



Ultra-relativistic gravity has properties associated with the strong force

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Abstract The equations of motion, as well as the potential energy V of a self-gravitating N -body system in the first post-Minkowskian approximation have recently been derived. Here, for the particular case of two equal masses, the ultra-relativistic limit of these equations is analysed. It is shown that the requirement that the component of the gravitational force along the vector connecting the two particles is attractive, implies that the ultra-relativistic gravitational force acting on these two particles has properties usually associated with the strong force, namely, confinement and asymptotic freedom. This surprising result may have implications in particle physics: if the typical length of the system is of the order of the size of the pion, and if the mass of each particle is of the order of the mass of a light quark, then the magnitude of the above force is of the order of the magnitude of the strong force, whereas the bound state of this two equal masses body system yields a particle of mass of the order of the mass of the pion.

1 Introduction

The motivation for this work is the book [1] and a series of papers [2–6] by Vayenas and others (including the author), where the iconoclastic idea is presented that ultra-relativistic gravity is consistent with the strong force. This idea is supported via the analysis of a Bohr-type model and the employment of the following simplistic formula for the relativistic gravitational force acting between two masses m_1 and m_2 at a distance r , moving with relativistic speeds v_1 and v_2 :

$$F_V = \frac{Gm_1m_2\gamma_1^3\gamma_2^3}{r^2}, \quad \gamma_j = [1 - (v_j/c)^2]^{-1/2}, \quad (1.1)$$

where, G denotes the gravitational constant.

In particular, in [1] a neutron is modeled as a composite particle involving three equal sub-particles (quarks) of mass m , placed at the vertices of an equilateral triangle; these particles are moving on a circular orbit with constant speed v . Employing identical arguments with those used by Bohr except that the Coulomb force is now replaced with the force given by the first of Eq. (1.1), and using the known values for the mass and the radius of the neutron, one finds the numerical value of m . Surprisingly, this value is within the range of the current experimental estimate of the mass of the electron-neutrino, m_ν . By considering other composite particles formed from electrons and electron-neutrinos, a similar analysis yields good approximations for the experimental values of these composite particles. This is remarkable because after fixing the value of m_ν via the neutron model, the subsequent models do not have any free parameters. Based on these elementary computations, the authors of [1] have suggested that ultra-relativistic gravity is consistent with the strong force and that light quarks have the mass of the electron-neutrino. It should be noted that the analysis presented in [1–6] implies that the associated γ values are very large, i.e. v/c is of order 1.

Of course, the above model has various weaknesses, including the use of the anachronistic Bohr-type model instead of Dirac's equation. However, it is well-known that the Bohr model provides the semi-classical approximation of the Schrodinger equation and similar considerations are also valid for the Dirac equation. Thus, perhaps employing a Bohr-type model is not fatal. Assuming that this is indeed the case, then the most obvious weakness of the above model is its reliance on Eq. (1.1). These equations are derived in [1] by starting with Newton's gravitational law and simply replacing in this law the mass m with the so-called *gravitational mass* given by $m\gamma^3$, where γ is the associated Lorentz factor. However, in the special theory of relativity, one simply replaces in the formula of acceleration t by the proper time τ defined by $\tau = t/\gamma$; there is no need for the introduction

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of any additional notions, such as the notion of the gravitational mass. Historically, this notion arose due to the remark of Einstein that in the particular case that the components of the velocity in the y and z directions vanish, the term $m \frac{dv_x}{dt}$, where v_x denotes the x -component of the velocity, can be re-written in the form $m\gamma^3 \frac{dv_x}{dt}$. Regarding the derivation of (1.1), it should also be noted that the correct law of gravity cannot be obtained by ad hoc substitutions in the original Newton's gravitational law, but its derivation requires the analysis of the equations of the general theory of relativity.

Taking into consideration the simplistic derivation of (1.1), it is the author's opinion that the proper framework of viewing [1–6] is the following: if the force between quarks is modelled by the ad hoc Eq. (1.1), then the above Bohr-type model makes theoretical predictions consistent with experimental values. In this connection, it is useful to recall that in 1917 the well-known physicist Silberstein claimed that Einstein's theory of general relativity was not needed for the computation of the perihelion shift: he modified Newton's gravitational law by an ad hoc factor γ^n , and by choosing $n = 5$, he could match the experimental data [7]. It turns out that this result can be easily explained: a simple computation shows that in the limit of large M/m , where M and m are the masses of the sun and Mercury respectively, the equations of general relativity indeed yield the formula for the force suggested by Silberstein, namely $F = GmM\gamma^5/r^2$, where r is the distance between the sun and Mercury.

Motivated by the above discussion and the remarkable fact that the Vayenas model, which does not have any free parameters, yields theoretical values for several composite masses that are in close agreement with experimental data, the author has speculated that: in the limit of small masses and large speeds, the general theory of relativity must yield results which somehow are in agreement with results obtained via the Vayenas model. It will be shown below that this is indeed the case.

In order to investigate the equations of the general theory of relativity in the simplest possible setting, we will consider a system of two equal masses. Unfortunately, even in this case the question of finding the correct analogue of Newton's gravitational law within the context of general relativity remains open. Naturally, the form of this law has been investigated by many physicists, starting with Einstein himself: since Einstein replaced in 1915 Newton's fundamental law with a new one, he was keen to know the precise form of his more general law.

After the failure to obtain an exact formulation for the analogue of Newton's gravitational law, many physicists have concentrated in the so-called *post-Newtonian approximation*, namely the case when v/c is small, where v is a typical speed. In particular, Blanchet [8] derived a formula valid up to order $(v/c)^7$ (this formula is 4.5 pages long). This result is very useful for the study of several important phenom-

ena, including gravitational waves, but unfortunately it is not useful for our purposes, since in our case v/c is of order 1. On the other hand, in our case one can employ the so-called *post-Minkowskian approximation* defined by the requirement that r_s/r is small, where r_s defined by $r_s = 2Gm/c^2$ is the Schwarzschild radius of a typical particle of mass m , and r is a typical length. The most well-known physical application of the post-Minkowskian approximation is the case of unbounded orbits encountered in the scattering of particles moving with high velocities and a small deviation angle; in this case, the smallness of r_s/r is achieved via the large values of r . Here, we will consider a different application, namely the case of bounded orbits associated with very small masses; in this case, the smallness of r_s/r is due to the small value of r_s . For example, in the case of a meson with a typical length r of the order of 10^{-16} m, for a typical mass m , say, of the order of the mass of the electron, it follows that r_s/r is of the order of 10^{-41} , which is indeed very small!

The problem of deriving the equations of motion of N self-gravitating massive particles (without spins) in the leading post-Minkowskian approximation was recently addressed in [9]. In the particular case of $N = 2$ the validity of the equations derived in [9] was verified in the Appendix C of [9] as follows: the dependence on v_j/c , $j = 1, 2$, was expanded up to terms of order $(v_j/c)^7$, and the resulting expressions were shown to be identical with the expressions obtained in the post-Newtonian approximation in [8] when keeping terms that are only linear in m_1 and m_2 .

The results of [9] provide the starting point of the work presented here. Indeed, in Sect. 2, the simplification of the equations of motion derived in [9] in the particular case that $m_1 = m_2$ is presented. In Sect. 3 the large γ limit of these equations is computed. This computation shows that in the ultra-relativistic limit the force between two equal masses in the framework of the post-Minkowskian approximation has features associated with the strong force, namely, confinement and asymptotic freedom. In Sect. 4 comparisons between the force obtained here and the first of Eq. (1.1) with $m_1 = m_2$ are presented, and also possible implications to particle physics are discussed. Important open questions and a summary of the main results are presented in Sect. 5.

