Eur. Phys. J. C (2018) 78:935
https://doi.org/10.1140/epjc/s10052-018-6422-8

THE EUROPEAN PHYSICAL JOURNAL C

Regular Article - Theoretical Physics

The Reissner–Nordström black hole with the fastest relaxation rate

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Received: 12 September 2018 / Accepted: 5 November 2018 / Published online: 14 November 2018

Abstract Numerous *numerical* investigations of the quasinormal resonant spectra of Kerr-Newman black holes have revealed the interesting fact that the characteristic relaxation times $\tau(\bar{a},\bar{Q})$ of these canonical black-hole spacetimes can be described by a two-dimensional function $\bar{\tau}\equiv\tau/M$ which increases monotonically with increasing values of the dimensionless angular-momentum parameter $\bar{a}\equiv J/M^2$ and, in addition, is characterized by a non-trivial (*non*-monotonic) functional dependence on the dimensionless charge parameter $\bar{Q}\equiv Q/M$. In particular, previous numerical investigations have indicated that, within the family of spherically symmetric charged Reissner–Nordström spacetimes, the black hole with $\bar{Q}\simeq 0.7$ has the *fastest* relaxation rate. In the present paper we use *analytical* techniques in order to investigate this intriguing non-monotonic functional dependence of the Reissner–Nordström black-hole relaxation rates on the dimensionless physical parameter $\bar{Q}$. In particular, it is proved that, in the eikonal (geometric-optics) regime, the black hole with $\bar{Q}=\frac{\sqrt{51-3\sqrt{33}}}{8}\simeq 0.73$ is characterized by the *fastest* relaxation rate (the smallest dimensionless relaxation time $\bar{\tau}$) within the family of charged Reissner–Nordström black-hole spacetimes.

1 Introduction

The mathematically elegant uniqueness theorems of Israel, Carter, and Hawking [1–4] (see also [5–7]) have revealed the physically important fact that all asymptotically flat stationary black-hole solutions of the non-linearly coupled Einstein-Maxwell field equations belong to the Kerr-Newman family of curved spacetimes [8–41]. The three-dimensional phase-space of these canonical black-hole spacetimes is described by conserved physical parameters that can be measured by asymptotic observers: the black-hole mass M, the black-hole electric charge Q, and the black-hole angular momentum $J\equiv Ma$.

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In accord with the results of the uniqueness theorems [1,3–7], the linearized dynamics of massless gravitational and electromagnetic fields in perturbed Kerr-Newman black-hole spacetimes are characterized by damped quasinormal resonant modes with the physically motivated boundary conditions of purely ingoing waves at the black-hole outer horizon and purely outgoing waves at spatial infinity [42]. These exponentially decaying black-hole-field oscillation modes describe the gradual dissipation of the external massless perturbation fields which are radiated into the black hole and to spatial infinity. As a result of this dissipation process, the perturbed spacetime gradually returns into a stationary Kerr-Newman black-hole solution of the Einstein-Maxwell field equations.

Interestingly, for a given set $\{\bar{Q}\equiv Q/M,\bar{a}\equiv J/M^2\}$ of the Kerr-Newman dimensionless physical parameters [43,44], the physically motivated boundary conditions at the black-hole horizon and at spatial infinity single out a discrete set $\{\bar{\omega}\equiv M\omega(\bar{Q},\bar{a};l,m,n)\}_{n=0}^{n=\infty}$ [45] of dimensionless complex resonant frequencies which characterize the relaxation dynamics of the massless fields in the curved black-hole spacetime (see [46–48] for excellent reviews on the physically exciting topic of black-hole resonant spectra). In particular, the fundamental resonant frequency $\bar{\omega}(n=0)$ [49] determines, through the relation

$$\bar{\tau}_{\text{relax}}\equiv\bar{\omega}_{\text{I}}^{-1}(n=0)\ , \qquad (1)$$

the characteristic dimensionless relaxation time of the perturbed black-hole spacetime.

