Strong decays of the $Y(4660)$ as a vector tetraquark state in solid quark-hadron duality

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Abstract In this article, we choose the $[sc]_P[\bar{s}\bar{c}]_A - [sc]_A[\bar{s}\bar{c}]_P$ type tetraquark current to study the hadronic coupling constants in the strong decays $Y(4660) \rightarrow J/\psi f_0(980), \eta_c \phi(1020), \chi_{c0} \phi(1020), D_s \bar{D}_s, D_s^* \bar{D}_s^*, D_s \bar{D}_s^*, D_s^* \bar{D}_s, \psi' \pi^+ \pi^-, J/\psi \phi(1020)$ with the QCD sum rules based on solid quark-hadron quality. The predicted width $\Gamma(Y(4660)) = 74.2_{-19.2}^{+29.2}$ MeV is in excellent agreement with the experimental data $68 \pm 11 \pm 1$ MeV from the Belle collaboration, which supports assigning the $Y(4660)$ to be the $[sc]_P[\bar{s}\bar{c}]_A - [sc]_A[\bar{s}\bar{c}]_P$ type tetraquark state with $J^{PC} = 1^{--}$. In calculations, we observe that the hadronic coupling constants $|G_{Y\psi'f_0}| \gg |G_{YJ/\psi f_0}|$, which is consistent with the observation of the $Y(4660)$ in the $\psi' \pi^+ \pi^-$ mass spectrum, and favors the $\psi' f_0(980)$ molecule assignment. It is important to search for the process $Y(4660) \rightarrow J/\psi \phi(1020)$ to diagnose the nature of the $Y(4660)$, as the decay is greatly suppressed.

1 Introduction

In 2007, the Belle collaboration observed the $Y(4360)$ and $Y(4660)$ in the $\pi^+ \pi^- \psi'$ invariant mass distribution with statistical significances 8.0σ and 5.8σ respectively in the process $e^+ e^- \rightarrow \gamma_{\text{ISR}} \pi^+ \pi^- \psi'$ between threshold and $\sqrt{s} = 5.5$ GeV using 673fb^{-1} of data collected with the Belle detector at KEKB [1]. In 2008, the Belle collaboration observed the $Y(4630)$ in the $\Lambda_c^+ \Lambda_c^-$ invariant mass distribution with a significance of 8.2σ in the exclusive process $e^+ e^- \rightarrow \gamma_{\text{ISR}} \Lambda_c^+ \Lambda_c^-$ with an integrated luminosity of 695fb^{-1} at the KEKB [2]. The values of the mass and width of the $Y(4630)$ are consistent within errors with that of the new charmonium-like state $Y(4660)$.

In 2014, the Belle collaboration measured the $e^+ e^- \rightarrow \gamma_{\text{ISR}} \pi^+ \pi^- \psi'$ cross section from 4.0 to 5.5 GeV with the full data sample of the Belle experiment using the ISR (initial state radiation) technique, and determined the parameters of

the $Y(4360)$ and $Y(4660)$ resonances and superseded previous Belle determination [3]. The masses and widths are shown explicitly in Table 1. Furthermore, the Belle collaboration studied the $\pi^+ \pi^-$ invariant mass distribution and observed that there are two clusters of events around the masses of the $f_0(500)$ and $f_0(980)$ corresponding to the $Y(4360)$ and $Y(4660)$, respectively. The J^{PC} quantum numbers of the final states accompanying the ISR photon(s) are restricted to $J^{PC} = 1^{--}$. According to potential model calculations [4–6], the $4^3S_1, 5^3S_1, 6^3S_1$ and 3^3D_1 charmonium states are expected to be in the mass range close to the two resonances $Y(4360)$ and $Y(4660)$, however, there are not enough vector charmonium candidates which can match those new Y states consistently.

Now, let us begin with discussing the nature of the $f_0(500)$ and $f_0(980)$ to explore the $Y(4660)$. In the scenario of conventional two-quark states, the structures of the $f_0(500)$ and $f_0(980)$ in the ideal mixing limit can be symbolically written as,

$$f_0(500) = \frac{\bar{u}u + \bar{d}d}{\sqrt{2}}, \quad f_0(980) = \bar{s}s. \quad (1)$$

While in the scenario of tetraquark states, the structures of the $f_0(500)$ and $f_0(980)$ in the ideal mixing limit can be symbolically written as [7–9],

$$f_0(500) = u\bar{d}\bar{u}\bar{d}, \quad f_0(980) = \frac{us\bar{u}\bar{s} + ds\bar{d}\bar{s}}{\sqrt{2}}. \quad (2)$$

In Ref. [10], we take the nonet scalar mesons below 1 GeV as the two-quark-tetraquark mixed states and study their masses and pole residues with the QCD sum rules in details. We determine the mixing angles, which indicate that the dominant components are the two-quark components. The $Y(4660)$ maybe have $\bar{s}s$ constituent. The decay $Y(4630) \rightarrow \Lambda_c^+ \Lambda_c^-$ has been observed, if the $Y(4660)$ and $Y(4630)$ are the same particle, the decay $Y(4630) \rightarrow \Lambda_c^+ \Lambda_c^-$ is Okubo-Zweig-Iizuka suppressed, there should be some rescattering mechanism to account for the decay.

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Table 1 The masses and widths from the different experiments

Year	Mass (MeV)	Width (MeV)	Experiment
2007			
$Y(4360)$	$4361 \pm 9 \pm 9$	$74 \pm 15 \pm 10$	Belle [1]
$Y(4660)$	$4664 \pm 11 \pm 5$	$48 \pm 15 \pm 3$	Belle [1]
2008			
$Y(4630)$	4634^{+8+5}_{-7-8}	92^{+40+10}_{-24-21}	Belle [2]
2014			
$Y(4360)$	$4347 \pm 6 \pm 3$	$103 \pm 9 \pm 5$	Belle [3]
$Y(4660)$	$4652 \pm 10 \pm 8$	$68 \pm 11 \pm 1$	Belle [3]

Table 2 The masses from the QCD sum rules with different quark structures, where the OPE denotes truncations of the operator product expansion up to the vacuum condensates of dimension n , the No denotes the vacuum condensates of dimension n' are not included

	Structures	OPE (No)	Mass (GeV)	References
$Y(4660)$	$\psi' f_0(980)$	10	4.71	[14]
$Y(4660)$	$\psi' f_0(980)$	6	4.67	[15]
$Y(4660)$	$[sc]_S[\bar{s}\bar{c}]_V + [sc]_V[\bar{s}\bar{c}]_S$	8 (7)	4.65	[20]
$Y(4660)$	$[sc]_S[\bar{s}\bar{c}]_V + [sc]_V[\bar{s}\bar{c}]_S$	10	4.68	[21]
$Y(4660)$	$[qc]_S[\bar{q}\bar{c}]_V + [qc]_V[\bar{q}\bar{c}]_S$	8 (7)	4.64	[22]
$Y(4360)$	$[qc]_S[\bar{q}\bar{c}]_V + [qc]_V[\bar{q}\bar{c}]_S$	10	4.34	[24]
$Y(4660)$	$[sc]_P[\bar{s}\bar{c}]_A - [sc]_A[\bar{s}\bar{c}]_P$	10	4.70	[23]
$Y(4660)$	$[sc]_P[\bar{s}\bar{c}]_A - [sc]_A[\bar{s}\bar{c}]_P$	10	4.66	[24]
$Y(4660)$	$[qc]_P[\bar{q}\bar{c}]_A - [qc]_A[\bar{q}\bar{c}]_P$	10	4.66	[23]
$Y(4660)$	$[qc]_P[\bar{q}\bar{c}]_A - [qc]_A[\bar{q}\bar{c}]_P$	10	4.59	[24]
$Y(4660)$	$[qc]_A[\bar{q}\bar{c}]_A$	10	4.66	[25]
$Y(4660)$	$[sc]_S[\bar{s}\bar{c}]_S$	6	4.69	[26]

The threshold of the $\psi' f_0(980)$ is 4676 MeV from the Particle Data Group [11], which is just above the mass $m_{Y(4660)} = 4652 \pm 10 \pm 8$ MeV from the Belle collaboration [3]. The $Y(4660)$ can be assigned to be a $\psi' f_0(980)$ molecular state [12–15] or a $\psi' f_0(980)$ hadro-charmonium [16]. Other assignments, such as a $2P[cq]_S[\bar{c}\bar{q}]_S$ tetraquark state [17, 18], a $\psi(6S)$ state [6], a $\psi(5S)$ state [19], a ground state P-wave tetraquark state [20–26] are also possible.

In Table 2, we list out the predictions of the masses of the vector tetraquark (tetraquark molecule) states based on the QCD sum rules [14, 15, 20–26], where the S , P , A and V denote the scalar (S), pseudoscalar (P), axialvector (A) and vector (V) diquark states. From the Table, we can see that it is not difficult to reproduce the experimental value of the mass of the $Y(4660)$ with the QCD sum rules. However, the quantitative predications depend on the quark structures, the input parameters at the QCD side, the pole contributions of the ground states, and the truncations of the operator product expansion.

In the QCD sum rules for the hidden-charm (or hidden-bottom) tetraquark states and molecular states, the integrals

$$\int_{4m_Q^2(\mu)}^{s_0} ds \rho_{QCD}(s, \mu) \exp\left(-\frac{s}{T^2}\right), \quad (3)$$

are sensitive to the energy scales μ , where the $\rho_{QCD}(s, \mu)$ are the QCD spectral densities, the T^2 are the Borel parameters, the s_0 are the continuum thresholds parameters, the predicted masses depend heavily on the energy scales μ . In Refs. [23, 27–30], we suggest an energy scale formula $\mu = \sqrt{M_{X/Y/Z}^2 - (2M_Q)^2}$ with the effective Q -quark mass M_Q to determine the ideal energy scales of the QCD spectral densities. The formula enhances the pole contributions remarkably, we obtain the pole contributions as large as (40–60)%, the largest pole contributions up to now. Compared to the old values obtained in Ref. [23], the new values based on detailed analysis with the updated parameters are preferred [24]. The energy scale formula also works well in the QCD sum rules for the hidden-charm pentaquark states [31–33].

For the correlation functions of the hidden-charm (or hidden-bottom) tetraquark currents, there are two heavy quark propagators and two light quark propagators, if each heavy quark line emits a gluon and each light quark line contributes a quark pair, we obtain an operator $G\bar{G}\bar{q}q\bar{q}q$, which is of dimension 10, we should take into account the vacuum condensates at least up to dimension 10 in the operator product expansion.

In Refs. [23–25, 34, 35], we study the mass spectrum of the vector tetraquark states in a comprehensive way by carrying out the operator product expansion up to the vacuum condensates of dimension 10, and use the energy scale formula $\mu = \sqrt{M_{X/Y/Z}^2 - (2M_c)^2}$ or modified energy scale formula $\mu = \sqrt{M_{X/Y/Z}^2 - (2M_c + 0.5 \text{ GeV})^2} = \sqrt{M_{X/Y/Z}^2 - (4.1 \text{ GeV})^2}$ to determine the ideal energy scales of the QCD spectral densities in a consistent way. In the scenario of tetraquark states, we observe that the preferred quark configurations for the $Y(4660)$ are the $[sc]_P[\bar{s}\bar{c}]_A - [sc]_A[\bar{s}\bar{c}]_P$ and $[qc]_A[\bar{q}\bar{c}]_A$. In this article, we choose the quark configuration $[sc]_P[\bar{s}\bar{c}]_A - [sc]_A[\bar{s}\bar{c}]_P$ to examine the nature of the $Y(4660)$.

In Ref. [36], we assign the $Z_c^\pm(3900)$ to be the diquark-antidiquark type axialvector tetraquark state, study the hadronic coupling constants $G_{Z_c J/\psi \pi}$, $G_{Z_c \eta_c \rho}$, $G_{Z_c D \bar{D}^*}$ with the QCD sum rules by taking into account both the connected and disconnected Feynman diagrams in the operator product expansion. We pay special attentions to matching the hadron side of the correlation functions with the QCD side of the correlation functions to obtain solid duality. The routine works well in studying the decays $X(4140/4274) \rightarrow J/\psi \phi(1020)$ [37, 38].

