Gradient flows for β functions via multi-scale renormalization group equations

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Abstract Renormalization schemes and cutoff schemes allow for the introduction of various distinct renormalization scales for distinct couplings. We consider the coupled renormalization group flow of several marginal couplings which depend on just as many renormalization scales. The usual β functions describing the flow with respect to a common global scale are assumed to be given. Within this framework one can always construct a metric and a potential in the space of couplings such that the β functions can be expressed as gradients of the potential. Moreover the potential itself can be derived explicitly from a prepotential which, in turn, determines the metric. Some examples of renormalization group flows are considered, and the metric and the potential are compared to expressions obtained elsewhere.

1 Introduction

Originally multi-scale renormalization group (RG) flows were introduced to deal with physical problems involving distinct energy scales [1]. On the other hand it is plausible to consider multi-scale RG flows motivated by purely formal arguments:

In dimensional regularization marginal couplings (i.e. dimensionless in $d = 4$) acquire a dimension $d - 4$ which requires the introduction of a scale μ , and in perturbation theory the corresponding renormalized couplings depend on $t \equiv \log(\mu^2/\mu_0^2)$ where μ_0 serves to define initial conditions for the running couplings. In the presence of several marginal couplings g_a , $a = 1 \dots n_g$, it is standard to introduce a single scale μ common to all couplings, since this allows to construct RG equations for Green functions with respect to an overall change of scale. However, a priori it is allowed and possible to introduce as many parameters μ_i or $\tau_i \equiv \log(\mu_i^2/\mu_{0i}^2)$, $i = 1 \dots n_g$. An overall change of scale

can still be defined provided all τ_i are related to an overall scale t .

In the presence of an ultraviolet (UV) cutoff Λ the renormalization group can also be used to describe the running of bare couplings with Λ keeping the renormalized couplings fixed. A UV cutoff Λ must not necessarily be universal: Consider, for example, a momentum space cutoff of propagators which decrease rapidly for $p^2 > \Lambda^2$. A priori it is possible to choose different cutoffs for different fields. Although the number of fields (counting multiplets as single fields) does not necessarily coincide with the number of marginal couplings one obtains again the possibility to introduce n_g parameters τ_i now defined as $\tau_i \equiv \log(\Lambda_i^2/\mu_0^2)$. Distinct momentum space cutoffs can also be introduced in the form of distinct form factors attached to the vertices corresponding to marginal couplings, as it happens automatically in the case of compositeness. Actually the so-called gradient flow in field space (not to be confused with the here considered gradient flow for couplings/ β functions), originally introduced for gauge fields on a lattice [2], serves also as a UV cutoff for correlation functions of composite operators and could be generalized to distinct cutoffs for distinct couplings. Finally Pauli–Villars regularization allows for several distinct cutoffs as well.

Subsequently we will use the idea of n_g scales τ_i independently from whether these refer to renormalization points μ_i or to UV cutoffs Λ_i .

Computing the radiative corrections to vertices associated to n_g marginal couplings the various couplings and scales will mix at least in higher loop order. Consequently, in general each coupling g_a will depend on each scale τ_i leading to a system of β functions

$$\beta_a^i(g) \equiv \frac{\partial g_a}{\partial \tau_i}. \quad (1.1)$$

Assuming as many couplings g_a as scales τ_i and linearly independent $\beta_a^i(g)$ this set of partial derivatives can formally be inverted to give $\frac{\partial \tau_i}{\partial g_a}(g)$.

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On the other hand it remains possible to define a universal overall scale (or a cutoff) t with respect to which the properties of a physical system change unless it is scale invariant. Varying t the couplings g_a satisfy standard (although scheme dependent) RG equations $\frac{\partial g_a}{\partial t} = \beta_a(g)$. We will assume that the scales τ_i are proportional to t such that

$$\frac{d\tau_i}{dt} \equiv \frac{\partial \tau_i}{\partial g_a} \frac{\partial g_a}{\partial t} \equiv \frac{\partial \tau_i}{\partial g_a} \beta_a(g) = C_i \quad (1.2)$$

where the constants C_i may differ from 1 for different scales τ_i . But since these drop out (cancel) in the interesting quantities below we will consider $C_i = 1$.