2 The leading order of the post-Minkowskian approximation for two equal masses

The authors of [9] start with Einstein's field equations in harmonic coordinates corresponding to N self-interacting particles, and they compute the first order post-Minkowskian approximation of these equations. This involves keeping terms linear in G . In more details, the retarded field is treated via the usual Lienard–Wickert procedure, see (2.5) in [9]; in order to obtain equations of motion expressed in terms

of ordinary variables at equal times, the retardations are expanded consistently with the leading post-Minkowskian approximation, see (2.8) in [9]. The solution is then inserted into the geodesic equation for the motion of the particles. The next important step is the regularisation of these equations; in this respect, the self-field of each particle is removed. In this way, the general equations of motion (2.12) in [9] in the first post-Minkowskian approximation (neglecting G^2 terms) are obtained, which are then rewritten using Newtonian like variables, see (2.13) in [9]. The validity of these equations is confirmed as follows: when $v/c \rightarrow 0$, the latter equations yield the terms of order G of the well-known results derived up to the fourth order in the post-Newtonian approximation.

Equation (2.3) below is the particular case of (2.13) of [9] for the case of two particles with equal masses. Indeed, letting $N = 2$ and $m_1 = m_2$ in these equations it follows that $\mathbf{v}_2 = -\mathbf{v}_1$.

We introduce the following notations:

$$\begin{aligned} m_1 = m_2 = m, \quad \mathbf{v}_1 = -\mathbf{v}_2 = \frac{\mathbf{v}}{2}, \quad v = |\mathbf{v}|, \\ \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2, \quad r = |\mathbf{r}_1 - \mathbf{r}_2|, \\ \hat{\mathbf{r}} = \frac{\mathbf{r}}{r}, \quad \dot{r} = \frac{dr}{dt}. \end{aligned} \tag{2.1}$$

Then, the definitions of $\gamma_1, \gamma_2, \gamma_{12}$ of [9] imply

$$\gamma_1 = \gamma_2, \quad \frac{1}{\gamma^2} = 1 - \frac{v^2}{4c^2}, \quad \frac{1}{\gamma_{12}^2} = 2 - \frac{1}{\gamma^2}. \tag{2.2}$$

The basic equations of motion yield the following single equation:

$$\frac{d}{d\tau}(\gamma\mathbf{v}) = -\frac{2Gm}{r^2y^{\frac{3}{2}}} \left[f_1(\gamma)\hat{\mathbf{r}} - \frac{\gamma^2}{2}f_2(\gamma)\lambda\frac{\mathbf{v}}{c^2} \right], \tag{2.3}$$

where τ denotes the proper time, $f_1(\gamma)$ and $f_2(\gamma)$ are defined by

$$\begin{aligned} f_1(\gamma) = 8\gamma^4 - 8\gamma^2 + 1, \quad f_2(\gamma) = 16\gamma^6 - 8\gamma^4 \\ + 6\gamma^2 - 1, \end{aligned} \tag{2.4}$$

and λ, y are defined by the equations

$$\lambda = \frac{\hat{\mathbf{r}} \cdot \mathbf{v}}{2c}, \quad y = 1 + \gamma^2\lambda^2. \tag{2.5}$$

The identity

$$\frac{d\mathbf{r}}{dt} = \frac{d}{dt}(\mathbf{r}_1 - \mathbf{r}_2) = \mathbf{v}_1 - \mathbf{v}_2 = \mathbf{v},$$

together with the equation $\mathbf{r} = r\hat{\mathbf{r}}$, imply

$$\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{dt}. \tag{2.6}$$

The equation $\hat{\mathbf{r}} \cdot \hat{\mathbf{r}} = 1$ yields $\hat{\mathbf{r}} \cdot d\hat{\mathbf{r}}/dt = 0$, thus (2.6) and the definition of λ imply

$$\lambda = \frac{\dot{r}}{2c}. \tag{2.7}$$

Using the identity

$$\frac{d}{d\tau} = \gamma\frac{d}{dt},$$

Equation (2.3) implies that the motion of two equal masses in the post-Minkowskian approximation is characterised by the equation

$$\begin{aligned} \frac{d}{dt}(\gamma\mathbf{v}) = -\frac{2Gm\gamma^{-1}}{r^2y^{\frac{3}{2}}} \left[(8\gamma^4 - 8\gamma^2 + 1)\hat{\mathbf{r}} - \frac{\gamma^2}{4}(16\gamma^6 \right. \\ \left. - 8\gamma^4 + 6\gamma^2 - 1)\frac{\dot{r}\mathbf{v}}{c} \right]. \end{aligned} \tag{2.8}$$

Recalling the definition of the force, \mathbf{f} ,

$$\mathbf{f} = m\frac{d}{dt}(\gamma\mathbf{v}),$$

we obtain

$$\begin{aligned} \mathbf{f} = -\frac{Gm^2}{r^2y^{\frac{3}{2}}} \left[(8\gamma^3 - 8\gamma + \gamma^{-1})\hat{\mathbf{r}} - \frac{1}{4}(16\gamma^7 - 8\gamma^5 \right. \\ \left. + 6\gamma^3 - \gamma)\frac{\dot{r}\mathbf{v}}{c} \right]. \end{aligned} \tag{2.9}$$

In the limit of small v , we find

$$\frac{\dot{r}}{c} \rightarrow 0, \quad \frac{v}{c} \rightarrow 0, \quad \gamma \rightarrow 1,$$

and Eq. (2.9) yields

$$\mathbf{f} \sim -\frac{Gm^2}{r^2}\hat{\mathbf{r}},$$

which is the form of the usual gravitational force between two equal masses.

In order to compute the large γ limit of \mathbf{f} it is useful to first obtain an equation for γ .

Proposition 2.1 Consider two self-interacting particles of equal mass m located at time t at \mathbf{r}_1 and \mathbf{r}_2 . In the leading order of the post-Minkowskian approximation the velocities of these two particles satisfy the simple equation

$$\mathbf{v}_2 = -\mathbf{v}_1, \quad \mathbf{v}_j = \frac{d\mathbf{r}_j}{dt}, \quad j = 1, 2. \tag{2.10}$$

Let

$$\mathbf{v} = 2\mathbf{v}_1, \quad v = |\mathbf{v}|, \quad \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2, \quad r = |\mathbf{r}|, \quad \dot{r} = \frac{dr}{dt}, \tag{2.11}$$

and

$$\gamma = \frac{1}{\sqrt{1 - v^2/4c^2}}, \quad y = 1 + \left(\frac{\dot{r}}{2c}\right)^2\gamma^2. \tag{2.12}$$

Then γ satisfies the ODE

$$\frac{d\gamma}{dr} = \frac{4r_s}{r^2 y^{3/2}} \left(\gamma^7 - \frac{3}{2} \gamma^5 + \frac{3}{8} \gamma^3 + \frac{1}{16} \gamma \right), \quad r_s = \frac{2Gm}{c^2}. \tag{2.13}$$

Furthermore, y satisfies the ODE

$$\frac{dy}{dr} + \frac{2}{r}y - \frac{2}{r}\gamma^2 = \frac{8r_s}{r^2 y^{3/2}} \left[y \left(\gamma^6 - \frac{\gamma^4}{2} + \frac{3}{8}\gamma^2 - \frac{1}{16} \right) - \left(\gamma^6 - \frac{\gamma^2}{8} \right) \right]. \tag{2.14}$$

Proof In order to derive the equation satisfied by γ , we will use the identity

$$\frac{d}{dt}(\gamma \mathbf{v} \cdot \gamma \mathbf{v}) = 2\gamma \mathbf{v} \cdot \frac{d}{dt}(\gamma \mathbf{v}). \tag{2.15}$$

Using in the lhs of the above equation the identity

$$\gamma \mathbf{v} \cdot \gamma \mathbf{v} = \gamma^2 v^2 = 4c^2(\gamma^2 - 1),$$

and replacing the term $d(\gamma \mathbf{v})/dt$ via Eqs. (2.8), (2.15) becomes

$$\frac{d\gamma^2}{dt} = \frac{16Gm\dot{r}\gamma^2}{r^2 y^{3/2} c^2} \left(\gamma^6 - \frac{3}{2}\gamma^4 + \frac{3}{8}\gamma^2 + \frac{1}{16} \right). \tag{2.16}$$

Replacing in (2.16) \dot{r} by dr/dt and then cancelling dt we find

$$\frac{d\gamma^2}{dr} = \frac{16Gm}{r^2 y^{3/2} c^2} \left(\gamma^8 - \frac{3}{2}\gamma^6 + \frac{3}{8}\gamma^4 + \frac{1}{16}\gamma^2 \right), \tag{2.17}$$

which can be rewritten in the form (2.13).

