Scalar field as a null dust

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Abstract We show that a canonical, minimally coupled scalar field which is non-self interacting and massless is equivalent to a null dust fluid (whether it is a test or a gravitating field), in a spacetime region in which its gradient is null. Under similar conditions, the gravitating and nonminimally coupled Brans-Dicke-like scalar of scalar-tensor gravity, instead, cannot be represented as a null dust unless its gradient is also a Killing vector field.

1 Introduction

Scalar fields give rise to the simplest field theory and are ubiquitous in cosmology, particle physics, and theories and models of classical and quantum gravity [1–10]. A minimally coupled scalar field ϕ with canonical kinetic energy is described by the action

$$S_{(\phi)} = \int d^4x \sqrt{-g} \left[-\frac{1}{2} \nabla^c \phi \nabla_c \phi - V(\phi) \right], \quad (1)$$

where g is the determinant of the spacetime metric g_{ab} , ∇_a denotes the covariant derivative operator of g_{ab} , and $V(\phi)$ is the scalar field potential (we use units in which the speed of light and Newton's constant are unity and we follow the notation of Ref. [1]). The scalar field stress-energy tensor is

$$\begin{aligned} T_{ab}^{(\phi)} &= -\frac{2}{\sqrt{-g}} \frac{\delta S_{(\phi)}}{\delta g^{ab}} \\ &= \nabla_a \phi \nabla_b \phi - \frac{1}{2} g_{ab} \nabla^c \phi \nabla_c \phi - V(\phi) g_{ab} \end{aligned} \quad (2)$$

and ϕ satisfies the Klein-Gordon equation obtained from the covariant conservation $\nabla^b T_{ab}^{(\phi)} = 0$,

$$\square \phi - \frac{dV}{d\phi} = 0, \quad (3)$$

where $\square \equiv g^{ab} \nabla_a \nabla_b$ is the curved space d'Alembertian. The question of whether this scalar field can be represented as an effective fluid has been posed, and answered, long ago [11–13]. If the gradient $\nabla^c \phi$ is timelike in a spacetime region, the scalar field stress-energy tensor (2) is equivalent to a perfect fluid stress-energy tensor [11–13] with fluid four-velocity

$$u_a = \frac{\nabla_a \phi}{\sqrt{-\nabla^c \phi \nabla_c \phi}}. \quad (4)$$

Spacelike scalar field gradients have also been considered [13, 14], although they are not physically very relevant. Here we focus on the remaining case, apparently not covered in the literature, in which $\nabla^c \phi$ is null on a spacetime region, $\nabla^c \phi \nabla_c \phi = 0$. A *dust* corresponds to a fluid with timelike four-velocity u^a , energy density ρ , and zero pressure described by the energy-momentum tensor $T_{ab}^{(dust)} = \rho u_a u_b$. Because there is no pressure gradient, the fluid elements of the dust follow timelike geodesics, as can be deduced from covariant conservation [1]. A *null dust* [15, 16] corresponds to the limit in which the four-velocity becomes null, and is described by

$$T_{ab}^{(nd)} = \rho k_a k_b, \quad k_c k^c = 0, \quad (5)$$

where $\rho > 0$ and the trace $T \equiv T^c_c$ vanishes. The covariant conservation of $T_{ab}^{(nd)}$ implies that k^c is also geodesic [15, 16]. The null dust is interpreted as a coherent zero rest mass field propagating at the speed of light in the null direction k^a . Naturally, a null dust can be realized by propagating electromagnetic [17] or gravitational [15, 18] waves. Here we consider the analogous problem for a massless scalar field. The null dust plays a non-negligible role in the literature on classical and quantum gravity, especially in the study of Vaidya [19, 20], *pp*-wave [15, 18, 21], Robinson-Trautman [22], and twisting [23, 24] solutions of the Einstein or Einstein-Maxwell equations, classical and quantum gravitational collapse, horizon formation, mass inflation [25–34], black hole evaporation [35–37], and canonical Lagrangians and Hamiltonians [16, 38, 39]. More recently, null dust has

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been studied in relation with the fluid-gravity correspondence and holography [39–48]. The collision of special scalar field-null dust solutions was studied long ago in [49,50] and scalar-Vaidya solutions are of interest in the AdS/CFT correspondence [39,48]. Since the null vector k^a can be rescaled by a positive function without changing its causal character, it is possible to find a representation of the stress-energy tensor (5) in which $\rho \doteq 1$ and $T_{ab}^{(nd)} \doteq k_a k_b$ (a dot over an equal sign denotes the fact that the equality is only valid in that representation). However, in general, in this representation, the null geodesics tangent to k^a are not affinely parametrized unless k^a is divergence-free [16].

