Tetraquarks, pentaquarks and dibaryons in the large N QCD

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Abstract We study the multi-quark hadrons in large N QCD under the 't-Hooft limit, extending Witten's picture of the baryons. We explore the decay widths of tetraquarks, pentaquarks and dibaryons. Based on the decay behaviors, we point out that in the $N \rightarrow \infty$ limit decay widths of tetraquarks stay constant, while those of pentaquarks and dibaryons above certain thresholds can diverge. In the large N limit, we find that the ground states of the three spectroscopic series are stable or narrow and that the excited states of pentaquarks and dibaryons above the indicated thresholds are not observables. We compare our results with those obtained in previous large N generalizations of tetraquarks.

1 Introduction

In three colour QCD, multi-quark hadrons beyond the classical $q\bar{q}$ and qqq configurations may be anticipated to exist. This expectation stands on a long-time observation that colour antisymmetric diquarks transform as a colour $\bar{3}$ representation, the same as antiquarks. Thus, starting from a colour singlet hadron, one or more substitutions

$$\bar{q} \rightarrow [q'q''] \quad (1)$$

applied to an antiquark (and/or the corresponding substitutions for a quark) generate new, more complex, colour singlet configurations (for earlier literature, see e.g. [2] and references therein, for recent reviews on exotic hadrons, see e.g. [3–8] and references therein). Explicitly, ignoring the flavor structure for simplicity of notation, one has

$$\begin{aligned} q\bar{q} \quad (\text{Meson}) &\rightarrow \begin{cases} qqq \quad (\text{Baryon}) \\ \bar{q}\bar{q}\bar{q} \quad (\text{Antibaryon}) \end{cases} \rightarrow [qq][\bar{q}\bar{q}] \quad (\text{Tetraquark}) \\ &\rightarrow [qq][q\bar{q}]\bar{q} \quad (\text{Pentaquark}) \rightarrow [qq][qq][q\bar{q}] \quad (\text{Dibaryon}) \end{aligned} \quad (2)$$

However, the extension of these concepts of multi-quarks to general N is not unique. Taking tetraquarks as the examples, one can construct the diquark-antidiquark structure in general N according to:

$$T^{(2q)} = [q^a q'^b - q^b q'^a] \times \bar{q}_a \bar{q}'_b, \quad (a, b = 1, \dots, N). \quad (3)$$

In this scheme, tetraquarks manifest themselves as poles in the correlation functions of four-quark operators [9–18]. Meanwhile a formulation *à la* Witten [19] for tetraquarks, the baryonium scheme, has been explored by Rossi and Veneziano [20–22]

$$\begin{aligned} T &= \epsilon_{aa_1 \dots a_{N-1}} (q^{[a_1} \dots q^{a_{N-1}]} \epsilon^{ab_1 \dots b_{N-1}} (\bar{q}_{[b_1} \dots \bar{q}_{b_{N-1}]}) \\ &\equiv (q^{[\dots} q^{A]}) (\bar{q}_{[\dots} \bar{q}_{B]}) \times \epsilon_{aA} \epsilon^{aB}, \end{aligned} \quad (4)$$

where brackets indicate antisymmetrization of the indices, A and B are sets of $N-1$ antisymmetrized indices and sum from 1 to N over repeated indices is understood.

The antisymmetric combination of $N-1$ quark fields – the generalised diquark – transforms according to the antiquark representation $\bar{N}$ and can replace any antiquark in the totally antisymmetric antibaryon. Thus, the construction of more structures can be performed, as in Fig. 1, the first substitution giving the tetraquark (4). A second substitution gives the large N extension of the $N=3$ pentaquark [25, 26]:

$$P = (q^{[\dots} q^{A]}) (q^{[\dots} q^{B]}) \bar{q}_{[\dots} \bar{q}_{C]} \times \epsilon_{aA} \epsilon_{bB} \times \epsilon^{abC}, \quad (5)$$

with C a set of $N-2$ antisymmetrized indices. Continuing in this way, we end with the generalised dibaryon [27, 28]:

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Fig. 1 Multi-quark hadrons generated by the substitution (1) in N colours QCD. Full boxes indicate the generalised diquarks in (4) and full dots the antiquarks; dotted lines are QCD strings, the crossed circle at the center represents the N dimensional ϵ symbol that joins the N representations $\bar{N}$ into an $SU(N)$ singlet. Hadrons resulting for $N = 3$ are indicated. For brevity, the same names are used in the text for the configurations with the same value of p in N colours QCD

$$D = \left(q^{[...} q^{A]} \right) \dots \left(q^{[...} q^{B]} \right) \times \epsilon_{aA} \dots \epsilon_{bB} \times \epsilon^{a...b}. \quad (6)$$

which binds N generalised diquarks.

Assigning the baryon number $1/N$ to one quark such that $B(\text{baryon}) = 1$, the baryon numbers of tetraquarks, pentaquarks and dibaryons are: 0, 1, $N - 1$, respectively.

In a world of two colours, the above structures disappear: $N = 2$ QCD is made only of mesons, $q\bar{q}$, “baryons”, qq , and molecules thereof. The new spectroscopic series start to appear at $N = 3$, (our world!) and can be extended to N colours.

In the following, we will analyse the decay widths of the new structures that appear at $N = 3$, Eqs. (4, 5, 6), dropping the qualification “generalised” for brevity. We will work in large N QCD under the ’t-Hooft limit [19]:

$$g_s^2 = \frac{\lambda^2}{N}, \quad N \rightarrow \infty, \quad (7)$$

where g_s is the strong coupling constant and λ , the ’t-Hooft coupling, is fixed. In this limit, some decays of generalised tetraquark in Witten’s picture have been considered in Refs. [23, 24].

