

# Dynamical analysis and cosmological viability of varying $G$ and $\Lambda$ cosmology

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Received: 13 June 2018 / Accepted: 17 August 2018 / Published online: 27 August 2018  
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**Abstract** The cosmological viability of varying  $G(t)$  and  $\Lambda(t)$  cosmology is discussed by determining the cosmological eras provided by the theory. Such a study is performed with the determination of the critical points while stability analysis is performed. The application of renormalization group in the ADM formalism of general relativity provides a modified second-order theory of gravity where varying  $G(t)$  plays the role of a minimally coupled field, different from that of scalar-tensor theories, while  $\Lambda(t) = \Lambda(G(t))$  is a potential term. We find that the theory provides two de Sitter phases and a tracking solution. In the presence of matter source, two new critical points are introduced, where the matter source contributes to the universe. One of those points describes the  $\Lambda$ CDM cosmology and in order for the solution at the point to be cosmologically viable, it has to be unstable. Moreover, the second point, where matter exists, describes a universe where the dark energy parameter for the equation of state has a different value from that of the cosmological constant.

## 1 Introduction

The detailed analysis of the cosmological data over the last years supports the assumptions that the universe is spatially flat, it has been through an inflation phase in the past prior to the radiation dominated era, and that, currently, the universe is in a second acceleration epoch [1–5]. The acceleration phase of the universe has been attributed to a matter source in the gravitational field equations which has an equation of state parameter with a negative value. The nature of this exotic matter source has led to the dark energy problem.

In the literature one can find various proposals/models to solve the dark energy problem. These proposals can be cate-

gorized in two different families; more specifically, in these where in the context of Einstein's general relativity, an energy momentum tensor is introduced to explain the acceleration phases [6–16], and these in which the Einstein–Hilbert action is modified, leading to the so-called modified/alternative theories of gravity, such that the origin of the acceleration to correspond to the gravitational theory, for instance, see [17–29] and references therein.

A common feature for some of the modified theories of gravity is that Newton's constant  $G$ , is varying; and it is a varying parameter. For instance, in Brans–Dicke theory and in  $f(R)$ -gravity someone can define the effective parameters  $G_{eff} = G\phi^{-1}$  and  $G_{eff} = Gf'(R)$  respectively [17, 18, 30]. Dealing with fundamental constants in physics as parameters is the main concept of the renormalization group [31–33]. Reuter and Weyer in [34] inspired by the property that Brans–Dicke action modify  $G$ , reconstructed the Brans–Dicke action with the use of the renormalization group in general relativity, by assuming that  $G$  and  $\Lambda$  (the cosmological constant) are varying. Various alternative gravitational theories [35–50] have been modified by the renormalization group in cosmological systems as also in strong gravitational systems [51–56].

A study which provides important analytical information about the existence of cosmological epochs (such as matter dominated era, acceleration phase and others) and the stability of those epochs is the analysis of critical points of the gravitational field equations [57, 58]. In the dark energy models, the analysis of the critical point provides results for the evolution of the universe [59] and the viability of each model being studied [60]. For some extended applications of the critical point analysis in modified theories of gravity, we refer the reader to [61–68] and references therein.

We are interested in the dynamical analysis of the gravitational field equations which follows from the renormaliza-

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tion group in the ADM Lagrangian of general relativity as described in [69]. Specifically, in [69] the authors assumed that  $G$  and  $\Lambda$  are varying parameters such that new degrees of freedom are introduced. The theory, remains of second-order and the variable  $G$  can be seen as a scalar field coupled to gravity, but different from that of Brans–Dicke or from the scalar–tensor theory. The reason for the latter lies in the starting point for the application of the renormalization group. This is the ADM Lagrangian and not the Einstein–Hilbert action as in [34]. Some exact solutions for that specific modified gravitational theory can be found in [70,71]. Cosmological constraints and comparison with the  $\Lambda$ CDM model are given in [72] where it was found that for this specific variable  $G$ ,  $\Lambda$  cosmology is compatible with some of the observational data and can explain the late acceleration phase of the universe.

More specifically, in this work, we study the existence of critical points in varying  $G$ ,  $\Lambda$  cosmology [69] in order to explore the possible cosmological eras provided by the theory. We define new dimensionless variables and in terms of the  $H$ -normalization [59] we study the critical points of the cosmological model. Because the resulting field equations of [69] have similarities with scalar–tensor theories, our analysis can be compared with the analysis performed for the Brans–Dicke theory in [65]. However, as we shall see, there are essential differences with the scalar–tensor theories. The plan of the paper follows.

In Sect. 2 we present the model of our consideration which belongs to the family of varying  $G$  and  $\Lambda$  cosmology. Section 3 includes the main material of our analysis where the analysis of the critical points for dimensionless variables and in the  $H$ -normalization is discussed. Our discussion of the results is given in Sect. 4, where we also draw our conclusions.

## 2 Field equations in varying $G$ and $\Lambda$ cosmology

In the ADM formalism of general relativity, Bonanno et al. [69] after the application of the renormalization group, proposed the following modification for the ADM Lagrangian of general relativity,

$$\begin{aligned}
 S = S_m + \frac{1}{16\pi} \int \frac{N\sqrt{h}}{G} (K_{ij}K^{ij} - K^2 + R^* - 2\Lambda(G))d^3x \\
 + \frac{\mu}{16\pi} \int \frac{N\sqrt{h}}{G} \left( N^{-2}(G_{,0})^2 - 2\frac{N^i}{N^2}G_{,0}G_{,i} \right. \\
 \left. - \left( h^{ij} - \frac{N^iN^j}{N^2} \right) G_{,i}G_{,j} \right) d^3x, \tag{1}
 \end{aligned}$$

where  $S_m$  describes the action integral of the matter source, and  $G$ ,  $\Lambda(G)$  are varying.

Furthermore, the line element of the background metric in the ADM formalism is expressed as [73]

$$\begin{aligned}
 ds^2 = -(N^2 - N_iN^i)dt^2 \\
 + 2N_idtdx^i + h_{ij}dx^i dx^j, \quad i, j = 1, 2, 3, \tag{2}
 \end{aligned}$$

in which  $N$  denotes the lapse function,  $N^i$  are the components of the shift vector,  $h_{ij}$  is the metric tensor three-dimensional surface [73,74].  $K_{ij}$  denotes the extrinsic curvature and  $R^*$  the curvature of the three-dimensional surface with metric tensor  $h_{ij}$ .

In the special consideration of a spatially flat isotropic and homogeneous universe, line element (2) is that of the Friedmann–Lemaître–Robertson–Walker (FLRW) geometry, that is,

$$ds^2 = -N^2 dt^2 + a^2(t) (dx^2 + dy^2 + dz^2). \tag{3}$$

Therefore, the action integral (1) is simplified and the following point-like Lagrangian can be extracted [69]

$$\begin{aligned}
 \mathcal{L}(N, a, \dot{a}, G, \dot{G}) = \frac{1}{N} \left( -\frac{3}{G} a \dot{a}^2 + \frac{\mu}{2G} a^3 \left( \frac{\dot{G}}{G} \right)^2 \right) \\
 - Na^3 V(G) + Na^3 \rho_m, \tag{4}
 \end{aligned}$$

where  $\Lambda(G) = GV(G)$  and  $\rho_m$  presents the contribution of the matter source.

