Eur. Phys. J. C (2018) 78:655
https://doi.org/10.1140/epjc/s10052-018-6147-8

THE EUROPEAN PHYSICAL JOURNAL C

CrossMark

Letter

On scale versus conformal symmetry in turbulence

Yaron Oz[a]

Raymond and Beverly Sackler School of Physics and Astronomy, Tel-Aviv University, Tel-Aviv 69978, Israel

Received: 19 May 2018 / Accepted: 9 August 2018 / Published online: 17 August 2018

Abstract We consider the statistical description of steady state fully developed incompressible fluid turbulence at the inertial range of scales in any number of spatial dimensions. We show that in the absence of condensates turbulence statistics exhibits scale but not conformal symmetry, with the only possible exception being the direct enstrophy cascade in two space dimensions. We argue that the same conclusions hold for compressible non-relativistic turbulence as well as for relativistic turbulence. We discuss the modification of our conclusions in the presence of vacuum expectation values of negative dimension operators (condensates). We consider the issue of non-locality of the stress-energy tensor of inertial range turbulence field theory.

Fully developed incompressible fluid turbulence is largely considered as the most important unsolved problem of classical physics. Most fluid motions in nature at all scales are turbulent. Aircraft motions, river flows, atmospheric phenomena, astrophysical flows and even blood flows are some examples of set-ups where turbulent flows occur. Despite centuries of research, we still lack an analytical description and understanding of fluid flows in the non-linear regime. Insights to turbulence hold a key to understanding the principles and dynamics of non-linear systems with a large number of strongly interacting degrees of freedom far from equilibrium.

One defines the inertial range to be the range of length scales $l \ll r \ll L$, where the scales l and L are determined by the viscosity and forcing, respectively. Experimental and numerical data suggest that turbulence at the inertial range of scales reaches a steady state that exhibits statistical homogeneity and isotropy and is characterized by universal anomalous scaling exponents that depend only on the number of space dimensions d [1,2] (for a proposal of anomalous scaling setup at low Reynolds number see [3]).

[a] e-mail: yaronoz@post.tau.ac.il

The aim of this letter is to consider the question whether turbulence statistics realized by the universal anomalous scaling exponents can exhibit, as many critical systems do, not only scale symmetry but rather conformal symmetry. Note, that by scale symmetry we do not mean a self-similar (non intermittent) structure of the turbulence statistics, but rather the emergence of anomalous scalings e.g. of the velocity structure functions.

We will use exact scaling relations to show that in the absence of condensates turbulence statistics exhibits scale but not conformal symmetry. We will find one possible exception, which is the direct enstrophy cascade in two space dimensions. Furthermore, we will argue that the same conclusions hold for compressible non-relativistic turbulence as well as for relativistic turbulence. We will discuss how the presence of vacuum expectation values of negative dimension operators (condensates) modifies these conclusions.

Under a d-dimensional conformal transformation:

$$x^i \to x'^i \; dx'^2 = \Omega^2(x) dx^2, \quad i = 1, ..., d \,, \tag{1}$$

and

$$\frac{\partial x'^i}{\partial x^j} = \Omega(x) R^i_j(x), \quad R^i_k(x) R^k_j(x) = \delta^i_j \,, \tag{2}$$

thus $R^i_j(x) \in SO(d)$. Raising and lowering indices is done with Kronecker delta δ_{ij}, δ^{ij}. Conformal transformations rescale lengths non-uniformly, while preserving the angles between vectors. The conformal group includes the Euclidean group that consists of translations and rotations as well as dilatations and special conformal transformations. Special conformal transformations are composed of an inversion $x^i \to \frac{x^i}{x^2}$ followed by a translation $x^i \to x^i + a^i$ and by a second inversion. They take the form:

$$x^i \to x'^i = \frac{x^i + a^i x^2}{1 + 2a \cdot x + a^2 x^2} \,. \tag{3}$$

Consider first the case of inertial range incompressible fluid turbulence that describes fluid flows with a low Mach number. We are interested in the statistics of the turbulent velocity vector field v^i that satisfies the incompressibility condition $\partial_i v^i = 0$, and hence in the correlation functions:

$$\langle v^{i_1}(x_1)\cdots v^{i_n}(x_n)\rangle \,, \tag{4}$$

where the separation between points x_{ij} in (4) is in the inertial range. We will work in the limit $l \to 0$ and x_{ij} fixed in which experimental and numerical data suggest that the correlation functions (4) are finite. Note, however, that this finiteness is not analytically established. The correlation functions have also a a finite limit when $x_{ij} \to 0$, after taking the limit $l \to 0$, which allows us to define composite operators $v^{i_1}(x)v^{i_2}(x)\cdots v^{i_k}(x)$ [4]. The dimension of these operators is k-times the dimension of v^i.