As we will show, we find that decay amplitudes from the ground states may diverge at large N . However such decays are generally forbidden by phase space and the divergent amplitudes do not affect the observability of such particles. At $N = \infty$, ground states of multi-quark hadrons are narrow or stable, particularly in the case of the dibaryon.

Results for the decay amplitudes of excited states are summarised in Table 1. For tetraquarks, we find decay amplitudes of the excited states that stay constant or decrease

Table 1 Decay amplitudes for the decay of multi-quark hadrons from states with only one quark excited, A^* . Amplitudes are normalised so that $|A^*|^2 = \Gamma$, thus including a $1/\sqrt{N!}$ factor from the phase space of $\mathcal{O}(N)$ indistinguishable particles, see text. Entries in the Table give the leading N dependence. The first two columns refer to decays obtained from Fig. 1 by cutting one or more QCD strings with a $q\bar{q}$ pair, the third column to the decay of an excited into the ground state by one meson emission. For multi-meson or multi-baryon decay amplitudes, polynomial in N appearing as prefactors of the exponentials are omitted. When giving the results for dibaryons, the diquarks inside are treated, at large N , as quasi classical particles

$T^* \rightarrow$	$B + \bar{B}$	$T + \text{Meson}$	Mesons
$A^* \propto$	N^0	N^0	$< e^{-\frac{N}{2}}$
$P^* \rightarrow$	$B + T$	$B + B + \bar{B}$	$P + \text{Meson}$
$A^* \propto$	N^0	$N^{1/2}$	$< e^{-\frac{N}{2}}$
$D^* \rightarrow$	$NB + \bar{B}$	$D + \text{Meson}$	$(N - 1)B$
$A^* \propto$	$> e^{+\frac{N}{2} \log N}$	N^0	$> e^{+\frac{N}{2} \log N}$

with N , thus confirming that the corresponding hadrons are observables. For excited pentaquarks and dibaryons, the de-excitation amplitudes into the ground state plus one meson are limited. However, at $N = \infty$, there are modes which have divergent amplitudes, namely $P^* \rightarrow B + B + \bar{B}$, $D^* \rightarrow NB + \bar{B}$ and $D^* \rightarrow (N - 1)B$. Taken literally, these results would imply sharp thresholds at $2B + \bar{B}$ and $(N - 1)B$ respectively, below which we expect observable pentaquarks and dibaryons, and above which we expect large, unobservable widths: a situation similar with charmonia above and below the open charm meson-antimeson threshold.

The rest of this paper is organized as follows. In Sect. 2 we briefly review meson-baryon couplings in Witten’s scheme to warm up and establish the notations. We consider the decays of generalised tetraquarks in Sect. 3, and extend to the decays of generalised pentaquarks and dibaryons in Sects. 4, 5. Discussion and comparison with previous large N analyses are contained in Sect. 6.

2 Meson-baryon couplings

As suggested by Witten, baryons become very simple in the limit $N \rightarrow \infty$: quarks move independently from each other in an effective, Hartree-Fock, potential which is N independent. In the ground state, all quarks are in the same wavefunction, $\phi_0(x)$, with an N independent baryon radius.

For general states, once colour antisymmetry is guaranteed, quarks behave like a set of bosons, distributed in single particle excited states, ϕ_r . The state is determined by the occupation numbers n_r , with,

$$M^* = NM_0 + \sum_r n_r \epsilon_r, \quad \sum_r n_r = N, \quad (8)$$

Fig. 2 Feynman diagrams for meson-baryon trilinear coupling in the large N limit. An excited quark emits a gluon, turning into a quark-antiquark pair with the factor λ^2/N

where M_0 and ϵ_r the excitation energy of ϕ_r , are N -independent [19]. We shall restrict to states with excitation energy that stays fixed when $N \rightarrow \infty$. For illustration, we focus on: (i) the ground state with $n_0 = N$ and all other occupation numbers vanishing and (ii) the single quark excited state with $n_0 = N - 1$ and $n_r = 1$ for some r .

The normalised wave function of the above baryon states can be written as,

Ground state:

$$\Psi_0^{1\ 2\ \dots N}(x_1, x_2, \dots N) = \epsilon^{12\dots N} \phi_0(x_1) \times \phi_0(x_2) \dots \phi_0(x_N) \quad (9)$$

Excited state:

$$\begin{aligned} \Psi_r^{1\ 2\ \dots N}(x_1, x_2, \dots N) &= \epsilon^{12\dots N} \times \frac{1}{\sqrt{N}} \\ &\times [\phi_r(x_1)\phi_0(x_2) \dots \phi_0(x_N) + \phi_0(x_1)\phi_r(x_2) \dots \phi_0(x_N) \\ &+ \dots + \phi_0(x_1)\phi_0(x_2) \dots \phi_r(x_N)]. \end{aligned} \quad (10)$$

Here $1, 2, \dots, N$ are colour indices. Since the flavour and spin indices are omitted, Ψ are antisymmetric in colour (to make a colour singlet) and fully symmetric in coordinate space.

The meson-baryon trilinear coupling is represented in Fig. 2. The initial quark wave function is indicated by ϕ_{in} , the final quark is in the ground state ϕ_0 . With $\phi_{in} = \phi_0$ or ϕ_r , we obtain the ground state meson-baryon coupling, e.g. $g_{N\bar{N}\pi}$, or the transition amplitude of the excited state, e.g. $A(\Delta \rightarrow N\pi)$.

