .

$$\text{For the last two subcases } P \text{ is } \begin{pmatrix} 0 & q & j \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} \frac{3}{2\alpha} & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

respectively with $j = \frac{\alpha\sqrt{36-\alpha^2}+\alpha^2-9}{2\alpha^2-9}$ and $q = \frac{-\alpha\sqrt{36-\alpha^2}+\alpha^2-9}{2\alpha^2-9}$.

This subsection concludes with the stability diagram in the $\alpha - \nu$ plane for both the C_7 and C_8 in Fig. 2.

Here $0 \leq \nu \leq 2$ and $\alpha \neq 0$. The cyan-shaded area is the area bounded by $\nu = 0$, $\nu = 1$ and $\nu = \frac{2\alpha^2}{9}$ represents stability. The yellow region represents saddle system (25)–(27). The ‘o’ marked lines are those parameter values for which the system (25)–(27) is center-stable.

Fig. 2 Stability diagram (C_7 , C_8)

4.9 Critical point C_9

In this case, it is necessary that $\alpha \neq 0$.

$$\text{For this case, } A = \begin{pmatrix} -\frac{3\nu}{2\alpha} \\ \frac{3\nu}{2\alpha} \\ \frac{2\alpha^2-9}{2\alpha^2} \end{pmatrix}.$$

The Jacobian matrix of the system (25)–(27) at C_9 has the characteristic polynomial as following:

$$\lambda^3 + \left(3\nu - \frac{3}{2}\right)\lambda^2 + \left[\frac{18\nu\alpha^2-81}{4\alpha^2}\right]\lambda - \frac{27(2\alpha^2-9)(\nu-1)}{4\alpha^2} = 0. \quad (36)$$

This polynomial has eigenvalues as $3(1-\nu)$, $\frac{-3\alpha+3\sqrt{36-7\alpha^2}}{4\alpha}$ and $\frac{-3\alpha-3\sqrt{36-7\alpha^2}}{4\alpha}$. The stability analysis when $\nu \neq 1$ and $\alpha \neq -\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}$ (hyperbolic cases) are easy.

Therefore we present the Table 15 containing all the non-hyperbolic subcases for C_9 . In the end we will present a diagram in $\alpha - \nu$ plane to show the complete stability analysis as usual.

Table 14 C_8 (reduced system)

ν	α	Center manifold	Reduced system
1	$\frac{3}{\sqrt{2}}$	id to subcase (e) of C_6	id to subcase (e) of C_6
1	$-\frac{3}{\sqrt{2}}$	id to subcase (g) of C_5	id to subcase (g) of C_5
$\neq 1$	$\nu = \frac{2\alpha^2}{9}, \alpha > 0$	id to subcase (i) of C_6	id to subcase (i) of C_6
$\neq 1$	$\nu = \frac{2\alpha^2}{9}, \alpha < 0$	id to subcase (i) of C_5	id to subcase (i) of C_5
1	$\nu \neq \frac{2\alpha^2}{9}$	$\bar{y} = O(\bar{x}^3), \bar{\Omega}_m = O(\bar{x}^3)$	$\bar{x}' = 0$
2	α	$\bar{x} = O(\ (\bar{y}, \bar{\Omega}_m)\ ^3)$	$\bar{y}' = \alpha \bar{y} \bar{\Omega}_m, \bar{\Omega}_m' = -\nu \bar{y}^2$
0	α	id to subcase (a) of C_1	id to subcase (a) of C_1

Table 15 C_9 (reduced system)

ν	α	Center manifold	Reduced system
1	$\frac{3}{\sqrt{2}}$	id to subcase (a) of C_7	id to subcase (a) of C_7
1	$-\frac{3}{\sqrt{2}}$	id to subcase (b) of C_7	id to subcase (b) of C_7
$\neq 1$	$\frac{3}{\sqrt{2}}$	id to subcase (f) of C_5	id to subcase (f) of C_5
$\neq 1$	$-\frac{3}{\sqrt{2}}$	id to subcase (h) of C_6	id to subcase (h) of C_6
1	$\alpha \neq -\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}$	$\bar{y} = O(\bar{x}^3), \bar{\Omega}_m = O(\bar{x}^3)$	$\bar{x}' = 0$