In order to derive the equation satisfied by y , we consider the following identities:

$$\frac{d}{dt}(\gamma \lambda) = \frac{1}{2c} \frac{d}{dt}(\hat{\mathbf{r}} \cdot \gamma \mathbf{v}) = \frac{1}{2c} \left(\frac{d\hat{\mathbf{r}}}{dt} \cdot \gamma \mathbf{v} + \hat{\mathbf{r}} \cdot \frac{d(\gamma \mathbf{v})}{dt} \right). \tag{2.18}$$

Equation (2.6) yields

$$v^2 = \left(\dot{r}\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{dt} \right) \cdot \mathbf{v} = \dot{r}^2 + r\frac{d\hat{\mathbf{r}}}{dt} \cdot \mathbf{v}. \tag{2.19}$$

The expressions

$$\frac{d\hat{\mathbf{r}}}{dt} \cdot \mathbf{v}, \quad \frac{d}{dt}\gamma \mathbf{v},$$

appearing in (2.18) can be obtained via Eqs. (2.19) and (2.8), respectively. Hence, Eq. (2.18) implies the following equation for y :

$$\frac{dy}{dt} = \frac{\gamma \lambda}{c} \left\{ \frac{\gamma}{r}(v^2 - \dot{r}^2) - \frac{2Gm\gamma^{-1}}{r^2 y^{3/2}} \left[(8\gamma^4 - 8\gamma^2 + 1) - \gamma^2(16\gamma^6 - 8\gamma^4 + 6\gamma^2 - 1)\frac{\dot{r}^2}{4c^2} \right] \right\}. \tag{2.20}$$

Replacing λ by $\dot{r}/2c$ and using the identities

$$\gamma^2 v^2 - \gamma^2 \dot{r}^2 = 4c^2(\gamma^2 - y), \quad \frac{\gamma^2 \dot{r}^2}{4c^2} = y - 1,$$

Equation (2.20) becomes

$$\frac{dy}{dt} = \dot{r} \left\{ \frac{2}{r}(\gamma^2 - y) + \frac{16Gm}{r^2 c^2 y^{3/2}} y \left(\gamma^6 - \frac{\gamma^4}{2} + \frac{3\gamma^2}{8} - \frac{1}{16} \right) - \frac{16Gm}{r^2 c^2 y^{3/2}} \left(\gamma^6 - \frac{\gamma^2}{8} \right) \right\}. \tag{2.21}$$

Hence, Eq. (2.14) follows. \square

Equations (2.13) and (2.14) determine the two unknown functions γ and y in term of r . Actually, it will be shown below that it is possible to express explicitly y in term of γ and r . Before deriving this expression we will first derive the expressions for the energy and for the angular momentum.

The energy E

The energy E is defined by

$$E = 2mc^2\gamma + V. \tag{2.22}$$

Hence, conservation of energy implies

$$\frac{dV}{dr} = -2mc^2 \frac{d\gamma}{dr}.$$

Using Eq. (2.13) we find

$$\frac{dV}{dr} = -\frac{16Gm^2}{r^2 y^{3/2}} \left(\gamma^7 - \frac{3}{2}\gamma^5 + \frac{3}{8}\gamma^3 + \frac{\gamma}{16} \right). \tag{2.23}$$

In order to integrate the above equation we first observe that since $d\gamma/dr = O(Gm)$, γ can be treated as ‘‘constant’’. Furthermore, we will employ the following important identity:

$$\frac{d}{dr} \frac{1}{ry^{1/2}} = -\frac{\gamma^2}{r^2 y^{3/2}} + O(Gm). \tag{2.24}$$

Indeed, using (2.14) we find

$$\frac{d}{dr} \frac{1}{ry^{1/2}} = -\frac{1}{r^2 y^{1/2}} - \frac{1}{2ry^{3/2}} \left[\frac{2}{r}(\gamma^2 - y) + O(Gm) \right],$$

and then Eq. (2.24) follows.

Treating γ as ‘‘constant’’ and using (2.24), Eq. (2.23) yields

$$V = \frac{16Gm^2}{ry^{1/2}} \left(\gamma^5 - \frac{3}{2}\gamma^3 + \frac{3}{8}\gamma + \frac{1}{16\gamma} \right). \tag{2.25}$$

If $\gamma = 1$, Eq. (2.25) becomes

$$V = -\frac{Gm^2}{r},$$

which is the form of the Newtonian gravitational potential for two equal masses.

The angular momentum J

The angular momentum J is defined in [9] by

$$J = m_1 \gamma_1 r_1 \times v_1 + m_2 \gamma_2 r_2 \times v_2 + \tilde{J}.$$

Using the equations

$$m_1 = m_2 = m, \quad \gamma_1 = \gamma_2 = \gamma, \quad v_1 = -v_2 = \frac{v}{2},$$

we find

$$J = m \frac{\gamma}{2} r \times v + \tilde{J}. \tag{2.26a}$$

Thus,

$$\frac{d\tilde{J}}{dt} = -\frac{m}{2} \frac{d}{dt} r \times \gamma v = -\frac{m}{2} r \times \frac{d}{dt} \gamma v.$$

Replacing $d\gamma v/dt$ by the rhs of (2.8) we obtain

$$\frac{d\tilde{J}}{dt} = -(r \times v) \frac{Gm^2}{c^2 r^2 y^{3/2}} \frac{\gamma}{4} (16\gamma^6 - 8\gamma^4 + 16\gamma^2 - 1).$$

This equation implies that $d(r \times \gamma v)/dt$ is of order $O(Gm)$, thus, we can integrate this equation treating γ and $(r \times v)$ as “constants”. Then, using (2.24), we find

$$\tilde{J} = (r \times v) \frac{Gm^2}{c^2 r y^{1/2}} \frac{1}{4} (16\gamma^6 - 8\gamma^4 + 16\gamma^2 - 1). \tag{2.26b}$$

Using Eq. (2.6) we find

$$|\hat{r} \times v| = \left| \hat{r} \times \left(\dot{r} \hat{r} + r \frac{d\hat{r}}{dt} \right) \right| = r \left| \hat{r} \times \frac{d\hat{r}}{dt} \right| = r \left| \frac{d\hat{r}}{dt} \right|, \tag{2.27}$$

where we have used the fact that \hat{r} is orthogonal to $d\hat{r}/dt$.

The definitions of $1/\gamma^2$ and of y , as well as the identity

$$v^2 = \dot{r}^2 + r^2 \left| \frac{d\hat{r}}{dt} \right|^2,$$

imply

$$\begin{aligned} r \left| \frac{d\hat{r}}{dt} \right| &= \sqrt{v^2 - \dot{r}^2} = \sqrt{4c^2 \left(1 - \frac{1}{\gamma^2} \right) - \frac{4c^2}{\gamma^2} (y - 1)} \\ &= 2c \sqrt{1 - \frac{y}{\gamma^2}}. \end{aligned}$$

Hence, the above equation together with Eq. (2.27) yield

$$\frac{m\gamma}{2} |r \times v| = m\gamma cr \sqrt{1 - \frac{y}{\gamma^2}}. \tag{2.28}$$

Using the fact that

$$\frac{dy}{dt} \approx \frac{2\dot{r}}{r} (\gamma^2 - y),$$

it is straightforward to verify that

$$\frac{d}{dt} \left(\gamma r \sqrt{1 - \frac{y}{\gamma^2}} \right) \approx 0.$$

Proposition 2.2 Under the assumptions and notations of Proposition 2.1, y is given by

$$y = \gamma^2 - \frac{k (\gamma^4 - \gamma^2 - \frac{1}{8})^{\frac{4}{3}}}{r^2 \gamma^2 (\gamma^2 - \frac{1}{2})^{\frac{2}{3}}}, \quad k \text{ constant}. \tag{2.29}$$

Furthermore, the force f between the two particles is given by

$$\begin{aligned} f &= -\frac{Gm^2}{r^2 y^{\frac{3}{2}}} \left[8\gamma^3 - 8\gamma + \gamma^{-1} - \frac{1}{4} (16\gamma^7 - 8\gamma^5 \right. \\ &\quad \left. + 6\gamma^3 - \gamma) \left(\frac{\dot{r}}{c} \right)^2 \right] \hat{r} \\ &\quad + \frac{Gm^2}{4r^2 y^{\frac{3}{2}}} (16\gamma^7 - 8\gamma^5 + 6\gamma^3 - \gamma) \frac{\dot{r}}{c} \frac{r}{c} \frac{d\hat{r}}{dt}, \quad \tilde{r} = \frac{r}{r}. \end{aligned} \tag{2.30}$$

Proof Let $x = \gamma^2$.