For the matter source, we assume that it describes a dust fluid which attributes the dark matter source of the universe and it is minimally coupled to gravity, that is  $p_m = 0$  and  $\rho_m = 8\pi\rho_{m0}a^{-3}$ . At this point, it is important to mention that we have assumed the comoving observer  $u^\mu = \frac{1}{N}\delta_0^a$ , such that  $u^\mu u_\mu = -1$ .

Lagrangian (4) describes a second-order theory with degrees of freedom  $\{N, a, G\}$ . Specifically, the variation with respect to the lapse function provides the constraint equation, while two second-order equations follow from the variation with respect to the rest parameters  $a(t)$  and  $G(t)$ . Parameter  $\mu$  denotes the interaction; its value is unknown and it is a dimensionless parameter [69]. It is analogue to the Brans–Dicke parameter. Furthermore, it is important parameter  $\mu$  to be nonzero in order the field equations to admit nontrivial solutions [69].

Variation with respect to the dependent variables  $\{N, a, G\}$  in Lagrangian (4) derives the modified gravitational field equations [69–71]

$$-\frac{3}{G} a \dot{a}^2 + \frac{\mu}{2G} a^3 \left( \frac{\dot{G}}{G} \right)^2 + a^3 V(G) = -8\pi\rho_{m0}, \tag{5}$$

$$\ddot{a} + \frac{1}{2a} \dot{a}^2 - \dot{a} \frac{\dot{G}}{G} + \frac{\mu}{4} a \frac{\dot{G}^2}{G^2} - \frac{1}{2} GaV = 0, \tag{6}$$

$$\ddot{G} - \frac{2}{\mu} G \left( \frac{\dot{a}}{a} \right)^2 + 3 \frac{\dot{a}}{a} \dot{G} - \frac{3}{2} \frac{\dot{G}^2}{G} + \mu G^3 V_{,G} = 0, \tag{7}$$

where without loss of generality we have set the lapse function to be constant, i.e.  $N(t) = 1$ .

Equations (5), (6) are the modified Friedmann’s equations, while equation (7) is the corresponding “Klein–Gordon” equation for the “field”  $G(t)$ . It is important to mention that  $G(t)$  does not belong to the family of scalar–tensor theories [75].

An equivalent way to write the field equations is with the use of the Hubble function  $H = \frac{\dot{a}}{a}$ , that is,

$$3H^2 = G_{eff}(\rho_m + \rho_G), \tag{8}$$

$$2\dot{H} + 3H^2 = -G_{eff}p_G, \tag{9}$$

and

$$\frac{\ddot{G}}{G} - \frac{2}{\mu}H^2 + 3\left(\frac{\dot{G}}{G}\right)H - \frac{3}{2}\left(\frac{\dot{G}}{G}\right)^2 + \mu G^2 V_{,G} = 0, \tag{10}$$

from where  $\rho_G(t)$ ,  $p_G(t)$  denotes the energy density and the pressure component related to the field  $G(t)$ , as follows

$$\rho_G = \frac{1}{2}\mu\left(\frac{\dot{G}}{G}\right)^2 + GV(G), \tag{11}$$

$$p_G = -2H\frac{\dot{G}}{G^2} + \frac{\mu}{2}\frac{\dot{G}^2}{G^3} - V(G), \tag{12}$$

while  $G_{eff} = 8\pi G(t)$  is the effective time-varying Newton’s “constant”. Furthermore, from (5) and (6) we observe that the Einstein field equations are recovered provided that  $G = G_0$ , i.e. a constant, and  $\Lambda(G_0)$  satisfies the equation

$$G_0 V_{,G}(G_0) - V(G_0) = 0. \tag{13}$$

The latter equation is always true for  $V(G) = V_0 G$ , that is,  $\Lambda(G) \simeq G^2$ .

In the following section the detailed analysis for the critical points of the field Eqs. (5)–(7) is presented

### 3 Dynamical analysis

In this Section, we study the existence and the stability of critical/fixed points for the gravitational field equations. In order to perform such an analysis we define dimensionless variables in the  $H$ -normalization, see [59]. The novelty of these coordinates is that any critical point corresponds to a power-law scale factor, i.e.  $a(t) = a_0 t^p$  or to a de Sitter universe with exponential scale factor, i.e.  $a(t) = a_0 e^{H_0 t}$ .

#### 3.1 Dimensionless variables

We continue by defining the new dimensionless variables in the  $H$ -normalization approach,

$$x = \frac{\mu}{6H^2}\left(\frac{\dot{G}}{G}\right)^2, \quad y = \frac{GV(G)}{3H^2}, \quad \Omega_m = G\frac{\rho_m}{3H^2}, \tag{14}$$

while the constraint Eq. (5) takes the algebraic form

$$1 - x - y = \Omega_m, \tag{15}$$

from where it follows that since  $\Omega_m \in [0, 1]$ , then  $0 \leq 1 - x - y \leq 1$ . Parameters are not necessarily positive. The sign of  $x$  depends on the interaction parameter  $\mu$ , while the sign of variable  $y$  depends on the sign of the varying  $\Lambda(G(t))$ . Moreover, the energy density of the field  $G$  is defined as  $\Omega_G = x + y$ .

Consider now the new independent parameter  $\tau = \ln a$ ; then second-order differential Eqs. (6) and (7) can be written as the first-order ordinary differential equations

$$\frac{dx}{d\tau} = -3x(1 - x + y) - \sqrt{\frac{6x}{\mu}}(1 - x - \lambda y), \tag{16}$$

$$\frac{dy}{d\tau} = 3y(1 + x - y) + \sqrt{\frac{6x}{\mu}}y(1 - \lambda), \tag{17}$$

$$\frac{d\lambda}{d\tau} = -\sqrt{\frac{6x}{\mu}}(1 - \lambda\Gamma(\lambda)), \tag{18}$$

in which the new parameter  $\lambda$  and function  $\Gamma(\lambda)$  are defined as

$$\lambda = G(\ln V)_{,G} \quad \text{and} \quad \Gamma(\lambda) = \frac{V_{,GG}V}{(V_{,G})^2} - 1. \tag{19}$$

As far as the equation of state parameter for the dark energy fluid term is concerned, from the definition of (11) and (12) with the use of the variables (14) we calculate

$$w_G(x, y) = -\left(\frac{x - y}{x + y}\right) - 2\sqrt{\frac{2}{3\mu}}\frac{\sqrt{x}}{x + y}. \tag{20}$$

The deceleration parameter,  $q = -1 - \frac{\ddot{H}}{H^2}$  is expressed as

$$q(x, y) = \frac{1}{2}\left(1 + 3(x - y) - 2\sqrt{\frac{6x}{\mu}}\right), \tag{21}$$

and the equation of state parameter for the total fluid is derived to be

$$w_{tot}(x, y) = (x - y) - 2\sqrt{\frac{2x}{3\mu}}. \tag{22}$$

The dynamical system (16)–(18) in general has dimension three. However, the dimension of the system is reduced by one in the vacuum, with the use of the algebraic equation

(15). Another possible case where the dimension is reduced is when  $\lambda$  is an identical constant, that is  $\lambda = \lambda_0$ , which corresponds to the power-law potential  $V(G) = V_0 G^{\lambda_0}$ , that is,  $\Lambda(G) = V_0 G^{\lambda_0+1}$ , with  $\lambda_0 \neq 0$ .