For the last subcase in the Table 15,

$$P = \begin{pmatrix} 0 & \frac{\alpha}{6} + \frac{\alpha^2(\alpha - \sqrt{36-7\alpha^2})}{12(2\alpha^2-9)} & \frac{\alpha}{6} + \frac{\alpha^2(\alpha + \sqrt{36-7\alpha^2})}{12(2\alpha^2-9)} \\ 0 & -\frac{\alpha}{6} + \frac{\alpha^2(\alpha - \sqrt{36-7\alpha^2})}{12(2\alpha^2-9)} & -\frac{\alpha}{6} + \frac{\alpha^2(\alpha + \sqrt{36-7\alpha^2})}{12(2\alpha^2-9)} \\ 1 & 1 & 1 \end{pmatrix}.$$

We find again that the stability diagram for C_9 and C_{10} is totally same. Hence it will be presented at the end of the next subsection of C_{10} .

4.10 Critical point C_{10}

This is the last subcase. In this case too it is necessary that $\alpha \neq 0$.

$$\text{Also, } A = \begin{pmatrix} -\frac{3\nu}{2\alpha} \\ \frac{3\nu}{2\alpha} \\ \frac{2\alpha^2-9}{2\alpha^2} \end{pmatrix}.$$

The Jacobian matrix of the system (25)–(27) at C_{10} has the characteristic polynomial

$$\lambda^3 + \left(3\nu - \frac{3}{2}\right)\lambda^2 + \left[\frac{18\nu\alpha^2 - 81}{4\alpha^2}\right]\lambda + \frac{27(2\alpha^2 - 9)(\nu - 1)}{4\alpha^2} = 0. \quad (37)$$

which has eigenvalues as $3(1 - \nu)$, $\frac{-3\alpha + 3\sqrt{36-7\alpha^2}}{4\alpha}$ and $\frac{-3\alpha - 3\sqrt{36-7\alpha^2}}{4\alpha}$. The stability analysis when $\nu \neq 1$ and $\alpha \neq -\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}$ (hyperbolic cases) are easy again.

Therefore we will present the Table 16 containing all the non-hyperbolic subcases for this critical point. At the end we will present the stability diagram in $\alpha - \nu$ plane.

For the last subcase in the Table 16,

$$P = \begin{pmatrix} 0 & \frac{\alpha}{6} + \frac{\alpha^2(\alpha - \sqrt{36-7\alpha^2})}{12(2\alpha^2-9)} & \frac{\alpha}{6} + \frac{\alpha^2(\alpha + \sqrt{36-7\alpha^2})}{12(2\alpha^2-9)} \\ 0 & -\frac{\alpha}{6} + \frac{\alpha^2(\alpha - \sqrt{36-7\alpha^2})}{12(2\alpha^2-9)} & -\frac{\alpha}{6} + \frac{\alpha^2(\alpha + \sqrt{36-7\alpha^2})}{12(2\alpha^2-9)} \\ 1 & 1 & 1 \end{pmatrix}.$$

Lastly, we provide the stability diagram in the $\alpha - \nu$ plane for C_9 and C_{10} in Fig. 3 and end this section.

Here it is necessary that $\alpha \neq 0$. The two squares in the top corners bounded by the lines $\nu = 1, \alpha = \frac{3}{\sqrt{2}}$ and $\nu = 1, \alpha = -\frac{3}{\sqrt{2}}$ respectively are cyan-shaded. They represent the parameter values for which the system (25)–(27) is stable. The rest of the region represents saddle system. The 'o' marked line represents center-stability.