In this article, we assign the $Y(4660)$ to be the $[sc]_P[\bar{s}\bar{c}]_A - [sc]_A[\bar{s}\bar{c}]_P$ type vector tetraquark state, and study the strong decays $Y(4660) \rightarrow J/\psi f_0(980)$, $\eta_c \phi(1020)$, $\chi_{c0} \phi(1020)$, $D_s \bar{D}_s$, $D_s^* \bar{D}_s^*$, $D_s \bar{D}_s^*$, $D_s^* \bar{D}_s$, $\psi' \pi^+ \pi^-$, $J/\psi \phi(1020)$ with the QCD sum rules based on the solid quark-hadron duality, and reexamine the assignment of the $Y(4660)$.

The article is arranged as follows: we illustrate how to calculate the hadronic coupling constants in the two-body strong decays of the tetraquark states with the QCD sum rules in Sect. 2, in Sect. 3, we obtain the QCD sum rules for the hadronic coupling constants $G_{YJ/\psi f_0}$, $G_{Y\eta_c \phi}$, $G_{Y\chi_{c0} \phi}$, $G_{YD_s \bar{D}_s}$, $G_{YD_s^* \bar{D}_s^*}$, $G_{YD_s \bar{D}_s^*}$, $G_{Y\psi' \pi^+ \pi^-}$, $G_{YJ/\psi \phi}$; Sect. 4 is reserved for our conclusion.

2 The hadronic coupling constants in the two-body strong decays of the tetraquark states

In this section, we illustrate how to calculate the hadronic coupling constants in the two-body strong decays of the tetraquark states with the QCD sum rules. We write down the three-point correlation functions $\Pi(p, q)$ firstly,

$$\Pi(p, q) = i^2 \int d^4x d^4y e^{ipx} e^{iqy} \langle 0 | T \{ J_B(x) J_C(y) J_A^\dagger(0) \} | 0 \rangle, \quad (4)$$

where the currents $J_A(0)$ interpolate the tetraquark states A , the $J_B(x)$ and $J_C(y)$ interpolate the conventional mesons B and C , respectively,

$$\begin{aligned} \langle 0 | J_A(0) | A(p') \rangle &= \lambda_A, \\ \langle 0 | J_B(0) | B(p) \rangle &= \lambda_B, \\ \langle 0 | J_C(0) | C(q) \rangle &= \lambda_C, \end{aligned} \quad (5)$$

the λ_A , λ_B and λ_C are the pole residues or decay constants.

At the phenomenological side, we insert a complete set of intermediate hadronic states with the same quantum numbers as the current operators $J_A(0)$, $J_B(x)$, $J_C(y)$ into the three-point correlation functions $\Pi(p, q)$ and isolate the ground state contributions to obtain the result [39–41],

$$\begin{aligned} \Pi(p, q) &= \frac{\lambda_A \lambda_B \lambda_C G_{ABC}}{(m_A^2 - p^2)(m_B^2 - p^2)(m_C^2 - q^2)} \\ &+ \frac{1}{(m_A^2 - p^2)(m_B^2 - p^2)} \int_{s_C^0}^{\infty} dt \frac{\rho_{AC'}(p^2, p^2, t)}{t - q^2} \\ &+ \frac{1}{(m_A^2 - p^2)(m_C^2 - q^2)} \int_{s_B^0}^{\infty} dt \frac{\rho_{AB'}(p^2, t, q^2)}{t - p^2} \\ &+ \frac{1}{(m_B^2 - p^2)(m_C^2 - q^2)} \end{aligned}$$

$$\begin{aligned} &\times \int_{s_A^0}^{\infty} dt \frac{\rho_{A'B}(t, p^2, q^2) + \rho_{A'C}(t, p^2, q^2)}{t - p^2} + \dots \\ &= \Pi(p'^2, p^2, q^2), \end{aligned} \quad (6)$$

where $p' = p + q$, the G_{ABC} are the hadronic coupling constants defined by

$$\langle B(p) C(q) | A(p') \rangle = i G_{ABC}, \quad (7)$$

the four functions $\rho_{AC'}(p'^2, p^2, t)$, $\rho_{AB'}(p'^2, t, q^2)$, $\rho_{A'B}(t', p^2, q^2)$ and $\rho_{A'C}(t', p^2, q^2)$ have complex dependence on the transitions between the ground states and the higher resonances or the continuum states.

We rewrite the correlation functions $\Pi_H(p'^2, p^2, q^2)$ at the hadron side as

$$\begin{aligned} \Pi_H(p'^2, p^2, q^2) &= \int_{(m_B+m_C)^2}^{s_A^0} ds' \int_{\Delta_s^2}^{s_B^0} ds \int_{\Delta_u^2}^{u_C^0} du \frac{\rho_H(s', s, u)}{(s' - p'^2)(s - p^2)(u - q^2)} \\ &+ \int_{s_A^0}^{\infty} ds' \int_{\Delta_s^2}^{s_B^0} ds \int_{\Delta_u^2}^{u_C^0} du \frac{\rho_H(s', s, u)}{(s' - p'^2)(s - p^2)(u - q^2)} + \dots, \end{aligned} \quad (8)$$

through dispersion relation, where the $\rho_H(s', s, u)$ are the hadronic spectral densities,

$$\begin{aligned} \rho_H(s', s, u) &= \lim_{\epsilon_3 \rightarrow 0} \lim_{\epsilon_2 \rightarrow 0} \lim_{\epsilon_1 \rightarrow 0} \\ &\times \frac{\text{Im}_{s'} \text{Im}_s \text{Im}_u \Pi_H(s' + i\epsilon_3, s + i\epsilon_2, u + i\epsilon_1)}{\pi^3}, \end{aligned} \quad (9)$$

where the Δ_s^2 and Δ_u^2 are the thresholds, the s_A^0 , s_B^0 , u_C^0 are the continuum thresholds.

Now we carry out the operator product expansion at the QCD side, and write the correlation functions $\Pi_{QCD}(p'^2, p^2, q^2)$ as

$$\Pi_{QCD}(p'^2, p^2, q^2) = \int_{\Delta_s^2}^{s_B^0} ds \int_{\Delta_u^2}^{u_C^0} du \frac{\rho_{QCD}(p'^2, s, u)}{(s - p^2)(u - q^2)} + \dots, \quad (10)$$

through dispersion relation, where the $\rho_{QCD}(p'^2, s, u)$ are the QCD spectral densities,

$$\begin{aligned} \rho_{QCD}(p'^2, s, u) &= \lim_{\epsilon_2 \rightarrow 0} \lim_{\epsilon_1 \rightarrow 0} \\ &\times \frac{\text{Im}_s \text{Im}_u \Pi_{QCD}(p'^2, s + i\epsilon_2, u + i\epsilon_1)}{\pi^2}. \end{aligned} \quad (11)$$

However, the QCD spectral densities $\rho_{QCD}(s', s, u)$ do not exist,

$$\rho_{QCD}(s', s, u) = \lim_{\epsilon_3 \rightarrow 0} \lim_{\epsilon_2 \rightarrow 0} \lim_{\epsilon_1 \rightarrow 0} \frac{\text{Im}_{s'} \text{Im}_s \text{Im}_u \Pi_{QCD}(s' + i\epsilon_3, s + i\epsilon_2, u + i\epsilon_1)}{\pi^3} = 0, \quad (12)$$

because

$$\lim_{\epsilon_3 \rightarrow 0} \frac{\text{Im}_{s'} \Pi_{QCD}(s' + i\epsilon_3, p^2, q^2)}{\pi} = 0. \quad (13)$$

Thereafter we will write the QCD spectral densities $\rho_{QCD}(p'^2, s, u)$ as $\rho_{QCD}(s, u)$ for simplicity.

We match the hadron side of the correlation functions with the QCD side of the correlation functions, and carry out the integral over ds' firstly to obtain the solid duality [36],

$$\begin{aligned} & \int_{\Delta_s^0}^{s_B^0} ds \int_{\Delta_u^2}^{u_C^0} du \frac{\rho_{QCD}(s, u)}{(s - p^2)(u - q^2)} \\ &= \int_{\Delta_s^0}^{s_B^0} ds \int_{\Delta_u^2}^{u_C^0} du \frac{1}{(s - p^2)(u - q^2)} \\ & \times \left[\int_{\Delta^2}^{\infty} ds' \frac{\rho_H(s', s, u)}{s' - p'^2} \right], \end{aligned} \quad (14)$$

the Δ^2 denotes the thresholds $(m_B + m_C)^2$. Now we write down the quark-hadron duality explicitly,

$$\begin{aligned} & \int_{\Delta_c^0}^{s_B^0} ds \int_{\Delta_u^2}^{u_C^0} du \frac{\rho_{QCD}(s, u)}{(s - p^2)(u - q^2)} \\ &= \int_{\Delta_c^0}^{s_B^0} ds \int_{\Delta_u^2}^{u_C^0} du \int_{(m_B + m_C)^2}^{\infty} ds' \frac{\rho_H(s', s, u)}{(s' - p'^2)(s - p^2)(u - q^2)} \\ &= \frac{\lambda_A \lambda_B \lambda_C G_{ABC}}{(m_A^2 - p'^2)(m_B^2 - p^2)(m_C^2 - q^2)} \\ & + \frac{C_{A'B} + C_{A'C}}{(m_B^2 - p^2)(m_C^2 - q^2)}. \end{aligned} \quad (15)$$

No approximation is needed, we do not need the continuum threshold parameter s_A^0 in the s' channel. The s' channel and s channel are quite different, we can not set the continuum threshold parameters in the s channel as $s_B^0 = s_A^0$, i.e. we can not set $s_B^0 = s_Y^0 = (5.15 \text{ GeV})^2$ in the present case, where the B denotes the J/ψ , η_c , $\bar{D}_s$, $\bar{D}_s^*$, because the contaminations from the excited states ψ' , η'_c , $\bar{D}'_s$, $\bar{D}'_s^*$ are out of control.

We can introduce the parameters $C_{AC'}$, $C_{AB'}$, $C_{A'B}$ and $C_{A'C}$ to parameterize the net effects,

$$\begin{aligned} C_{AC'} &= \int_{s_C^0}^{\infty} dt \frac{\rho_{AC'}(p'^2, p^2, t)}{t - q^2}, \\ C_{AB'} &= \int_{s_B^0}^{\infty} dt \frac{\rho_{AB'}(p'^2, t, q^2)}{t - p^2}, \end{aligned}$$

$$\begin{aligned} C_{A'B} &= \int_{s_A^0}^{\infty} dt \frac{\rho_{A'B}(t, p^2, q^2)}{t - p'^2}, \\ C_{A'C} &= \int_{s_A^0}^{\infty} dt \frac{\rho_{A'C}(t, p^2, q^2)}{t - p'^2}. \end{aligned} \quad (16)$$

In numerical calculations, we take the relevant functions $C_{A'B}$ and $C_{A'C}$ as free parameters, and choose suitable values to eliminate the contaminations from the higher resonances and continuum states to obtain the stable QCD sum rules with the variations of the Borel parameters.