### 3.2 Critical points in the vacuum

Consider the vacuum scenario,  $\Omega_m = 0$ , where from the constraint Eq. (15) it follows  $y = 1 - x$ . Therefore, the reducing dynamical system is

$$\frac{dx}{d\tau} = -6(1-x)x - \sqrt{\frac{6x}{\mu}}(1-x)(1-\lambda_0), \tag{23}$$

$$\frac{d\lambda}{d\tau} = -\sqrt{\frac{6x}{\mu}}(1-\lambda\Gamma(\lambda_0)), \tag{24}$$

while, as we have discussed before for a power-law potential in which  $\lambda = const.$ , the latter dynamical system reduced to the one-dimensional system (23).

We continue by assuming two special forms for the potential, (a) power-law potential  $V_{(a)}(G) = V_0 G^{\lambda_0}$ , where  $\lambda_0$  is a constant, and (b) exponential potential  $V_{(b)}(G) = V_0 \exp(\lambda_0 G)$ , such that  $\Gamma(\lambda) = (\lambda_0)^{-1}$  is a constant parameter. The critical points of these two potentials are the only physically different possible points. It is possible for another potential the dynamical system (23), (24) to admit more critical points from the potentials  $V_{(a)}$ ,  $V_{(b)}$ ; however, the physical properties will be on that of the points of potentials  $V_{(a)}$  and  $V_{(b)}$ .

#### 3.2.1 Power-law potential

Consider the power-law potential,  $V_{(a)}(G) = V_0 G^{\lambda_0}$ , then the equilibrium points of Eq. (23) are

$$P_1 : x = 0, \quad P_2 : x = 1 \quad \text{and} \quad P_3 : x = \frac{(1-\lambda_0)^2}{6\mu}.$$

where point  $P_3$  depends on the value of the constant  $\lambda_0$ .

Below we discuss the physical properties and the stability of each point.

- Point  $P_1$  corresponds to the epoch in which the potential  $V(G)$  dominates the universe and the  $\dot{G} = 0$ , that is,  $V(G)$  is the cosmological constant. Hence,  $w_{tot}(P_1) = w_G(P_1) = -1$ , and describes  $P_1$  is a de Sitter point, which can describe the past inflationary epoch when  $P_1$  is unstable; or it can be a future attractor in the evolution of the universe when  $P_1$  is a stable point. The stability of the point depends on the value of the power  $\lambda_0$ . In particular for values of  $\mu$ , in which  $\mu x$  is positive close to the limit  $x \rightarrow 0$ , point  $P_1$  is unstable for  $\lambda_0 > 1$ , while for  $\lambda_0 \leq 1$  the eigenvalue has a negative limit and the point is stable. On the other hand when  $\mu$  is negative close to the limit

$x \rightarrow 0$ ; for instance for  $\mu < 0$  and  $x \rightarrow 0^+$  point  $P_1$  describes a stable spiral.

- Point  $P_2$  corresponds to the epoch in which the kinetic term dominates the universe and  $\rho_G(P_2) = \frac{1}{2}\mu \left(\frac{\dot{G}}{G}\right)^2$ . The equation of state parameter is calculated to be  $w_{tot}(P_2) = w_G(P_2) = 1 - 2\sqrt{\frac{2}{3\mu}}$ , which is real for positive values of the parameter  $\mu$ . The point describes an accelerated universe, i.e.  $w_G(P_2) < -\frac{1}{3}$ , for values of  $\mu$  in the range  $0 < \mu < \frac{3}{2}$ , while  $w_G(P_2) \geq -1$  for  $\frac{2}{3} \leq \mu < \frac{3}{2}$ . Hence, the scale factor at the point  $P_2$  is exponential for  $\mu = \frac{2}{3}$ , and power-law for other values of  $\mu$ . For  $\mu = \frac{8}{3}$  and  $\mu = 6$  point  $P_2$  corresponds to eras where the field  $G$  behaves like dust or radiation fluids respectively. It is important to mention that there is not any finite value of  $\mu$  such that the geometric matter source,  $\rho_G$ , has the equation of state parameter of the stiff fluid. Finally, point  $P_2$  is stable for all the values of  $\mu$  which are defined where  $\lambda_0 < 1 + \sqrt{6\mu}$ .
- Point  $P_3$  exists when  $\lambda \neq 1$ , from where we calculate that  $w_{tot}(P_3) = w_G(P_3) = -1 - \frac{2(1-\lambda_0)}{3\mu} + \frac{(1-\lambda_0)^2}{3\mu}$ . The stability of the point depends on the parameters  $\mu$  and  $\lambda$ ; specifically, the point is stable when  $\mu < 0$  or  $\mu > \frac{(1-\lambda_0)^2}{6}$ . The point  $P_3$  describes acceleration for ranges of the free parameters  $\lambda_0, \mu$  in which: (i)  $\{\lambda_0 < 1, \lambda_0 > 3\}, \mu < 0$  or  $2\mu > (2 - \lambda_0)^2 - 1$ ; (ii)  $1 < \lambda_0 \leq 3, \mu > 0$  or  $2\mu > (2 - \lambda_0)^2 - 1$ . Finally, for  $\lambda = 3$ ,  $P_3$  describes a de Sitter universe. Thus, it is clear that except from the coordinates of point  $P_3$ , the eigenvalue of the point depends on the constant  $\lambda_0$  which, in general, can take any value except for zero. In Fig. 1 the surface where point  $P_3$  is stable and describes an accelerated universe such that  $w_G(P_3) \in [-1, 1/3]$  is plotted in the space of the parameters  $\{\mu, \lambda_0\}$  for  $-2 < \lambda_0 < 0, 0 < \lambda_0 < 4$ , and  $0 < |\mu| < 3$ .