One can ask: Within the family of charged and rotating Kerr-Newman black-hole spacetimes, which black hole has the longest relaxation time $\bar{\tau}_{\text{relax}}(\bar{Q},\bar{a})$? Interestingly, it has previously been proved that the answer to this question is *not* unique. In particular, previous analytical [50–55] and numerical [56–61] studies of the Kerr and Kerr-Newman quasinormal resonance spectra have explicitly proved that the characteristic black-hole relaxation times grow unboundedly as the extremal limit $\bar{T}_{\text{BH}} \to 0$ is approached [62]:

$$\bar{\tau}_{\text{relax}} \propto \bar{T}_{\text{BH}}^{-1} \to \infty \quad \text{for} \quad \bar{T}_{\text{BH}} \to 0 \tag{2}$$

where $\bar{T}_{\text{BH}} \equiv MT_{\text{BH}}$ is the characteristic dimensionless Bekenstein-Hawking temperature of the black-hole spacetime [63–65].

In the present paper we shall focus on the opposite regime of Reissner–Nordström black holes which *minimize* the dimensionless relaxation time (1). In particular, we here raise the following physically interesting question: Within the family of spherically symmetric charged Reissner–Nordström black holes, which black hole has the *fastest* relaxation rate [that is, the shortest relaxation time $\bar{\tau}_{\text{relax}}(\bar{Q})$]? Intriguingly, below we shall explicitly prove that, as opposed to the case of near-extremal spacetimes which maximize the black-hole relaxation times [see Eq. (2)], the answer to the black-hole minimization question is *unique*.

2 Review of former analytical and numerical studies

Former analytical and numerical studies of the Kerr-Newman resonance spectra [50–61] have revealed the fact that, for a given value of the black-hole electric charge $\bar{Q}$, the characteristic black-hole relaxation time $\bar{\tau}_{\text{relax}}(\bar{Q}, \bar{a})$ is a monotonically increasing function of the rotation parameter $\bar{a}$. Hence, within the canonical family of Kerr-Newman black-hole spacetimes, the relaxation rate $\bar{\tau}^{-1}_{\text{relax}}(\bar{Q}, \bar{a})$ can be maximized by considering non-spinning ($\bar{a} = 0$) black holes [66]. In the present paper we shall therefore focus our attention on spherically symmetric charged Reissner–Nordström black-hole spacetimes.

In addition, former numerical studies [56,60,61,67–69] of the resonance spectra of charged black holes have revealed the interesting fact that the black-hole relaxation times $\bar{\tau}_{\text{relax}}(\bar{Q}, \bar{a})$ are characterized by a non-trivial (*non-monotonic*) functional dependence on the dimensionless physical parameter $\bar{Q}$. In particular, detailed numerical computations [57,58,69] indicate that, in the physically interesting case of coupled gravitational-electromagnetic quadrupole perturbations fields with $l = 2$, the black-hole relaxation time (1) is minimized for

$$\bar{Q}_{\text{min}} \simeq 0.7 \quad \text{for} \quad l = 2. \tag{3}$$

The main goal of the present compact paper is to shed light on the intriguing non-monotonic functional dependence of the characteristic Reissner–Nordström black-hole relaxation times $\bar{\tau}_{\text{relax}}(\bar{Q})$ on the dimensionless charge parameter $\bar{Q}$. In particular, in order to facilitate a fully *analytical* treatment of the black-hole minimization problem, we shall study in the present paper the eikonal (geometric-optics) regime of the quasinormal resonance spectra which characterize the composed black-hole-field system. Interestingly, below we shall explicitly demonstrate that the analytical results obtained in the eikonal $l \gg 1$ regime agree remarkably well with the numerical results [57,58,69] of the quadrupole $l = 2$ perturbation fields.

