On the status of the hoop conjecture in charged curved spacetimes

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Abstract The status and regime of validity of the famous Thorne hoop conjecture in spatially regular *charged* curved spacetimes are clarified.

1 Introduction

The hoop conjecture [1] has attracted the attention of physicists and mathematicians since its introduction by Thorne almost five decades ago [1,2]. This mathematically elegant and physically influential conjecture asserts that a self-gravitating matter configuration of mass M will form an engulfing horizon if its circumference radius $R = C/2\pi$ is equal to (or less than) the corresponding Schwarzschild radius $2M$ [3]. That is, the hoop conjecture states that [1]

$$C \leq 4\pi M \implies \text{Black-hole horizon exists.} \quad (1)$$

It is widely believed that the hoop conjecture reflects a fundamental aspect of classical general relativity. In particular, the conjecture is supported by several studies (see [4–6] and references therein). Intriguingly, however, there are also some claims in the physics literature that the Thorne hoop conjecture can be violated in *charged* curved spacetimes [7,8].

The main goal of the present compact paper is to clarify the status of the Thorne hoop conjecture in spatially regular charged spacetimes. In particular, below we shall explicitly demonstrate that the hoop conjecture is valid in charged curved spacetimes provided the mass parameter on the r.h.s of the hoop relation (1) is appropriately interpreted as the gravitational mass $M(R)$ contained within the engulfing hoop of radius R and *not* as the total (asymptotically measured) mass M_∞ of the entire spacetime.

2 Validity of the hoop conjecture in spatially regular charged spacetimes

It has been argued in [7,8] that the hoop conjecture (1) can be violated in spatially regular horizonless charged spacetimes. In particular, Ref. [7] analyzed the compactness of spherically symmetric fluid matter configurations with uniform charge densities and concluded that, taking the mass parameter in the r.h.s of the hoop relation (1) as the *total* mass M_∞ of the system (as measured by asymptotic observers), the hoop conjecture can be violated. As an illustrative example, Ref. [7] constructed a uniformly charged horizonless ball which is characterized by the dimensionless relations

$$\frac{M_\infty}{R} = 0.65 \quad \text{and} \quad \frac{Q^2}{R^2} = 0.39. \quad (2)$$

This spatially regular *horizonless* charged matter configurations is characterized by the dimensionless ratio

$$\frac{C}{4\pi M_\infty} \simeq 0.769 < 1, \quad (3)$$

and, as claimed in [7], it therefore violates the hoop conjecture (1).

However, we believe that in the Thorne hoop conjecture (1), which relates the mass parameter of the system to its circumference radius $R = C/2\pi$, it is physically more appropriate to interpret M as the gravitational mass contained *within* the engulfing hoop and not as the total mass of the entire curved spacetime.

In particular, since the exterior ($r > R$) electromagnetic energy density associated with a charged ball of radius R and electric charge Q is $T_0^0(r > R) = Q^2/8\pi r^4$ [9], the electromagnetic energy $E_{\text{elec}}(r > R) = \int_R^\infty T_0^0 4\pi r^2 dr$ outside the charged ball is given by the simple expression

$$E_{\text{elec}}(r > R) = \frac{Q^2}{2R}. \quad (4)$$

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Thus, for a charged ball of radius R , electric charge Q , and total mass (energy) M_∞ as measured by asymptotic observers, the gravitational mass contained *within* ($r \leq R$) the ball is given by

$$M(r \leq R) = M_\infty - \frac{Q^2}{2R}. \quad (5)$$

From Eqs. (2) and (5) one obtains the dimensionless relation

$$\frac{M(r \leq R)}{R} = 0.455 \quad (6)$$

for the horizonless charged matter configuration considered in [7]. Taking cognizance of Eqs. (1) and (6), one finds

$$\frac{C(R)}{4\pi M(r \leq R)} \simeq 1.099 > 1. \quad (7)$$

The dimensionless ratio (7) implies, in particular, that the uniformly charged matter configurations studied in [7] do *not* violate the Thorne hoop conjecture (1).

3 Summary

In this compact paper we have explored the (in)validity of the Thorne hoop conjecture [1] in spatially regular *charged* curved spacetimes. Our analysis is motivated by the intriguing claims made in the physics literature (see e.g. [7,8]) according to which this famous conjecture, which is widely believed to reflect a fundamental aspect of classical general relativity, can be violated by horizonless charged matter configurations.

The present analysis clearly demonstrates the fact that, as opposed to the claims made in [7,8], the Thorne hoop conjecture is valid in charged spacetimes provided that, for a given radius R of the engulfing hoop, the mass parameter in the hoop relation (1) is appropriately interpreted as the gravitational mass $M(R)$ contained *within* the hoop (sphere) of radius R and not as the total mass M_∞ of the entire spacetime.

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