It is remarkable that the polynomial in x appearing in (2.17) can be written in terms of the two polynomials appearing in (2.21):

$$\begin{aligned} x^4 - \frac{3}{2}x^3 + \frac{3}{8}x^2 + \frac{x}{16} &= x \left(x^3 - \frac{x^2}{2} \right. \\ &\quad \left. + \frac{3}{8}x - \frac{1}{16} \right) - \left(x^3 - \frac{x}{8} \right). \end{aligned}$$

Thus, (2.17) can be rewritten in the form

$$\begin{aligned} \frac{dx}{dr} &= \frac{16Gmx}{r^2 y^{\frac{3}{2}} c^2} \left(x^3 - \frac{x^2}{2} + \frac{3}{8}x - \frac{1}{16} \right) \\ &\quad - \frac{16Gm}{r^2 y^{\frac{3}{2}} c^2} \left(x^3 - \frac{x}{8} \right). \end{aligned} \tag{2.31}$$

Subtracting Eqs. (2.14), (2.31) we find

$$\begin{aligned} \frac{d}{dr} (x - y) &= -\frac{2}{r} (x - y) \\ &\quad + \frac{16Gm(x - y)}{r^2 y^{\frac{3}{2}} c^2} \left(x^3 - \frac{x^2}{2} + \frac{3}{8}x - \frac{1}{16} \right). \end{aligned}$$

Multiplying this equation by r^2 we obtain

$$\begin{aligned} \frac{d}{dr} [r^2(x - y)] &= \frac{16Gm}{r^2 y^{\frac{3}{2}} c^2} [r^2(x - y)] \\ &\quad \times \left(x^3 - \frac{x^2}{2} + \frac{3}{8}x - \frac{1}{16} \right). \end{aligned}$$

Dividing the above Eq. (2.17) we find

$$\frac{d[r^2(x - y)]}{r^2(x - y)} = \frac{x^3 - \frac{x^2}{2} + \frac{3}{8}x - \frac{1}{16}}{x^4 - \frac{3x^3}{2} + \frac{3}{8}x^2 + \frac{x}{16}} dx. \tag{2.32}$$

Using the fact that the denominator of the rhs of (2.32) factorises,

$$x^4 - \frac{3x^3}{2} + \frac{3}{8}x^2 + \frac{x}{16} = x \left(x - \frac{1}{2} \right) \left(x^2 - x - \frac{1}{8} \right),$$

and integrating (2.32) we find

$$r^2(x - y) = \frac{k(x^2 - x - \frac{1}{8})^{\frac{4}{3}}}{x(x - \frac{1}{2})^{\frac{2}{3}}}, \quad k \text{ constant}, \quad (2.33)$$

which implies (2.29).

The force is given by Eq. (2.9). Replacing in this equation v by (2.6) we find the alternative representation for f given by (2.30). \square

3 The large γ computation

If γ is large, the bracket multiplying \hat{r} in Eq. (2.30) is approximated by

$$8\gamma^3 \left[1 - 2\gamma^4 \left(\frac{\dot{r}}{2c} \right)^2 \right].$$

Thus, the requirements of large γ and of an attractive force imply that $\dot{r}/2c$ must be small. Actually, if γ is of order $O(\frac{1}{\epsilon})$, $\epsilon \rightarrow 0$, and $(\dot{r}/2c)^2$ is of order $O(\epsilon^\nu)$, then $\nu - 4 \geq 0$. Hence, we introduce the following notations:

$$r = \tilde{r}\delta, \quad \gamma = \frac{\tilde{\gamma}}{\epsilon}, \quad \left(\frac{\dot{r}}{2c} \right)^2 = \tilde{R}\epsilon^\nu, \quad \nu \geq 4, \quad \epsilon \rightarrow 0, \quad (3.1)$$

where \tilde{r} , $\tilde{\gamma}$, \tilde{R} are ϵ -dependent dimensionless variables of order $O(1)$, as $\epsilon \rightarrow 0$.

We will first show that the ϵ -independent part of \tilde{r} is constant. In this direction we note that the rhs of (2.29) can be simplified. Indeed,

$$\begin{aligned} & \frac{(\gamma^4 - \gamma^2 - \frac{1}{8})^{\frac{4}{3}}}{(\gamma^2 - \frac{1}{2})^{\frac{2}{3}}} \\ &= \frac{(\gamma^4)^{\frac{4}{3}}}{(\gamma^2)^{\frac{2}{3}}} \left(1 - \gamma^{-2} - \frac{1}{8}\gamma^{-4} \right)^{\frac{4}{3}} \left(1 - \frac{1}{2}\gamma^{-2} \right)^{-\frac{2}{3}} \\ &= \gamma^4 \left(1 - \gamma^{-2} - \frac{1}{4}\gamma^{-4} + O(\gamma^{-6}) \right), \quad \gamma \rightarrow \infty. \end{aligned}$$

Thus,

$$y = \gamma^2 - \gamma^2 \frac{k}{r^2} \left[1 - \frac{1}{\gamma^2} - \frac{1}{4\gamma^4} + O\left(\frac{1}{\gamma^6}\right) \right], \quad \gamma \rightarrow \infty, \quad (3.2)$$

where the above bracket does not contain a term of order $O(\gamma^{-5})$.

Using in (3.2) the definition of y given by the second of equations (2.12), and dividing by γ^2 we find

$$\frac{1}{\gamma^2} + \left(\frac{\dot{r}}{2c} \right)^2 = 1 - \frac{k}{r^2} + \frac{k}{r^2} \left[\frac{1}{\gamma^2} + \frac{1}{4\gamma^4} + O\left(\frac{1}{\gamma^6}\right) \right].$$

Solving this equation for $(\dot{r}/2c)^2$ we obtain

$$\begin{aligned} \left(\frac{\dot{r}}{2c} \right)^2 &= 1 - \frac{k}{r^2} - \left(1 - \frac{k}{r^2} \right) \frac{1}{\gamma^2} \\ &+ \frac{k}{r^2} \left(\frac{1}{4\gamma^4} \right) + \frac{k}{r^2} O\left(\frac{1}{\gamma^6}\right). \end{aligned}$$

Rewriting the term k/r^2 multiplying by $1/4\gamma^4$ in the above equation as $(k/r^2 - 1) + 1$, we find

$$\begin{aligned} \left(\frac{\dot{r}}{2c} \right)^2 &= 1 - \frac{k}{r^2} - \left(1 - \frac{k}{r^2} \right) \frac{1}{\gamma^2} - \left(1 - \frac{k}{r^2} \right) \frac{1}{4\gamma^4} \\ &+ \frac{1}{4\gamma^4} + \frac{k}{r^2} O\left(\frac{1}{\gamma^6}\right). \end{aligned}$$