It is the aim of the present paper to show that the concept of different scales τ_i leads naturally to the definition of a gradient flow

$$\eta^{ab}(g)\beta_b(g) = \frac{\partial \Phi(g)}{\partial g_a}. \quad (1.3)$$

In addition we find that the potential $\Phi(g)$ is related to a prepotential P via

$$\Phi(g) = \frac{dP(g(t))}{dt} = \beta_a \frac{\partial P(g)}{\partial g_a}. \quad (1.4)$$

In principle such a prepotential can always be constructed if one solves the system of coupled RG equations for $g_a(t)$, inserts the solutions into the potential $\Phi(g(t))$, integrates with respect to t and re-expresses t in terms of $g_a(t)$. In practice these steps are hardly feasible, whereas within the present approach the prepotential is related to the metric η^{ab} (see the next section) which allows for its construction.

The possibility to express β functions in terms of a metric $\eta^{ab}(g)$ and a potential $\Phi(g)$ was observed first by Wallace and Zia [3,4] for a multi-component φ^4 theory. The consideration of Weyl consistency conditions for local couplings in a gravitational background in dimensional regularization led Osborn and Jack to explicit expressions for a metric $\eta^{ab}(g)$ and a potential $\Phi(g)$ [5–9]; the symmetry of the metric matrix is possibly spoiled, however, in higher order in perturbation theory.

A candidate η_Z^{ab} for a metric is the correlation function of two composite operators $l^{2d} \langle O^a(x) O^b(0) \rangle|_{|x|=l}$ (l denotes an UV cutoff) where the composite operators O^a , O^b are dual to the couplings g_a , g_b respectively. Such a metric was introduced by Zamolodchikov [10] in order to show the irreversibility of the RG flux in $d = 2$ dimensional field theory where the positivity of η_Z^{ab} can be shown.

It turned out to be difficult to demonstrate the irreversibility of the RG flow in $d = 4$ [5,6,11–21]. In particular there remains the possibility of limit cycles [22,23], i.e. recurrent trajectories related to non-vanishing β functions. Such field theories are nevertheless conformal but the irreversible flow concerns functions which differ from β functions [20].

Couplings g_a can be considered as sources for composite operators O^a , at least if promoted to local quantities $g_a(x)$. Then a functional $G(g_a)$ can be defined such that derivatives of $G(g_a)$ with respect to g_a generate correlation functions of operators O^a [24]. This allows to relate the Zamolodchikov metric $\eta_Z^{ab} \sim \langle O^a O^b \rangle$ to the second derivative of G , $\eta_Z^{ab} \sim \frac{\partial^2 G}{\partial g_a \partial g_b}$. We are not very precise here since, within the present framework of multiple scales, we find a somewhat different expression for the metric η^{ab} in (1.3).

The starting point of our approach is purely algebraic and could find applications for RG flows beyond quantum field theory. We will compare, however, our results for gradient flows in some simple field theory models to those obtained elsewhere.

2 Gradient flow from multiple scales

As stated in the Introduction we consider n_g marginal couplings g_a depending on n_g scales τ_i . We assume that the matrix of partial derivatives $\frac{\partial g_a}{\partial \tau_i}(g)$ can be inverted such that $\frac{\partial \tau_i}{\partial g_a}(g)$ exists, and that Eq. (1.2) holds. (In case all τ_i are replaced by a single scale t as it might be suggested by (1.2) with $C_i = 1$, $\frac{dg_a}{dt}(g)$ could not be inverted.)

We consider a prepotential $P(\tau(g))$ (omitting indices of g_a and τ_i if these appear as arguments of functions); its total derivative with respect to an overall scale t will be identified with the potential $\Phi(\tau(g))$:

$$\begin{aligned} \Phi(\tau(g)) &= \frac{dP(\tau(g))}{dt} \\ &= \frac{\partial P(\tau(g))}{\partial g_a} \beta_a = \frac{\partial P(\tau(g))}{\partial \tau_i} \frac{\partial \tau_i}{\partial g_a} \beta_a \end{aligned} \quad (2.1)$$