The basic transition occurs via one gluon exchange: $q^i(\phi_{in}) \rightarrow q^i(\phi_0) + q^\ell + \bar{q}^\ell$. Projecting over the colour singlet meson state,

$$M = \frac{1}{\sqrt{N}} \sum_\ell \bar{q}_\ell q^\ell, \quad (11)$$

one finds the effective operator for the baryon to baryon transition,

$$O = \frac{\lambda^2}{N} \sqrt{N} \bar{q} \mathcal{O}(x) q. \quad (12)$$

In the above, the $\mathcal{O}(x)$ is a N -independent operator acting on the single quark wave functions $\phi(x)$ in (9) and (10) and connecting ϕ_{in} to ϕ_0 .

The transition operator applied to (9) gives N equal terms and we obtain [19]:

$$g_{B\bar{B}M} \sim \frac{\lambda^2}{N} \sqrt{N} N \propto \sqrt{N}. \quad (13)$$

When applied to (10), the transition operator has to operate on ϕ_r only, to obtain a non-vanishing result when the scalar product with (9) is taken. We obtain N equal factors, divided by the normalisation $\sqrt{N}$, so that,

$$A(B^* \rightarrow B + M) \sim \frac{\lambda^2}{N} \sqrt{N} \frac{N}{\sqrt{N}} \propto N^0. \quad (14)$$

It is interesting to notice that the tree-level meson-baryon low energy scattering amplitude obtained from (13) is $\mathcal{O}(1)$ since the baryon's propagator brings in a factor N^{-1} due to the baryon's mass [19].

3 Tetraquark decays

We first consider the natural decay of the tetraquark, the baryonium mode:

$$T \rightarrow B + \bar{B}, \quad (15)$$

which is depicted in Fig. 3. The Feynman diagram is basically the same of the previous Section, but we must be careful with the normalizations of initial and final states, which differ by N dependent factors.

We normalise the mesons M , baryons B and tetraquarks T as,

$$\langle 0 | M M^\dagger | 0 \rangle = \langle 0 | B B^\dagger | 0 \rangle = \langle 0 | T T^\dagger | 0 \rangle = 1. \quad (16)$$

For mesons the above normalization is realized in (11) while for baryons, we have,

$$B = q^1 q^2 \dots q^N, \quad \bar{B} = \bar{q}_1 \bar{q}_2 \dots \bar{q}_N, \quad (17)$$

which is consistent with Eq. (9).

For tetraquarks, we use the wave function:

$$T = \frac{1}{\sqrt{N}} B_a \bar{B}^a, \quad (18)$$

Fig. 3 Feynman diagrams for the tetraquark decays into a pair of baryons, $T \rightarrow B + \bar{B}$

where $B_a = \frac{\partial}{\partial q^a} B$ is the operator B with q^a suppressed, similarly for $\bar{B}^a$ with $\bar{q}_a$ and the sum over $a = 1, \dots, N$ is understood.

Due to the anticommutation properties of quark fields, there might be a minus sign in the definition of B_a , depending from the position of q^a in (17). However, one has the identity,

$$q^a B_a = q^a \frac{\partial}{\partial q^a} B = B, \quad \text{for any } a \quad (19)$$

since $q^a \partial / \partial q^a$ is a bosonic operator.

Consider in Fig. 3 the case where quark lines in the initial diquark correspond to B_1 . Then $i_N = 1$ and the pair created by the gluon interaction provides the missing q^1 and $\bar{q}_1$ to the product $B_1 \bar{B}^1$. We need to add all diagrams where the gluon is emitted by the other quark lines, which gives a factor $N - 1$. Finally, summing over a gives a factor of N , since all the terms of the sum are equal to the one with $a = 1$.

Given various anticommutation signs, one can suspect that terms arising from different values of a come with different signs, but one can show that it is not so. The gluon interaction from the i th quark produces the substitution:

$$q^i \rightarrow q^i q^a \bar{q}_a \quad (20)$$

Since $q^a \bar{q}_a$ is a bosonic operator, the result from the term $B_a \bar{B}^a$ can be written as,

$$\begin{aligned} B_a \bar{B}^a \text{ (no sum on } a) &\rightarrow B_a \cdot q^a \bar{q}_a \cdot \bar{B}^a \\ &= (-1)^{(N-1)} q^a \frac{\partial}{\partial q^a} B \cdot \bar{q}_a \frac{\partial}{\partial \bar{q}_a} \bar{B} \\ &= (-1)^{(N-1)} \cdot B \bar{B}, \end{aligned} \quad (21)$$

which is independent on a .

Adding the diagrams in which gluon emission occurs from an antiquark line gives a factor two to the final result.

Fig. 4 Feynman diagrams for the decay of an excited tetraquark to the ground state by emitting a light meson. The diagrams with the deexcitation from the antiquark only produce a factor of two

In conclusion, putting all together, we find the scaling:

$$A(T \rightarrow B + \bar{B}) = \frac{1}{\sqrt{N}} \frac{\lambda^2}{N} (N - 1) N \propto \sqrt{N}. \quad (22)$$

The decay in Eq. (15) is unlikely to occur for the ground state tetraquark, since the $B\bar{B}$ state has one pair of constituent quarks compared to the initial state, and there is not enough phase space.

Instead, the decay in $B\bar{B}$ will occur for an excited (radial or orbital) state T^* , where the excitation energy $\epsilon_r - \epsilon_0$ can be used to create the additional pair that will transform the tetraquark into $B\bar{B}$. A consequence is that only the excited quark emits the gluon in Fig. 3 and we loose a factor of $\sqrt{N - 1}$ as in Sect. 2:

$$A(T^* \rightarrow B + \bar{B}) \propto N^0. \quad (23)$$

The analogy with Eqs. (14) and (13) is evident.