A natural question arises: is a scalar field ϕ with null gradient in a region of spacetime equivalent to an effective null dust? Special solutions with this property are known [18,49,50]. As expected, the answer is affirmative but only if ϕ is massless and there is no potential $V(\phi)$. Moreover, being generated by a scalar, this effective null dust is irrotational, as expected. A second question also arises naturally. Given that scalar fields play an important role in theories of gravity alternative to Einstein's general relativity (GR) searched for in tests of gravity [51–54], the simplest alternative being Brans-Dicke theory [55] and its scalar-tensor generalizations [56–58] which contain a gravitational scalar field Φ (approximately equivalent to the inverse of the effective gravitational coupling strength), is it possible that this scalar is equivalent to an effective null dust if $\nabla^c \Phi \nabla_c \Phi = 0$? The scalar-tensor action is [55–58]

$$S_{(ST)} = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left[\Phi R - \frac{\omega(\Phi)}{\Phi} \nabla^c \Phi \nabla_c \Phi - V(\Phi) \right] + S_{(m)}, \quad (6)$$

where the function $\omega(\Phi)$ (which was a constant parameter in the original Brans-Dicke theory [55]) is the “Brans-Dicke coupling”, $V(\Phi)$ is a scalar field potential (absent in the original Brans-Dicke theory), and $S_{(m)} = \int d^4x \sqrt{-g} \mathcal{L}_{(m)}$ describes the matter sector of the theory. The scalar-tensor field equations obtained from the variation of this action take the form of effective Einstein equations,

$$R_{ab} - \frac{1}{2} g_{ab} R = \frac{8\pi}{\Phi} T_{ab}^{(m)} + \frac{\omega}{\Phi^2} \left(\nabla_a \Phi \nabla_b \Phi - \frac{1}{2} g_{ab} \nabla_c \Phi \nabla^c \Phi \right) + \frac{1}{\Phi} (\nabla_a \nabla_b \Phi - g_{ab} \square \Phi) - \frac{V}{2\Phi} g_{ab}, \quad (7)$$

$$\square \Phi = \frac{1}{2\omega + 3} \left(\frac{8\pi T^{(m)}}{\Phi} + \Phi \frac{dV}{d\Phi} - 2V - \frac{d\omega}{d\Phi} \nabla^c \Phi \nabla_c \Phi \right), \quad (8)$$

where $T^{(m)} \equiv g^{ab} T_{ab}^{(m)}$ is the trace of the matter stress-energy tensor $T_{ab}^{(m)}$. It is well known that, if the gradient $\nabla^c \Phi$ is

timelike, the Brans-Dicke-like scalar Φ acts as an effective fluid source which, however, is not a perfect fluid as in the minimally coupled case, but is instead an (effective) imperfect fluid with a heat flux density, described by an effective stress-energy tensor of the form [12,59–62]

$$T_{ab}^{(\Phi)} = (P_{(\Phi)} + \rho_{(\Phi)}) u_a u_b + P_{(\Phi)} g_{ab} + q_a^{(\Phi)} u_b + q_b^{(\Phi)} u_a. \quad (9)$$

When $\nabla^c \Phi \nabla_c \Phi < 0$, the effective fluid four-velocity is

$$u_a = \frac{\nabla_a \Phi}{\sqrt{-\nabla^c \Phi \nabla_c \Phi}}, \quad u_c u^c = -1, \quad (10)$$

and

$$q_c^{(\Phi)} q_c^{(\Phi)} > 0, \quad q_c^{(\Phi)} u^c = 0. \quad (11)$$

Explicit expressions of the effective fluid quantities ρ_Φ , P_Φ , and $q_a^{(\Phi)}$ are given in Refs. [59,62].

An important conceptual step has been taken in moving the question from the minimally coupled scalar ϕ to the Brans-Dicke-like field Φ : the former can be a test field or a matter source of the Einstein equations, while the latter always contributes to sourcing the metric in the scalar-tensor field Eq. (7). This distinction will have to be kept in mind in the following sections.

It turns out that, contrary to its minimally coupled counterpart ϕ , a Brans-Dicke-like field Φ cannot be regarded as a null dust, which is a perfect fluid. In fact, in the stress-energy $T_{ab}^{(\Phi)}$ given by Eq. (9), the terms $\nabla_a \nabla_b \Phi$, $\square \Phi$ linear in the second covariant derivatives of Φ always introduce an (effective) imperfect fluid component, *i.e.*, a heat flux. By contrast, the canonical terms $\nabla_a \Phi \nabla_b \Phi$, $\nabla^c \Phi \nabla_c \Phi$ quadratic in the first order covariant derivatives correspond to (effective) perfect fluid terms.