At the end of this section, we will present the phase plane diagrams of the autonomous system (25)–(27) for various values of the parameter ν and α in Figs. 4, 5, 6 and 7. The last diagram (Fig. 8) depicts the vector fields of the autonomous system (25)–(27).

5 Cosmological interpretations and conclusion

The present work deals with a cosmological model consisting of three non-interacting fluids namely the baryonic matter in the form of perfect fluid ($p_b = (\nu - 1)\rho_b$), dark matter in the form of dust and dark energy as a scalar field respectively. This cosmological model has been studied in the framework

Table 16 C_{10} (reduced system)

ν	α	Center manifold	Reduced system
1	$\frac{3}{\sqrt{2}}$	id to subcase (a) of C_8	id to subcase (a) of C_8
1	$-\frac{3}{\sqrt{2}}$	id to subcase (b) of C_8	id to subcase (b) of C_8
$\neq 1$	$\frac{3}{\sqrt{2}}$	id to subcase (f) of C_6	id to subcase (f) of C_6
$\neq 1$	$-\frac{3}{\sqrt{2}}$	id to subcase (h) of C_5	id to subcase (h) of C_5
1	$\alpha \neq -\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}$	$\bar{y} = O(\bar{x}^3), \bar{\Omega}_m = O(\bar{x}^3)$	$\bar{x}' = 0$

Fig. 3 Stability diagram (C_9, C_{10}) **Fig. 5** C_2, C_6 **Fig. 4** C_1, C_6 **Fig. 6** C_3

of dynamical system analysis by forming the evolution equations (Einstein's field equations) into an autonomous system with suitable transformation of the variables. In Table 1, it has been shown that there are 10 equilibrium points (C_1 – C_{10}) of the autonomous system. The values of the relevant cosmological parameters at the equilibrium points have also been presented in Table 2.

The equilibrium point C_1 is completely dominated by the baryonic matter and as expected in standard cosmology, the model will be in decelerating phase if the baryonic fluid is normal (ie, non-exotic: $\nu > \frac{2}{3}$) and it will be in the accelerating era for exotic nature of the matter (ie, $\nu < \frac{2}{3}$).

Fig. 7 C_4, C_5, C_6, C_8

Fig. 8 Vector field diagram of (25)

The equilibrium points C_2 and C_4 have identical nature and both of them are not interesting from the cosmological viewpoint as there is deceleration (of unusual magnitude) when matter is completely dominated by DE. The point C_3 describes a known result of standard cosmology: In the dust era there is deceleration with the value of the deceleration parameter $\frac{1}{2}$.

The equilibrium point C_5 and C_6 correspond to cosmological era dominated by the dark energy (DE). The condition ($\alpha^2 < 3$) restricts the DE to be exotic and there is acceleration. By choosing α appropriately it is possible to match the model with the recent observations.

Both the equilibrium points C_7 and C_8 in the phase space have identical cosmological behaviors. For real points in the phase space as well as for realistic baryonic fluid, ν should be restricted as $0 < \nu < 2$. The cosmological model is dominated by both the DE and the baryonic matter. In fact, the cosmic phase described by these phase space points (ie C_7

and C_8) has dominance of baryon over DE if $\frac{2}{3} < \nu < 2$ and there is deceleration. While if $0 < \nu < \frac{2}{3}$ then DE dominates the cosmic evolution and there is acceleration.

The scaling solutions represented by the equilibrium points C_9 and C_{10} are dominated by both the dark matter (DM) and DE. But they are not of much physical interest as they always correspond to dust era of evolution.

Thus, from the dynamical system analysis, the equilibrium points describe various cosmological eras and some of them are interesting from the cosmological viewpoints with recent observed data. Therefore, one may conclude that from complicated cosmological models one may get cosmological inferences without solving the evolution equations, rather by analyzing with dynamical system approach and the present work is an example of this inference.

Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors' comment: The results of these papers are not of data analysis nature, therefore there is no data.]

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