Solving for $1 - k/r^2$ we obtain

$$\begin{aligned} \left(1 - \frac{k}{r^2} \right) \left[1 - \frac{1}{\gamma^2} - \frac{1}{4\gamma^4} + O\left(\frac{1}{\gamma^6}\right) \right] &= \left(\frac{\dot{r}}{2c} \right)^2 \\ &- \frac{1}{4\gamma^4} + O\left(\frac{1}{\gamma^6}\right). \end{aligned}$$

Hence,

$$\begin{aligned} 1 - \frac{k}{r^2} &= \left(\frac{\dot{r}}{2c} \right)^2 \left[1 - \frac{1}{\gamma^2} - \frac{1}{4\gamma^4} + O\left(\frac{1}{\gamma^6}\right) \right]^{-1} \\ &+ \left[1 - \frac{1}{\gamma^2} - \frac{1}{4\gamma^4} + O\left(\frac{1}{\gamma^6}\right) \right]^{-1} \\ &\times \left[-\frac{1}{4\gamma^4} + O\left(\frac{1}{\gamma^6}\right) \right], \end{aligned}$$

or

$$\begin{aligned} 1 - \frac{k}{r^2} &= \left(\frac{\dot{r}}{2c} \right)^2 \left[1 + \frac{1}{\gamma^2} + \frac{5}{4\gamma^4} + O\left(\frac{1}{\gamma^6}\right) \right] \\ &- \frac{1}{4\gamma^4} + O\left(\frac{1}{\gamma^6}\right). \end{aligned} \quad (3.3)$$

Using the third of Eq. (3.1) it follows that the rhs of Eq. (3.3) is of $O(\epsilon^\nu, \epsilon^4) = O(\epsilon^4)$, thus $k/r^2 - 1 = O(\epsilon^4)$. Hence,

$$\lim_{\epsilon \rightarrow 0} r^2 = \lim_{\epsilon \rightarrow 0} k = \text{constant}.$$

Therefore, \tilde{r} can be represented in the form

$$\tilde{r} = \tilde{r}_0 + \check{r}\epsilon^\lambda, \quad \lambda > 0,$$

where \tilde{r}_0 is a constant independent of ϵ and the ϵ -dependent variable \check{r} is of $O(1)$ as $\epsilon \rightarrow 0$. Since r usually appears in the denominator, it is convenient to represent \tilde{r} in the form

$$\tilde{r} = \tilde{r}_0(1 - \rho\epsilon^\lambda)^{-1}, \quad \lambda > 0, \quad (3.4)$$

where \tilde{r}_0 is an ϵ -independent constant and ρ is an ϵ -dependent variable which is of $O(1)$ as $\epsilon \rightarrow 0$.

We will next prove that the ϵ -independent term of $\tilde{\gamma}$ is constant. For this purpose we represent γ in the form

$$\gamma = \frac{\tilde{\gamma}_0}{\epsilon}(1 + \Gamma\epsilon^\mu), \quad \mu > 0, \tag{3.5}$$

where $\tilde{\gamma}_0$ is ϵ -independent, whereas the ϵ -dependent variable Γ is of $O(1)$ as $\epsilon \rightarrow 0$.

Recalling that $\gamma = O(\epsilon^{-1})$ and that $(\dot{r}/2c)^2 = O(\epsilon^\nu)$, the second of equations (2.12) defining y , implies

$$y = 1 + \gamma^2 \left(\frac{\dot{r}}{2c}\right)^2 = 1 + O(\epsilon^{\nu-2}). \tag{3.6}$$

Hence, (2.13) becomes

$$\frac{d\gamma}{dr} = \frac{4r_s\gamma^7}{r^2} \frac{[1 + O(\epsilon^2)]}{[1 + O(\epsilon^{\nu-2})]}. \tag{3.7}$$

Using in this equation the relations

$$\begin{aligned} \gamma &= \frac{\tilde{\gamma}_0}{\epsilon}[1 + O(\epsilon^\mu)], \quad r = \tilde{r}_0(1 - \rho\epsilon^\lambda)^{-1}\delta, \\ dr &= \tilde{r}_0(1 - \rho\epsilon^\lambda)^{-2}\delta\epsilon^\lambda d\rho, \end{aligned} \tag{3.8}$$

we find that the leading order terms of (3.7) yield the equation

$$\frac{d\tilde{\gamma}_0}{d\rho_0} = \frac{4}{\tilde{r}_0} \frac{r_s\tilde{\gamma}_0^7}{\delta} \epsilon^{\lambda-6},$$

where ρ_0 denotes the ϵ -independent part of ρ . Hence,

$$\tilde{\gamma}_0 = (A - \alpha\rho_0)^{\frac{1}{6}}, \quad \alpha = \frac{24}{\tilde{r}_0} \frac{r_s}{\delta} \epsilon^{\lambda-6}, \tag{3.9}$$

where α and A are ϵ -independent constants.

Combining Eqs. (2.18) and (2.21) and recalling the definition $r_s = 2Gm/c^2$, the definition (2.22) implies

$$\frac{E}{2mc^2} = \gamma + \frac{4r_s}{ry^{1/2}} \left(\gamma^5 - \frac{3}{2}\gamma^3 + \frac{3}{8}\gamma + \frac{1}{16\gamma} \right).$$

Using in this equation the representations for $r = \tilde{r}\delta$, γ and y , given by Eqs. (3.4), (3.5) and (3.6) respectively, we find

$$\begin{aligned} \frac{E}{2mc^2} &= \frac{\tilde{\gamma}_0}{\epsilon}(1 + \Gamma\epsilon^\mu) \\ &+ \frac{4r_s\tilde{\gamma}_0^5}{\tilde{r}_0\delta\epsilon^5} (1 - \rho\epsilon^\lambda)(1 + \Gamma\epsilon^\mu) \frac{[1 + O(\epsilon^2)]}{[1 + O(\epsilon^{\nu-2})]^{1/2}}. \end{aligned}$$

Simplifying we obtain

$$\begin{aligned} \frac{E}{2mc^2} &= \frac{\tilde{\gamma}_0}{\epsilon} + \tilde{\gamma}_0\Gamma\frac{\epsilon^\mu}{\epsilon} \\ &+ \frac{\alpha\tilde{\gamma}_0^5}{6}\epsilon^{1-\lambda} \left(1 + O(\epsilon^2, \epsilon^{\nu-2}, \epsilon^\mu, \epsilon^\lambda) \right), \end{aligned} \tag{3.10}$$

where we have used the second of Eq. (3.9) to express $r_s/\tilde{r}_0\delta$ in terms of $\epsilon^{6-\lambda}$.

Equation (3.10) implies that the leading order term of $E/2mc^2$ is one of the following:

$$\frac{\tilde{\gamma}_0}{\epsilon}, \quad \lambda < 2; \quad \alpha\frac{\tilde{\gamma}_0^5}{6}\epsilon^{1-\lambda}, \quad \lambda > 2; \quad \frac{\tilde{\gamma}_0 + \alpha\tilde{\gamma}_0^5/6}{\epsilon}, \quad \lambda = 2.$$

Thus, $\tilde{\gamma}_0$ cannot be a variable.

The proposition below summarises the above results and also expresses the consequence of the basic Eqs. (2.13) and (2.29) using the new representation for r , γ and $(\dot{r}/2c)^2$.

Proposition 3.1 *Under the assumptions and notations of Proposition 2.1, the variables r , γ and $(\dot{r}/2c)^2$ can be represented in the following form:*

$$\begin{aligned} r &= \tilde{r}_0(1 - \rho\epsilon^\lambda)^{-1}\delta, \quad \lambda > 0; \quad \gamma = \frac{\tilde{\gamma}_0}{\epsilon}(1 + \Gamma\epsilon^\mu), \\ \mu > 0; \quad \left(\frac{\dot{r}}{2c}\right)^2 &= \tilde{R}\epsilon^\nu, \quad \nu \geq 4, \end{aligned} \tag{3.11}$$

where \tilde{r}_0 and $\tilde{\gamma}_0$ are ϵ -independent constants, whereas the ϵ -dependent variables ρ , Γ and \tilde{R} are of $O(1)$ as $\epsilon \rightarrow 0$. The ϵ -independent parts of the variables ρ and Γ denoted by ρ_0 and Γ_0 satisfy the relation

$$\Gamma_0 = \beta\rho_0 + B, \tag{3.12}$$

where β and B are ϵ -independent constants.