If the B are charmonium or bottomonium states, we set $p'^2 = p^2$ and perform the double Borel transform with respect to the variables $P^2 = -p^2$ and $Q^2 = -q^2$, respectively to obtain the QCD sum rules,

$$\begin{aligned} & \frac{\lambda_A \lambda_B \lambda_C G_{ABC}}{m_A^2 - m_B^2} \left[\exp\left(-\frac{m_B^2}{T_1^2}\right) \right. \\ & \quad \left. - \exp\left(-\frac{m_A^2}{T_1^2}\right) \right] \exp\left(-\frac{m_C^2}{T_2^2}\right) \\ & + (C_{A'B} + C_{A'C}) \exp\left(-\frac{m_B^2}{T_1^2} - \frac{m_C^2}{T_2^2}\right) \\ &= \int_{\Delta_s^0}^{s_B^0} ds \int_{\Delta_u^2}^{u_C^0} du \rho_{QCD}(s, u) \exp\left(-\frac{s}{T_1^2} - \frac{u}{T_2^2}\right), \end{aligned} \quad (17)$$

where the T_1^2 and T_2^2 are the Borel parameters. If the B are open-charm or open-bottom mesons, we set $p'^2 = 4p^2$ and perform the double Borel transform with respect to the variables $P^2 = -p^2$ and $Q^2 = -q^2$, respectively to obtain the QCD sum rules,

$$\begin{aligned} & \frac{\lambda_A \lambda_B \lambda_C G_{ABC}}{4(\tilde{m}_A^2 - m_B^2)} \left[\exp\left(-\frac{m_B^2}{T_1^2}\right) \right. \\ & \quad \left. - \exp\left(-\frac{\tilde{m}_A^2}{T_1^2}\right) \right] \exp\left(-\frac{m_C^2}{T_2^2}\right) \\ & + (C_{A'B} + C_{A'C}) \exp\left(-\frac{m_B^2}{T_1^2} - \frac{m_C^2}{T_2^2}\right) \\ &= \int_{\Delta_s^0}^{s_B^0} ds \int_{\Delta_u^2}^{u_C^0} du \rho_{QCD}(s, u) \exp\left(-\frac{s}{T_1^2} - \frac{u}{T_2^2}\right), \end{aligned} \quad (18)$$

where $\tilde{m}_A^2 = \frac{m_A^2}{4}$.

3 The width of the $Y(4660)$ as a vector tetraquark state

Now we write down the three-point correlation functions for the strong decays $Y(4660) \rightarrow J/\psi f_0(980)$, $\eta_c \phi(1020)$, $\chi_{c0} \phi(1020)$, $D_s \bar{D}_s$, $D_s^* \bar{D}_s^*$, $D_s \bar{D}_s^*$, $D_s^* \bar{D}_s$, $\psi' \pi^+ \pi^-$,

$J/\psi\phi(1020)$, respectively, and apply the method presented in previous section to obtain the QCD sum rules for the hadronic coupling constants $G_{YJ/\psi f_0}$, $G_{Y\eta_c\phi}$, $G_{Y\chi_{c0}\phi}$, $G_{YD_s\bar{D}_s}$, $G_{YD_s^*\bar{D}_s^*}$, $G_{YD_s\bar{D}_s^*}$, $G_{Y\psi'f_0}$, $G_{YJ/\psi\phi}$.

For the two-body strong decays $Y(4660) \rightarrow J/\psi f_0(980)$, $\psi' f_0(980)^*$, the correlation function is

$$\begin{aligned} \Pi_{\mu\nu}(p, q) &= i^2 \int d^4x d^4y e^{ipx} e^{iqy} \langle 0 | T \{ J_{J/\psi, \mu}(x) J_{f_0}(y) J_\nu^\dagger(0) \} | 0 \rangle, \\ &\quad (19) \end{aligned}$$

where

$$\begin{aligned} J_{J/\psi, \mu}(x) &= \bar{c}(x) \gamma_\mu c(x), \\ J_{f_0}(y) &= \bar{s}(y) s(y), \\ J_\nu(0) &= \frac{\varepsilon^{ijk} \varepsilon^{imn}}{\sqrt{2}} \left[s^{Tj}(0) C c^k(0) \bar{s}^m(0) \gamma_\nu C \bar{c}^{Tn}(0) \right. \\ &\quad \left. - s^{Tj}(0) C \gamma_\nu c^k(0) \bar{s}^m(0) C \bar{c}^{Tn}(0) \right]. \end{aligned} \quad (20)$$

For the two-body strong decay $Y(4660) \rightarrow \eta_c \phi(1020)$, the correlation function is

$$\begin{aligned} \Pi_{\mu\nu}(p, q) &= i^2 \int d^4x d^4y e^{ipx} e^{iqy} \langle 0 | T \{ J_{\eta_c}(x) J_{\phi, \mu}(y) J_\nu^\dagger(0) \} | 0 \rangle, \\ &\quad (21) \end{aligned}$$

where

$$\begin{aligned} J_{\eta_c}(x) &= \bar{c}(x) i \gamma_5 c(x), \\ J_{\phi, \mu}(y) &= \bar{s}(y) \gamma_\mu s(y). \end{aligned} \quad (22)$$

For the two-body strong decay $Y(4660) \rightarrow \chi_{c0} \phi(1020)$, the correlation function is

$$\begin{aligned} \Pi_{\mu\nu}(p, q) &= i^2 \int d^4x d^4y e^{ipx} e^{iqy} \langle 0 | T \{ J_{\chi_{c0}}(x) J_{\phi, \mu}(y) J_\nu^\dagger(0) \} | 0 \rangle, \\ &\quad (23) \end{aligned}$$

where

$$J_{\chi_{c0}}(x) = \bar{c}(x) c(x). \quad (24)$$

For the two-body strong decay $Y(4660) \rightarrow D_s \bar{D}_s$, the correlation function is

$$\begin{aligned} \Pi_\nu(p, q) &= i^2 \int d^4x d^4y e^{ipx} e^{iqy} \langle 0 | T \{ J_{D_s}^\dagger(x) J_{D_s}(y) J_\nu^\dagger(0) \} | 0 \rangle, \\ &\quad (25) \end{aligned}$$

where

$$J_{D_s}(y) = \bar{s}(y) i \gamma_5 c(y). \quad (26)$$

For the two-body strong decay $Y(4660) \rightarrow D_s^* \bar{D}_s^*$, the correlation function is

$$\begin{aligned} \Pi_{\alpha\beta\nu}(p, q) &= i^2 \int d^4x d^4y e^{ipx} e^{iqy} \langle 0 | T \{ J_{D_s^*, \alpha}^\dagger(x) J_{D_s^*, \beta}(y) J_\nu^\dagger(0) \} | 0 \rangle, \\ &\quad (27) \end{aligned}$$

where

$$J_{D_s^*, \beta}(y) = \bar{s}(y) \gamma_\beta c(y). \quad (28)$$

For the two-body strong decay $Y(4660) \rightarrow D_s \bar{D}_s^*$, the correlation function is

$$\begin{aligned} \Pi_{\mu\nu}(p, q) &= i^2 \int d^4x d^4y e^{ipx} e^{iqy} \langle 0 | T \{ J_{D_s^*, \mu}^\dagger(x) J_{D_s}(y) J_\nu^\dagger(0) \} | 0 \rangle. \\ &\quad (29) \end{aligned}$$

For the two-body strong decay $Y(4660) \rightarrow J/\psi \phi(1020)$, the correlation function is

$$\begin{aligned} \Pi_{\alpha\beta\nu}(p, q) &= i^2 \int d^4x d^4y e^{ipx} e^{iqy} \langle 0 | T \{ J_{J/\psi, \alpha}^\dagger(x) J_{\phi, \beta}(y) J_\nu^\dagger(0) \} | 0 \rangle. \\ &\quad (30) \end{aligned}$$

At the phenomenological side, we insert a complete set of intermediate hadronic states with the same quantum numbers as the current operators into the three-point correlation functions and isolate the ground state contributions to obtain the hadron representation [39–41].

For the decays $Y(4660) \rightarrow J/\psi f_0(980)$, $\psi' f_0(980)^*$, the correlation function can be written as

$$\begin{aligned} \Pi_{\mu\nu}(p, q) &= \frac{f_{J/\psi} m_{J/\psi} f_{f_0} m_{f_0} \lambda_Y G_{YJ/\psi f_0}}{(p'^2 - m_Y^2) (p^2 - m_{J/\psi}^2) (q^2 - m_{f_0}^2)} \\ &\quad \times \left(-g_{\mu\alpha} + \frac{p_\mu p_\alpha}{p^2} \right) \left(-g_\nu^\alpha + \frac{p'_\nu p'^\alpha}{p'^2} \right) \\ &\quad + \frac{f_{\psi'} m_{\psi'} f_{f_0} m_{f_0} \lambda_Y G_{Y\psi' f_0}}{(p'^2 - m_Y^2) (p^2 - m_{\psi'}^2) (q^2 - m_{f_0}^2)} \\ &\quad \times \left(-g_{\mu\alpha} + \frac{p_\mu p_\alpha}{p^2} \right) \left(-g_\nu^\alpha + \frac{p'_\nu p'^\alpha}{p'^2} \right) + \dots \\ &= \Pi(p'^2, p^2, q^2) g_{\mu\nu} + \dots \end{aligned} \quad (31)$$

For the decay $Y(4660) \rightarrow \eta_c \phi(1020)$, the correlation function can be written as

$$\begin{aligned} \Pi_{\mu\nu}(p, q) &= \frac{f_{\eta_c} m_{\eta_c}^2}{2m_c} \frac{f_\phi m_\phi \lambda_Y G_{Y\eta_c\phi} \varepsilon_{\alpha\beta\rho\sigma} q^\alpha p'^\rho}{(p'^2 - m_Y^2)(p^2 - m_{\eta_c}^2)(q^2 - m_\phi^2)} \\ &\quad \times \left(-g_\mu^\beta + \frac{q_\mu q^\beta}{q^2} \right) \left(-g_\nu^\sigma + \frac{p'_\nu p'^\sigma}{p'^2} \right) + \dots \\ &= \Pi(p'^2, p^2, q^2) \varepsilon_{\mu\nu\alpha\beta} p^\alpha q^\beta + \dots \end{aligned} \quad (32)$$

For the decay $Y(4660) \rightarrow \chi_{c0} \phi(1020)$, the correlation function can be written as

$$\begin{aligned} \Pi_{\mu\nu}(p, q) &= \frac{f_{\chi_{c0}} m_{\chi_{c0}} f_\phi m_\phi \lambda_Y G_{Y\chi_{c0}\phi}}{(p'^2 - m_Y^2)(p^2 - m_{\chi_{c0}}^2)(q^2 - m_\phi^2)} \\ &\quad \times \left(-g_{\mu\alpha} + \frac{q_\mu q_\alpha}{q^2} \right) \left(-g_\nu^\alpha + \frac{p'_\nu p'^\alpha}{p'^2} \right) + \dots \\ &= \Pi(p'^2, p^2, q^2) g_{\mu\nu} + \dots \end{aligned} \quad (33)$$

For the decay $Y(4660) \rightarrow D_s \bar{D}_s$, the correlation function can be written as

$$\begin{aligned} \Pi_\nu(p, q) &= \frac{f_{D_s}^2 m_{D_s}^4}{(m_c + m_s)^2} \\ &\quad \times \frac{\lambda_Y G_{YD_s\bar{D}_s}}{(p'^2 - m_Y^2)(p^2 - m_{D_s}^2)(q^2 - m_{D_s}^2)} (p - q)^\alpha \\ &\quad \times \left(-g_{\alpha\nu} + \frac{p'_\alpha p'_\nu}{p'^2} \right) + \dots \\ &= \Pi(p'^2, p^2, q^2) (-p_\nu) + \dots \end{aligned} \quad (34)$$

For the decay $Y(4660) \rightarrow D_s^* \bar{D}_s^*$, the correlation function can be written as

$$\begin{aligned} \Pi_{\alpha\beta\nu}(p, q) &= \frac{f_{D_s^*}^2 m_{D_s^*}^2 \lambda_Y G_{YD_s^*\bar{D}_s^*}}{(p'^2 - m_Y^2)(p^2 - m_{D_s^*}^2)(q^2 - m_{D_s^*}^2)} (p - q)^\sigma \\ &\quad \times \left(-g_{\nu\sigma} + \frac{p'_\nu p'_\sigma}{p'^2} \right) \left(-g_{\alpha\rho} + \frac{p_\alpha p_\rho}{p^2} \right) \\ &\quad \times \left(-g_\beta^\rho + \frac{q_\beta q^\rho}{q^2} \right) + \dots \\ &= \Pi(p'^2, p^2, q^2) (-g_{\alpha\beta} p_\nu) + \dots \end{aligned} \quad (35)$$