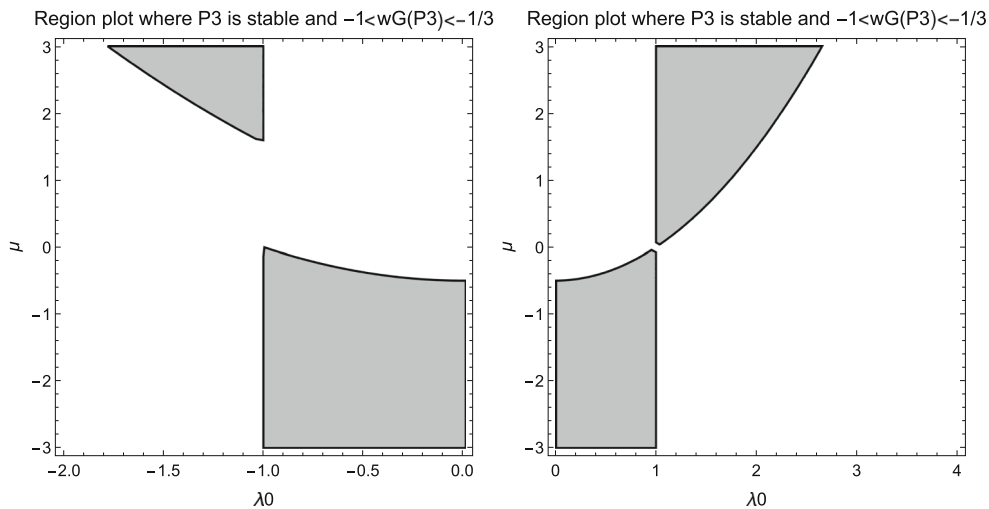
#### 3.2.2 Exponential potential

In the case of the exponential potential  $V_{(b)}(G)$ , where  $\Gamma(\lambda)$  is constant, i.e.  $\Gamma(\lambda) = (\lambda_0)^{-1}$ , the stationary points  $Q(x, \lambda)$  for the dynamical system (23) and (24) are

$$Q_1 = (0, \lambda_0), \quad Q_2 = (1, \lambda_0), \quad Q_3 = \left(\frac{(1-\lambda_0)^2}{6\mu}, \lambda_0\right),$$

$$Q_4 = \left(\frac{1}{6\mu}, 0\right) \quad \text{and} \quad Q_5 = (1, 0).$$

Points  $Q_{1-3}$  are the points  $P_{1-3}$  for the power-law potential in which  $\lambda = \lambda_0, \lambda_0 \neq 0$ , and have the same physical properties. Points  $Q_4$  and  $Q_5$  are new points. The discussion on the physical properties and the stability of the critical points follows.



**Fig. 1** Region plot in the space  $\{\mu, \lambda_0\}$  for  $0 < |\mu| < 3$  and  $-2 < \lambda_0 < 0$  (left figure),  $0 < \lambda_0 < 4$  (right figure), where point  $P_3$  is stable and describes an accelerated universe with  $-1 \leq w_G(P_3) < -\frac{1}{3}$

- Point  $Q_1$  actually describes invariant manifold of the dynamical system rather than a stationary point in the space  $\{x, \lambda\}$ . Any point on the line,  $x = 1$ , has the same physical properties with  $P_1$ , that is the universe is dominated by the potential  $V(G)$ , which plays the role of the cosmological constant because  $\dot{G} = 0$ ; thus  $w_{tot}(Q_1) = w_{tot}(Q_2) = -1$ . In order to study the stability of the point we apply the central manifold theorem where we find that the family of solutions are stable for values of  $\lambda$  as they are given by the stability of point  $P_1$ .
- Point  $Q_2$  has the same physical properties with  $P_2$  and  $\mu$  is necessarily positive. However, the stability of the point is different; the eigenvalues of the linearized system are calculated to  $e_1(Q_2) = \sqrt{\frac{6}{\mu}}$ ,  $e_2(Q_2) = 6 + \sqrt{\frac{6}{\mu}}(1 - \lambda_0)$ ; hence,  $Q_2$  is a hyperbolic (unstable) point, where the field  $G(t)$  has a constant equation of state parameter, i.e.  $w_G(Q_2) = 1 - 2\sqrt{\frac{2}{3\mu}}$ .
- The eigenvalues for the linearized system close to the point  $Q_3$  are calculated to be  $e_1(Q_3) = -\frac{1-\lambda_0}{\mu}$  and  $e_2(Q_3) = -3 + \frac{(1-\lambda_0)^2}{2\mu}$ , which means that point  $Q_3$  is stable when (a)  $\lambda_0 > 1$  and  $\mu < 0$  or (b)  $\lambda_0 < 1$  and  $\mu < \frac{(1-\lambda_0)^2}{6}$ . As for the physical description of the solution at the point  $Q_3$ , that is exactly the same as that of point  $P_3$  for the power-law potential.
- Point  $Q_4$ , describes a solution where potential  $V(G)$  and the kinetic term of the field  $G$  contribute to the universe. The equation of state parameter is calculated to be  $w_{tot}(Q_4) = w_G(Q_4) = -1 - \frac{1}{3\mu}$  which means that it describes an accelerated universe for  $\mu < -\frac{1}{2}$  or  $\mu > 0$ . To determine the stability of the point, we calculate eigenvalues which are  $e_1(Q_4) = -\frac{1}{\mu}$ ,  $e_2(Q_4) = -9 + \frac{7}{2\mu}$ . Hence, for  $\mu > \frac{7}{18}$  both eigenvalues are negative and the

point is stable. Moreover, for  $\mu > 0$  someone can calculate that  $w_{tot}(Q_4) < -1$ , which means that the parameter for the total equation of state crosses the phantom divided line.

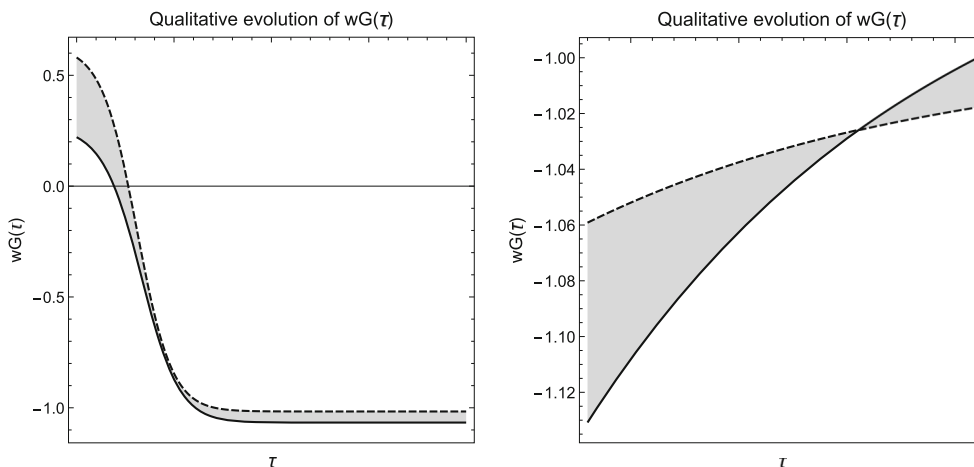
- Point  $Q_5$ , can be seen as a special case of point  $Q_2$  where  $\lambda_0 = 0$ . The physical properties are the same as point  $P_1$ , that is,  $w_{tot}(Q_5) = w_G(Q_5) = 1 - 2\sqrt{\frac{2}{3\mu}}$ . We calculate the eigenvalues of the linearized system, that is,  $e_1(Q_5) = -\sqrt{\frac{6}{\mu}}$ ,  $e_2(Q_5) = 6 + \sqrt{\frac{6}{\mu}}$ , where we conclude that because eigenvalue  $e_2(Q_5)$  has always a real positive value, the solution which is described by point  $Q_5$  is unstable.