The effective imperfect fluid description of scalar-tensor gravity has been applied recently to elucidate anomalies in the limit to GR of electrovacuum Brans-Dicke theory [63]. However, it is not possible to do so for the corresponding scalar-tensor solutions describing null fields because a null dust description of scalar-tensor gravity is missing in this case.

2 Massless canonical scalar field as an effective null dust in GR

In this section we restrict to Einstein's theory. In GR, a canonical scalar field (whether it is a test field or a gravitating one) has stress-energy tensor (2) and satisfies the Klein-Gordon equation (3). We assume that $\nabla^c \phi \nabla_c \phi = 0$ in a certain space-time region, to which we will implicitly limit ourselves in the rest of this work. Then, the requirement that $T_{ab}^{(\phi)}$ assumes the null dust form $T_{ab} = \rho k_a k_b$ with k^c null, necessarily

implies $T^{(\phi)} = 0$ and $V(\phi) = 0$, as expected. Then, the Klein-Gordon equation reduces to $\square\phi = 0$ and (2) reduces to $T_{ab}^{(\phi)} = \nabla_a\phi\nabla_b\phi$. *A priori*, there are two possibilities to identify this stress-energy tensor with that of a null dust. One could choose the representation in which

$$T_{ab}^{(\phi)} = \nabla_a\phi\nabla_b\phi \doteq k_ak_b, \quad k_a \doteq \nabla_a\phi, \quad \rho \doteq 1. \quad (12)$$

For a general null dust, this representation excludes the affine parametrization for the null geodesics tangent to k^a . However, here these two representations become compatible because of the irrotationality of this scalar field-dust. In fact, the covariant conservation equation $\nabla^b T_{ab}^{(\phi)} = 0$ yields

$$k^b\nabla_b k_c = -(\nabla^b k_b)k_c, \quad (13)$$

but the divergence of k^c vanishes because k_c is a pure gradient. Indeed, $\nabla^a k_a = \square\phi = 0$ by virtue of the Klein-Gordon equation and Eq. (13) reduces to the affinely parametrized geodesic equation $k^b\nabla_b k_c = 0$. For a scalar field-null dust, therefore, the representation with unit energy density $\rho = 1$ coincides with the affine parametrization along the null geodesics tangent to k^a , due to irrotationality.

The second *a priori* possibility consists of keeping ρ general and choosing

$$T_{ab}^{(\phi)} = \nabla_a\phi\nabla_b\phi = \rho k_ak_b, \quad k_a = \frac{\nabla_a\phi}{\sqrt{\rho}}. \quad (14)$$

Covariant conservation then gives

$$\nabla^b T_{ab}^{(\phi)} = (k^b\nabla_b\rho + \rho\nabla^b k_b)k_a + \rho k^b\nabla_b k_a = 0. \quad (15)$$

Using the Klein-Gordon equation $\square\phi = 0$, we now have

$$\nabla^a k_a = -\frac{\nabla^a\rho\nabla_a\phi}{2\rho^{3/2}} \quad (16)$$

so that Eq. (15) becomes

$$k^b\nabla_b k_c = \frac{1}{\rho} \left(-k^b\nabla_b\rho + \frac{k^b\nabla_b\rho}{2\sqrt{\rho}} \right) k_c. \quad (17)$$

Also in this representation it could seem that affine parametrization is incompatible with the choice $\rho \equiv 1$ but, again, this is not the case. Let us adopt the representation $\rho \doteq 1$; then we have $k^c k_c = 0$,

$$k^a\nabla_b k_a = 0 \quad (18)$$

(which follows from taking the covariant derivative of the normalization $k^c k_c = 0$), and

$$\nabla^c k_c = 0. \quad (19)$$

At this point it is easy to derive also a wave equation for the null vector field k^c ,

$$\square k_a - R_{ab}k^b = 0, \quad (20)$$

in which $k^c = \nabla^c\phi$ couples explicitly to the curvature (to the Ricci tensor R_{ab}) even though ϕ does not couple (to the Ricci scalar R) in the Klein-Gordon equation. To obtain Eq. (20), first note that

$$\square k_a = \nabla^b\nabla_b k_a = \nabla^b(\nabla_b\nabla_a\phi) = \nabla^b(\nabla_a\nabla_b\phi) = \nabla^b\nabla_a k_b \quad (21)$$

and that the identity [1]

$$[\nabla_a, \nabla_b]k_c = R_{abc}{}^d k_d = R_{abcd}k^d \quad (22)$$

yields

$$\nabla_b\nabla_a k_c = \nabla_a\nabla_b k_c - R_{abcd}k^d, \quad (23)$$

so that

$$\nabla_b\nabla_a k^b = \nabla_a\nabla_b k^b - R_{ab}{}^b{}_d k^d \equiv R_{ad}k^d \quad (24)$$

and $\square k_a = \nabla_b\nabla_a k^b = R_{cd}k^d$.