Note, that here we are using the fact that the dimension of the velocity vector field is positive in units of length. Hence, the operator product of two velocity vector fields is expected to be regular. This is not true when we consider operators that are derivatives of the velocity vector field. Such operators have negative dimensions in units of length and their operator product is singular. These UV divergences have important observable consequences such as the dissipative anomaly (see e.g. [5,6]).

Consider Kolmogorov's law [7,8]:

$$\langle v^{ij}(x_1)v^k(x_2)\rangle = \epsilon\left(\delta^{ik}x^j_{12} + \delta^{jk}x^i_{12} - \frac{2}{d}\delta^{ij}x^k_{12}\right), \tag{5}$$

where ϵ is related, up to a numerical multiplicative factor that depends on the number of dimensions, to the mean rate of energy dissipation due to viscosity, and $v^{ij}(x)$ is a traceless symmetric 2-tensor of $SO(d)$

$$v^{ij}(\vec{x}) = v^i(x)v^j(x) - \frac{v^2(x)}{d}\delta^{ij} \,. \tag{6}$$

Kolmogorov's law (5) is an exact relation in statistical turbulence, and thus provides a consistency check on any proposal for such a description. In the following we will show that (5) is not compatible with conformal symmetry.

The third moment of the velocity vector field is called Kolmogorov's law. However, the third moment that was derived by Kolmogorov in the second paper in [7,8] is for the differences of the velocity vector field at two separate points. In order to recast it in the tensor form (5) one uses the assumption of statistical homogeneity and isotropy (see e.g. [4,9]).

The mean rate of energy dissipation ϵ in (5) is a dimensionful constant that depends on the forcing scale L. This is generally case with the coefficients of turbulence structure functions. Thus, the scale versus conformal symmetry that we consider corresponds to the universal scaling exponents of the structure functions.

The operators of conformal field theories are classified as primary or descendants. A conformal primary operator of dimension Δ in an irreducible representation R^I_J of $SO(d)$ transforms under a conformal transformation (1) as:

$$O_I(x) \to O'_I(x') = \Omega(x)^{-\Delta}R^J_I(x)O_J(x) \,. \tag{7}$$

Descendants are derivatives of the primary operators and their transformation can be deduced from (2) and (7). Consider two primary operators O_I and O_J with scaling dimensions Δ_I and Δ_J, respectively. Scale (dilatation) symmetry $x^i \to \lambda x^i$, $\lambda = const$, restricts the form of their two-point function:

$$\langle O_I(x_1)O_J(x_2)\rangle = \frac{c_{IJ}(x_{12})}{x_{12}^{\Delta_I+\Delta_J}} \,, \tag{8}$$

where c_{IJ} is dimensionless. Special conformal symmetry sets further restrictions. Under a conformal transformation (1) we have:

$$x^2_{12} \to x'^2_{12} = \Omega(x_1)\Omega(x_2)x^2_{12} \,. \tag{9}$$

Using (7), (8) and (9) one gets that conformal covariance of two-point function (8) requires $\Delta_I = \Delta_J$. This is clearly not the case in (5), since the dimension of v^i is not equal to the dimension of v^{ij}. In fact the orthogonality theorem requires that the two operators transform in the same irreducible $SO(d)$ representation [10] and

$$c_{IJ}(x_{12}) \to R^K_I(x_1)R^L_J(x_2)c_{KL}(x_{12}) \,. \tag{10}$$

However, v^i and v^{ij} transform in different irreducible representations of $SO(d)$.

It is possible that the operators v^i or v^{ij} (or both) are not primary operators. Indeed, consider the divergence free velocity vector v^i. The two-point function of a conserved spin one operator J^i, $\partial_i J^i = 0$ of dimension Δ reads:

$$\langle J^i(\vec{x}_1)J^j(\vec{x}_2)\rangle = \frac{\delta^{ij} + a\frac{x^i x^j}{x^2}}{x_{12}^{2\Delta}}, \quad a = \frac{2\Delta}{d-2\Delta-1} \,. \tag{11}$$

The requirement to be a conformal primary (7) implies that $a = -2$ in (11), hence $\Delta = d-1$ [11]. Thus, for v^i to be a conformal primary operator its dimension should have been $d-1$. However, its dimension is experimentally close to its Kolmogorov (K41) linear scaling dimension [7,8] $\Delta_{K41}[v^i] = -\frac{1}{3}$ (in inverse length units) [12].