3 The composed black-hole-field quasinormal resonance spectra in the eikonal (geometric-optics) $l \gg 1$ regime

We consider a Reissner–Nordström black-hole spacetime of mass M and electric charge Q which is characterized by the curved line element [8,43]

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{4}$$

where

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}. \tag{5}$$

The radii [8]

$$r_\pm = M \pm \sqrt{M^2 - Q^2} \tag{6}$$

of the inner and outer horizons of the charged black-hole spacetime (4) are determined by the zeroes of the radially-dependent metric function (5)

Massless perturbations fields of the black-hole spacetime (4) are governed by the Schrödinger-like ordinary differential equation [8]

$$\frac{d^2\psi}{dy^2} + V_l[r(y)]\psi = 0 \tag{7}$$

where the tortoise radial coordinate y is defined by the differential relation

$$dy = \frac{dr}{f(r)} \tag{8}$$

and, in the eikonal (geometric-optics) $l \gg 1$ limit, the effective black-hole-field curvature potential in (7) is given by the functional relation [8]

$$V_l(r) = \omega^2 - l(l+1)\frac{f(r)}{r^2}. \tag{9}$$

The quasinormal resonant modes (damped oscillations) which characterize the linearized relaxation dynamics of the perturbed black-hole spacetime are determined by the Schrödinger-like radial equation (7) with the physically motivated boundary conditions of purely ingoing waves at the black-hole event horizon [which is characterized by $y \to -\infty$] and purely outgoing waves at spatial infinity [which is characterized by $y \to \infty$]:

$$\psi \sim \begin{cases} e^{-i\omega y} & \text{for } \ r \to r_+ \ \ (y \to -\infty)\,; \\ e^{i\omega y} & \text{for } \ r \to \infty \ \ (y \to \infty). \end{cases} \tag{10}$$

We shall now use analytical techniques in order to determine the characteristic quasinormal resonant frequencies which characterize the black-hole spacetime (4) in the eikonal large-l regime. To this end, we shall use the results of the elegant WKB analysis presented in [70,71], according to which the quasinormal resonance spectrum which characterizes the Schrödinger-like ordinary differential equation (7) with the boundary conditions (10) is determined, in the eikonal $l \gg 1$ regime, by the simple leading-order resonance equation [70,71]

$$i\frac{V_0}{\sqrt{2V_0^{(2)}}} = n + \frac{1}{2} + O(l^{-1}); \qquad n = 0, 1, 2, \ldots \tag{11}$$

Here the composed black-hole-field curvature potential V_0 and its radial derivatives $V_0^{(k)} \equiv d^kV/dy^k$ [see Eq. (9)] are evaluated at the radial peak $y = y_0$, which is characterized by the simple functional relation

$$\frac{dV}{dy} = 0 \quad \text{for} \quad y = y_0 \ \ (r = r_0). \tag{12}$$

Substituting the effective black-hole-field radial potential (9) into (12), one finds the simple expression [69]

$$\bar{r}_0 \equiv \frac{r_0}{M} = \frac{1}{2}\Big(3 + \sqrt{9 - 8\bar{Q}^2}\Big) \tag{13}$$

for the dimensionless radius which characterizes the peak of the curvature potential in the eikonal large-l regime. Substituting (13) into the WKB resonance equation (11) and using the differential relation (8), one obtains the (rather cumbersome) functional expressions

$$\bar{\omega}_{\text{R}} = \frac{(2l+1)\sqrt{2}\sqrt{3 + \sqrt{9 - 8\bar{Q}^2} - 2\bar{Q}^2}}{\big[3 + \sqrt{9 - 8\bar{Q}^2}\big]^2} \tag{14}$$

and

$$\bar{\omega}_{\text{I}} = -\Big(n + \frac{1}{2}\Big) \cdot \frac{4\sqrt{3}\sqrt{\big[3+\sqrt{9-8\bar{Q}^2}-2\bar{Q}^2\big]\big[3+\sqrt{9-8\bar{Q}^2}-\frac{8}{3}\bar{Q}^2\big]}}{\big[3+\sqrt{9-8\bar{Q}^2}\big]^3}; \quad n = 0, 1, 2, \ldots \tag{15}$$

for the characteristic dimensionless resonant frequencies of the charged Reissner–Nordström black-hole spacetime (4).