Taking the amplitudes in (22) and (23) as proportional to the couplings of $g_{TB\bar{B}}$ and $g_{T^*B\bar{B}}$, we may estimate the large N behaviour of the scattering amplitude $B + \bar{B} \rightarrow B + \bar{B}$ via tetraquark intermediate state. We find:

$$\begin{aligned} A(B + \bar{B} \rightarrow B + \bar{B}) &\propto g_{TB\bar{B}} \frac{1}{M_T^2} g_{TB\bar{B}} + \sum_{T^*} g_{T^*B\bar{B}} \frac{1}{M_{T^*}^2} g_{T^*B\bar{B}} \\ &\propto \mathcal{O}(N^{-1}) + \sum_{T^*} \mathcal{O}(N^{-2}). \end{aligned} \quad (24)$$

since $M_T^2 \sim M_{T^*}^2 \propto N^2$. We find a leading contribution of order N^{-1} from the ground state in agreement with the estimate in [21, 22], see their Eq. (26), and a nonleading one from the excited states of order N^{-2} .

We now consider the decay of an excited tetraquark to the ground state by emitting a meson, which is shown in Fig. 4. There is a factor $(1/\sqrt{N})^2$ from the normalization of the initial and final tetraquarks, a factor $1/\sqrt{N}$ for the normalization of the meson, a factor N for the number of $q^l \bar{q}_l$ pairs, a factor N for the number of diquarks with different colours, each of which gives a factor $\sqrt{N - 1}$ for the transition from

Fig. 5 Feynman diagrams for the tetraquark decays into $N - 1$ mesons

an excited quark. Multiplying by the factor $1/N$ from the gluon interaction, we obtain,

$$A(T^* \rightarrow T + M) \propto N^0. \quad (25)$$

The diagrams with the deexcitation from the antiquark only produce a factor of two.

We continue the discussion on tetraquarks with the decay into mesons:

$$T \rightarrow M_1 + \dots + M_{N-1}. \quad (26)$$

Quarks lines in the diquark are paired with the corresponding antiquark lines, to form $N - 1$ quark-antiquark mesons, depicted in Fig. 5. The complex of these diagrams lead to the effective Lagrangian,

$$\mathcal{L}_{eff} = \frac{g(N)}{(2m)^{N-4}} M^{N-1} T, \quad (27)$$

and,

$$g(N) \propto \frac{N}{\sqrt{N}} \left(\frac{\lambda^2}{\sqrt{N}} \right)^{N-1} \propto N^{\frac{1}{2}(N-1)}, \quad (28)$$

m is the quark mass and the dimensional factor, needed to makes $g(N)$ adimensional, will be cancelled by similar dimensional factors appearing in the computation of the rate see Appendix. We find, omitting finite powers of N ,

$$\begin{aligned} \Gamma &\propto \frac{g(N)^2}{(2m)^{2(N-4)}} \frac{(N-1)!^2}{(N-1)!} \\ &\times \frac{1}{2M_T} \int \frac{dk_1 dk_2 \dots dk_{N-1}}{(2\mu)^{N-1}} \delta \left(Q_T - \sum_i \frac{k_i^2}{2\mu} \right) \\ &\times \delta^{(3)} \left(\sum_i \mathbf{k}_i \right), \end{aligned} \quad (29)$$

the $(N-1)!^2$ in the numerator comes from the contraction of the operator M^{N-1} with the external meson states, the $(N-1)!$ in the denominator from the phase space of $N - 1$

identical mesons. In the phase space integral, μ is the meson mass and Q_T the Q value of the decay, using Eq. (8):

$$Q_T = M(T) - (N-1)\mu = (2M_0 - \mu)(N-1) \quad (30)$$

The phase space integral is discussed in Appendix and shown to give a decreasing contribution at large N . An upper bound to the rate is obtained by keeping factorials only. Omitting polynomial prefactors, we find,

$$\begin{aligned} \Gamma &< g(N)^2 (N-1)! \sim \frac{N!}{N^N} = e^{-N}; \\ A(T \rightarrow M_1 + \dots + M_{N-1}) &= \sqrt{\Gamma} < e^{-\frac{N}{2}}. \end{aligned} \quad (31)$$

Up to a polynomial in N , the result holds for the decay amplitude of the excited tetraquark, A^* , as given in Table. 1.

Finally, as a way to check our method, we compute the amplitude for the decay of a tetraquark to a complex $T_{-2} = (N-2)q + (N-2)\bar{q}$, using the diagram in Fig. 5 restricted to the exchange of one gluon.

Following Eq. (18), we define,

$$B_{ab} = \frac{\partial}{\partial q^a} \frac{\partial}{\partial q^b} B = -B_{ba}. \quad (32)$$

with B_{ab} antisymmetric for any value of N . The normalized T_{-2} is defined as,

$$T_{-2} = \frac{1}{\sqrt{N(N-1)/2}} \sum_{a < b} B_{ab} \bar{B}^{ab}, \quad a, b = 1 \dots, N. \quad (33)$$

From Fig. 5 we find:

$$\begin{aligned} T &= \frac{1}{\sqrt{N}} \sum_a B_a \bar{B}^a \\ &\rightarrow \frac{1}{\sqrt{N}} \frac{\lambda^2}{N} \sum_{b > a} B_{ab} \bar{B}_{ab} \sum_{\ell} q^{\ell} \bar{q}_{\ell} \\ &= \frac{1}{\sqrt{N}} \frac{\lambda^2}{N} \sqrt{N} M \sqrt{\frac{N(N-1)}{2}} T_{-2} \\ &\propto N^0 M \cdot T_{-2}, \end{aligned} \quad (34)$$

in agreement with [24].