For the decay $Y(4660) \rightarrow D_s \bar{D}_s^*$, the correlation function can be written as

$$\begin{aligned} \Pi_{\mu\nu}(p, q) &= \frac{f_{D_s} m_{D_s}^2}{m_c + m_s} \frac{f_{D_s^*} m_{D_s^*} \lambda_Y G_{YD_s\bar{D}_s^*} \varepsilon_{\alpha\beta\rho\sigma} p^\alpha p'^\rho}{(p'^2 - m_Y^2)(p^2 - m_{D_s^*}^2)(q^2 - m_{D_s}^2)} \\ &\quad \times \left(-g_\mu^\beta + \frac{p_\mu p^\beta}{p^2} \right) \left(-g_\nu^\sigma + \frac{p'_\nu p'^\sigma}{p'^2} \right) + \dots \\ &= \Pi(p'^2, p^2, q^2) (-\varepsilon_{\mu\nu\alpha\beta} p^\alpha q^\beta) + \dots \end{aligned} \quad (36)$$

For the decay $Y(4660) \rightarrow J/\psi \phi(1020)$, the correlation function can be written as

$$\begin{aligned} \Pi_{\alpha\beta\nu}(p, q) &= \frac{f_{J/\psi} m_{J/\psi} f_\phi m_\phi \lambda_Y G_{YJ/\psi\phi}}{(p'^2 - m_Y^2)(p^2 - m_{J/\psi}^2)(q^2 - m_\phi^2)} (p - q)^\sigma \\ &\quad \times \left(-g_{\nu\sigma} + \frac{p'_\nu p'_\sigma}{p'^2} \right) \left(-g_{\alpha\rho} + \frac{p_\alpha p_\rho}{p^2} \right) \\ &\quad \times \left(-g_\beta^\rho + \frac{q_\beta q^\rho}{q^2} \right) + \dots \\ &= \Pi(p'^2, p^2, q^2) (-g_{\alpha\beta} p_\nu) + \dots \end{aligned} \quad (37)$$

In calculations, we observe that the hadronic coupling constant $G_{YJ/\psi\phi}$ is zero at the leading order approximation, and we will neglect the process $Y(4660) \rightarrow J/\psi \phi(1020)$.

In Eqs. (31)–(37), we have used the following definitions for the decay constants and hadronic coupling constants,

$$\begin{aligned} \langle 0 | J_{J/\psi, \mu}(0) | J/\psi(p) \rangle &= f_{J/\psi} m_{J/\psi} \xi_\mu^{J/\psi}, \\ \langle 0 | J_{\psi', \mu}(0) | \psi'(p) \rangle &= f_{\psi'} m_{\psi'} \xi_\mu^{\psi'}, \\ \langle 0 | J_{f_0}(0) | f_0(p) \rangle &= f_{f_0} m_{f_0}, \\ \langle 0 | J_{\eta_c}(0) | \eta_c(p) \rangle &= \frac{f_{\eta_c} m_{\eta_c}^2}{2m_c}, \\ \langle 0 | J_{\phi, \mu}(0) | \phi(p) \rangle &= f_\phi m_\phi \xi_\mu^\phi, \\ \langle 0 | J_{\chi_{c0}}(0) | \chi_{c0}(p) \rangle &= f_{\chi_{c0}} m_{\chi_{c0}}, \\ \langle 0 | J_{D_s}(0) | D_s(p) \rangle &= \frac{f_{D_s} m_{D_s}^2}{m_c + m_s}, \\ \langle 0 | J_{D_s^*, \mu}(0) | D_s^*(p) \rangle &= f_{D_s^*} m_{D_s^*} \xi_\mu^{D_s^*}, \\ \langle 0 | J_\mu(0) | Y(p) \rangle &= \lambda_Y \xi_\mu^Y, \end{aligned} \quad (38)$$

$$\begin{aligned} \langle J/\psi(p) f_0(q) | X(p') \rangle &= i \xi_{J/\psi}^{\alpha*} \xi_{f_0}^Y G_{YJ/\psi f_0}, \\ \langle \psi'(p) f_0(q) | X(p') \rangle &= i \xi_{\psi'}^{\alpha*} \xi_{f_0}^Y G_{Y\psi' f_0}, \\ \langle \eta_c(p) \phi(q) | X(p') \rangle &= i \varepsilon^{\alpha\beta\rho\sigma} q_\alpha \xi_\beta^{\phi*} p'_\rho \xi_\sigma^Y G_{Y\eta_c\phi}, \\ \langle \chi_{c0}(p) \phi(q) | X(p') \rangle &= i \xi_\phi^{\alpha*} \xi_{\chi_{c0}}^Y G_{Y\chi_{c0}\phi}, \\ \langle \bar{D}_s(p) D_s(q) | X(p') \rangle &= i (p - q)^\alpha \xi_\alpha^Y G_{YD_s\bar{D}_s}, \\ \langle \bar{D}_s^*(p) D_s^*(q) | X(p') \rangle &= i (p - q)^\alpha \xi_\alpha^Y \xi_\beta^{\bar{D}_s^*} \xi_\beta^{D_s^*} G_{YD_s^*\bar{D}_s^*}, \\ \langle \bar{D}_s^*(p) D_s(q) | X(p') \rangle &= i \varepsilon^{\alpha\beta\rho\sigma} p_\alpha \xi_\beta^{\bar{D}_s^*} p'_\rho \xi_\sigma^Y G_{YD_s\bar{D}_s^*}, \\ \langle J/\psi(p) \phi(q) | X(p') \rangle &= i (p - q)^\alpha \xi_\alpha^Y \xi_\beta^{J/\psi*} \xi_\beta^{\phi*} G_{YJ/\psi\phi}, \end{aligned} \quad (39)$$

where the $\xi_\mu^{J/\psi}$, $\xi_\mu^{\psi'}$, ξ_μ^ϕ , $\xi_\mu^{D_s^*}$, ξ_μ^Y are the polarization vectors, the $G_{YJ/\psi f_0}$, $G_{Y\psi' f_0}$, $G_{Y\eta_c\phi}$, $G_{Y\chi_{c0}\phi}$, $G_{YD_s\bar{D}_s}$, $G_{YD_s^*\bar{D}_s^*}$, $G_{YD_s\bar{D}_s^*}$, $G_{YJ/\psi\phi}$ are the hadronic coupling constants.

We study the components $\Pi(p'^2, p^2, q^2)$ of the correlation functions, and carry out the operator product expansion up to the vacuum condensates of dimension 5 and neglect the tiny contributions of the gluon condensate. Then we obtain

the QCD spectral densities through dispersion relation and use Eqs. (17) and (18) to obtain the QCD sum rules for the hadronic coupling constants,

$$\begin{aligned}
 & \frac{f_{J/\psi} m_{J/\psi} f_{f_0} m_{f_0} \lambda_Y G_{Y J/\psi f_0}}{m_Y^2 - m_{J/\psi}^2} \left[\exp\left(-\frac{m_{J/\psi}^2}{T_1^2}\right) \right. \\
 & \quad \left. - \exp\left(-\frac{m_Y^2}{T_1^2}\right) \right] \exp\left(-\frac{m_{f_0}^2}{T_2^2}\right) \\
 & \quad + (C_{Y' J/\psi} + C_{Y' f_0}) \exp\left(-\frac{m_{J/\psi}^2}{T_1^2} - \frac{m_{f_0}^2}{T_2^2}\right) \\
 & = -\frac{1}{32\sqrt{2}\pi^4} \int_{4m_c^2}^{s_{J/\psi}^0} ds \int_0^{s_{f_0}^0} du u s \sqrt{1 - \frac{4m_c^2}{s}} \left(1 + \frac{2m_c^2}{s}\right) \\
 & \quad \times \exp\left(-\frac{s}{T_1^2} - \frac{u}{T_2^2}\right) \\
 & \quad - \frac{m_s \langle \bar{s}s \rangle}{4\sqrt{2}\pi^2} \int_{4m_c^2}^{s_{J/\psi}^0} ds s \sqrt{1 - \frac{4m_c^2}{s}} \left(1 + \frac{2m_c^2}{s}\right) \exp\left(-\frac{s}{T_1^2}\right) \\
 & \quad - \frac{m_s \langle \bar{s}g_s \sigma Gs \rangle}{12\sqrt{2}\pi^2 T_2^2} \int_{4m_c^2}^{s_{J/\psi}^0} ds \sqrt{1 - \frac{4m_c^2}{s}} (s + 2m_c^2) \exp\left(-\frac{s}{T_1^2}\right) \\
 & \quad + \frac{m_s \langle \bar{s}g_s \sigma Gs \rangle}{48\sqrt{2}\pi^2} \int_{4m_c^2}^{s_{J/\psi}^0} ds \frac{s - 12m_c^2}{\sqrt{s(s - 4m_c^2)}} \exp\left(-\frac{s}{T_1^2}\right), \quad (40)
 \end{aligned}$$

$$\begin{aligned}
 & \frac{f_{\eta_c} m_{\eta_c}^2 f_{\phi} m_{\phi} \lambda_Y G_{Y \eta_c \phi}}{2m_c (m_Y^2 - m_{\eta_c}^2)} \left[\exp\left(-\frac{m_{\eta_c}^2}{T_1^2}\right) \right. \\
 & \quad \left. - \exp\left(-\frac{m_Y^2}{T_1^2}\right) \right] \exp\left(-\frac{m_{\phi}^2}{T_2^2}\right) \\
 & \quad + (C_{Y' \eta_c} + C_{Y' \phi}) \exp\left(-\frac{m_{\eta_c}^2}{T_1^2} - \frac{m_{\phi}^2}{T_2^2}\right) \\
 & = \frac{3m_s m_c}{16\sqrt{2}\pi^4} \int_{4m_c^2}^{s_{\eta_c}^0} ds \int_0^{s_{\phi}^0} du \sqrt{1 - \frac{4m_c^2}{s}} \exp\left(-\frac{s}{T_1^2} - \frac{u}{T_2^2}\right) \\
 & \quad - \frac{m_c \langle \bar{s}s \rangle}{2\sqrt{2}\pi^2} \int_{4m_c^2}^{s_{\eta_c}^0} ds \sqrt{1 - \frac{4m_c^2}{s}} \exp\left(-\frac{s}{T_1^2}\right) \\
 & \quad + \frac{m_c \langle \bar{s}g_s \sigma Gs \rangle}{6\sqrt{2}\pi^2 T_2^2} \int_{4m_c^2}^{s_{\eta_c}^0} ds \sqrt{1 - \frac{4m_c^2}{s}} \exp\left(-\frac{s}{T_1^2}\right) \\
 & \quad - \frac{m_c \langle \bar{s}g_s \sigma Gs \rangle}{24\sqrt{2}\pi^2} \int_{4m_c^2}^{s_{\eta_c}^0} ds \frac{1}{\sqrt{s(s - 4m_c^2)}} \exp\left(-\frac{s}{T_1^2}\right), \quad (41)
 \end{aligned}$$