Before we proceed to our analysis with the case in which we include matter source, in Fig. 2 we present the qualitative evolution of the parameter for the equation of state  $w_G(\tau)$ , for positive and negative values of the parameter  $\mu$  and for  $\lambda_0 = 2$ . For positive values of  $\mu$ , the initial condition is for  $x(0) \simeq 1$ , while we observe that the final attractor describes an accelerated universe close to the de Sitter point. On the other hand, for negative values of  $\mu$ , i.e.  $x < 0$  and initial condition  $x(0) \simeq -0.01$ , the final attractor is again close to the de Sitter universe. The value of the parameter  $\mu$  is unknown, and Fig. 2 provides a qualitative evolution of the equation of state parameter. From the numerical simulation, we observe that the equation of state parameter can cross the phantom divide line which does not contradict the observations [5].

### 3.3 Critical points with matter source

As in the case of vacuum, we perform the same analysis for power-law and exponential potential.





**Fig. 2** Qualitative evolution for the equation of state parameter for various values of the free parameter  $\mu$ , and for  $\lambda_0 = 2$ . Left figure is for  $\mu > 0$ , and specifically solid line is for  $\mu = 5$ , dashed line for  $\mu = 20$ . Right figure is for negative values of  $\mu$  and in particular, solid line is

for  $\mu = -5$ , dashed line for  $\mu = -100$ . From the figure, it is clear that positive values of  $\mu$  are preferred in order for the  $w_G$  parameter to be free of singularities in the past

### 3.3.1 Power-law potential

For the power-law potential  $V_a(G)$ , where  $\lambda = const.$ , i.e.  $\lambda = \lambda_0$ , the dynamical (16), (17) admits the critical points of the form  $A = (x(A), y(A))$ ,

$$A_0 = (0, 0), \quad A_1 = (0, 1), \quad A_2 = (1, 0),$$

$$A_3 = \left( \frac{(1 - \lambda_0)^2}{6\mu}, 1 - \frac{(1 - \lambda_0)^2}{6\mu} \right), \quad A_4 = \left( \frac{2}{3\mu}, 0 \right),$$

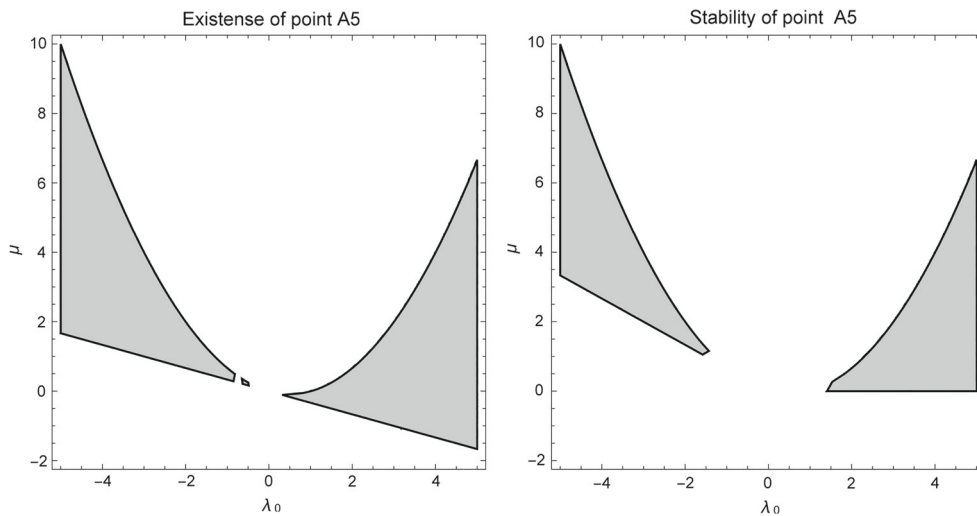
$$A_5 = \left( \frac{3\mu}{2\lambda_0^2}, \frac{2\lambda_0 + 3\mu}{2\lambda_0^2} \right).$$

We observe that points  $A_1, A_2$  and  $A_3$  have the coordinates of  $P_1, P_2$  and  $P_3$  respectively, while the new critical points are the  $A_0, A_4$  and  $A_5$ . More specifically for each critical point we have:

- Point  $A_0$  corresponds to the matter dominated era where  $\Omega_m(A_0) = 1, \Omega_G(A_0) = 0$  and  $w_{tot} = 0$ . One of the eigenvalues of the linearized system close to the critical point is positive which means that the point is unstable.
- At the point  $A_1$  the potential  $V(G)$  dominates the universe while  $\dot{G} = 0$ , that is  $\Omega_m(A_1) = 0$  and  $\Omega_G(A_1) = 1$ , while  $w_{tot}(A_1) = w_G(A_1) = -1$ . The stability of the point is explicitly that which is described for the point  $P_1$ .
- The discussion of the physical properties for point  $A_2$  is exactly that for point  $P_2$ , because  $\Omega_m(A_2) = 0$  and  $\Omega_G(A_2) = 1$ . However, the eigenvalues are calculated to be  $e_1(A_2) = 3 + \sqrt{\frac{6}{\mu}}$ , and  $e_2(A_2) = 6 + (1 - \lambda_0)\sqrt{\frac{6}{\mu}}$ , that is, the point is always unstable because  $\text{Re}(e_1(A_2)) > 0$ .
- Point  $A_3$  exists for all the values  $\lambda$  and  $\mu$ , while the physical solution is that described by point  $P_3$ . The eigenvalues

of the point are calculated to be  $e_1(A_3) = -3 + \frac{(1 - \lambda_0)^2}{2\mu}$ , and  $e_2(A_3) = -3 - \frac{(1 - \lambda_0)\lambda_0}{\mu}$ . The stability of the point depends on the values of the parameters  $\lambda_0, \mu$  and specifically the eigenvalues are positive, that is,  $A_3$  is unstable when (a)  $\lambda_0 \leq -1$  or  $\lambda_0 > 1$  and  $0 < \mu < \frac{(1 - \lambda_0)^2}{6}$ ; (b)  $-1 < \lambda_0 < 0$  and  $0 < \mu < \frac{\lambda_0(\lambda_0 - 1)}{3}$ .