with

$$\beta_a = \frac{dg_a}{dt} \quad (2.2)$$

assumed to be known. Next we consider the derivative of (2.1) with respect to g_a :

$$\begin{aligned} \frac{\partial}{\partial g_a} \Phi(\tau(g)) &= \left(\frac{\partial}{\partial g_a} \frac{\partial P(\tau(g))}{\partial \tau_i} \right) \frac{\partial \tau_i}{\partial g_b} \beta_b \\ &\quad + \frac{\partial P(\tau(g))}{\partial \tau_i} \frac{\partial}{\partial g_a} \left(\frac{\partial \tau_i}{\partial g_b} \beta_b \right). \end{aligned} \quad (2.3)$$

Due to (1.2) the second term on the right hand side of (2.3) vanishes. The first term on the right hand side of (2.3) can be rewritten as

$$\frac{\partial^2 P(\tau(g))}{\partial \tau_j \partial \tau_i} \frac{\partial \tau_j}{\partial g_a} \frac{\partial \tau_i}{\partial g_b} \beta_b \equiv \eta^{ab} \beta_b, \quad (2.4)$$

hence (2.3) assumes the form of a gradient flow,

$$\frac{\partial}{\partial g_a} \Phi(\tau(g)) = \eta^{ab} \beta_b \quad (2.5)$$

with

$$\eta^{ab} = \frac{\partial^2 P(\tau(g))}{\partial \tau_j \partial \tau_i} \frac{\partial \tau_j}{\partial g_a} \frac{\partial \tau_i}{\partial g_b}. \quad (2.6)$$

The metric (2.6) is manifestly symmetric and covariant under redefinitions $g \rightarrow g'(g)$. Note that η^{ab} differs from $\frac{\partial^2 P}{\partial g_a \partial g_b}$; the difference are terms of the form $\frac{\partial P}{\partial \tau_i} \frac{\partial^2 \tau_i}{\partial g_a \partial g_b}$. From (2.6) positivity of the metric depends now on the positivity of $\frac{\partial^2 P}{\partial \tau_j \partial \tau_i}$ and properties of $\frac{\partial \tau_i}{\partial g_a}$ on which we cannot make general statements.

Independently from the positivity of η^{ab} the above arguments allow to formulate a potential flow for a general system of β functions. We obtain no constraints on terms in the β functions in the form of Weyl consistency conditions as in dimensional regularization [5–9, 20]. The explicit construction of the above gradient flow from a given set β functions with respect to an overall scale t requires, however, to consider some subtleties.

Given a set of n_g β functions β_a the first task is to find n_g independent solutions of (1.2) for $\tau_i(g)$,

$$\frac{\partial \tau_i(g)}{\partial g_a} \beta_a(g) = C_i, \quad (2.7)$$

for nonzero constants C_i which may all be taken as 1 since a constant rescaling of τ_i cancels in η^{ab} . If the system is not degenerate there exist n_g independent solutions for $\tau_i(g)$ which involve arbitrary functions of $n_g - 1$ expressions $\varphi_k(g)$; $\varphi_k(g)$ are independent solutions of the set of corresponding homogeneous ($C_i = 0$) equations (2.7).

In cases where the lowest order terms of β_a are of the form $\beta_a = b_a g_a^n + \dots$ (with n an integer $\neq 1$, no sum over a) it is natural to take $\tau_i(g) = -\delta_i^a \frac{1}{b_a(n-1)} g_a^{1-n} + \dots$ such that $\tau_i(g) = t$ to lowest order, and to construct the higher order terms subsequently. (If the β functions are known to a given order in perturbation theory it can be useful to supplement them with formally higher order terms in g to find analytic expressions for $\frac{\partial \tau_i}{\partial g_a}$ satisfying (2.7). Explicit expressions for $\tau_i(g)$ which require to integrate $\frac{\partial \tau_i(g)}{\partial g_a}$ are actually never required.) In other cases of β_a one has some freedom in the construction of $\frac{\partial \tau_i}{\partial g_a}$, but such redefinitions in the space of τ_i drop out in the final quantities which depend on g_a only.