4 Pentaquarks

In analogy to Eqs. (18) and (33), one can simplify the expression describing normalized pentaquarks according to,

$$P = \frac{1}{\sqrt{N(N-1)/2}} \sum_{a < b} B_a B_b \bar{B}^{ab}, \quad a, b = 1 \dots, N. \quad (35)$$

The antisymmetry in colour has to be matched to the symmetry under exchange of the two diquarks in Eqs. (5) and (35), which are fermions or bosons for $N = \text{even}$ or odd, respectively.

To see how this works, we smear out the coordinates of the generalized diquarks with a function $F(x, y, z)$, according to,

$$P = \frac{1}{\sqrt{2N(N-1)}} \int dx dy dz F(x, y, z) \times \sum_{a \leq b} B_a(x) B_b(y) \bar{B}^{ab}(z). \quad (36)$$

Thus, Bose or Fermi symmetry under the exchange: $x, a \leftrightarrow y, b$ implies,

$$F(y, x, z) = (-1)^N F(x, y, z). \quad (37)$$

For simplicity, we have assumed diquarks with equal flavour and spin distribution, in which case (37) implies diquarks in relative P -wave for N odd. Unequal flavour and spin properties, however, allow S and P -wave diquarks, as considered in [26].

A case in point is the calculation of the decay into a baryon and a tetraquark, which goes through the process (20), where the gluon is radiated from either one or the other diquark. The process transforms the initial state (35) according to (overall signs are ignored)

$$P \rightarrow \frac{1}{\sqrt{N(N-1)/2}} \left(\frac{\lambda^2}{N} \right) (N-1) \times \sum_{a < b} \left[\bar{q}_a q^a B_a B_b + (-1)^N B_a \bar{q}_b q^b B_b \right] \bar{B}^{ab}, \quad (38)$$

where $(-1)^N$ is the statistical factor associated to the exchange of diquark coordinates, (37). Using Eqs. (19, 32), we have,

$$\begin{aligned} & \sum_{a < b} \left[\bar{q}_a q^a B_a B_b + (-1)^N B_a \bar{q}_b q^b B_b \right] \bar{B}^{ab} \\ &= -B \sum_{a < b} \left(B_b \bar{B}^b + B_a \bar{B}^a \right) \\ &= -B \left[\sum_{b=1}^N B_b \bar{B}^b (b-1) + \sum_{a=1}^N B_a \bar{B}^a (N-a-1) \right] \\ &= -(N-2)\sqrt{N} B \cdot T. \end{aligned} \quad (39)$$

One should add pair creation from the antiquarks in B^{ab} , with the antiquark completing the antidiquark and the quark absorbed by one diquark to make the baryon. This process gives an amplitude of the same form as (38) (dictated by colour conservation and statistics) and the same N dependence, so that,

$$A(P \rightarrow B + T) \propto \sqrt{N}. \quad (40)$$

As before, decay from the ground state is forbidden by phase space while the decay amplitude from an excited pentaquark is reduced by a factor $\sqrt{N-1}$,

$$A(P^* \rightarrow B + T) \propto N^0. \quad (41)$$

For other decay modes, we obtain similarly:

$$A(P \rightarrow B + B + \bar{B}) = \mathcal{O}(N), \quad (42)$$

$$A(P^* \rightarrow P + M) \propto N^0, \quad (43)$$

$$A(P, P^* \rightarrow B + \text{Mesons}) < e^{-\frac{N}{2}}. \quad (44)$$

For decay of a pentaquark excited state, P^* , to the final state in (42), we have a reduction of a factor $\sqrt{N}$ in amplitude, Table 1, which, however, is still divergent for $N \rightarrow \infty$.

5 Dibaryons

Starting from the definition, Eq. (6), we smear the coordinates of each generalised diquark with a function $F(x_1, \dots, x_N)$,

$$D = \int dx_1 \dots dx_N F(x_1, \dots, x_N) \times \epsilon^{\alpha \dots \beta} B_\alpha(x_1) \dots B_\beta(x_N). \quad (45)$$

For diquarks with identical quark spin and flavour composition, $F(x_1, x_2, \dots, x_N)$ must be symmetric (antisymmetric) in the exchange of any two coordinates for $N = \text{even}$ (odd).

For $N = 3$, the role of spin and flavour composition is well illustrated by the first example of a dibaryon considered in the literature, the so-called H -dibaryon introduced by Jaffe [27] and based on the antisymmetric scalar diquarks of $SU(3)_{\text{flavor}}$,

$$\begin{aligned} H &= \epsilon^{\alpha\beta\gamma} ([ds]_0)_\alpha(x_1) ([su]_0)_\beta(x_2) ([ud]_0)_\gamma(x_3) \\ &= \epsilon^{\alpha\beta\gamma} \phi_{\alpha 1}(x_1) w_{\beta 2}(x_2) w_{\gamma 3}(x_3) \\ &\sim \epsilon^{\alpha\beta\gamma} \epsilon^{ijk} \phi_{\alpha i}(x_1) w_{\beta j}(x_2) w_{\gamma k}(x_3), \end{aligned} \quad (46)$$

where the subscript 0 indicates the total spin of the diquarks, i, j, k are $SU(3)_{\text{flavor}}$ indices. An S -wave dibaryon, fully symmetric under coordinate exchanges, is made possible by the antisymmetry in flavour of the bosonic diquarks.