Inspecting the analytically derived geometric-optics expression (15), one reveals the interesting fact that the function $\bar{\omega}_{\text{I}}(\bar{Q})$ has a non-trivial (non-monotonic) dependence on the dimensionless physical parameter $\bar{Q}$. In particular, analyzing the function $\bar{\omega}_{\text{I}}^{(0)}(\bar{Q})$ [see Eq. (15)] for the fundamental ($n = 0$) resonant frequency in the eikonal regime, we find that the characteristic dimensionless relaxation time $\bar{\tau}(\bar{Q}) \equiv 1/\bar{\omega}_{\text{I}}^{(0)}$ of the composed black-hole-field system is *minimized* for the particular value

$$\bar{Q}_{\text{min}} = \frac{\sqrt{51 - 3\sqrt{33}}}{8} \simeq 0.726 \quad \text{for} \quad l \gg 1 \tag{16}$$

of the dimensionless black-hole charge parameter.

Interestingly, one finds that the analytically derived expression (16) for the dimensionless charge parameter $\bar{Q}_{\text{min}}$, which characterizes the Reissner–Nordström black hole with the fastest relaxation rate (the shortest relaxation time) in the eikonal large-l regime, agrees remarkably well with the numerically computed value $\bar{Q}^{\text{numerical}}_{\text{min}} \simeq 0.7$ [56–61,69] [see Eq. (3)] for the quadrupole $l = 2$ perturbation fields.

4 Summary and discussion

The characteristic quasinormal resonance spectra of black holes have attracted the attention of physicists and mathematicians during the last five decades [46–48]. In the present paper we have raised the following physically interesting questions: Within the canonical family of charged and rotating Kerr-Newman black holes, which black hole has the slowest relaxation rate and which black hole has the *fastest* relaxation rate?

Using analytical techniques we have shown that, while the answer to the first question is not unique [see the characteristic relation (2) for the family of near-extremal black holes with diverging relaxation times in the $\bar{T}_{\text{BH}} \to 0$ limit], the answer to the second question is unique within the phase space of spherically symmetric charged Reissner–Nordström black-hole spacetimes. In particular, based on previous analytical and numerical studies of the black-hole resonance spectra [50–61] which revealed the interesting fact that, for a given value of the dimensionless charge parameter $\bar{Q}$, the characteristic black-hole relaxation time $\bar{\tau}(\bar{a},\bar{Q})$ is a monotonically increasing function of the dimensionless angular-momentum parameter $\bar{a}$, we have focused our attention on the relaxation properties of non-spinning ($\bar{a}=0$) charged Reissner–Nordström black-hole spacetimes. In addition, in order to facilitate a fully *analytical* exploration of the intriguing non-monotonic functional dependence of the black-hole relaxation rates $\bar{\tau}^{-1}(\bar{Q})$ on the dimensionless physical parameter $\bar{Q}$, we have considered in the present paper the eikonal (geometric-optics) regime of the composed black-hole-field resonance spectrum.

Using standard analytical techniques [70,71], we have explicitly demonstrated that the composed black-hole-field relaxation spectrum is characterized by a non-trivial (non-monotonic) functional dependence on the dimensionless charge parameter $\bar{Q}$. This analytically derived result supports the numerical data presented in [57,58,69]. Moreover, it is interesting to note that the *analytically* calculated black-hole charge parameter [see Eq. (16)]

$$\left(\frac{Q}{M}\right)^{\text{analytical}}_{\text{min}}=\frac{\sqrt{51-3\sqrt{33}}}{8}\ , \tag{17}$$

which minimizes the dimensionless relaxation time of the composed Reissner–Nordström-black-hole-field system in the eikonal $l\gg 1$ regime, agrees remarkably well with the *numerically* computed charge parameter $\bar{Q}^{\text{numerical}}_{\text{min}}\simeq 0.7$ [see Eq. (3)] which minimizes the black-hole relaxation time in the canonical case of coupled gravitational-electromagnetic quadrupole perturbation fields with $l=2$ [57,58,69].

Acknowledgements This research is supported by the Carmel Science Foundation. I would like to thank Yael Oren, Arbel M. Ongo, Ayelet B. Lata, and Alona B. Tea for helpful discussions.

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