$$\begin{aligned}
 & \frac{f_{\chi_{c0}} m_{\chi_{c0}} f_{\phi} m_{\phi} \lambda_Y G_{Y \chi_{c0} \phi}}{m_Y^2 - m_{\chi_{c0}}^2} \left[\exp\left(-\frac{m_{\chi_{c0}}^2}{T_1^2}\right) \right. \\
 & \quad \left. - \exp\left(-\frac{m_Y^2}{T_1^2}\right) \right] \exp\left(-\frac{m_{\phi}^2}{T_2^2}\right) \\
 & \quad + (C_{Y' \chi_{c0}} + C_{Y' \phi}) \exp\left(-\frac{m_{\chi_{c0}}^2}{T_1^2} - \frac{m_{\phi}^2}{T_2^2}\right) \\
 & = \frac{1}{32\sqrt{2}\pi^4} \int_{4m_c^2}^{s_{\chi_{c0}}^0} ds \int_0^{s_{\phi}^0} du u s \sqrt{1 - \frac{4m_c^2}{s}} \\
 & \quad \times \left(1 - \frac{4m_c^2}{s}\right) \exp\left(-\frac{s}{T_1^2} - \frac{u}{T_2^2}\right) \\
 & \quad - \frac{m_s \langle \bar{s}s \rangle}{4\sqrt{2}\pi^2} \int_{4m_c^2}^{s_{\chi_{c0}}^0} ds s \sqrt{1 - \frac{4m_c^2}{s}} \\
 & \quad \times \left(1 - \frac{4m_c^2}{s}\right) \exp\left(-\frac{s}{T_1^2}\right) \\
 & \quad + \frac{m_s \langle \bar{s}g_s \sigma Gs \rangle}{48\sqrt{2}\pi^2 T_2^2} \int_{4m_c^2}^{s_{\chi_{c0}}^0} ds s \sqrt{1 - \frac{4m_c^2}{s}} \left(1 - \frac{4m_c^2}{s}\right) \exp\left(-\frac{s}{T_1^2}\right) \\
 & \quad - \frac{m_s \langle \bar{s}g_s \sigma Gs \rangle}{24\sqrt{2}\pi^2} \int_{4m_c^2}^{s_{\chi_{c0}}^0} ds \frac{s - 6m_c^2}{\sqrt{s(s - 4m_c^2)}} \exp\left(-\frac{s}{T_1^2}\right), \quad (42)
 \end{aligned}$$

$$\begin{aligned}
 & \frac{f_{D_s}^2 m_{D_s}^4 \lambda_Y G_{Y D_s \bar{D}_s}}{(m_c + m_s)^2 4 (\tilde{m}_Y^2 - m_{D_s}^2)} \left[\exp\left(-\frac{m_{D_s}^2}{T_1^2}\right) \right. \\
 & \quad \left. - \exp\left(-\frac{\tilde{m}_Y^2}{T_1^2}\right) \right] \exp\left(-\frac{m_{D_s}^2}{T_2^2}\right) \\
 & \quad + (C_{Y' \bar{D}_s} + C_{Y' D_s}) \exp\left(-\frac{m_{D_s}^2}{T_1^2} - \frac{m_{D_s}^2}{T_2^2}\right) \\
 & = \frac{3m_c}{64\sqrt{2}\pi^4} \int_{m_c^2}^{s_{D_s}^0} ds \int_{m_c^2}^{s_{D_s}^0} du u \left(1 - \frac{m_c^2}{s}\right)^2 \\
 & \quad \times \left(1 - \frac{m_c^2}{u}\right) \exp\left(-\frac{s}{T_1^2} - \frac{u}{T_2^2}\right) \\
 & \quad + \frac{3m_s}{64\sqrt{2}\pi^4} \int_{m_c^2}^{s_{D_s}^0} ds \int_{m_c^2}^{s_{D_s}^0} du u \left(1 - \frac{m_c^2}{s}\right) \\
 & \quad \times \left(1 - \frac{m_c^2}{u}\right) \left(1 + \frac{m_c^2}{s} + \frac{m_c^2}{u} - \frac{3m_c^4}{us}\right) \exp\left(-\frac{s}{T_1^2} - \frac{u}{T_2^2}\right) \\
 & \quad - \frac{\langle \bar{s}s \rangle}{8\sqrt{2}\pi^2} \int_{m_c^2}^{s_{D_s}^0} du \left[u \left(1 - \frac{m_c^2}{u}\right)^2 \right. \\
 & \quad \left. + 2m_s m_c \left(1 - \frac{m_c^2}{u}\right) \right] \exp\left(-\frac{m_c^2}{T_1^2} - \frac{u}{T_2^2}\right) \\
 & \quad - \frac{\langle \bar{s}s \rangle}{8\sqrt{2}\pi^2} \int_{m_c^2}^{s_{D_s}^0} ds \left[m_c^2 \left(1 - \frac{m_c^2}{s}\right)^2 \right. \\
 & \quad \left. + m_s m_c \left(1 - \frac{m_c^4}{s^2}\right) \right] \exp\left(-\frac{s}{T_1^2} - \frac{m_c^2}{T_2^2}\right)
 \end{aligned}$$

$$\begin{aligned}
& + \frac{m_s m_c \langle \bar{s}s \rangle}{16\sqrt{2}\pi^2 T_1^2} \int_{m_c^2}^{s_{D_s^*}^0} duu \left(1 - \frac{m_c^2}{u}\right)^2 \exp\left(-\frac{m_c^2}{T_1^2} - \frac{u}{T_2^2}\right) \\
& + \frac{m_s m_c \langle \bar{s}s \rangle}{16\sqrt{2}\pi^2} \int_{m_c^2}^{s_{D_s^*}^0} ds \left(1 - \frac{m_c^2}{s}\right)^2 \\
& \times \left(1 + \frac{m_c^2}{T_2^2}\right) \exp\left(-\frac{s}{T_1^2} - \frac{m_c^2}{T_2^2}\right) \\
& + \frac{m_c^2 \langle \bar{s}g_s \sigma Gs \rangle}{32\sqrt{2}\pi^2 T_1^4} \int_{m_c^2}^{s_{D_s^*}^0} du \left[u \left(1 - \frac{m_c^2}{u}\right)^2\right. \\
& \left. + 2m_s m_c \left(1 - \frac{m_c^2}{u}\right)\right] \exp\left(-\frac{m_c^2}{T_1^2} - \frac{u}{T_2^2}\right) \\
& - \frac{\langle \bar{s}g_s \sigma Gs \rangle}{16\sqrt{2}\pi^2 T_2^2} \int_{m_c^2}^{s_{D_s^*}^0} ds \left[m_c^2 \left(1 - \frac{m_c^2}{s}\right)^2\right. \\
& \left. + m_s m_c \left(1 - \frac{m_c^4}{s^2}\right)\right] \left(1 - \frac{m_c^2}{2T_2^2}\right) \exp\left(-\frac{s}{T_1^2} - \frac{m_c^2}{T_2^2}\right) \\
& - \frac{m_s m_c^3 \langle \bar{s}g_s \sigma Gs \rangle}{96\sqrt{2}\pi^2 T_1^6} \int_{m_c^2}^{s_{D_s^*}^0} duu \left(1 - \frac{m_c^2}{u}\right)^2 \exp\left(-\frac{m_c^2}{T_1^2} - \frac{u}{T_2^2}\right) \\
& - \frac{m_s m_c^5 \langle \bar{s}g_s \sigma Gs \rangle}{96\sqrt{2}\pi^2 T_2^6} \int_{m_c^2}^{s_{D_s^*}^0} ds \left(1 - \frac{m_c^2}{s}\right)^2 \exp\left(-\frac{s}{T_1^2} - \frac{m_c^2}{T_2^2}\right) \\
& - \frac{\langle \bar{s}g_s \sigma Gs \rangle}{192\sqrt{2}\pi^2} \int_{m_c^2}^{s_{D_s^*}^0} du \left[\left(1 - \frac{m_c^2}{u}\right)\left(3 - \frac{m_c^2}{u}\right)\right. \\
& \left. + \frac{6m_s m_c}{u}\right] \exp\left(-\frac{m_c^2}{T_1^2} - \frac{u}{T_2^2}\right) \\
& - \frac{\langle \bar{s}g_s \sigma Gs \rangle}{96\sqrt{2}\pi^2} \int_{m_c^2}^{s_{D_s^*}^0} ds \left[\frac{m_c^2}{s} \left(1 - \frac{m_c^2}{s}\right)\left(6 - \frac{m_c^2}{s}\right)\right. \\
& \left. - \frac{3m_s m_c^3}{s^2}\right] \exp\left(-\frac{s}{T_1^2} - \frac{m_c^2}{T_2^2}\right) \\
& - \frac{\langle \bar{s}g_s \sigma Gs \rangle}{192\sqrt{2}\pi^2} \int_{m_c^2}^{s_{D_s^*}^0} du \left[\left(3 - \frac{m_c^4}{u^2}\right)\right. \\
& \left. + \frac{6m_s m_c}{u} \frac{u + m_c^2}{u - m_c^2}\right] \exp\left(-\frac{m_c^2}{T_1^2} - \frac{u}{T_2^2}\right) \\
& - \frac{\langle \bar{s}g_s \sigma Gs \rangle}{96\sqrt{2}\pi^2} \int_{m_c^2}^{s_{D_s^*}^0} ds \left(\frac{m_c^6}{s^3}\right. \\
& \left. + \frac{3m_s m_c}{s^2} \frac{s^2 + m_c^4}{s - m_c^2}\right) \exp\left(-\frac{s}{T_1^2} - \frac{m_c^2}{T_2^2}\right), \quad (43)
\end{aligned}$$

$$\begin{aligned}
& + \left(C_{Y'D_s^*} + C_{Y'D_s^*}\right) \exp\left(-\frac{m_{D_s^*}^2}{T_1^2} - \frac{m_{D_s^*}^2}{T_2^2}\right) \\
& = \frac{m_c}{64\sqrt{2}\pi^4} \int_{m_c^2}^{s_{D_s^*}^0} ds \int_{m_c^2}^{s_{D_s^*}^0} du \left(2u + m_c^2\right) \left(1 - \frac{m_c^2}{s}\right)^2 \\
& \times \left(1 - \frac{m_c^2}{u}\right)^2 \exp\left(-\frac{s}{T_1^2} - \frac{u}{T_2^2}\right) \\
& + \frac{m_s}{64\sqrt{2}\pi^4} \int_{m_c^2}^{s_{D_s^*}^0} ds \int_{m_c^2}^{s_{D_s^*}^0} duu \left(1 - \frac{m_c^2}{s}\right) \left(1 - \frac{m_c^2}{u}\right) \\
& \times \left(2 + \frac{2m_c^2}{s} + \frac{5m_c^2}{u} - \frac{m_c^4}{u^2}\right. \\
& \left. - \frac{7m_c^4}{us} - \frac{m_c^6}{us^2}\right) \exp\left(-\frac{s}{T_1^2} - \frac{u}{T_2^2}\right) \\
& - \frac{\langle \bar{s}s \rangle}{24\sqrt{2}\pi^2} \int_{m_c^2}^{s_{D_s^*}^0} du \left[\left(2u + m_c^2\right) \left(1 - \frac{m_c^2}{u}\right)^2\right. \\
& \left. + 6m_s m_c \left(1 - \frac{m_c^2}{u}\right)\right] \exp\left(-\frac{m_c^2}{T_1^2} - \frac{u}{T_2^2}\right) \\
& - \frac{\langle \bar{s}s \rangle}{8\sqrt{2}\pi^2} \int_{m_c^2}^{s_{D_s^*}^0} ds \left[m_c^2 \left(1 - \frac{m_c^2}{s}\right)^2\right. \\
& \left. + m_s m_c \left(1 - \frac{m_c^4}{s^2}\right)\right] \exp\left(-\frac{s}{T_1^2} - \frac{m_c^2}{T_2^2}\right) \\
& + \frac{m_s m_c \langle \bar{s}s \rangle}{48\sqrt{2}\pi^2 T_1^2} \int_{m_c^2}^{s_{D_s^*}^0} du \left(2u + m_c^2\right) \\
& \times \left(1 - \frac{m_c^2}{u}\right)^2 \exp\left(-\frac{m_c^2}{T_1^2} - \frac{u}{T_2^2}\right) \\
& + \frac{m_s m_c^3 \langle \bar{s}s \rangle}{16\sqrt{2}\pi^2 T_2^2} \int_{m_c^2}^{s_{D_s^*}^0} ds \left(1 - \frac{m_c^2}{s}\right)^2 \exp\left(-\frac{s}{T_1^2} - \frac{m_c^2}{T_2^2}\right) \\
& + \frac{\langle \bar{s}g_s \sigma Gs \rangle}{288\sqrt{2}\pi^2 T_1^2} \int_{m_c^2}^{s_{D_s^*}^0} du \left[\left(2u + m_c^2\right) \left(1 - \frac{m_c^2}{u}\right)^2\right. \\
& \left. + 6m_s m_c \left(1 - \frac{m_c^2}{u}\right)\right] \left(4 + \frac{3m_c^2}{T_1^2}\right) \exp\left(-\frac{m_c^2}{T_1^2} - \frac{u}{T_2^2}\right) \\
& + \frac{m_c^2 \langle \bar{s}g_s \sigma Gs \rangle}{32\sqrt{2}\pi^2 T_2^4} \int_{m_c^2}^{s_{D_s^*}^0} ds \left[m_c^2 \left(1 - \frac{m_c^2}{s}\right)^2\right. \\
& \left. + m_s m_c \left(1 - \frac{m_c^4}{s^2}\right)\right] \exp\left(-\frac{s}{T_1^2} - \frac{m_c^2}{T_2^2}\right) \\
& - \frac{m_s m_c^3 \langle \bar{s}g_s \sigma Gs \rangle}{288\sqrt{2}\pi^2 T_1^6} \int_{m_c^2}^{s_{D_s^*}^0} du \left(2u + m_c^2\right) \\
& \times \left(1 - \frac{m_c^2}{u}\right)^2 \exp\left(-\frac{m_c^2}{T_1^2} - \frac{u}{T_2^2}\right)
\end{aligned}$$