Starting from (45), we restrict from now on to equal spin and flavour composition. We can reduce all terms multiplying the fully antisymmetric tensor to the basic permutation and write

$$D = \int dx_1 \dots dx_N F(x_1, \dots, x_N) B_1(x_1) \dots B_N(x_N). \quad (47)$$

We consider first the decays into many baryons, $D \rightarrow NB + \bar{B}$. The decay is produced by the emission of one $q\bar{q}$ pair from each diquark, Eq. (20), with each quark joining

its diquark to form a baryon and the antiquarks forming the antibaryon. Starting from (47) we obtain

$$D \rightarrow \left(\frac{\lambda^2}{N}\right)^N (N-1)^N \int dx_1 \dots dx_N F(x_1, \dots, x_N) \times [\bar{q}_1(0)q^1(x_1)B_1(x_1)] \dots [\bar{q}_N(0)q^N(x_N)B_N(x_N)]. \quad (48)$$

After using Eq. (19) and ignoring overall signs, we obtain

$$D \rightarrow \left(\frac{\lambda^2}{N}\right)^N (N-1)^N \int dx_1 \dots dx_N F(x_1, \dots, x_N) \times \bar{B}(0)B(x_1) \dots B(x_N). \quad (49)$$

The effective Lagrangian for the $D \rightarrow NB + \bar{B}$ can be constructed as:

$$\mathcal{L}_{eff} = g(N) \times \begin{cases} (M_0)^{-\frac{3N}{2} + \frac{3}{2}}, & (N = \text{odd}) \\ (M_0)^{-N+2}, & (N = \text{even}) \end{cases} \times (B^{\dagger N} \bar{B}^{\dagger} D + \text{h.c.});$$

$$g(N) \propto N^0. \quad (50)$$

To compute the decay rate, we consider first the case of $N = \text{odd}$, in particular $N = 3$, where baryons are fermions. The extension to even N , when F is symmetric, is obvious and it gives the same result.

Suppose that in the dibaryon, diquarks are in single diquark wave functions $w_0(x)$, $w_1(x) \dots$. We use a different symbol for the wave functions, to distinguish the motion of diquarks, which are quasi classical particles, with energy levels that vanish at $N = \infty$, from the motion of quarks in the diquark, which are fully quantum with energy level spacing of order $\mathcal{O}(\Lambda_{QCD})$.

The effective lagrangian has baryons in the same wave functions as the original diquarks, and one needs to consider the fully antisymmetric combination, represented by the so-called Slater determinant. We write explicitly the $N = 3$ case,

$$F(x_1, x_2, x_3 | a, b, c) = C \times \text{Det} \begin{bmatrix} w_a(x_1) & w_b(x_1) & w_c(x_1) \\ w_a(x_2) & w_b(x_2) & w_c(x_2) \\ w_a(x_3) & w_b(x_3) & w_c(x_3) \end{bmatrix}. \quad (51)$$

In the above, C is a normalization constant, the determinant is the sum of $3!$ products $w_{a'}(x_1)w_{b'}(x_2)w_{c'}(x_3)$ where a', b', c' is a permutation of a, b, c , multiplied by $+1$ (-1) if the permutation is even (odd). The w are orthogonal and these terms make an orthonormal system, so that the normalization is $C = 1/\sqrt{3!}$.

Omitting proportionality constants independent from N , the transition matrix element is,

$$T = \langle B(k_1), B(k_2), B(k_3), \bar{B}(p) | \mathcal{L}_{eff} | D \rangle = g(3) \frac{1}{\sqrt{3!}} 3! \times \tilde{F}(k_1, k_2, k_3 | a, b, c) \times f(k), \quad (52)$$

where $f(k)$ is a function of the momenta and $\tilde{F}$ contains the Fourier transforms of the ϕ s.

The $3!$ in the numerator arises because each monomial, say $B(x_1)^{\dagger}B(x_2)^{\dagger}B(x_3)^{\dagger}w_a(x_1)w_b(x_2)w_c(x_3)$, when contracted with the external baryons gives rise to the full Slater determinant $\tilde{F}(k_1, k_2, k_3 | a, b, c)$ with the appropriate sign.

Given the orthonormality of $\tilde{F}$ and passing to general $N = \text{odd}$, one has:

$$\Gamma = |A|^2 \propto g(N)^2 N!^2 \frac{1}{N!} \times \int dk_1 dk_2 \dots dk_{N+1} \delta \left(Q_D - \sum_i \frac{k_i^2}{2M} \right) f(k) \quad (53)$$

where M is the nucleon mass and Q_D the Q value. Extending Eq. (8) to the ground state, one would obtain a negative Q value, so we consider directly the excited state, where

$$Q_D = M_D^* - (N+1)M = M_0^*(N-1)N - N^2 M_0 \sim (M_0^* - M_0)N^2 \quad (54)$$

As shown in Appendix, the phase space integral is divergent for large N so that a *lower bound* to the rate is obtained by taking the factorials only and we find:

$$A(D^* \rightarrow NB + \bar{B}) = \sqrt{\Gamma} > e^{+\frac{N}{2} \log N}. \quad (55)$$

as given in Table 1, up to polynomial prefactors. It is necessary to mention that the results are based on the treatment that the diquarks are quasi classical particles.

We next consider $D \rightarrow (N-1)B$. The basic element is reported in Fig. 6, which illustrates the exchange of quark q^2 between the generalised diquarks B_1 and B_2 , completing the latter into a baryon and transforming the former in B_{12} . Transferring q^3 , q^4 , etc. to B_3 , B_4 etc. one obtains a final state with $N-1$ baryons. The amplitude of Fig. 6, taking into account the different options for quarks i and ℓ , is,

$$A(B_1 + B_2 \rightarrow B_{12} + B) \propto \left(\frac{\lambda^2}{N}\right)^2 (N-2)(N-1). \quad (56)$$

One can proceed considering the transfer of q^3 to B_3 and so on. Finally, one replaces the role of B_1 with B_2 , etc.