With $\frac{\partial \tau_i}{\partial g_a}(g)$ and its inverse $\frac{\partial g_a}{\partial \tau_i}(g)$ at hand one can proceed with the construction of a metric η^{ab} . η^{ab} has to satisfy integrability conditions which can be derived as follows. Consider the following derivatives of the prepotential $P(\tau(g))$:

$$\frac{\partial}{\partial g_a} \frac{\partial P(\tau(g))}{\partial \tau_i} = \frac{\partial^2 P(\tau(g))}{\partial \tau_i \partial \tau_j} \frac{\partial \tau_j}{\partial g_a} = \eta^{ab} \frac{\partial g_b}{\partial \tau_i} \quad (2.8)$$

which imply the integrability conditions

$$\frac{\partial}{\partial g_c} \left(\eta^{ab} \frac{\partial g_b}{\partial \tau_i} \right) = \frac{\partial}{\partial g_a} \left(\eta^{cb} \frac{\partial g_b}{\partial \tau_i} \right). \quad (2.9)$$

In order to solve (2.9) it can be helpful to expand the derivatives such that (2.9) becomes

$$\frac{\partial \eta^{ab}}{\partial g_c} \frac{\partial g_b}{\partial \tau_i} + \eta^{ab} \frac{\partial}{\partial g_c} \frac{\partial g_b}{\partial \tau_i} = \frac{\partial \eta^{cb}}{\partial g_a} \frac{\partial g_b}{\partial \tau_i} + \eta^{cb} \frac{\partial}{\partial g_a} \frac{\partial g_b}{\partial \tau_i}. \quad (2.10)$$

Contracting (2.10) with $\frac{\partial \tau_i}{\partial g_d}$ leads to

$$\frac{\partial \eta^{ad}}{\partial g_c} - \frac{\partial \eta^{cd}}{\partial g_a} = \eta^{cb} L^ad_b - \eta^{ab} L^cd_b \quad (2.11)$$

with

$$L^ad_b = \frac{\partial \tau_i}{\partial g_d} \frac{\partial}{\partial g_a} \frac{\partial g_b}{\partial \tau_i} = -\frac{\partial g_b}{\partial \tau_i} \frac{\partial^2 \tau_i}{\partial g_a \partial g_d}. \quad (2.12)$$

In the last step we have used

$$0 = \frac{\partial}{\partial g_a} \delta^d_b = \frac{\partial}{\partial g_a} \left(\frac{\partial g_b}{\partial \tau_i} \frac{\partial \tau_i}{\partial g_d} \right) = L^ad_b + \frac{\partial g_b}{\partial \tau_i} \frac{\partial^2 \tau_i}{\partial g_a \partial g_d}. \quad (2.13)$$

Given $\frac{\partial \tau_i}{\partial g_a}(g)$ and its inverse $\frac{\partial g_a}{\partial \tau_i}(g)$ it is straightforward to compute L^ad_b from the last term in (2.12).

Note that there are more integrability conditions (2.11) than those which follow from (2.5) alone and read

$$\frac{\partial}{\partial g_c} \left(\eta^{ab} \beta_b \right) = \frac{\partial}{\partial g_a} \left(\eta^{cb} \beta_b \right). \quad (2.14)$$

However not all (symmetric) solutions η^{ab} of (2.14) guarantee that η^{ab} is covariant under redefinitions $g \rightarrow g'(g)$. On the other hand this is guaranteed by solutions η^{ab} of (2.11); it suffices to contract the last two terms in (2.8) with $\frac{\partial \tau_i}{\partial g_d}$. Once a metric satisfying (2.11) has been obtained a potential $\Phi(g)$ can be found by integration of (2.5), and a prepotential can be found by integration of (2.8).

Again the solutions of the system of partial differential equations (2.11) are not unique. In the considered cases we found no obstruction for diagonal metrics $\eta^{ab} \sim \delta^{ab} f_a(g)$, but such ansätze do not always lead to the simplest expressions for the diagonal elements $f_a(g)$ of η^{ab} . These ambiguities are not related to redefinitions in the space of couplings since redefinitions would also affect the β functions; these have been taken as fixed inputs, however. In the next Section we consider some examples.