Thus far, the equations of this section apply whether ϕ is a test or a gravitating field. In the case in which ϕ is the only matter source of the Einstein equations, one can go further and note that in this special situation $T^{(\phi)} = 0$ yields $R = 0$ and the Einstein equations reduce to $R_{ab} = 8\pi\nabla_a\phi\nabla_b\phi$, from which it follows that $R_{ab}k^b = 8\pi k_ak_b k^b = 0$ and

$$\square k^c = 0. \quad (25)$$

3 Scalar-tensor gravity

Let us consider now scalar-tensor gravity. In this case, the Brans-Dicke-like field Φ cannot be a test field and always gravitates. Assuming the gradient $\nabla^c\Phi$ to be null, the effective stress-energy tensor $T_{ab}^{(\Phi)}$ cannot be a null dust. In fact, assume $V(\Phi) \equiv 0$ and $T^{(m)} = 0$, then Eq. (8) gives $\square\Phi = 0$ and (9) reduces to

$$T_{ab}^{(\Phi)} = \frac{\omega}{\Phi^2} \nabla_a\Phi\nabla_b\Phi + \frac{\nabla_a\nabla_b\Phi}{\Phi}, \quad (26)$$

which satisfies $T^{(\Phi)} = 0$. One can set $k_a = \nabla_a \ln \Phi$ and

$$T_{ab}^{(\Phi)} = \omega k_ak_b + \frac{\nabla_a\nabla_b\Phi}{\Phi}. \quad (27)$$

Then k^c is null and is also divergence-free since

$$\nabla^c k_c = \frac{\square\Phi}{\Phi} - \frac{\nabla^a\Phi\nabla_a\Phi}{\Phi^2} = 0, \quad (28)$$

where the first term vanishes because of the field equation for Φ . Using the identity

$$\frac{\nabla_a\nabla_b\Phi}{\Phi} = \nabla_{(a} \left(\frac{\nabla_{b)}\Phi}{\Phi} \right) + \frac{\nabla_a\Phi\nabla_b\Phi}{\Phi^2}, \quad (29)$$

one obtains

$$T_{ab}^{(\Phi)} = (\omega + 1) k_ak_b + \frac{\nabla_ak_b + \nabla_bk_a}{2}, \quad (30)$$

which is not the stress-energy tensor of a null dust due to the second term on the right hand side. Unless k^a is a (null) Killing vector field [1] satisfying the Killing equation $\nabla_a k_b + \nabla_b k_a = 2\nabla_a \nabla_b \Phi = 0$, this second term does not vanish. It is clear that this restriction can be obeyed only by very special geometries and, therefore, we will not pursue it further.

By proceeding in analogy with Sec. 2, one derives the wave equation for k^c

$$\square k_a + R_{ab} k^b = 0 \quad (31)$$

(note the opposite sign to Eq. (20) in the coupling to the Ricci tensor). In fact, the covariant conservation $\nabla^b T_{ab}^{(\Phi)} = 0$ gives

$$(\omega + 1) \left(k_a \nabla^b k_b + k^b \nabla_b k_a \right) + \frac{1}{2} \left(\square k_a + \nabla_b \nabla_a k^b \right) = 0, \quad (32)$$

where the first bracket vanishes because k^a is divergence-free and because of the affinely parameterized geodesic equation. The use of Eq. (24) then yields Eq. (31). If, in addition, $T^{(m)} = 0$, Eq. (31) simplifies to $\square k^c = 0$.

4 Conclusions

We have filled a gap in the literature regarding the equivalence between a scalar field ϕ and an effective null dust when the gradient $\nabla^c \phi$ of this scalar is a null vector field over a region of spacetime. A canonical, minimally coupled, free and massless scalar field with null gradient is equivalent to an irrotational null dust. When attempting to generalize this property to a gravitating Brans-Dicke-like scalar field Φ in scalar-tensor gravity [55–58], we have found that the equivalence does not carry over, unless the null gradient of Φ is also a Killing vector. This is a very strong restriction, which makes this situation rather uninteresting from the physical point of view. There does not seem to be much room to apply or extend the null fluid picture. However, it is true that non-trivial topological null fluid solutions exist which can be mapped into knotted solutions of source-free electromagnetism [64]. Perhaps analogous topological configurations exist in the scalar-tensor gravity equivalent of a null dust. This issue will be explored in the future.

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