- Point  $A_4$  describes a universe where  $\Omega_m(A_4) = 1 - \frac{2}{3\mu}$  and  $\Omega_G(A_4) = \frac{2}{3\mu}$ ; while for the parameters of the equation of state  $w_{tot}(A_4) = -\frac{2}{3\mu}$  and  $w_G(A_4) = -1$  follows. Hence, field  $G$  behaves like a cosmological constant and, specifically, that point corresponds to the  $\Lambda$ CDM cosmology, where parameter  $\mu$  is related to the energy density of the dark energy. It is important to mention that the point exists only for values of  $\mu$  where  $0 \leq \Omega_m(A_4) \leq 1$ , that is,  $\mu \geq \frac{2}{3}$ . The corresponding eigenvalues of the linearized system are  $e_1(A_4) = -\frac{9}{2} + \frac{7}{\mu}$ ,  $e_2(A_4) = -3 + \frac{2(2 - \lambda_0)}{\mu}$ ; thus, we conclude that point  $A_4$  is stable for  $\lambda > \frac{13}{3}$  and  $\frac{14}{9} < \mu < \frac{2(\lambda_0 - 2)}{3}$ . However, when point  $A_4$  is stable, we calculate  $\Omega_m(A_4) > \frac{4}{7}$ , which is bigger value from the observable one, i.e.  $\Omega_{m0} \simeq 0.28$ . Therefore, in order for the model to be cosmologically viable, point  $A_4$  has to be unstable.
- Point  $A_5$  exists when (a)  $\lambda_0 < 0$  with  $-\frac{\lambda_0}{3} < \mu < \frac{\lambda_0(\lambda_0 - 1)}{3}$ ; (b)  $\lambda_0 > 0$  with  $-\frac{\lambda_0}{3} < \mu < \frac{\lambda_0(\lambda_0 - 1)}{3}$ , these ranges are given in Fig. 3. Point  $A_5$  describes a universe where  $\Omega_m(A_5) = 1 + \frac{1}{\lambda_0} \left( \frac{3\mu}{\lambda_0} - 1 \right)$ ; and  $\Omega_G(A_5) = \frac{1}{\lambda_0} \left( \frac{3\mu}{\lambda_0} - 1 \right)$  with equation of state parameters  $w_G(A_5) = -\frac{3\lambda_0}{3\mu + \lambda_0}$  and  $w_{tot}(A_5) = -\frac{3}{\lambda_0}$ ,



**Fig. 3** Region plot on the space of the parameters  $\lambda_0$  and  $\mu$  where point  $A_5$  exists (left figure) and the point is stable (right figure)

where  $w_{tot}(A_5) < -\frac{1}{3}$  for  $0 < \lambda_0 < 9$ . The point describes a universe with radiation and dark matter when  $\mu = -\frac{10\lambda_0}{3}$  and  $\lambda_0 \leq 9$ . It is an interesting point because it can describe a phase where radiation dominates the universe for  $\lambda \rightarrow 9^+$ . Point  $A_5$  is stable when (i)  $\lambda_0 < -1$  with  $-\frac{2\lambda_0}{3} < \mu < \frac{\lambda_0(\lambda_0-1)}{3}$  and (ii)  $\lambda_0 > 1$ ,  $0 < \mu < \frac{\lambda_0(\lambda_0-1)}{3}$  as they are given in Fig. 3.

We continue our analysis with the scenario of the exponential potential  $V_{(b)}(G)$ .

### 3.3.2 Exponential potential

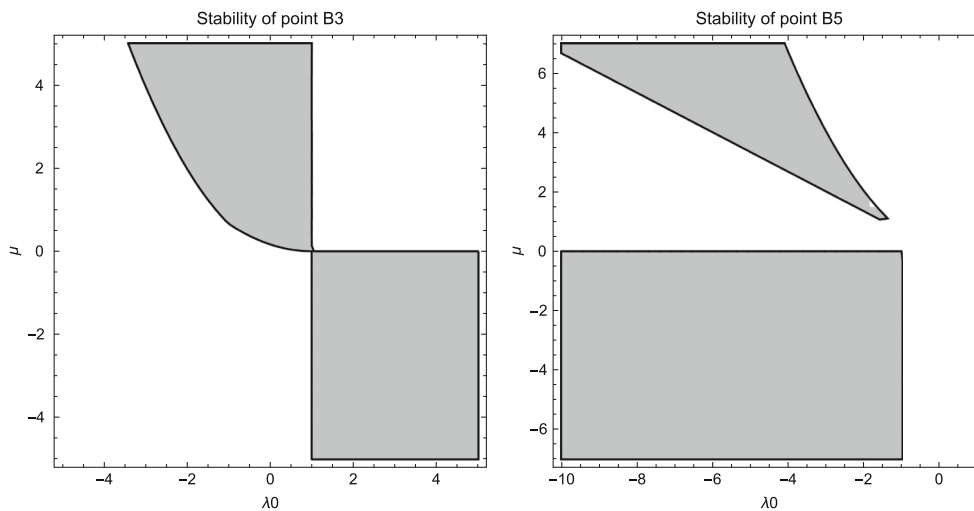
Consider now the exponential potential  $V_{(b)}(G)$ . The dynamical system (16)–(18) has dimension three and the critical points are of the form  $B = B(x(B), y(B), \lambda(B))$ , in particular

$$\begin{aligned}
 B_0 &= (0, 0, \lambda), & B_1 &= (0, 1, \lambda), & B_2 &= (1, 0, \lambda_0), \\
 B_3 &= \left( \frac{(1-\lambda_0)^2}{6\mu}, 1 - \frac{(1-\lambda_0)^2}{6\mu}, \lambda_0 \right), \\
 B_4 &= \left( \frac{2}{3\mu}, 0, \lambda_0 \right), & B_5 &= \left( \frac{3\mu}{2\lambda^2}, \frac{2\lambda + 3\mu}{2\lambda^2}, \lambda_0 \right), \\
 B_6 &= \left( \frac{1}{6\mu}, 1 - \frac{1}{6\mu}, 0 \right), & B_7 &= (1, 0, 0), \\
 B_8 &= \left( \frac{2}{3\mu}, 0, 0 \right).
 \end{aligned}$$

Points  $B_0$ – $B_5$  are specifically points  $A_0$ – $A_5$  respectively, while  $B_6$  and  $B_7$  are related to  $Q_4$  and  $Q_5$  in the vacuum scenario, and  $B_8$  is the only new point which is a special of point  $A_4$  with  $\lambda$  zero. Because of that correspondence, it is not necessary to discuss the physical properties of the points;

therefore, we continue with the discussion of the stability conditions.