3 Examples

First we consider a system of 3 two-loop β functions for gauge couplings where fermion loops generate mixings at the two-loop level as in the Standard Model. We maintain the notation g_1, g_2, g_3 of the previous sections where g_a are related to the usual gauge couplings α by $g_a = \frac{\alpha_a}{4\pi}$. The β

functions are written as

$$\begin{aligned}\beta_1 &= b_{10}g_1^2 + b_{11}g_1^3 + b_{12}g_1^2g_2 + b_{13}g_1^2g_3, \\ \beta_2 &= b_{20}g_2^2 + b_{21}g_2^2g_1 + b_{22}g_2^3 + b_{23}g_2^2g_3, \\ \beta_3 &= b_{30}g_3^2 + b_{31}g_3^2g_1 + b_{32}g_3^2g_2 + b_{33}g_3^3.\end{aligned}\quad (3.1)$$

In the Standard Model we have [25]

$$\begin{aligned}b_{10} &= \frac{41}{6}, \quad b_{11} = \frac{199}{18}, \quad b_{12} = \frac{9}{2}, \quad b_{13} = \frac{44}{3}, \\ b_{20} &= -\frac{19}{6}, \quad b_{21} = \frac{3}{4}, \quad b_{22} = \frac{35}{4}, \quad b_{23} = 12, \\ b_{30} &= -7, \quad b_{31} = \frac{11}{6}, \quad b_{32} = \frac{9}{2}, \quad b_{33} = -26.\end{aligned}\quad (3.2)$$

It is fairly easy to find $\tau_i(g)$ which satisfy (2.7) to the considered order with $C_i = 1$ and $\tau_i = t$ to lowest order:

$$\begin{aligned}\tau_1 &= -\frac{1}{b_{10}g_1} \\ &\quad -\frac{1}{b_{10}}\left(\frac{b_{11}}{b_{10}}\log g_1 + \frac{b_{12}}{b_{20}}\log g_2 + \frac{b_{13}}{b_{30}}\log g_3\right), \\ \tau_2 &= -\frac{1}{b_{20}g_2} \\ &\quad -\frac{1}{b_{20}}\left(\frac{b_{21}}{b_{10}}\log g_1 + \frac{b_{22}}{b_{20}}\log g_2 + \frac{b_{23}}{b_{30}}\log g_3\right), \\ \tau_3 &= -\frac{1}{b_{30}g_3} \\ &\quad -\frac{1}{b_{30}}\left(\frac{b_{31}}{b_{10}}\log g_1 + \frac{b_{32}}{b_{20}}\log g_2 + \frac{b_{33}}{b_{30}}\log g_3\right).\end{aligned}\quad (3.3)$$

The quantities $\frac{\partial \tau_i}{\partial g_a}(g)$ and $\frac{\partial g_a}{\partial \tau_i}(g)$ can now be obtained straightforwardly. The integrability conditions (2.11) admit solutions corresponding to an expansion of the metric η^{ab} around the unit matrix:

$$\begin{aligned}\eta^{11} &= 1 + \frac{b_{21}g_2^3 + b_{31}g_3^3}{3b_{10}g_1^2}, \\ \eta^{22} &= 1 + \frac{b_{12}g_1^3 + b_{32}g_3^3}{3b_{20}g_2^2}, \\ \eta^{33} &= 1 + \frac{b_{13}g_1^3 + b_{23}g_2^3}{3b_{30}g_3^2}.\end{aligned}\quad (3.4)$$

With this metric one finds a potential $\Phi(g)$ of the form

$$\begin{aligned}\Phi(g) &= \frac{1}{3}\left(g_1^3\left(b_{10} + \frac{3}{4}b_{11}g_1 + b_{12}g_2 + b_{13}g_3\right) \right. \\ &\quad \left. + g_2^3\left(b_{20} + \frac{3}{4}b_{22}g_2 + b_{21}g_1 + b_{23}g_3\right) \right. \\ &\quad \left. + g_3^3\left(b_{30} + \frac{3}{4}b_{33}g_3 + b_{31}g_1 + b_{32}g_2\right)\right).\end{aligned}\quad (3.5)$$

By construction $\Phi(g)$ can be derived from a prepotential $P(g)$ as in (2.1), $\Phi(g) = \frac{\partial P(g)}{\partial g_a}\beta_a$, with

$$\begin{aligned}P(g) &= \frac{1}{6}(g_1^2 + g_2^2 + g_3^2) \\ &\quad - \frac{1}{36}\left(\frac{b_{11}g_1^3}{b_{10}} + \frac{b_{22}g_2^3}{b_{20}} + \frac{b_{33}g_3^3}{b_{30}}\right).\end{aligned}\quad (3.6)$$

It is remarkable that the prepotential $P(g)$ does not depend on the mixing terms in the β functions.