$$\begin{aligned}
& \frac{f_{D_s^*}^2 m_{D_s^*}^2 \lambda_Y G_{YD_s^* \bar{D}_s^*}}{4(\tilde{m}_Y^2 - m_{D_s^*}^2)} \left[\exp\left(-\frac{m_{D_s^*}^2}{T_1^2}\right) \right. \\
& \left. - \exp\left(-\frac{\tilde{m}_Y^2}{T_1^2}\right) \right] \exp\left(-\frac{m_{D_s^*}^2}{T_2^2}\right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{m_s m_c \langle \bar{s} g_s \sigma G s \rangle}{96 \sqrt{2} \pi^2 T_2^2} \int_{m_c^2}^{s_{D_s^*}^0} ds \\
& \times \left(1 - \frac{m_c^2}{s} \right)^2 \left(1 + \frac{m_c^2}{T_2^2} - \frac{m_c^4}{T_2^4} \right) \exp \left(-\frac{s}{T_1^2} - \frac{m_c^2}{T_2^2} \right) \\
& - \frac{\langle \bar{s} g_s \sigma G s \rangle}{96 \sqrt{2} \pi^2} \int_{m_c^2}^{s_{D_s^*}^0} du \left[\left(1 - \frac{m_c^2}{u} \right) \right. \\
& \left. + \frac{3 m_s m_c}{u} \right] \exp \left(-\frac{m_c^2}{T_1^2} - \frac{u}{T_2^2} \right) \\
& - \frac{\langle \bar{s} g_s \sigma G s \rangle}{96 \sqrt{2} \pi^2} \int_{m_c^2}^{s_{D_s^*}^0} ds \left[\frac{m_c^2}{s} \left(1 - \frac{m_c^2}{s} \right) \left(6 - \frac{m_c^2}{s} \right) \right. \\
& \left. - \frac{3 m_s m_c^3}{s^2} \right] \exp \left(-\frac{s}{T_1^2} - \frac{m_c^2}{T_2^2} \right) \\
& + \frac{\langle \bar{s} g_s \sigma G s \rangle}{96 \sqrt{2} \pi^2} \int_{m_c^2}^{s_{D_s^*}^0} du \left(1 + \frac{m_s m_c u + m_c^2}{u} \right) \\
& \times \exp \left(-\frac{m_c^2}{T_1^2} - \frac{u}{T_2^2} \right) \\
& + \frac{\langle \bar{s} g_s \sigma G s \rangle}{96 \sqrt{2} \pi^2} \int_{m_c^2}^{s_{D_s^*}^0} ds \left(\frac{m_c^4}{s^2} \right. \\
& \left. + \frac{m_s m_c}{s^2} \frac{s^2 + m_c^4}{s - m_c^2} \right) \exp \left(-\frac{s}{T_1^2} - \frac{m_c^2}{T_2^2} \right), \quad (44)
\end{aligned}$$

$$\begin{aligned}
& \frac{f_{D_s} m_{D_s}^2}{m_c + m_s} \frac{f_{D_s^*} m_{D_s^*}^2}{4 (\tilde{m}_Y^2 - m_{D_s^*}^2)} \lambda_Y G_{Y D_s \bar{D}_s^*} \left[\exp \left(-\frac{m_{D_s^*}^2}{T_1^2} \right) \right. \\
& \left. - \exp \left(-\frac{\tilde{m}_Y^2}{T_1^2} \right) \right] \exp \left(-\frac{m_{D_s}^2}{T_2^2} \right) \\
& + (C_{Y' \bar{D}_s^*} + C_{Y' D_s}) \exp \left(-\frac{m_{D_s^*}^2}{T_1^2} - \frac{m_{D_s}^2}{T_2^2} \right) \\
& = -\frac{\langle \bar{s} g_s \sigma G s \rangle}{32 \sqrt{2} \pi^2} \int_{m_c^2}^{s_{D_s}^0} du \left[\frac{m_c}{u} \left(1 - \frac{m_c^2}{u} \right) \right. \\
& \left. + \frac{2 m_s m_c^2}{3 u^2} \right] \exp \left(-\frac{m_c^2}{T_1^2} - \frac{u}{T_2^2} \right) \\
& + \frac{\langle \bar{s} g_s \sigma G s \rangle}{32 \sqrt{2} \pi^2} \int_{m_c^2}^{s_{D_s}^0} ds \left[\frac{m_c}{s} \left(1 - \frac{m_c^2}{s} \right) \right. \\
& \left. - \frac{2 m_s m_c^2}{3 s^2} \right] \exp \left(-\frac{s}{T_1^2} - \frac{m_c^2}{T_2^2} \right) \\
& - \frac{\langle \bar{s} g_s \sigma G s \rangle}{48 \sqrt{2} \pi^2} \int_{m_c^2}^{s_{D_s}^0} du \left(\frac{m_c^3}{u^2} + \frac{m_s}{2 u^2} \frac{u^2 + m_c^4}{u - m_c^2} \right) \\
& \times \exp \left(-\frac{m_c^2}{T_1^2} - \frac{u}{T_2^2} \right) - \frac{\langle \bar{s} g_s \sigma G s \rangle}{96 \sqrt{2} \pi^2} \\
& \times \int_{m_c^2}^{s_{D_s}^0} ds \left(\frac{m_s}{s^2} \frac{s^2 + m_c^4}{s - m_c^2} \right) \exp \left(-\frac{s}{T_1^2} - \frac{m_c^2}{T_2^2} \right), \quad (45)
\end{aligned}$$

$$\begin{aligned}
& \frac{f_{J/\psi} m_{J/\psi} f_{f_0} m_{f_0} \lambda_Y G_{Y J/\psi f_0}}{m_Y^2 - m_{J/\psi}^2} \left[\exp \left(-\frac{m_{J/\psi}^2}{T_1^2} \right) \right. \\
& \left. - \exp \left(-\frac{m_Y^2}{T_1^2} \right) \right] \exp \left(-\frac{m_{f_0}^2}{T_2^2} \right) \\
& + (C_{Y' J/\psi} + C_{Y' f_0}) \exp \left(-\frac{m_{J/\psi}^2}{T_1^2} - \frac{m_{f_0}^2}{T_2^2} \right) \\
& + \frac{f_{\psi'} m_{\psi'} f_{f_0} m_{f_0} \lambda_Y G_{Y \psi' f_0}}{m_Y^2 - m_{\psi'}^2} \left[\exp \left(-\frac{m_{\psi'}^2}{T_1^2} \right) \right. \\
& \left. - \exp \left(-\frac{m_Y^2}{T_1^2} \right) \right] \exp \left(-\frac{m_{f_0}^2}{T_2^2} \right) \\
& + (C_{Y' \psi'} + \tilde{C}_{Y' f_0}) \exp \left(-\frac{m_{\psi'}^2}{T_1^2} - \frac{m_{f_0}^2}{T_2^2} \right) \\
& = -\frac{1}{32 \sqrt{2} \pi^4} \int_{4 m_c^2}^{s_{\psi'}^0} ds \int_0^{s_{f_0}^0} du u s \sqrt{1 - \frac{4 m_c^2}{s}} \\
& \times \left(1 + \frac{2 m_c^2}{s} \right) \exp \left(-\frac{s}{T_1^2} - \frac{u}{T_2^2} \right) \\
& - \frac{m_s \langle \bar{s} s \rangle}{4 \sqrt{2} \pi^2} \int_{4 m_c^2}^{s_{\psi'}^0} ds s \sqrt{1 - \frac{4 m_c^2}{s}} \left(1 + \frac{2 m_c^2}{s} \right) \exp \left(-\frac{s}{T_1^2} \right) \\
& - \frac{m_s \langle \bar{s} g_s \sigma G s \rangle}{12 \sqrt{2} \pi^2 T_2^2} \int_{4 m_c^2}^{s_{\psi'}^0} ds \sqrt{1 - \frac{4 m_c^2}{s}} (s + 2 m_c^2) \exp \left(-\frac{s}{T_1^2} \right) \\
& + \frac{m_s \langle \bar{s} g_s \sigma G s \rangle}{48 \sqrt{2} \pi^2} \int_{4 m_c^2}^{s_{\psi'}^0} ds \frac{s - 12 m_c^2}{\sqrt{s(s - 4 m_c^2)}} \exp \left(-\frac{s}{T_1^2} \right), \quad (46)
\end{aligned}$$

where $\tilde{m}_Y^2 = \frac{m_Y^2}{4}$. In calculations, we observe that there appears divergence due to the endpoint $s = 4 m_c^2, s = m_c^2$ and $u = m_c^2$, we can avoid the endpoint divergence with the simple replacement $\frac{1}{s - 4 m_c^2} \rightarrow \frac{1}{s - 4 m_c^2 + 4 m_s^2}, \frac{1}{u - m_c^2} \rightarrow \frac{1}{u - m_c^2 + 4 m_s^2}$ and $\frac{1}{s - m_c^2} \rightarrow \frac{1}{s - m_c^2 + 4 m_s^2}$ by adding a small squared s -quark mass $4 m_s^2$.

