Lower bound on the radii of circular orbits in the extremal Kerr black-hole spacetime

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Abstract It is often stated in the physics literature that maximally-spinning Kerr black-hole spacetimes are characterized by near-horizon co-rotating circular geodesics of radius r_{circular} with the property $r_{\text{circular}} \rightarrow r_{\text{H}}^+$, where r_{H} is the horizon radius of the extremal black hole. Based on the famous Thorne hoop conjecture, in the present compact paper we provide evidence for the existence of a non-trivial lower bound $\frac{r_{\text{circular}} - r_{\text{H}}}{r_{\text{H}}} \gtrsim (\mu/M)^{1/2}$ on the radii of circular orbits in the extremal Kerr black-hole spacetime, where μ/M is the dimensionless mass ratio which characterizes the composed black-hole-orbiting-particle system.

1 Introduction

The circular geodesic motions of test particles in rotating Kerr black-hole spacetimes have attracted the attention of physicists and mathematicians during the last five decades (see [1–6] and references therein). The characteristic radii of these astrophysically important orbits are bounded from below by the radius of the equatorial null circular geodesic which, for a given value of the black-hole angular momentum, is characterized by the smallest possible circumference.

Intriguingly, as discussed by many authors [1–9], in the maximally-spinning (extremal) Kerr black-hole spacetime, there exist circular orbits of radius r_{circular} which are characterized by the limiting near-horizon behavior $r_{\text{circular}}/r_{\text{H}} \rightarrow 1^+$ [10, 11].

The main goal of the present compact paper is to provide evidence, which is based on the famous Thorne hoop conjecture [12], for the existence of a non-trivial lower bound $r_{\text{circular}} > r_{\text{min}} > r_{\text{H}}$ on the radii of circular orbits in the maximally-spinning Kerr black-hole spacetime. In particular, as we shall show below, according to the hoop conjecture the smallest possible circular radius $r_{\text{min}} = r_{\text{min}}(\mu/M)$ is

determined by the dimensionless mass ratio μ/M [13] which characterizes the composed extremal-black-hole-orbiting-particle system.

2 Circular orbits in the maximally-spinning Kerr black-hole spacetime

We consider a particle of proper mass μ which orbits around a maximally-spinning (extremal) Kerr black-hole of mass M (with $\mu/M \ll 1$) and angular momentum $J = M^2$. The extremal curved black-hole spacetime is characterized by the line element [1],

$$ds^2 = -(1 - 2Mr/\Sigma)dt^2 - (4M^2r \sin^2 \theta / \Sigma)dt d\phi + (\Sigma/\Delta)dr^2 + \Sigma d\theta^2 + (r^2 + M^2 + 2M^3r \sin^2 \theta / \Sigma) \sin^2 \theta d\phi^2, \quad (1)$$

where (t, r, θ, ϕ) are the familiar Boyer–Lindquist spacetime coordinates and the metric functions in (1) are defined by the relations,

$$\Delta \equiv (r - M)^2; \quad \Sigma \equiv r^2 + M^2 \cos^2 \theta. \quad (2)$$

The degenerate horizon of the extremal black-hole spacetime is defined by the radial functional relation $\Delta(r = r_{\text{H}}) = 0$, which yields the simple relation

$$r_{\text{H}} = M. \quad (3)$$

The dimensionless energy ratio $E(r)/\mu$ (energy per unit mass as measured by asymptotic observers) associated with a circular orbit of radius r in the equatorial plane of the extremal (maximally-spinning) Kerr black-hole spacetime is given by the functional expression [1],

$$\frac{E(r)}{\mu} = \frac{r \pm M^{1/2}r^{1/2} - M}{r^{3/4}(r^{1/2} \pm 2M^{1/2})^{1/2}}. \quad (4)$$

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The upper/lower signs in the dimensionless energy expression (4) correspond to co-rotating/counter-rotating circular orbits in the black-hole spacetime, respectively. From Eq. (4) one finds the near-horizon co-rotating energy ratio [14],

$$\frac{E(x)}{\mu} = \frac{1}{\sqrt{3}} + \frac{2}{3\sqrt{3}} \cdot x + O(x^2), \quad (5)$$

where the dimensionless parameter x (with the near-horizon property $x \ll 1$) is defined by the relation,

$$r \equiv M(1 + x). \quad (6)$$

3 The Thorne hoop conjecture and a lower bound on the radii of circular orbits in the extremal black-hole spacetime

The total energy (as measured by asymptotic observers) of the composed extremal-Kerr-black-hole-orbiting-particle system is given by

$$E_{\text{total}} = M + E_{\text{ISCO}} + O(E_{\text{ISCO}}^2/M). \quad (7)$$

According to the Thorne hoop conjecture [12], the composed system will form an engulfing horizon if it can be placed inside a ring whose circumference C is equal to (or smaller than) $4\pi E_{\text{total}}$. That is, Thorne's famous hoop conjecture asserts that [12]

$$C(E_{\text{total}}) \leq 4\pi E_{\text{total}} \implies \text{Black-hole horizon exists.} \quad (8)$$