The metric (3.4) and the potential (3.5) differ from the ones for the same system of β functions in [22] where the potential consists in quartic terms in g_a only (to two-loop order). They differ also from the metric η_{JO} obtained by Jack and Osborn from Weyl consistency conditions [6]. In the space of gauge couplings their metric η_{JO} is also diagonal, but of the form $\eta_{JO}^{aa} \sim \frac{N_a}{g_a^2}$ with constants N_a to two-loop order. As a consequence consistency conditions among the two-loop terms of the β functions (in dimensional regularisation and minimal subtraction) can be derived, see also [26]. We found, however, that an expansion of η^{ab} around η_{JO}^{aa} cannot satisfy the integrability conditions (2.11). (We recall that the metric η_{JO}^{ab} is not guaranteed to be symmetric to higher loop order.) Here, on the other hand, we obtain the potential from a simple prepotential.

The other example is more involved already to one-loop order. It concerns a scalar with quartic self interaction and a Yukawa coupling to a Fermion, like the Higgs-top sector of the Standard Model with a quartic Higgs coupling $\lambda|H|^4$ and a top quark Yukawa coupling h_t . Our notation is

$$g_1 = \frac{h_t^2}{16\pi^2}, \quad g_2 = \frac{\lambda}{16\pi^2}.\quad (3.7)$$

The general one-loop β functions are

$$\beta_1 = a_1g_1^2, \quad \beta_2 = b_1g_2^2 + b_2g_1g_2 + b_3g_1^2\quad (3.8)$$

where in the Standard Model

$$a_1 = \frac{9}{4}, \quad b_1 = 12, \quad b_2 = 6, \quad b_3 = -3.\quad (3.9)$$

The general solution of Eq. (2.7) (again with $C_i = 1$) for $\tau_i(g)$ is of the form

$$\tau_i = -\frac{1}{a_1g_1} + F_i(X)\quad (3.10)$$

where $F_i(X)$ is an arbitrary function of

$$\begin{aligned}X &= \frac{a_1}{w}\log\left(\frac{w-\alpha}{w+\alpha}\right) - \log g_1 \quad \text{where} \\ w &= \sqrt{(b_2 - a_1)^2 - 4b_1b_3}, \quad \alpha = 2b_1\frac{g_2}{g_1} + b_2 - a_1.\end{aligned}\quad (3.11)$$

(The argument of the root w is positive for $b_3 < 0$, $b_1 > 0$ as in the Standard Model.)

We have studied various ansätze for $F_i(X)$ without observing substantial differences in the final results (since related by redefinitions of τ_i); subsequently we consider the simplest possibility

$$\tau_1 = -\frac{1}{a_1 g_1}, \quad \tau_2 = -\frac{1}{a_1 g_1} + X. \quad (3.12)$$

Among the solutions of the integrability conditions (2.11) for the metric η^{ab} we discuss the one which allow for expansions of the potential $\Phi(g)$ and the prepotential $P(g)$ in powers of couplings (without logarithms or dilogarithms). This metric is off-diagonal and, using β_2 from (3.8), can be written as

$$\begin{aligned} \eta^{11} &= \frac{1}{3a_1 g_1^5} \beta_2^3 - \frac{g_2}{2g_1^4} \beta_2^2 \\ &\quad + \left(\frac{3}{10} b_1^2 g_2^4 - \frac{1}{6} (b_2^2 + 2b_1 b_3) g_1^2 g_2^2 + \frac{3}{2} b_3^2 g_1^4 \right) \\ &\quad \frac{g_2}{g_1^4} + \frac{b_2 a_1 g_2^2}{3g_1^2} - \frac{b_3^3 g_1}{3a_1}, \\ \eta^{22} &= \frac{1}{g_1^2} (\beta_2 - a_1 g_1 g_2) (2b_1 g_2 + (b_2 - a_1) g_1), \\ \eta^{12} &= \frac{1}{g_1^3} (\beta_2 - a_1 g_1 g_2) (b_3 g_1^2 - b_1 g_2^2). \end{aligned} \quad (3.13)$$