- Point  $B_0$  is always unstable because one of the eigenvalues is always positive.
- Point  $B_1$  has two zero eigenvalues, hence central manifold theorem has to be applied. In particular, the coordinates of  $B_1$  describe a line in the space  $\{x, y, \lambda\}$ . We find that the stability and instability of the solution corresponds explicitly to the conditions given by point  $P_1$ .
- Point  $B_2$  has the eigenvalues  $e_1(B_2) = 3 + \sqrt{\frac{6}{\mu}}$ ,  $e_2(B_2) = 3 + \sqrt{\frac{6}{\mu}}(1 - \lambda)$  and  $e_3(B_2) = \sqrt{\frac{6}{\mu}}$ ; hence the point is always unstable.
- Point  $B_3$  is found to be stable when parameters  $\lambda_0$  and  $\mu$  are given by the following set of ranges: (a) For  $\lambda \leq -1$ ,  $\mu > \frac{\lambda_0(\lambda_0-1)}{3}$ ; (b) for  $-1 < \lambda_0 < 1$ ,  $\mu > \frac{(1-\lambda_0)^2}{6}$ , and (c) for  $\lambda > 1$ ,  $\mu < 0$ . The surface in the space of variables  $\{\lambda_0, \mu\}$  in which point  $B_3$  is stable is presented in Fig. 4.
- At the point  $B_4$ , the eigenvalues of the linearized system have the simple expressions,  $e_1(B_4) = \frac{2}{\mu}$ ,  $e_2(B_4) = -\frac{9}{2} + \frac{7}{\mu}$  and  $e_3(B_4) = 3 + \frac{2(2-\lambda_0)}{\mu}$ , which means that the point is stable when  $\lambda_0 < 0$  and  $2\lambda_0 - 4 < 3\mu < 0$ .
- Point  $B_5$  is stable when (a)  $\lambda_0 < -1$  and  $\mu < 0$  or  $-\frac{2\lambda_0}{3}\mu < \frac{\lambda_0(\lambda_0-1)}{3}$  as it is presented in Fig. 4.
- Point  $B_6$  provides the eigenvalues  $e_1(B_6) = -\sqrt{\frac{6}{\mu}}$ ,  $e_2(B_6) = 3 + \sqrt{\frac{6}{\mu}}$  and  $e_3(B_6) = 6 + \sqrt{\frac{6}{\mu}}$  which means that the solution at the point is always unstable.
- Close to point  $B_7$  the eigenvalues of the linearized system are  $e_1(B_7) = -\frac{1}{\mu}$ ,  $e_2(B_7) = -6 + \frac{7}{2\mu}$  and  $e_3(B_7) = -3 + \frac{2}{3\mu}$ , from where it follows that the point is always stable for every value of  $\mu > \frac{2}{3}$ .



**Fig. 4** Region plot on the space of the parameters  $\lambda_0$  and  $\mu$  where points  $B_3$  (left figure) and  $B_5$  (right figure) are stable

- The eigenvalues at point  $B_8$  are derived to be  $e_1(B_8) = -\frac{2}{\mu}$ ,  $e_2(B_8) = 3 + \frac{4}{\mu}$  and  $e_3(B_8) = -\frac{9}{2} + \frac{7}{\mu}$ , which means that the point is always unstable.

It is important to mention that our study for the power-law and the exponential potentials covers all the possible physical states which can be determined by the dynamical system (16)–(18). The only differences will be on the stability of the points. Therefore, it is not necessary to extend the present analysis for other kind of potentials.

In order to explain the latter statement, we note that any stationary point corresponds to a value  $\lambda_0$  such that  $\Gamma(\lambda_0) = const$ . Now we can always rescale a new variable  $\bar{\lambda}_0$ , such that these points to be described by the exponential potential. For instance, consider the hyperbolic potential for the minimally coupled scalar field studied in [76–78]. The admitted critical points [79] correspond to eras where the hyperbolic potential mimics the exponential potential or the power-law potential [80].

#### 4 Conclusions

In this work, we applied the method of fixed point analysis in order to study the cosmological viability of a gravitational theory with varying  $G$  and  $\Lambda$ , which was proposed in [69]. In the renormalization group approach, there is not a unique way to perform the modification of the fundamental “constants”. In [69] the authors proposed the modification to be done in the ADM Lagrangian, which leads to the introduction of a field  $G$  different from that of the scalar–tensor theories. On the other hand, as it has been found in [34], the modification of  $G$  and  $\Lambda$  in Einstein–Hilbert Action can lead to Brans–Dicke like gravitational theory. Another equivalent way to

reproduce the field equations of [69] is the renormalization group to be applied in field equation’s of Einstein’s general relativity.

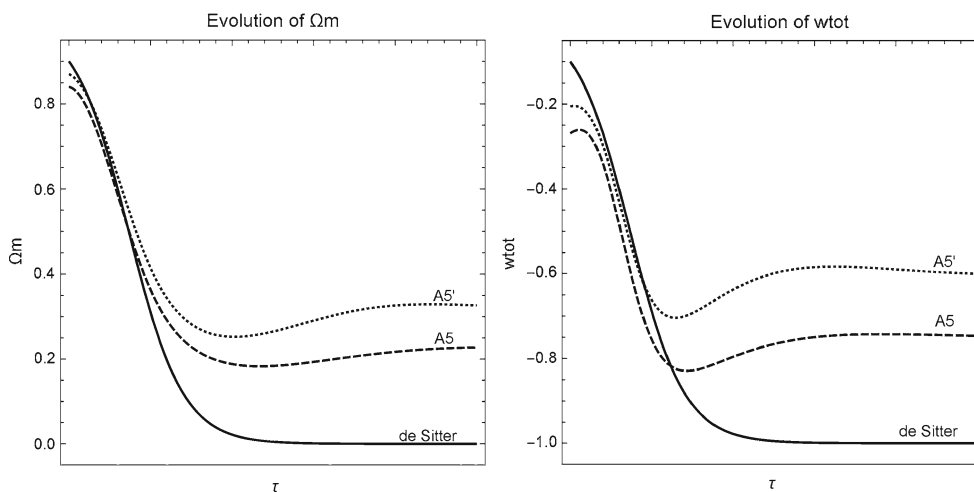
In the cosmological scenario of a spatially flat FLRW universe, the resulting field equations are of second-order with free variables the scale factor  $a(t)$  and the field  $G(t)$ , where the cosmological constant plays the role of the potential for the field  $G$ , that is, we considered  $\Lambda(t) = \Lambda(G(t))$ . Furthermore, in our cosmological scenario, minimally coupled pressureless matter source has been introduced.

In order to perform the dynamical analysis, we define new dimensionless variables while the field equations were rewritten as an algebraic-differential system consisted by three first-order differential equations. For two exact forms of the “potential” term  $\Lambda(G(t))$  the critical/fixed points for the reduced system of algebraic-differential equations ARE determined. The exact forms of the potentials that we selected cover all the possible different families of points with the same physical properties, which can be provided by the theory for any other form of the “potential”  $\Lambda(G(t))$ .

For the vacuum scenario and for power-law potential, we determined three critical points. Two of the points, namely  $P_2$  and  $P_3$ , provide (in general) power-law scale factors corresponding to ideal gas solutions while the physical solution for the third point,  $P_1$ , describes a de Sitter universe. For the exponential potential, in addition to the above, two new critical points are determined,  $Q_4$  and  $Q_5$ , which describe singular solutions of the form  $a(t) = a_0 t^\kappa$ , with  $\kappa = \kappa(\mu, \lambda)$ .