The hadronic parameters are taken as $m_{J/\psi} = 3.0969$ GeV, $m_\phi = 1.019461$ GeV, $m_{\eta_c} = 2.9839$ GeV, $m_{f_0} = 0.990$ GeV, $m_{D_s} = 1.969$ GeV, $m_{D_s^*} = 2.1122$ GeV, $m_{\chi_{c0}} = 3.41471$ GeV, $m_{\psi'} = 3.686097$ GeV, $m_{\pi^+} = 0.13957$ GeV, $f_{\psi'} = 0.295$ GeV, $\sqrt{s_{J/\psi}^0} = 3.6$ GeV, $\sqrt{s_{\eta_c}^0} = 3.5$ GeV, $\sqrt{s_{\psi'}^0} = 4.0$ GeV, $\sqrt{s_{D_s}^0} = 2.5$ GeV, $\sqrt{s_{D_s^*}^0} = 2.6$ GeV, $\sqrt{s_{\chi_{c0}}^0} = 3.9$ GeV [11], $f_{J/\psi} = 0.418$ GeV, $f_{\eta_c} = 0.387$ GeV [42], $f_\phi = 0.253$ GeV, $\sqrt{s_\phi^0} = 1.5$ GeV [43], $f_{f_0} = 0.180$ GeV, $\sqrt{s_{f_0}^0} = 1.3$ GeV [44], $f_{D_s} = 0.240$ GeV, $f_{D_s^*} = 0.308$ GeV [45, 46], $f_{\chi_{c0}} = 0.359$ GeV [47], $m_Y = 4.652$ GeV [3], $\lambda_Y = 6.72 \times 10^{-2}$ GeV⁵ [24]. In Ref. [24], we obtain the values

Table 3 The Borel windows, hadronic coupling constants, partial decay widths of the $Y(4660)$

	$T^2(\text{GeV}^2)$	$ G_{ABC} $	$\Gamma(\text{MeV})$
$Y(4660) \rightarrow J/\psi f_0(980)$	3.2–4.2	$1.37^{+1.16}_{-1.06} \text{ GeV}$	$3.5^{+8.5}_{-3.4}$
$Y(4660) \rightarrow \eta_c \phi(1020)$	4.3–5.3	$0.98^{+0.27}_{-0.25} \text{ GeV}^{-1}$	$31.6^{+19.8}_{-14.0}$
$Y(4660) \rightarrow \chi_{c0} \phi(1020)$	3.6–4.6	$1.17^{+1.07}_{-0.95} \text{ GeV}$	$1.7^{+4.5}_{-1.6}$
$Y(4660) \rightarrow D_s \bar{D}_s$	1.9–2.7	$1.36^{+0.39}_{-0.33}$	$8.6^{+5.6}_{-3.7}$
$Y(4660) \rightarrow D_s^* \bar{D}_s^*$	2.5–3.3	$1.57^{+0.53}_{-0.50}$	$22.5^{+17.8}_{-11.7}$
$Y(4660) \rightarrow D_s \bar{D}_s^*$	2.5–3.3	$0.11^{+0.23}_{-0.11} \text{ GeV}^{-1}$	$0.4^{+1.7}_{-0.4}$
$Y(4660) \rightarrow \psi' f_0(980) \rightarrow \psi' \pi^+ \pi^-$	4.4–5.4	$7.00^{+2.24}_{-2.20} \text{ GeV}$	$5.5^{+4.1}_{-2.9}$
$Y(4660) \rightarrow J/\psi \phi(1020)$		~ 0	~ 0

$m_Y = 4.66 \text{ GeV}$ and $\lambda_Y = 6.74 \times 10^{-2} \text{ GeV}^5$. In this article, we choose a slightly smaller value $\lambda_Y = 6.72 \times 10^{-2} \text{ GeV}^5$, which corresponds to $m_Y = 4.65 \text{ GeV}$. For more literatures on the decay constants of the charmonium or bottomonium states, one can consult Ref. [48].

At the QCD side, we take the vacuum condensates to be the standard values $\langle \bar{q}q \rangle = -(0.24 \pm 0.01 \text{ GeV})^3$, $\langle \bar{s}s \rangle = (0.8 \pm 0.1) \langle \bar{q}q \rangle$, $\langle \bar{s}g_s \sigma Gs \rangle = m_0^2 \langle \bar{s}s \rangle$, $m_0^2 = (0.8 \pm 0.1) \text{ GeV}^2$ at the energy scale $\mu = 1 \text{ GeV}$ [39–41, 49], and take the $\overline{MS}$ masses $m_c(m_c) = (1.275 \pm 0.025) \text{ GeV}$ and $m_s(\mu = 2 \text{ GeV}) = (0.095 \pm 0.005) \text{ GeV}$ from the Particle Data Group [11]. Moreover, we take into account the energy-scale dependence of the quark condensate, mixed quark condensate and $\overline{MS}$ masses from the renormalization group equation,

$$\begin{aligned}
 \langle \bar{s}s \rangle(\mu) &= \langle \bar{s}s \rangle(1 \text{ GeV}) \left[\frac{\alpha_s(1 \text{ GeV})}{\alpha_s(\mu)} \right]^{\frac{12}{33-2n_f}}, \\
 \langle \bar{s}g_s \sigma Gs \rangle(\mu) &= \langle \bar{s}g_s \sigma Gs \rangle(1 \text{ GeV}) \left[\frac{\alpha_s(1 \text{ GeV})}{\alpha_s(\mu)} \right]^{\frac{2}{33-2n_f}}, \\
 m_c(\mu) &= m_c(m_c) \left[\frac{\alpha_s(\mu)}{\alpha_s(m_c)} \right]^{\frac{12}{33-2n_f}}, \\
 m_s(\mu) &= m_s(2 \text{ GeV}) \left[\frac{\alpha_s(\mu)}{\alpha_s(2 \text{ GeV})} \right]^{\frac{12}{33-2n_f}}, \\
 \alpha_s(\mu) &= \frac{1}{b_0 t} \left[1 - \frac{b_1 \log t}{b_0^2 t} + \frac{b_1^2 (\log^2 t - \log t - 1) + b_0 b_2}{b_0^4 t^2} \right], \quad (47)
 \end{aligned}$$

where $t = \log \frac{\mu^2}{\Lambda^2}$, $b_0 = \frac{33-2n_f}{12\pi}$, $b_1 = \frac{153-19n_f}{24\pi^2}$, $b_2 = \frac{2857 - \frac{5033}{9}n_f + \frac{325}{27}n_f^2}{128\pi^3}$, $\Lambda = 210 \text{ MeV}$, 292 MeV and 332 MeV for the flavors $n_f = 5, 4$ and 3 , respectively [11, 50], and evolve all the input parameters to the optimal energy scale μ with $n_f = 4$ to extract the hadronic coupling constants.

In the QCD sum rules for the mass of the $Y(4660)$, the optimal energy scale of the QCD spectral density is $\mu = 2.9 \text{ GeV}$ [24], which is determined by the energy

scale formula $\mu = \sqrt{M_{X/Y/Z}^2 - (2\mathbb{M}_c)^2}$ with the updated value of the effective c -quark mass $\mathbb{M}_c = 1.82 \text{ GeV}$ [25]. In the present QCD sum rules, if we choose the energy scale $\mu = 2.9 \text{ GeV}$, we obtain an energy scale as large as the masses of the η_c and J/ψ and much larger than the masses of the D_s and D_s^* , it is a too large energy scale. In this article, we take the energy scales of the QCD spectral densities to be $\mu = \frac{m_{\eta_c}}{2} = 1.5 \text{ GeV}$, which is acceptable for the mesons D and J/ψ [51]. We set the Borel parameters to be $T_1^2 = T_2^2 = T^2$ for simplicity. The unknown parameters are chosen as $C_{X'J/\psi} + C_{X'f_0} = -0.012 \text{ GeV}^8$, $C_{X'\eta_c} + C_{X'\phi} = 0.0016 \text{ GeV}^6$, $C_{X'\chi_{c0}} + C_{X'\phi} = 0.0135 \text{ GeV}^8$, $C_{X'D_s} + C_{X'\bar{D}_s} = 0.0038 \text{ GeV}^7$, $C_{X'D_s^*} + C_{X'\bar{D}_s^*} = 0.006 \text{ GeV}^7$, $C_{X'D_s} + C_{X'\bar{D}_s^*} = 0.001 \text{ GeV}^6$, $C_{X'\psi'} + \tilde{C}_{X'f_0} = -0.018 \text{ GeV}^8$ to obtain platforms in the Borel windows, which are shown in Table 3 explicitly. The Borel windows $T_{\max}^2 - T_{\min}^2 = 1.0 \text{ GeV}^2$ for the charmonium decays and $T_{\max}^2 - T_{\min}^2 = 0.8 \text{ GeV}^2$ for the open-charm decays, where the $T_{\max}^2$ and $T_{\min}^2$ denote the maximum and minimum of the Borel parameters, respectively. In the Borel windows, the platforms are flat enough, see the central values in Figs. 1 and 2.

In Figs. 1 and 2, we plot the hadronic coupling constants G_{YBC} with variations of the Borel parameters T^2 at much larger intervals than the Borel windows. From the figures, we can see that there appear platforms in the Borel windows indeed. After taking into account the uncertainties of the input parameters, we obtain the hadronic coupling constants, which are shown explicitly in Table 3. Now it is straightforward to calculate the partial decay widths of the $Y(4660) \rightarrow J/\psi f_0(980)$, $\eta_c \phi(1020)$, $\chi_{c0} \phi(1020)$, $D_s \bar{D}_s$, $D_s^* \bar{D}_s^*$, $D_s \bar{D}_s^*$, $D_s^* \bar{D}_s$ with formula,

$$\Gamma(Y(4660) \rightarrow BC) = \frac{p(m_Y, m_B, m_C)}{24\pi m_Y^2} |T^2|, \quad (48)$$

where $p(a, b, c) = \frac{\sqrt{[a^2 - (b+c)^2][a^2 - (b-c)^2]}}{2a}$, the T are the scattering amplitudes defined in Eq. (39), the numerical values of the partial decay widths are shown in Table 3.

Fig. 1 The hadronic coupling constants with variations of the Borel parameters T^2 , where the A, B, C, D, E and F denote the $G_{YJ/\psi f_0}$, $G_{Y\eta_c\phi}$, $G_{Y\chi_{c0}\phi}$, $G_{YD_s\bar{D}_s}$, $G_{YD_s^*\bar{D}_s^*}$ and $G_{YD_s\bar{D}_s^*}$, respectively, the regions between the two perpendicular lines are the Borel windows

The decay $Y(4660) \rightarrow \psi' f_0(980)$ is kinematically forbidden, but the decay $Y(4660) \rightarrow \psi' \pi^+ \pi^-$ can take place through a virtual intermediate $f_0(980)^*$, the partial decay width can be written as,

$$\begin{aligned} \Gamma(Y \rightarrow \psi' \pi^+ \pi^-) &= \int_{4m_\pi^2}^{(m_Y - m_{\psi'})^2} ds |T|^2 \\ &\times \frac{p(m_Y, m_{\psi'}, \sqrt{s}) p(\sqrt{s}, m_\pi, m_\pi)}{192\pi^3 m_Y^2 \sqrt{s}}, \\ &= 5.5_{-2.9}^{+4.1} \text{ MeV}, \end{aligned} \quad (49)$$

Fig. 2 The hadronic coupling constant $G_{Y\psi'f_0}$ with variation of the Borel parameter T^2 , the region between the two perpendicular lines is the Borel window

where

$$|T|^2 = \frac{(M_Y^2 - s)^2 + 2(5M_Y^2 - s)M_{\psi'}^2 + M_{\psi'}^4}{4M_Y^2 M_{\psi'}^2} \times G_{Y\psi'f_0}^2 \frac{1}{(s - m_{f_0}^2)^2 + s\Gamma_{f_0}^2(s)} G_{f_0\pi\pi}^2, \\ \Gamma_{f_0}(s) = \Gamma_{f_0}(m_{f_0}^2) \frac{m_{f_0}^2}{s} \sqrt{\frac{s - 4m_\pi^2}{m_{f_0}^2 - 4m_\pi^2}}, \\ \Gamma_{f_0}(m_{f_0}^2) = \frac{G_{f_0\pi\pi}^2}{16\pi m_{f_0}^2} \sqrt{m_{f_0}^2 - 4m_\pi^2}, \quad (50)$$

$\Gamma_{f_0}(m_{f_0}^2) = 50 \text{ MeV}$ [11], the hadronic coupling constant $G_{f_0\pi\pi}$ is defined by $\langle \pi^+(p)\pi^-(q) | f_0(p') \rangle = iG_{f_0\pi\pi}$.