If $\nu = 4$, then $\lambda = 4$ and also the ϵ -independent part of the variable \tilde{R} denoted by \tilde{R}_0 , satisfies the relation below:

$$\lambda = \nu = 4, \quad \tilde{R}_0 = 2\rho + C, \tag{3.13}$$

where C is an ϵ -independent constant. If $\nu > 4$, then there exist the following three cases:

$$\lambda = \nu, \quad \mu > \nu - 4, \quad \tilde{R}_0 = 2\rho_0 + C, \tag{3.14a}$$

$$\nu < \lambda < \nu + 2, \quad \mu = \nu - 4, \quad \tilde{R}_0 = -\frac{\beta}{4\tilde{\gamma}_0^4}\rho_0 + C, \tag{3.14b}$$

$$\lambda = \nu, \quad \mu = \nu - 4, \quad \tilde{R}_0 = \left(2 - \frac{\beta}{4\tilde{\gamma}_0^4}\right)\rho_0 + C, \tag{3.14c}$$

where C is an ϵ -independent constant.

Furthermore, the small parameters r_s/δ and ϵ are related by

$$\epsilon^\chi = \alpha\frac{r_s}{\delta}, \quad \chi = 6 + \mu - \lambda, \tag{3.15}$$

where $\alpha = 4\tilde{\gamma}_0^6/\beta\tilde{r}_0$ is an ϵ -independent constant of order 1; for (3.13) and (3.14a) $\chi > 0$, for (3.14b) $0 < \chi < 2$ and for (3.14c) $\chi = 2$.

Proof Using in the ODE (3.7) satisfied by γ the representations for r and γ given by the first two Eq. (3.1), we find

$$\frac{d\Gamma_0}{d\rho_0} = 4\frac{\tilde{\gamma}_0^6}{\tilde{r}_0} \frac{R_s}{\delta} \epsilon^{-\chi}, \quad \chi = 6 + \mu - \lambda.$$

Hence, we find Eq. (3.12) as well as Eq. (3.15).

Equation (3.5) implies

$$\frac{1}{\gamma^4} = \frac{\epsilon^4}{\tilde{\gamma}_0^4} [1 - 4\Gamma\epsilon^\mu + O(\epsilon^{\tilde{\mu}})], \quad \tilde{\mu} > \mu, \quad \epsilon \rightarrow 0.$$

Indeed, if $\Gamma = \Gamma_0 + O(\epsilon^{\mu_1})$, then the order of the next term in the above bracket is $\epsilon^{\tilde{\mu}}$, where $\tilde{\mu} = \min(\mu_1 + \mu, 2\mu)$.

Employing in Eq. (3.3) the above equation and recalling that $(\dot{r}/2c)^2 = \tilde{R}\epsilon^{\nu}$, we find

$$1 - \frac{k}{\tilde{r}_0^2 \delta^2} (1 - \rho e^{\lambda})^2 = \tilde{R}\epsilon^{\nu} [1 + O(\epsilon^2)] - \frac{1}{4} \frac{\epsilon^4}{\tilde{\gamma}_0^4} + \frac{\Gamma \epsilon^{4+\mu}}{\tilde{\gamma}_0^4} + O(\epsilon^6, \epsilon^{4+\tilde{\mu}}), \quad \epsilon \rightarrow 0. \tag{3.16}$$

The lhs of Eq. (3.16) simplifies to

$$1 - \frac{k}{\tilde{r}_0^2 \delta^2} + \frac{2k}{\tilde{r}_0^2 \delta^2} \rho \epsilon^{\lambda} + O(\epsilon^{\lambda}).$$

Since the leading order of $1 - k/\tilde{r}_0^2 \delta^2$ vanishes, the leading order of $k/\tilde{r}_0^2 \delta^2$ is 1, thus Eq. (3.16) simplifies to the equation

$$\text{constant} + 2\rho_0 \epsilon^{\lambda} \sim \tilde{R}_0 \epsilon^4 - \frac{1}{4} \frac{\epsilon^4}{\tilde{\gamma}_0^4} + \frac{\Gamma_0 \epsilon^{4+\mu}}{\tilde{\gamma}_0^4}, \quad \epsilon \rightarrow 0. \tag{3.17}$$

If $\nu = 4$, the term ϵ^{ν} can only be matched with the term ϵ^{λ} as well as with the constant, and then (3.13) follows. If $\nu > 4$, then ϵ^{ν} can be matched with either ϵ^{λ} and/or $\epsilon^{4+\mu}$: if $\lambda = \nu$, then (3.14a) follows; if $\mu + 4 = \nu$ and $\lambda > \nu$, then the expression for \tilde{R}_0 given in (3.14b) follows; if $\nu = \lambda = \mu + 4$, then the expression for \tilde{R}_0 given in (3.14c) follows.

The first of Eq. (3.15) implies the constraint

$$\chi = 6 + \mu - \lambda > 0. \tag{3.18}$$

In the case (3.14a), since $\lambda = \nu$ and $\mu > \nu - 4$, Eq. (3.18) is satisfied. In the case (3.14b), replacing in (3.18) μ by $\nu - 4$, we find $\lambda < \nu + 2$. Also, $6 + \mu - \lambda = 2 - (\lambda - \nu)$, thus in this case

$$\nu < \lambda < \nu + 2, \quad 0 < \chi < 2.$$

In the case (3.14c), Eq. (3.18) is superceded by the equation $\mu = \nu - 4$. Also, $6 + \mu - \lambda = 6 + \nu - 4 - \nu$, thus $\chi = 2$. \square

The large γ expression of the force

For large γ , Eq. (2.30) implies that \mathbf{f} can be approximated by the expression

$$\mathbf{f} \sim -\frac{8Gm^2}{r^2 y^{3/2}} \left\{ \gamma^3 \left[1 - 2\gamma^4 \left(\frac{\dot{r}}{2c} \right)^2 \right] \hat{\mathbf{r}} - \gamma^7 \left(\frac{\dot{r}}{c} \right) \frac{r}{2c} \frac{d\hat{\mathbf{r}}}{dt} \right\}, \quad \gamma \rightarrow \infty. \tag{3.19}$$

In order to estimate \mathbf{f} , we will next derive the following equation:

$$\frac{r}{2c} \left| \frac{d\hat{\mathbf{r}}}{dt} \right| = 1 - \frac{1}{2} \frac{\epsilon^2}{\tilde{\gamma}_0^2} + O(\epsilon^{2+\mu}, \epsilon^{\nu}), \quad \epsilon \rightarrow \infty. \tag{3.20}$$

For this derivation we note that the definition of \mathbf{v} implies

$$\mathbf{v} = \frac{d}{dt} r \hat{\mathbf{r}} = \dot{r} \hat{\mathbf{r}} + r \frac{d\hat{\mathbf{r}}}{dt}. \tag{3.21}$$

Hence, using the identity $\hat{\mathbf{r}} \cdot \frac{d\hat{\mathbf{r}}}{dt} = 0$, we find

$$\frac{v^2}{4c^2} = \left(\frac{\dot{r}}{2c} \right)^2 + \frac{r^2}{4c^2} \left| \frac{d\hat{\mathbf{r}}}{dt} \right|^2.$$

Replacing in the above equation $(v/2c)^2$ by $1 - \gamma^{-2}$ and then solving for the term involving $|d\hat{\mathbf{r}}/dt|$ we find

$$\frac{r}{2c} \left| \frac{d\hat{\mathbf{r}}}{dt} \right| = \sqrt{1 - \frac{1}{\gamma^2} - \left(\frac{\dot{r}}{2c} \right)^2}. \tag{3.22}$$

Recalling that

$$\frac{1}{\gamma^2} = \frac{\epsilon^2}{\tilde{\gamma}_0^2} + O(\epsilon^{2+\mu}), \quad \epsilon \rightarrow 0; \quad \left(\frac{\dot{r}}{2c} \right)^2 = \tilde{R}\epsilon^{\nu},$$

Equation (3.22) implies Eq. (3.20).