Since v^i cannot be a primary operator let us assume that it is a descendant of a primary operator. It is clearly not a descendant of v^{ij}, $v^i \neq \partial_j v^{ij}$, where as discussed above, the composite operator v^{ij} is well defined and we can take its derivative. Thus, the two-point function of v^i and v^{ij} can be nonzero only if they are both descendants of the same primary operator, and in such a case their dimensions should differ by an integer number. This is not case: Their K41 linear scaling dimensions differ by $-\frac{1}{3}$, while the actual experimental value may differ slightly from this value [12]. Note, that we are not assuming the Kolmogorov linear scaling but rather that the anomalous scaling of these objects is not very far from it.

As a second example consider the exact scaling relation of incompressible fluid turbulence derived in [13]:

$$\langle v^i(x_1)p(x_1)v^2(x_2)\rangle = Cx_{12}^i \ , \tag{12}$$

where C is a constant related to the mean rate of energy dissipation, and p is the fluid pressure. In this case the two operators are $v^i(x)p(x)$ and $v^2(x)$ and their dimensions differ but not by an integer, where we use the regularity of the operator product of v^i and p since both have a positive dimension in units of length. Thus, the two-point function (12) is not conformally covariant.

Two-dimensional fluid turbulence is special since there are two cascades: the direct enstrophy (vorticity squared) cascade and the inverse energy cascade. The analysis of the inverse energy cascade works as above and it is not conformally covariant. In [14] the isovorticity lines of two-dimensional inverse cascade turbulence have been studied numerically and have been identified as SLE_κ curves with $\kappa = 6$ (for an SLE review see e.g. [15]). This result hints that there may be an underlying two-dimensional conformal structure in inverse cascade turbulence theory, however, the jury is still out on this issue. If there is indeed a conformal structure, it seems to be in contradiction with our analysis. We will discuss this later when we consider the issue of condensates.

In the enstrophy cascade one can derive the exact scaling relation:

$$\langle \omega(x_1)\omega(x_2)v^i(x_2)\rangle = Cx_{12}^i \ , \tag{13}$$

where $\omega = \epsilon^{ij}\partial_i v_j$ is the vorticity pseudoscalar and C is a constant related to the mean rate of enstrophy dissipation. In this case the two operators are $\omega(x)$ and $\omega(x)v^i(x)$. Their dimensions may differ by one (the dimension of v^i is -1 or close to it) and it is still possible that they are both descendants of the same primary operator. This is the case considered by Polyakov in [6].

Consider next compressible non-relativistic fluid flows, where the fluid density $\rho(x)$ is not constant. In this case, one can derive an exact scaling relation that takes the form [13] :

$$\langle T^{0i}(x_1)T^{ij}(x_2)\rangle = \epsilon x_{12}^j \ , \tag{14}$$

where we sum over the index i, ϵ is a constant related to the mean rate of energy dissipation and

$$T^{0i} = \rho v^i, \quad T^{ij} = \rho v^i v^j + p\delta^{ij} \ , \tag{15}$$

satisfies the ideal compressible fluid equation $\partial_t T^{0j} + \partial_i T^{ij} = 0$. The two-point function (14) reduces to the Kolmogorov law (5) in the incompressible case $\rho = const$, when using $\langle p(x_1)v^i(x_2)\rangle = 0$ which follows from incompressibility, and using isotropy to recast it in the manifestly isotropic form (5). We have in (14) two different operators that do not have dimensions that differ by an integer. Hence (14) cannot be a two-point function of a confomally invariant theory.

In two space dimensions there is an enstrophy cascade of compressible fluid flows similar to the incompressible fluid one. One can derive an exact scaling relation for such turbulent flows that takes the form [16]:

$$\langle \omega^j(x_1)\omega(x_2)\rangle = Cx_{12}^j \ , \tag{16}$$

where

$$\omega^j = \epsilon^{ik}\partial_k T_i^j, \quad \omega = \epsilon^{ik}\partial_k T_{0i} \ , \tag{17}$$

and C is a constant related to the mean rate of enstrophy dissipation. The scaling relation (16) has been checked numerically in [16,17] and it reduces to (13) in the limit of non-relativistic fluid flows. In this case the difference between the dimensions of the two operators which is the dimension of v^i may be an integer as in the incompressible limit, thus it is possible that it is conformally covariant.