Collecting all factors, one obtains the effective lagrangian,

$$\mathcal{L}_{eff} = g(N) \times \begin{cases} (M_0)^{-\frac{3}{2}(N-1)+3}, & (N = \text{odd}) \\ (M_0)^{-N+4}, & (N = \text{even}) \end{cases} \times (B^{\dagger(N-1)} D + \text{h.c.}),$$

$$g(N) \propto \frac{N!}{N^N} \sim e^{-N}, \quad (57)$$

Fig. 6 Feynman diagrams for the dibaryon decays into $N - 1$ baryons, $D \rightarrow (N - 1)B$

and, proceeding as in the previous case

$$A(D \rightarrow (N - 1)B) > \frac{N!^{3/2}}{N^N} \sim e^{+\frac{N}{2} \log N}. \quad (58)$$

The result holds for decays of the excited dibaryons as well, in our approximation of neglecting polynomial prefactors.

Finally, we consider $D^* \rightarrow D + \text{Meson}$. The basic process is similar with Fig. 2, adapted to the excited dibaryon initial state. In the $N \rightarrow \infty$ limit, diquarks have infinite mass and can be considered as classical objects. We may assume that the excited diquark is B_1 :

- a factor $\sqrt{N - 1}$ for the gluon emitted by a quark in B_1 with the transition $w_r \rightarrow w_0$;
- a factor $\sqrt{N}$ for the meson;
- a factor λ^2/N for gluon exchange.

In total, we have,

$$A(D^* \rightarrow D + M) \propto N^0. \quad (59)$$

If allowed by phase space, the results (55) and (58) would make the width of the dibaryon ground state unobservably large. However, as observed in [27] for the H -dibaryon, colour and spin-spin attraction may push the mass of the ground state below the N baryons threshold as well, and make the ground state dibaryon to be stable under strong decay¹. A similar phenomenon has been advocated for diquarks made by two heavy quarks [31] to predict doubly heavy tetraquarks to be stable against strong decays [32–34].

In these scenarios, the amplitude (58) is ineffective, the ground state decays weakly and the de-excitation amplitudes

¹ An even lighter, stable H -dibaryon has been advocated in [29] as the source of the astrophysical dark matter, see [30] for an updated discussion.

of lightly excited dibaryons go to a constant value at large N , Eq. (59).

6 Discussions

In summary, we have extended Witten's description of baryons in large N QCD [19] to the multi-quark hadrons generated by replacing one or more antiquarks in an antibaryon with the generalised diquark made by $N - 1$ quarks in the anti-symmetric colour combination. The first step reproduces the generalised tetraquark considered by Rossi and Veneziano [21, 22]. Successive substitutions produce the large N generalisation of multi-quark configurations considered for $N = 3$: pentaquarks [26] and dibaryons [27, 28].

We have studied in the present paper the decays into conventional baryons and mesons of ground and low-lying excited states, namely states with a finite energy difference with respect to the ground state, in the limit $N \rightarrow \infty$.

Decay amplitudes for the ground states generally diverge with N . However, we have argued that the final configurations with divergent amplitudes are forbidden by phase space. In this case, we would expect ground states with narrow widths, for tetra and pentaquarks where multi-meson states are available, or decay by weak interactions, in the case of dibaryons where also N baryon states are expected to be phase-space forbidden.

The decay amplitudes of multi-quark low lying excited states are reported in Table 1. The first two columns refers to decays obtained from Fig. 1 by cutting one or more QCD strings with a $q\bar{q}$ pair, the third column to the decay of an excited into the ground state by meson emission. The last column refers to decays obtained by reorganising the quark-antiquark pairs of the initial state into a multi-meson state or in redistributing the quarks of one diquark to the other diquarks, to form a set of $N - 1$ baryons.

A few remarks are given in order.

- *Excited tetraquarks* The amplitudes for the decay of the excited states do vanish or remain constant for $N \rightarrow \infty$, therefore leading to observable states in this limit;
- Tetraquark de-excitation amplitudes and $B\bar{B}$ decay amplitudes are of the same order;
- for $N = 3$ and flavour composition $[cu][\bar{c}\bar{u}]$ the threshold for two-baryon decay is $2M(\Lambda_c) \sim 4570$ MeV; in Ref. [35] it is argued that $X(4660)$ is a P -wave tetraquark decaying predominantly into $\Lambda_c \bar{\Lambda}_c$ in addition to the mode into $\psi(2S)\pi\pi$ [36].
- Tetraquark-charmonium mixing is exponentially suppressed;
- excited pentaquarks and dibaryons: de-excitation amplitudes into the ground state and a meson remain limited for large N ;

- at $N = \infty$ there are modes which give divergent amplitudes, namely $P^* \rightarrow B + B + \bar{B}$ and $D^* \rightarrow NB + \bar{B}$ or $(N - 1)B$;
- taken literally, these results, imply sharp thresholds at $2B + \bar{B}$ and $(N - 1)B$ respectively, below which we expect observable pentaquarks and dibaryons, and above which we expect large, unobservable widths: a situation similar to charmonia above and below the open charm-anticharm meson threshold.
- For $N = 3$ and pentaquark with flavour composition: $[cu][ud]\bar{c}$, corresponding to the states observed by LHCb [25], the threshold for “non-observability” would be $2M(\Lambda_c) + M(P) \sim 5510$ MeV, for a double charmed dibaryon with flavour $[cu][cd][ud]$ the threshold would be at: $2M(\Lambda_c)$.

For tetraquarks, we agree with [24] for multiquark decays and for the decay into $T_{-2} + \text{Meson}$.