Using Eq. (3.20) and recalling the equations

$$\gamma^4 \left(\frac{\dot{r}}{2c} \right)^2 = O(\epsilon^{\nu-4}), \quad y = 1 + O(\epsilon^{\nu-2}), \quad \nu \geq 4, \quad \epsilon \rightarrow 0,$$

Equation (3.19) yields

$$f \sim \frac{8Gm^2}{r^2} \gamma^7 \frac{\dot{r}}{c}, \quad \epsilon \rightarrow 0. \tag{3.23}$$

Thus, using $\frac{\dot{r}}{c} \sim 2\epsilon^{\frac{\nu}{2}} \sqrt{\tilde{R}_0}$, we find

$$f \sim \frac{16Gm^2 \tilde{\gamma}_0^7}{\tilde{r}_0^2 \delta^2 \epsilon^{7-\frac{\nu}{2}}} \sqrt{\tilde{R}_0}, \quad \epsilon \rightarrow 0. \tag{3.24}$$

Hence, the dependence of f on ρ is only via the term $\sqrt{\tilde{R}_0}$.

It was shown in Proposition 3.1 that in the case (3.14b) $0 < \chi < 2$ and in the case (3.14c) $\chi = 2$. It will be shown in the next section that the case of $0 < \chi \leq 2$ is not relevant to particle physics, thus we next concentrate on the remaining two cases, namely (3.13) and (3.14a). In both of these cases, if $\rho_0 > 0$, then $\sqrt{\tilde{R}_0}$ is an increasing function of ρ_0 and hence f has the *fundamental property of confinement*. Furthermore, if $C = 0$, f satisfies the *fundamental property of asymptotic freedom*. It is important to note that if f possesses the property of asymptotic freedom, then C must vanish and hence, since \tilde{R}_0 is always positive, f also satisfies confinement. Hence, asymptotic freedom implies confinement.

The ratio of f with the Newtonian gravitational force, f_N , is given by equation

$$\frac{f}{f_N} \sim 8\gamma^7 \frac{\dot{r}}{c} \sim \frac{16\tilde{\gamma}_0^7 \sqrt{\tilde{R}_0}}{\epsilon^{7-\frac{\nu}{2}}}. \tag{3.25}$$

4 Possible implications for particle physics

Equation (3.15) imply

$$\epsilon = (2\alpha)^{\frac{1}{\chi}} \left(\frac{mG}{c^2\delta} \right)^{\frac{1}{\chi}}, \quad \chi = 6 + \mu - \lambda, \quad (4.1)$$

where $\alpha = 8\tilde{\gamma}_0^6/\tilde{r}_0\beta$, is an ϵ -independent constant.

Let m_c be the mass of the composite particle formed by the two particles of equal mass m . Conservation of energy yields

$$m_c c^2 = 2mc^2\gamma + V. \quad (4.2)$$

For large γ , the expression (2.25) for V together with the relations $y \sim 1$, $2Gm = c^2r_s$ and $\gamma = \tilde{\gamma}_0/\epsilon$ yield

$$\frac{V}{2mc^2\gamma} \sim \frac{4R_s}{r} \gamma^4 \sim \frac{4\tilde{\gamma}_0^4 R_s}{\tilde{r}_0} \frac{1}{\delta \epsilon^4}. \quad (4.3)$$

Equation (3.15) states that r_s/δ is proportional to ϵ^χ , thus if $\chi = 4$ then V is of the same order of magnitude as $mc^2\gamma$. If $\chi > 4$, V is much smaller than $2mc^2\gamma$, thus V can be neglected in Eq. (4.2) and then this equation yields

$$m_c \sim 2m\gamma = 2m \frac{\tilde{\gamma}_0}{\epsilon} = 2m\tilde{\gamma}_0(2\alpha)^{-\frac{1}{\chi}} \left(\frac{c^2\delta}{mG} \right)^{\frac{1}{\chi}}.$$

Hence,

$$m_c \approx m^{1-\frac{1}{\chi}} \left(\frac{c^2\delta}{G} \right)^{\frac{1}{\chi}}, \quad (4.4)$$

where the notation $A \approx B$ means $A = CB$ where the constant C is of order 1.

If $\chi < 4$, which always occurs for cases (3.14b) and (3.14c), then V is much larger than $2mc^2\gamma$, thus the latter term can be neglected in Eq. (4.2) and then using $V \sim 16m^2G\gamma^5/r$, Eq. (4.2) yields

$$m_c \sim \frac{16m^2G\gamma^5}{c^2r} \sim \frac{16\tilde{\gamma}_0^5}{\tilde{r}_0} \frac{m^2G}{c^2\delta\epsilon^5}.$$

Hence, using (4.1) we find

$$m_c \approx m^{2-\frac{5}{\chi}} \left(\frac{c^2\delta}{G} \right)^{\frac{5}{\chi}-1}. \quad (4.5)$$

For a given composite particle, m_c is known and δ is of the order of the radius of the particle. Thus, Eqs. (4.4) and (4.5) provide relations between m (the mass of the quark) and χ . In particular, if m is known, then χ can be determined. For the case of pion we have

$$2G \approx 10^{-10} m^3/s^2 kg, \quad \delta \approx 10^{-16} m, \quad c \approx 10^8 m/s, \\ m_{pion} \approx 10^{-28} kg. \quad (4.6)$$

For $\chi = 1, 2, 3$, Eq. (4.5) yields unrealistic values for m , thus the cases (3.14b) and (3.14c) are *not* relevant to particle physics. On the other hand, Eq. (4.4) becomes

$$10^{-28} \approx m^{1-\frac{1}{\chi}} 10^{\frac{10}{\chi}}.$$

Hence, for $\chi = 5$, $m \approx 10^{37.5} \text{ kg} \approx 10^{-2} \text{ eV}/c^2$, which is of the order of the mass of the electron–neutrino. This is consistent with the fact that for $\chi = 5$, the form of Eq. (4.4) is precisely of the form obtained via the Vayenas model. Indeed, the basic equation of the latter model is

$$\gamma m \frac{v^2}{r} = \frac{Gm^2\gamma^6}{r^2}.$$

Thus, using $v \sim c$, the above equation yields $\gamma \sim (c^2r/Gm)^{\frac{1}{5}}$. Vayenas uses conservation of energy in the form $m_c c^2 = 2mc^2\gamma$, thus $m_c = 2m\gamma$. Replacing in this equation γ by the above expression and recalling that $r \sim \delta$ we find

$$m_c \approx m^{1-\frac{1}{5}} \left(\frac{c^2\delta}{G} \right)^{\frac{1}{5}}, \quad (4.7)$$

which is Eq. (4.4) with $\chi = 5$.

The larger the value of χ , the larger the value of m . For example, for $\chi = 6$ we find a value of the order 10^2 times the value of $\chi = 5$, whereas for $\chi = 4$ we find a value of the order 10^{-3} times the value of $\chi = 5$ (perhaps a mass of the order of the mass of the lightest neutrino).

Regarding the Vayenas model it is also noted that if $\nu = 4$, then the force given by (3.24) becomes

$$f \approx \frac{Gm^2}{\delta^2\epsilon^6}, \quad (4.8)$$

which is precisely of the form of the Vayenas model. In summary, in the particular cases of $\chi = 5$ and $\nu = 4$ there is a qualitative agreement between general relativity and the results obtained via Vayenas’s model.

5 Conclusions

The main result derived in this paper is that the ultra-relativistic limit of the force between two equal masses in the framework of the post-Minkowskian approximation of general relativity yields a force with the properties of confinement and asymptotic freedom. It appears, that this result is interesting on its own right, independently of possible implications to particle physics.