Let us discuss now relativistic hydrodynamics defined by the conservation of a relativistic stress-energy $T^{\mu\nu}$, $\mu, \nu = 0, ..., d$, $\partial_\mu T^{\mu\nu} = 0$. The stress-energy tensor of an ideal relativistic fluid reads:

$$T^{\mu\nu} = (\varepsilon + p)u^\mu u^\nu + p\eta^{\mu\nu} \ , \tag{18}$$

where ε is the energy density, p is the relativistic pressure, $u^\mu = (\gamma, \gamma\frac{v^i}{c})$, $\gamma = (1 - \frac{v^2}{c^2})^{-\frac{1}{2}}$, is the relativistic fluid velocity vector $u_\mu u^\mu = -1$ and $\eta^{\mu\nu} = diag[-, +, ..., +]$ is the Minkowskian metric. One can derive an exact scaling [18] of the form (14) with (18), which reduces to the Kolmogorov law (5) in the limit of non-relativistic fluid flow $v \ll c$. The difference between the dimensions of the two operators in (18) is the dimension of v^i, which is unlikley to be an integer (it is not an integer in the non-relativistic limit). This suggests that also relativistic turbulence is not conformally covariant.

In two space dimensions there is also an enstrophy cascade of relativistic fluid flows similar to the incompressible fluid one. One can derive an exact scaling relation for such turbulent flows that takes the form (16), (17) and (18) [16], which reduces to (13) in the limit of non-relativistic fluid flows. In this case the difference between the dimensions of the two operators which is the dimension of v^i may be an integer as in the non-relativistic limit, thus it is possible that it is conformally covariant.

Consider the K41 theory of turbulence [7,8]. Since K41 theory neglects intermittency, the dimension of a general composite operator of the form $v^{i_1}(x)v^{i_2}(x)\cdots v^{i_k}(x)$ is $\frac{k}{3}$. Similarly to the above analysis, we conclude that K41 theory is scale but not conformally covariant, by e.g. using the Kolmogorov law. In [19] we proposed a field theory of turbulence at the inertial range of scales and derived the formula for anomalous scalings of the longitudinal structure functions proposed in [20]. The field theory is based on dressing the K41 mean field theory by a conformal field theory of a gapless dilaton mode, and we related the intermittency to the boundary conformal anomaly coefficient. The discussion here suggests that we should include in the intermittency also the scale anomaly coefficients.

In our analysis we assumed that the vacuum expectation values of the turbulence field theory operators are zero. However, since we have operators O_K with negative dimensions $\Delta_K < 0$ they may acquire expectation values [1]:

$$\langle O_K \rangle = c_K L^{-\Delta_K} \ , \tag{19}$$

where c_K are nonzero constants and L is the infrared forcing scale. As noticed by A. Zamolodchikov, the expectation values (19) break spontaneously conformal symmetry and modify the correlation functions of conformal field theories [21]. For instance, the two-point correlation function (8) will be modified due to such expectation values by dominating terms of the form [4,6,21]:

$$\frac{c_{IJK}(x_{12})}{x_{12}^{\Delta_I+\Delta_J}} \left(\frac{L}{x_{12}}\right)^{|\Delta_K|} , \tag{20}$$

where c_{IJK} is dimensionless. In such a case one cannot use the orthogonality theorem [10]. The issue whether such nonzero expectation values exist requires a study of the infrared boundary conditions and dynamics of the turbulent system. We do not have much to add on this crucial point, except that it may be a way to resolve the inconsistency between our analysis of the two-dimensional inverse cascade of the incompressible fluid and the numerical evidence [14] if indeed an underlying conformal structure exists.

Finally, let us make a comment about the stress-energy tensor of the field theory of inertial range incompressible fluid turbulence. A local stress-energy tensor is a dimension d conserved 2-tensor of $SO(d)$. It is easy to see on dimensional grounds that we cannot make such an object from the velocity vector v^i and the pressure p in the energy cascades, hence the stress-energy tensor of a field theory description of turbulence will necessarily be non-local. This is not in contradiction with field theory axioms that require the existence of energy and momentum but do not require the existence of their densities. In the enstrophy cascade in two space dimensions it is possible that such a local stress-energy tensor exists.

Acknowledgements I would like to thank G. Falkovich, V. Mukhanov, A. Polyakov and V. Yakhot for discussions. This work is supported in part by the I-CORE program of Planning and Budgeting Committee (grant number 1937/12), the US-Israel Binational Science Foundation, GIF and the ISF Center of Excellence.