As we shall now show, the hoop relation (8) provides compelling evidence for the existence of a lower bound (with $r_{\text{circular}} > r_{\text{min}} > r_{\text{H}}$) on the radii of circular orbits in the maximally-spinning (extremal) Kerr black-hole spacetime. In particular, using Eqs. (1) and (2) one obtains the functional expression [15,16]

$$C(r) = 2\pi\sqrt{r^2 + M^2 + 2M^3/r} \quad (9)$$

for the circumference $C = C(r)$ of an equatorial circular orbit in the extremal Kerr black-hole spacetime. Substituting (6) into (9), one finds the simple near-horizon relation,

$$C(x \ll 1) = 4\pi M \cdot \left[1 + \frac{3}{8} \cdot x^2 + O(x^3) \right]. \quad (10)$$

Taking cognizance of Eqs. (5), (7), (8), and (10), one obtains the dimensionless lower bound

$$x_{\text{circular}} > x_{\text{min}} = \left(\frac{8}{3\sqrt{3}} \cdot \frac{\mu}{M} \right)^{1/2} \quad (11)$$

on the scaled radii of circular orbits in the composed extremal-Kerr-black-hole-orbiting-particle system. In particular, according to the Thorne hoop conjecture [12], composed black-hole-particle configurations whose circular orbits are characterized by the relation $x_{\text{circular}} \leq x_{\text{min}}$ are expected to be engulfed by a larger horizon with $r_{\text{horizon}} \geq r_{\text{circular}}$ [see Eqs. (8) and (11)].

4 Summary

A remarkable feature of the maximally-spinning (extremal) Kerr black-hole spacetime, which has been discussed in the physics literature by many authors (see [1–9] and references therein), is the existence of co-rotating circular orbits which are characterized by the limiting radial behavior $r_{\text{circular}}/r_{\text{H}} \rightarrow 1^+$.

In the present compact paper we have used the famous Thorne hoop conjecture [12] in order to provide evidence for the possible existence of a larger horizon (with $r_{\text{horizon}} \geq r_{\text{circular}} \geq r_{\text{H}}$) that engulfs composed extremal-Kerr-black-hole-orbiting-particle configurations which violate the dimensionless relation (11). In particular, our analysis has revealed the intriguing fact that circular orbits in the extremal Kerr black-hole spacetime are restricted to the radial region [see Eqs. (3), (6), and (11)]

$$r_{\text{circular}} > r_{\text{H}} \cdot \left[1 + \left(\frac{8}{3\sqrt{3}} \cdot \frac{\mu}{M} \right)^{1/2} + O\left(\frac{\mu}{M}\right) \right]. \quad (12)$$

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10. Note that maximally spinning Kerr black-hole spacetimes are characterized by the relations $M = a = r_H$, where $\{M, a, r_H\}$ are respectively the mass, angular momentum per unit mass, and the degenerate horizon radius of the extremal black hole.
11. We shall use natural units in which $G = c = 1$.
12. K.S. Thorne, in *Magic Without Magic: John Archibald Wheeler*, ed. by J. Klauder (W. H. Freeman, San Francisco, 1972)
13. Here μ is the proper mass (with $\mu \ll M$) of the orbiting particle
14. We shall henceforth focus our attention on the *co-rotating* circular orbits of the Kerr black-hole spacetime. These are the orbits which approach the black-hole horizon ($r_{\text{circular}}^{\text{co-rotating}}/r_H \rightarrow 1^+$) in the extremal $a/M \rightarrow 1^-$ limit.
15. Here we have substituted the characteristic relations $dt = dr = d\theta = 0$, $\theta = \pi/2$, and $\Delta\phi = 2\pi$ for equatorial circular orbits in the line element (1) of the extremal Kerr black-hole spacetime.
16. It is worth noting that the circumference of an engulfing hoop which is perpendicular to the equatorial plane (that is, with the properties $dt = dr = d\phi = 0$ and $\Delta\theta = 2\pi$) of the maximally-spinning (extremal) black hole is smaller than the calculated circumference (9) which characterizes the equatorial ($\theta = \pi/2$) engulfing hoop [this simple fact stems from Eqs. (1) and (2) with the characteristic property $\Sigma < r^2 + M^2 + 2M^3/r$]. One therefore concludes that if the equatorial hoop (9) is characterized by the inequality (8), then the perpendicular engulfing hoop will also be characterized by the same inequality.