From the mathematical point of view, the main open question is whether the Minkowskian approximation remains valid in the large γ limit. This question can be easily addressed provided that the error term appearing in the leading expansion of the post-Minkoswian approximation is known: one simply needs to check that this error term remains small for large values of γ . Unfortunately, the computation of this term is rather complicated, and it is work in progress. The relevant methodology of how to estimate a typical error

term in the large γ limit is illustrated in the Appendix using as an example the expression for the energy.

By employing the values for the mass and the radius of pion, it is argued in Sect. 4 that the ultra-relativistic calculations presented here may be relevant to particle physics. In this connection it should be noted that our analysis makes crucial use of the ultra-relativistic approximation $\gamma \rightarrow \infty$. Thus, this analysis is not relevant for the formation of bound states between any two particles, but only for those particles which travel with speed close to c . In particular, the mechanism presented here is not applicable to the formation of bound states by heavy quarks. Furthermore, our results are valid only for the formation of a bound state of two particles of *equal* masses. In this connection it is noted that by considering different values for χ , the present analysis could be applicable to bound states formed by a neutrino and an anti-neutrino, as well as for an electron and a positron. Our analysis can be extended to the case that the two particles have different masses but the relevant formulation is harder. The question of computing the large γ limit of the case of 3 particles remains open.

Assuming that the results obtained here are indeed relevant to particle physics, then since the potential V is known (given by Eq. (2.25)), it is straightforward to derive the associated Dirac equation:

$$i\hbar\gamma^\mu\partial_\mu\Psi - (2mc + V)\Psi = 0, \quad (5.1)$$

where γ^μ , $\mu = 0, 1, 2, 3$, are the Dirac matrices, $\partial_0 = \partial_t/c$, and $\partial_j = \partial_{x_j}$, $j = 1, 2, 3$. Local phase invariance applied to the associate Lagrangian introduces the electromagnetic potential A^μ , where ∂_μ is replaced by the covariant derivative $\bar{\partial}_\mu = \partial_\mu + iqA_\mu/\hbar c$.

In order to derive (5.1) we note that the equation defining E implies $(E - V)^2/c^2 = 4m^2c^2\gamma^2$. Also, $|\mathbf{P}|^2 = m^2\gamma^2v^2 = 4m^2c^2\gamma^2 - 4m^2c^2$. Hence, $(E - V)^2/c^2 = |\mathbf{P}|^2 + 4m^2c^2$, which implies (5.1).

An interesting direction is to compute the ultra-relativistic limit of alternative formulations of general relativity aiming at addressing questions of quantum gravity. For example, such a formulation is presented in [10], where Newton's gravitational constant, G , and Einstein's cosmological constant, Λ , are allowed to depend on spacetime; this is done in two stages, first the above constants are allowed to depend on k , and then $G(k)$ and $\Lambda(k)$ are converted into scalar functions of spacetime by means of a cutoff identification $k = k(x)$ (in [10], for spherically symmetric systems this is achieved via a renormalization group calculation).

Other attempts to incorporate quantum gravity corrections in Einstein's equations involve the inclusion of extra curvature terms such as Riemann², Ricci², R^2 . However, the curvature contains at least a factor G , thus these terms will be of higher order in G ; hence they will not contribute to the leading post-Minkowskian approximation. The question

of whether these terms can also be neglected in the ultra-relativistic limit, should be investigated following the same approach suggested earlier (of estimating the relevant error terms).

Regarding the implications to particle physics of the main results presented here the main challenge is whether the analysis of (5.1) can reproduce some of the spectacular results of the standard model, such as the astonishing agreement of the effective coupling constant $\alpha_s(Q^2)$ with experimental measurements [11], as well as the computation of the hadron spectrum via lattice QCD [12]. The author does not have the expertise to address such questions, but hopefully the present paper will generate sufficient interest in the physics community that such questions will be investigated.

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Appendix A

Consistency of asymptotics for the energy E

It will be shown below that if $\mu > \frac{\lambda}{2} - 1$, then the applicability of large γ asymptotics for the expression for E can be justified.

Differentiating Eq. (2.22) and using (2.25) we find

$$\frac{1}{2mc^2} \frac{dE}{dr} = \frac{d\gamma}{dr} + 4R_s \frac{d}{dr} \left(\frac{1}{ry^{\frac{1}{2}}} \right) F(\gamma) + \frac{4R_s}{ry^{\frac{1}{2}}} \frac{dF}{dr}, \quad (A.1)$$

where F is defined by

$$F = \gamma^5 - \frac{3}{2}\gamma^3 + \frac{3}{8}\gamma + \frac{1}{16\gamma}. \quad (A.2)$$

The expression for dy/dr given in (2.14) implies the following identity:

$$\begin{aligned} & \frac{d}{dr} \left(\frac{1}{ry^{\frac{1}{2}}} \right) \\ &= -\frac{1}{r^2 y^{\frac{1}{2}}} - \frac{1}{2ry^{\frac{3}{2}}} \left[-\frac{2}{r}y + \frac{2}{r}\gamma^2 + \frac{8R_s}{r^2 y^{\frac{3}{2}}} \tilde{F}(y, \gamma) \right], \end{aligned} \tag{A.3}$$

where \tilde{F} is defined by

$$\tilde{F} = \gamma^6(y - 1) + y \left(-\frac{\gamma^4}{2} + \frac{3}{8}\gamma^2 - \frac{1}{16} \right) + \frac{\gamma^2}{8}. \tag{A.4}$$

Replacing in the rhs of (A.1) the terms $d\gamma/dr$ and $d\left(\frac{1}{ry^{1/2}}\right)/dr$ by equations (2.13) and (A.3) respectively, and noting that the γ -polynomial in the rhs of (2.13) equals $\gamma^2 F$, we find

$$\begin{aligned} \frac{1}{2mc^2} \frac{dE}{dr} &= -\frac{16R_s^2}{r^3 y^2} F(\gamma) \tilde{F}(y, \gamma) \\ &+ \frac{16R_s^2}{r^3 y^3} \left(\gamma^7 - \frac{3}{2}\gamma^5 + \frac{3}{8}\gamma^3 + \frac{1}{16}\gamma \right) \\ &\times \left(5\gamma^4 - \frac{9}{2}\gamma^2 + \frac{3}{8} - \frac{1}{16\gamma^2} \right). \end{aligned} \tag{A.5}$$

The expression $E/2mc^2$ involves γ , thus we represent E in the form \tilde{E}/ϵ . Hence,

$$\frac{dE}{dr} = \frac{(1 - \rho\epsilon^\lambda)^2}{\tilde{r}_0 \delta \epsilon^{\lambda+1}} \frac{d\tilde{E}}{d\rho}.$$

Also,

$$\begin{aligned} F(\gamma) \tilde{F}(y, \gamma) &= \frac{\tilde{\gamma}_0^5}{\epsilon^5} \left[1 + O\left(\frac{1}{\epsilon^2}\right) \right] \left[\frac{\tilde{\gamma}_0^6}{\epsilon^6} \tilde{R}\epsilon^\nu + O\left(\frac{1}{\epsilon^4}\right) \right] \\ &= O\left(\frac{1}{\epsilon^9}\right). \end{aligned}$$

Furthermore, the product of the γ -polynomials in the second term of the rhs of (A.5) is of order $O(\gamma^{11})$. Hence, equation (A.5) yields

$$\frac{1}{2mc^2} \frac{d\tilde{E}}{d\rho} = \frac{80\tilde{\gamma}_0^{11}}{\tilde{r}_0^2} \left(\frac{R_s}{\delta} \right)^2 \epsilon^{\lambda-10} \left(1 + O(\epsilon^2) \right).$$

Thus, recalling that R_s/δ is proportional to $\epsilon^{6+\mu-\lambda}$ the requirement that $d\tilde{E}/d\rho$ vanishes, imposes the constraint $\mu > \frac{\lambda}{2} - 1$.

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