[1] I thank A. Polyakov for pointing this out to me.

Funded by SCOAP^3.

References

1. L.D. Landau, E.M. Lifshitz, *Fluid Mechanics* (Butterworth-Heinemann, Oxford, 2000)
2. U. Frisch, *Turbulence: The Legacy of A. N. Kolmogorov* (Cambridge University Press, Cambridge, 1995)
3. V. Yakhot, D. Donzis, Emergence of Multi-Scaling in a Random Force-Stirred Fluid. Phys. Rev. Lett. **119**, 044501 (2017)
4. G. Falkovich, A. Zamolodchikov, Operator product expansion and multi-point correlations in turbulent energy cascades. J. Phys. A **48**, 18FT02 (2015)
5. A.M. Polyakov, Conformal turbulence. arXiv:hep-th/9209046
6. A. M. Polyakov, The Theory of turbulence in two-dimensions. Nucl. Phys. B **396**, 367 (1993). https://doi.org/10.1016/0550-3213(93)90656-A. arXiv:hep-th/9212145
7. A.N. Kolmogorov, The local structure of turbulence in incompressible viscous fluid for very large Reynolds number. Dokl. Akad. Nauk. SSSR **30**, 299 (1941) (reprinted. Proc. R. Soc. London, Ser. A **434**, 9 (1991))
8. A.N. Kolmogorov, Dissipation of energy in locally isotropic turbulence. Dokl. Akad. Nauk. SSSR 32, 16 (1941) (reprinted. Proc. R. Soc. London, Ser. A **434**, 15 (1991))
9. K. Gawedzki, Easy turbulence. arXiv:chao-dyn/9907024
10. S. Ferrara, A.F. Grillo, G. Parisi, R. Gatto, Canonical scaling and conformal invariance. Phys. Lett. **38B**, 333 (1972). https://doi.org/10.1016/0370-2693(72)90259-6
11. H. Osborn, A.C. Petkou, Implications of conformal invariance in field theories for general dimensions. Ann. Phys. **231**, 311 (1994). https://doi.org/10.1006/aphy.1994.1045. arXiv:hep-th/9307010
12. R. Benzi, S. Ciliberto, C. Baudet, G. Ruiz Chavarria, On the scaling of three-dimensional homogeneous and isotropic turbulence, Phys. D **80**, 385 (1995)
13. G. Falkovich, I. Fouxon, Y. Oz, New relations for correlation functions in Navier-Stokes turbulence. J. Fluid Mech. **644**, 465 (2010). arXiv:0909.3404 [nlin.CD]
14. D. Bernard, G. Boffetta, A. Celani, G. Falkovich, Conformal invariance in two-dimensional turbulence. Nat. Phys. **2**, 124 (2006)
15. J.L. Cardy, SLE for theoretical physicists. Ann. Phys. **318**, 81 (2005). https://doi.org/10.1016/j.aop.2005.04.001. arXiv:cond-mat/0503313
16. J.R. Westernacher-Schneider, L. Lehner, Y. Oz, Scaling relations in two-dimensional relativistic hydrodynamic turbulence. JHEP **1512**, 067 (2015). https://doi.org/10.1007/JHEP12(2015)067. arXiv:1510.00736 [hep-th]
17. J.R. Westernacher-Schneider, L. Lehner, Numerical measurements of scaling relations in two-dimensional conformal fluid turbulence. JHEP **1708**, 027 (2017). https://doi.org/10.1007/JHEP08(2017)027. arXiv:1706.07480 [hep-th]
18. I. Fouxon, Y. Oz, Exact scaling relations in relativistic hydrodynamic turbulence. Phys. Lett. B **694**, 261 (2010). https://doi.org/10.1016/j.physletb.2010.09.067. arXiv:0909.3574 [hep-th]
19. Y. Oz, Spontaneous symmetry breaking, conformal anomaly and incompressible fluid turbulence. https://doi.org/10.1007/JHEP11(2017)040. arXiv:1707.07855 [hep-th]

20. C. Eling, Y. Oz, The anomalous scaling exponents of turbulence in general dimension from random geometry. JHEP **1509**, 150 (2015). https://doi.org/10.1007/JHEP09(2015)150. arXiv:1502.03069 [nlin.CD]
21. A.B. Zamolodchikov, Two point correlation function in scaling Lee-Yang model. Nucl. Phys. B **348**, 619 (1991). https://doi.org/10.1016/0550-3213(91)90207-E