The corresponding potential $\Phi(g)$ is

$$\begin{aligned} \Phi(g) &= \frac{\beta_2^3}{3g_1^2} - \frac{b_3^3 g_1^4}{12} \\ &\quad - \frac{a_1 g_2^2}{g_1} \left(\frac{4}{5} b_1^2 g_2^3 + \frac{3}{2} b_1 b_2 g_1 g_2^2 \right. \\ &\quad \left. + \frac{2}{3} g_2 g_1^2 (b_2^2 + 2b_1 b_3) + b_2 b_3 g_1^3 \right) \\ &\quad + a_1^2 g_2^3 \left(\frac{1}{2} b_1 g_2 + \frac{1}{3} b_2 g_1 \right). \end{aligned} \quad (3.14)$$

It can be derived as in (2.1) from the prepotential

$$\begin{aligned} P(g) &= \frac{1}{9} (2b_1 b_3 + b_2^2 - 2b_2 a_1 + a_1^2) g_2^3 \\ &\quad + \frac{1}{3} b_3 (b_2 - a_1) g_1 g_2^2 + \frac{1}{3} b_3^2 g_1^2 g_2 \\ &\quad - \frac{b_3^3 g_1^3}{36a_1} + \frac{b_1 (b_2 - a_1) g_2^4}{6g_1} + \frac{b_1^2 g_2^5}{15g_1^2}. \end{aligned} \quad (3.15)$$

Note that the matching of the various coefficients in $\Phi(g) = \beta_1 \frac{\partial P(g)}{\partial g_1} + \beta_2 \frac{\partial P(g)}{\partial g_2}$ is highly nontrivial, and that the expression for $P(g)$ is actually somewhat simpler than the one for $\Phi(g)$. But both expressions for the metric and the potential differ considerably from the ones in [6] and [22].

4 Conclusions

Using the formalism of multi-scale RG equations we have shown how a potential flow for a set of n_g couplings and corresponding β functions can always be constructed. Irreversibility of the RG flow depends on the positivity of the metric. Even within the present framework, constructions of a metric and a corresponding potential are not unique since the integrability conditions (2.12) have different solutions. The existence of one solution leading to a positive metric would imply the irreversibility of the RG flow of the corresponding system. This cannot be expected in general since the present formalism holds equally for systems with limit cycles. On the other hand we see, at present, no systematic way to search for (or to exclude) solutions of the integrability conditions (2.12) leading to a positive metric. It would be desirable to derive conditions on the functions β_a considered here for the existence of a positive metric.

A particular feature of the present construction is that the potential $\Phi(g)$ derives always from a prepotential $P(g)$ as in (1.4), related to the metric as in (2.6). Contracting (1.3) with β_a and using (1.4) one obtains

$$\beta_a \eta^{ab} \beta_b = \frac{d^2 P(g(t))}{dt^2} \quad (4.1)$$

which may be helpful for the study of global features of the RG flow.

A holographic formulation of the RG flow via Hamilton-Jacobi equations for generic quantum field theories leads always to a gradient flow for β functions [27]. Conversely a gradient flow for β functions is a pre-requisite for a holographic formulation of the RG flow. The present approach may thus find applications in this direction, but also in contexts beyond quantum field theory.

In order to extend the range of possible applications of the present formalism it will be useful to generalise it towards non-marginal couplings such as mass terms. Then, within mass dependent subtraction schemes, the β functions may depend explicitly on the scale(s) which cases require further studies.

Finally the present approach requires as many scales τ_i as couplings g_a . If this assumption is relaxed the reversibility of the matrices of partial derivatives and/or the construction of a metric η^{ab} imply constraints on the β functions which merit further investigations.

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