Now it is easy to obtain the total decay width,

$$\Gamma(Y(4660)) = 74.2^{+29.2}_{-19.2} \text{ MeV}. \quad (51)$$

The predicted width $\Gamma(Y(4660)) = 74.2^{+29.2}_{-19.2} \text{ MeV}$ is in excellent agreement with the experimental data $68 \pm 11 \pm 1 \text{ MeV}$ from the Belle collaboration [3], which also supports assigning the $Y(4660)$ to be the $[sc]_P[\bar{s}\bar{c}]_A - [sc]_A[\bar{s}\bar{c}]_P$ type tetraquark state with $J^{PC} = 1^{--}$.

From Table 3, we can see that the hadronic coupling constants $|G_{Y\psi'f_0}| = 7.00^{+2.24}_{-2.20} \text{ GeV} \gg |G_{YJ/\psi f_0}| = 1.37^{+1.16}_{-1.04} \text{ GeV}$, which indicates that the coupling $Y(4660)\psi'f_0(980)$ is very strong, and consistent with the observation of the $Y(4660)$ in the $\psi'\pi^+\pi^-$ mass spectrum, and favors the $\psi'f_0(980)$ molecule assignment [12–15], as the strong coupling maybe lead to some $\psi'f_0(980)$ component. Now we perform Fierz re-arrangement to the vector current $J_\mu(x)$ both in the color and Dirac-spinor spaces, and obtain the result,

$$J_\mu = \frac{1}{2\sqrt{2}} \left\{ \bar{c}\gamma^\mu c \bar{s}s - \bar{c}c \bar{s}\gamma^\mu s + i\bar{c}\gamma^\mu \gamma_5 s \bar{s}i\gamma_5 c \right. \\ \left. - i\bar{c}i\gamma_5 s \bar{s}\gamma^\mu \gamma_5 c \right. \\ \left. - i\bar{c}\gamma_\nu \gamma_5 c \bar{s}\sigma^{\mu\nu} \gamma_5 s + i\bar{c}\sigma^{\mu\nu} \gamma_5 c \bar{s}\gamma_\nu \gamma_5 s \right. \\ \left. - i\bar{s}\gamma_\nu c \bar{c}\sigma^{\mu\nu} s + i\bar{s}\sigma^{\mu\nu} c \bar{c}\gamma_\nu s \right\}. \quad (52)$$

The $J_\mu(x)$ can be taken as a special superposition of color singlet-singlet type currents, which couple potentially to the meson-meson pairs or molecular states. The first term $\bar{c}\gamma^\mu c \bar{s}s$ is the molecule current chosen in Refs. [14, 15], which couples potentially to the $\psi'f_0(980)$ molecular state. There does not exist a term $\bar{c}\sigma_{\mu\nu} c \bar{s}\gamma^\nu s$, which couples potentially to the $J/\psi\phi(1020)$ or $\psi'\phi(1020)$ molecular state or scattering state. In calculations, we observe that the QCD side of the component $\Pi(p^2, p^2, q^2)$ in the correlation function $\Pi_{\alpha\beta\nu}(p, q)$ in Eq. (37) is zero at the leading order approximation, the hadronic coupling constant $G_{YJ/\psi\phi} \approx 0$. The decay $Y(4660) \rightarrow J/\psi\phi(1020)$ is greatly suppressed and can take place only through rescattering mechanism. It is important to search for the process $Y(4660) \rightarrow J/\psi\phi(1020)$ to diagnose the structure of the $Y(4660)$.

In Ref. [21], Sundu, Agaev and Azizi choose the $[sc]_S[\bar{s}\bar{c}]_V + [sc]_V[\bar{s}\bar{c}]_S$ type current to study the mass and width of the $Y(4660)$, and obtain the values $m_Y = 4677^{+71}_{-63} \text{ MeV}$ and $\Gamma_Y = (64.8 \pm 10.8) \text{ MeV}$ by saturating the width with the decays $Y(4660) \rightarrow J/\psi f_0(500)$, $J/\psi f_0(980)$, $\psi'f_0(500)$, $\psi'f_0(980)$. If the experimental value $m_Y = 4652 \pm 10 \pm 8 \text{ MeV}$ is taken, the decay $Y(4660) \rightarrow \psi'f_0(980)$ is kinematically forbidden, and can only take place through the upper tail of the mass distribution, the prediction $\Gamma(Y(4660) \rightarrow \psi'f_0(980)) = 30.2 \pm 8.5 \text{ MeV}$ is too large. Furthermore, other decay channels should be taken into account.

4 Conclusion

In this article, we illustrate how to calculate the hadronic coupling constants in the strong decays of the tetraquark states based on solid quark-hadron quality, then study the hadronic coupling constants $G_{YJ/\psi f_0}$, $G_{Y\eta_c\phi}$, $G_{Y\chi_{c0}\phi}$, $G_{YD_s\bar{D}_s}$, $G_{YD_s^*\bar{D}_s^*}$, $G_{YD_s\bar{D}_s^*}$, $G_{Y\psi'f_0}$, $G_{YJ/\psi\phi}$ in the decays $Y(4660) \rightarrow J/\psi f_0(980)$, $\eta_c\phi(1020)$, $\chi_{c0}\phi(1020)$, $D_s\bar{D}_s$, $D_s^*\bar{D}_s^*$, $D_s\bar{D}_s^*$, $\psi'\pi^+\pi^-$, $J/\psi\phi(1020)$ with the QCD sum rules in a systematic way. The predicted width $\Gamma(Y(4660)) = 74.2^{+29.2}_{-19.2} \text{ MeV}$ is in excellent agreement with the experimental data $68 \pm 11 \pm 1 \text{ MeV}$ from the Belle collaboration, which supports assigning the $Y(4660)$ to be the $[sc]_P[\bar{s}\bar{c}]_A - [sc]_A[\bar{s}\bar{c}]_P$ type tetraquark state with $J^{PC} = 1^{--}$. In calculations, we observe that the hadronic coupling constants $|G_{Y\psi'f_0}| \gg |G_{YJ/\psi f_0}|$, which indicates that the coupling $Y(4660)\psi'f_0(980)$ is very strong, and consistent

with the observation of the $Y(4660)$ in the $\psi'\pi^+\pi^-$ mass spectrum, and favors the $\psi'f_0(980)$ molecule assignment, as there may be appear some $\psi'f_0(980)$ component due to the strong coupling. The decay $Y(4660) \rightarrow J/\psi\phi(1020)$ is greatly suppressed and can take place only through rescattering mechanism. It is important to search for the process $Y(4660) \rightarrow J/\psi\phi(1020)$ to diagnose the nature of the $Y(4660)$.

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References

1. X.L. Wang et al., Phys. Rev. Lett. **99**, 142002 (2007)
2. G. Pakhlova et al., Phys. Rev. Lett. **101**, 172001 (2008)
3. X.L. Wang et al., Phys. Rev. D **91**, 112007 (2015)
4. T. Barnes, S. Godfrey, E.S. Swanson, Phys. Rev. D **72**, 054026 (2005)
5. W.J. Deng, H. Liu, L.C. Gui, X.H. Zhong, Phys. Rev. D **95**, 034026 (2017)
6. B.Q. Li, K.T. Chao, Phys. Rev. D **79**, 094004 (2009)
7. F.E. Close, N.A. Tornqvist, J. Phys. G **28**, R249 (2002)
8. C. Amsler, N.A. Tornqvist, Phys. Rept. **389**, 61 (2004)
9. L. Maiani, F. Piccinini, A.D. Polosa, V. Riquer, Phys. Rev. Lett. **93**, 212002 (2004)
10. Z.G. Wang, Eur. Phys. J. C **76**, 427 (2016)
11. M. Tanabashi et al., Phys. Rev. D **98**, 030001 (2018)
12. F.K. Guo, C. Hanhart, U.G. Meissner, Phys. Lett. B **665**, 26 (2008)
13. F.K. Guo, J. Haidenbauer, C. Hanhart, U.G. Meissner, Phys. Rev. D **82**, 094008 (2010)
14. Z.G. Wang, X.H. Zhang, Commun. Theor. Phys. **54**, 323 (2010)
15. R.M. Albuquerque, M. Nielsen, R. Rodrigues da Silva, Phys. Rev. D **84**, 116004 (2011)
16. S. Dubynskiy, M.B. Voloshin, Phys. Lett. B **666**, 344 (2008)
17. D. Ebert, R.N. Faustov, V.O. Galkin, Eur. Phys. J. C **58**, 399 (2008)
18. G. Cotugno, R. Faccini, A.D. Polosa, C. Sabelli, Phys. Rev. Lett. **104**, 132005 (2010)
19. L.Y. Xiao, X.Z. Weng, Q.F. Lu, X.H. Zhong, S.L. Zhu, Eur. Phys. J. C **78**, 605 (2018)
20. R.M. Albuquerque, M. Nielsen, Nucl. Phys. A **815**, 532009 (2009) [Erratum-ibid. A **857**, 48 (2011)]
21. H. Sundu, S.S. Agaev, K. Azizi, Phys. Rev. D **98**, 054021 (2018)
22. W. Chen, S.L. Zhu, Phys. Rev. D **83**, 034010 (2011)
23. Z.G. Wang, Eur. Phys. J. C **74**, 2874 (2014)
24. Z.G. Wang, Eur. Phys. J. C **78**, 518 (2018)
25. Z.G. Wang, Eur. Phys. J. C **76**, 387 (2016)
26. J.R. Zhang, M.Q. Huang, Phys. Rev. D **83**, 036005 (2011)
27. Z.G. Wang, T. Huang, Nucl. Phys. A **930**, 63 (2014)
28. Z.G. Wang, Commun. Theor. Phys. **63**, 325 (2015)
29. Z.G. Wang, T. Huang, Eur. Phys. J. C **74**, 2891 (2014)
30. Z.G. Wang, Eur. Phys. J. C **74**, 2963 (2014)
31. Z.G. Wang, Eur. Phys. J. C **76**, 70 (2016)
32. Z.G. Wang, T. Huang, Eur. Phys. J. C **76**, 43 (2016)
33. Z.G. Wang, Nucl. Phys. B **913**, 163 (2016)
34. Z.G. Wang, Eur. Phys. J. C **78**, 933 (2018)
35. Z.G. Wang, Eur. Phys. J. C **79**, 29 (2019)
36. Z.G. Wang, J.X. Zhang, Eur. Phys. J. C **78**, 14 (2018)
37. Z.G. Wang, Z.Y. Di, Eur. Phys. J. C **79**, 72 (2019)
38. Z.G. Wang, [arXiv:1812.04503](https://arxiv.org/abs/1812.04503)
39. M.A. Shifman, A.I. Vainshtein, V.I. Zakharov, Nucl. Phys. B **147**, 385 (1979)
40. M.A. Shifman, A.I. Vainshtein, V.I. Zakharov, Nucl. Phys. B **147**, 448 (1979)
41. L.J. Reinders, H. Rubinstein, S. Yazaki, Phys. Rept. **127**, 1 (1985)
42. D. Becirevic, G. Duplancic, B. Klajn, B. Melic, F. Sanfilippo, Nucl. Phys. B **883**, 306 (2014)
43. Z.G. Wang, Eur. Phys. J. C **77**, 174 (2017)
44. Z.G. Wang, W.M. Yang, S.L. Wan, Eur. Phys. J. C **37**, 223 (2004)
45. Z.G. Wang, W.M. Yang, S.L. Wan, Nucl. Phys. A **744**, 156 (2004)
46. Z.G. Wang, Eur. Phys. J. C **75**, 427 (2015)
47. V.A. Novikov, L.B. Okun, M.A. Shifman, A.I. Vainshtein, M.B. Voloshin, V.I. Zakharov, Phys. Rept. **41**, 1 (1978)
48. Z.G. Wang, W.M. Yang, S.L. Wan, Phys. Lett. B **615**, 79 (2005)
49. P. Colangelo, A. Khodjamirian, [arXiv:hep-ph/0010175](https://arxiv.org/abs/hep-ph/0010175)
50. S. Narison, R. Tarrach, Phys. Lett. **125B**, 217 (1983)
51. Z.G. Wang, T. Huang, Phys. Rev. D **89**, 054019 (2014)