We have added the decay into $B + \bar{B}$ which brings in a divergent behaviour for the ground state. The divergence at large N is not relevant for the width and the observability of the ground state, which is below threshold for the decay, but it makes it dominant as intermediate state in elastic $B + \bar{B}$ scattering, see Eq. (24). The $1/N$ behaviour we find for the latter amplitude is in agreement with the result given in [21, 22].

Also novel is the result about the de-excitation of a tetraquark to the ground state plus a meson, expected to have a constant limit for $N \rightarrow \infty$ and to be phenomenologically important.

Finally, it is interesting to compare the results for tetraquarks with the analysis based on the large N generalisation of diquarks given in (3). The results in [18] feature,

- a suppressed decay amplitude of the ground state into two mesons, of order N^{-2} ; for large N this is larger than the exponentially suppressed amplitude in Table 1, but it takes $N \geq 6$ for the power suppression to win over the exponential suppression (see [37] for a further suppression of two meson decay due to the potential barrier opposing $q\bar{q}$ annihilation in the diquark-antidiquark picture);
- amplitude of order $N^{-1/2}$ for the de-excitation into the ground state by meson emission;
- tetraquark-charmonium mixing occurs to order $N^{-3/2}$;
- the decay of an excited tetraquark into $B\bar{B}$ cannot be computed.

The similarities of two very different multiquark generalisations encourage us to think that they support the $N = 3$ interpretation of exotic hadrons as reflections of diquark dynamics.

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Appendix A: Phase space integral at large N

In connection with Eq. (29), we study here the behaviour of n -particles phase space integral for large n . In the relativistic case and for large n we approximate the momentum δ -function as,

$$\delta^{(3)}\left(\sum_i \mathbf{k}_i\right) \sim \delta^{(3)}(0) \sim (2mQ)^{-\frac{3}{2}}, \quad (\text{A1})$$

and consider:

$$\mathcal{I}(n) = \int dk_1 dk_2 \dots dk_n (2mQ)^{-\frac{3}{2}} \delta\left(Q - \sum_i \frac{k_i^2}{2m}\right), \quad (\text{A2})$$

Defining,

$$K^2 = \sum_i k_i^2 \quad (\text{A3})$$

the integral is,

$$\begin{aligned} \mathcal{I}(n) &= (2mQ)^{-\frac{3}{2}} \int dk_1 dk_2 \dots dk_n \delta\left(K^2 - \sum_i \frac{k_i^2}{2m}\right) \\ &= m(2mQ)^{-\frac{3}{2}} C(3n) K^{(3n-2)}, \end{aligned} \quad (\text{A4})$$

where $C(3n)K^{3n-1}$ is the surface of the hypersphere of radius K in $3n$ dimensions, that is,

$$C(3n) = \frac{2\pi^{\frac{3n}{2}}}{\Gamma\left(\frac{3n}{2}\right)} \quad (\text{A5})$$

so that,

$$\mathcal{I} = \frac{2m\pi^{\frac{3n}{2}}}{\Gamma\left(\frac{3n}{2}\right)} (2mQ)^{\frac{3n}{2} - \frac{5}{2}}. \quad (\text{A6})$$

(a) *Tetraquark to mesons* From (30) we read

$$Q_T = (2M_0 - \mu)(N - 1) = \delta Q_T(N - 1), \quad (\text{A7})$$

δQ_T is essentially the difference between the masses of two quarks bound in a nucleon or in the meson, $\delta Q_T < 2\mu$.

Setting $n = N - 1$ and neglecting non leading terms except in dimensionfull terms, we find

$$\begin{aligned} \mathcal{I} &= (2\mu)^{3N-7} \left(\frac{\delta Q_T}{3\mu} \right)^{\left(\frac{3N}{2}\right)} \frac{\left(\frac{3N}{2}\right)^{\left(\frac{3N}{2}\right)}}{\Gamma\left(\frac{3N}{2}\right)} \\ &\sim (2\mu)^{(3N-7)} \left(\frac{\delta Q_T}{3\mu} \right)^{\left(\frac{3N}{2}\right)}, \end{aligned} \quad (\text{A8})$$

having used Stirling's fomula. The factor $(2\mu)^{(3N-7)}$ is absorbed by corresponding factors in the formula for the rate and in the dimensions of the coupling, Eqs. (27) and (29), to yield the correct dimension of the decay rate. In conclusion,

$$\Gamma \propto g(N)^2 e^{-N \left[\frac{3}{2} \log\left(\frac{3\mu}{\delta Q_T}\right) + C \right]}, \quad (\text{A9})$$

where C is the logarithm of all the adimensional factors like $2\pi s$, meson to quark mass ratio etc.. For small δQ_T ,

$$\Gamma \propto e^{-N \left[1 + \frac{3}{2} \log\left(\frac{3\mu}{\delta Q_T}\right) \right]} < e^{-N}, \quad (\text{A10})$$

as reported in Eq. (31) and Table 1.

(b) *Dibaryon to $NB + \bar{B}$* In this case, $n = N + 1$ and, see Eqs. (8) and (54),

$$Q_D \sim (M_0^* - M_0) N^2 = \delta Q_D N^2, \quad (\text{A11})$$

where δQ_D is essentially the mass difference of a quark mass in the dibaryon or in the baryon. Using (50), (A6) and inserting the normalization factors for $N + 1$ fermions or bosons as appropriate, one finds from Eq. (53),

$$\Gamma \sim g(N)^2 N! \frac{(M_0)^2}{M_D} \left(\frac{\delta Q_D}{M_0} \right)^{\frac{3}{2}} \times \begin{cases} N^{3N}, & (N = \text{odd}) \\ N^{2N}, & (N = \text{even}) \end{cases}. \quad (\text{A12})$$

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