In the presence of matter, new critical points are determined, where the matter source contributes to the final state of the universe. For the power law potential, the points with the new physical solutions are the  $A_0, A_4$  and  $A_5$ . Point  $A_0$  describes the matter dominated era where  $\Omega_m = 1$  while the solution for the scale factor  $a(t) = a_0 t^{\frac{2}{3}}$  is always unsta-





**Fig. 5** Qualitative evolution of the energy density  $\Omega_m$  and the equation of state parameter  $w_{tot}$  for initial conditions close to the point  $A_0$ , for the power-law potential  $V(a)(G)$ . Solid lines are for initial condition with  $x(0) = 0$ , and for values  $(\lambda_0, \mu) = (3, 1)$ , where it is clear that the final state of the universe is the de Sitter solution. The dashed and

the dotted lines are for initial conditions with  $x(0) \neq 0$ , and for the set of variables  $(\lambda_0, \mu) = (4, 2.8)$  (dashed),  $(\lambda_0, \mu) = (5, 4)$  dotted. The free parameters have been chosen such point  $A_5$  to be the attractor in the evolution of the universe

ble. On the other hand, at the points  $A_4, A_5$ , the field  $G$  and the pressureless matter contribute to the evolution of the universe, that is  $\Omega_m(\mu, \lambda) \neq 0$ , and  $\Omega_G(\mu, \lambda) \neq 0$ . At point  $A_4$  the parameter for the equation of state has value  $-1$ , which means that IT mimics the cosmological constant and the point describes the limit of the  $\Lambda$ CDM universe. However, in order for the point to be physically accepted and to be in comparison with the observations, it has to be unstable. Moreover, at point  $A_5$ , field  $G$  acts as an ideal gas and it is possible to describe an epoch with radiation and matter sources.

Numerical simulations for the evolution of the energy density parameter  $\Omega_m$  and the equation of state parameter  $w_{tot}$  are presented in Fig. 5 for initial conditions close to the point  $A_0$ , for different values of the parameters  $\lambda$  and  $\mu$  such that point  $A_5$  is an attractor.

Finally, in the case of the exponential potential, only one extra point was found, namely  $B_7$ , (including those listed above) which has the same physical properties with point  $A_4$ . However, stability analysis provides that the solution at point  $B_7$  is always unstable.

From our analysis, it is clear that the theory provides the basic cosmological eras in the evolution of the universe. However, there are differences with other second-order theories, such as the scalar–tensor theories. In particular, the role of the interaction parameter  $\mu$  is actually unknown but we can see that it can be related to the energy density  $\Omega_G$  as some of the critical points, while from our results, it is clear that its possible values can be demanding the existence and stability of some specific critical points.

There are various similarities of the critical points with that of Brans–Dicke theory [65, 81]. For instance, in the case

of vacuum and for a power-law potential, both theories admit three critical points [81] while the physical properties of the critical points/solutions depend on the Brans–Dicke parameter or parameter  $\mu$  respectively. However, while the theory of our consideration always admits the de Sitter universe (point  $P_1$ ) as a critical point for arbitrary power-law potential, for the Brans–Dicke theory that is true, if and only if, the power-law potential is the quadratic. Other differences between the two theories appear when we include matter source, or generalize the form of the potential.

Consider the coordinate transformation

$$a \rightarrow A\phi, \quad G \rightarrow \phi^2 \tag{25}$$

Hence, Lagrangian (4) becomes

$$\mathcal{L}(a, \dot{a}, \phi, \dot{\phi}) = \frac{1}{N} \left( -3A\phi\dot{A}^2 - 3A^2\dot{A}\dot{\phi} + \frac{1}{2} \frac{2\mu - 6}{\phi} A^3 \dot{\phi}^2 \right) - Na^3 \phi^3 V(\phi) \tag{26}$$

which is the Lagrangian describes the field equations for the Brans–Dicke theory

$$S = \int dx^4 \sqrt{-g} \left[ \frac{1}{2} \phi R - \frac{1}{2} \frac{\omega_{BD}}{\phi} g^{\mu\nu} \phi_{;\mu} \phi_{;\nu} - V_{BD}(\phi) \right], \tag{27}$$

for the line element

$$ds^2 = -N^2 dt^2 + A^2(t) (dx^2 + dy^2 + dz^2) \tag{28}$$

where  $\omega_{BD} = 2\mu - 6$  and  $V_{BD} = \phi^3 V(\phi)$ . Recall that transformation (25) is not a conformal transformation, consequently the two Lagrangians (4), (26) are not conformal equivalents, it is just the same Lagrangian in different coordinates. However, these two point-like Lagrangians describe the field equations for two different gravitational theories for the line elements (3) and (28). Transformation (25) is important because we can transform solutions of one theory into solutions of the other theory. Another important observation is that when  $\mu = 3$ , Lagrangian (26) describes the gravitational field equations of  $f(R)$ -gravity, for details see [82] and references therein.

Without loss of generality we select  $N(t) = 1$ ; then for the power law potential  $V(G) = V_0 G^Q$  in (4) and in the case of vacuum, i.e.  $\rho_{m0} = 0$ , we determine the exact solution for the varying  $G$  and  $\Lambda$  theory

$$a(t) = a_0 t^{\frac{2\mu}{Q-1}}, \quad G(t) = G_0 t^{-\frac{2}{1+Q}} \quad (29)$$

with  $V_0 = (1+Q)^{-2} G_0^{-1-Q} ((1-Q)^2 - 6\mu)\mu$ . The latter solution describes a perfect fluid solution with equation of state parameter  $w_{G(t)} = -1 + \frac{Q-3\mu}{3\mu}$ .

Moreover, under the coordinate transformation (25) we find the Brans–Dicke equivalent potential to be  $V_{BD}(\phi) = V_0 \phi^{3+2Q}$ , while the exact solution becomes

$$A(t) = a_0 t^{\frac{2\mu(1+Q)+(Q-1)}{Q^2-1}}, \quad \phi(t) = G_0 t^{-\frac{1}{1+Q}} \quad (30)$$

which corresponds to a perfect fluid solution with equation of state parameter  $w_{BD} = -\frac{6\mu(1+Q)+3Q-(1+2Q^2)}{3(2\mu(1+Q)+(Q-1))}$ .

Hence, in order to see the differences between the two solutions we set  $\mu = 1$ , where we find that  $|w_{G(t)}| < 1$  for  $1 < Q < 7$ , while for the Brans–Dicke solution we determine that  $|w_{BD}| < 1$  when  $-1 < Q < \frac{9}{2} - \frac{\sqrt{97}}{2}$  and  $1 < Q < \frac{9}{2} + \frac{\sqrt{97}}{2}$ .

A more detailed analysis and comparison with cosmological data are necessary in order for the role of parameter  $\mu$  to be determined. Such an analysis extends the scope of this work and will be published elsewhere.

**Acknowledgements** AP acknowledges the financial support of FONDECYT Grant no. 3160121 and thanks the University of Athens and the  $KEΠB/\Sigma\Lambda$  for the hospitality provided while part of this work was performed.

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