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Axial gravity: a non-perturbative approach to split anomalies

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Abstract In a theory of a Dirac fermion field coupled to a metric-axial-tensor (MAT) background, using a Schwinger-DeWitt heat kernel technique, we compute non-perturbatively the two (odd parity) trace anomalies. A suitable collapsing limit of this model corresponds to a theory of chiral fermions coupled to (ordinary) gravity. Taking this limit on the two computed trace anomalies we verify that they tend to the same expression, which coincides with the already found odd parity trace anomaly, with the identical coefficient. This confirms our previous results on this issue.

1 Introduction

This paper is a follow up of [1] where a new version of modified gravity was introduced, a metric-axial-tensor gravity. That is, beside the usual metric, the model is endowed with an additional symmetric tensor that interacts chirally with fermions. The purpose there was not (or not yet) to describe a new phenomenological model of gravity, but to permit a more accurate investigation of the relation between gravity and chiral fermions. It is often stated in the literature that gravity is chirally blind, meaning that the relevant charge, the mass, is positive, and is thus different from the typical case of a U(1) interaction. This is certainly a basic peculiarity of gravity with several important consequences. However one should reflect on the fact that the coupling between gravity and matter is given by the juxtaposition of the metric and the

energy-momentum tensor, and the energy-momentum tensors of fermions with opposite chiralities are different.

One can suspect therefore that at some stage differences might emerge between fermions with opposite chiralities in their interaction with gravity. A privileged place where such differences may show up are the anomalies. And in this case the candidate is the trace anomaly, because it involves precisely the coupling between the metric and the energy-momentum tensor. The difficulty is how to make this difference emerge. As will be argued below, one should be careful to preserve the definite fermion chirality throughout the calculation. There is no direct way to do it, basically because the Dirac operator for a Weyl fermion contains a chiral projector. Therefore one has to resort to some indirect method. Like in many other cases in physics, the best way to avoid similar problems is to embed the system in a larger setup containing more variables and/or parameters. The metric-axial-tensor (MAT) gravity is designed to do this. It is formulated for Dirac fermions coupled to the usual metric and to an axial symmetric tensor. In this case the operator involved is the usual Dirac operator. The situation appropriate for Weyl fermions is recovered in a specific limit, the collapsing limit.

As mentioned above, MAT has already been introduced and used to compute the odd-parity trace anomaly in [1]. There the approach was perturbative, we calculated the Feynman diagrams at the lowest significant order. What we want to do in this paper is to show that the same result can be obtained non-perturbatively, by means of the heat kernel method and using different regularizations. Hereafter is a qualitative, but more detailed, presentation of both the problem we wish to solve and the method we use.

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1.1 Split and non-split anomalies

A basic differentiation between anomalies in fermionic field theories is the separation between *split* and *non-split* anomalies. Split anomalies have an opposite sign for opposite fermion chiralities. Non-split anomalies have the same sign for opposite chiralities. An example of the first are the consistent chiral gauge or gravity anomalies. They may of course arise only in the presence of chiral asymmetry. These anomalies undermine the consistency of theories in which they are present, and, as a consequence, they have been used as an exclusion criterion. An example of non-split anomalies are the covariant gauge or gravity anomalies, such as the Kimura–Delbourgo–Salam anomaly or the anomaly that is utilized to explain the decay of a π^0 into two γ's. But the examples are manifold. In the family of trace anomalies, the even ones are non-split, while the odd trace anomaly, which is the main character of this paper, is split.

Split and non-split anomalies differ also for the difficulties one comes across when computing them. While there are several tested techniques to compute non-split anomalies, the calculation of the split ones is rather non-trivial. In many of the latter cases one may avail oneself of such a powerful tool as the family index theorem (for instance for consistent gauge and gravity anomalies). But, like for the odd trace anomaly, this is not always so, and, in any case, it is important to be able to derive such anomalies with independent field-theoretical methods. If one resorts to path integral methods, one has to integrate out the fermion field(s), in which case the origin of the difficulties resides in the functional measure. As discussed in [1], a basic ingredient for the calculation is the functional integration measure which, for chiral fermions, is not well-defined. On the other hand, to get the correct result, it is imperative to preserve throughout the calculation the information that the fermion field, which is being integrated out, has a definite chirality. One is then obliged to either use indirect methods or to elude a direct intrusion of the functional measure in the calculation. The second alternative refers to the use of Feynman diagrams, in which case the chirality of fermions is preserved by vertices containing the appropriate chiral projector. This is the method employed in [1–3] together with dimensional regularization. In the present paper however, we focus on an indirect method of calculation, first used by Bardeen, [4], for chiral gauge anomalies. He considered a theory of Dirac fermions coupled to two external non-Abelian (vector V_μ and axial A_μ) gauge potentials. Clearly this poses no problems from the point of view of the functional measure and the derivation of the anomaly goes through without difficulties. Eventually one takes the collapsing limit $V \to \frac{V}{2}$ and $A \to \frac{V}{2}$ and verifies that, in such a limit, the anomaly becomes the desired consistent gauge anomaly. For the sake of clarity we present a summary of this derivation in Appendix A.

This approach has already been introduced and applied in [1] for the odd trace anomaly. To this end we introduced there a modification of ordinary gravity, the metric-axial-tensor (MAT) gravity: beside the usual metric $g_{\mu\nu}$ we introduced an axial symmetric 2-tensor $f_{\mu\nu}$, and coupled it to a Dirac fermion. Then we computed the trace of the energy-momentum tensor and of its axial companion and, eventually, we took the limit $g \to \frac{g}{2}$ and $f \to \frac{g}{2}$ and obtained the desired result. The limit of that derivation is that it relies on Feynman diagram techniques, and, so, it is perturbative. In fact we calculated only the lowest order of the odd trace anomaly and then covariantized it. This is of course permitted provided we are sure that there are no anomalies of the diffeomorphisms. With a MAT background this verification is exceedingly complicated and in [1] we did not do it and contented ourselves with an analogous but simpler verification carried out in [3]. It is clear that to prevent any objection we have to guarantee that diffeomorphisms are respected throughout the derivation. This can be done with DeWitt's method, [5,6]. This method is based on point-splitting. Therefore one needs a regularization in order to get rid of divergences, but the point-splitting is along a geodesic, thus guaranteeing covariance under diffeomorphisms. Our aim here is to combine DeWitt's with Bardeen's method. This requires a formulation of MAT more accurate than in [1]. For this reason the anomaly calculation proper needs to be preceded by a long introduction on the so-called hypercomplex calculus, which is the appropriate framework for MAT gravity.

Organization of the paper Section 2 is a short introduction of axial-complex numbers and axial-complex analysis. In Sect. 3 we deal with the axial-complex analysis of geodesics in an axial-complex space. We introduce normal coordinates, define the world function and the coincidence limit (i.e. the limit for vanishing geodetic distance), the VVM determinant and the parallel displacement matrix for tensors and for spinors. The (pseudo)Riemannian geometry of an axial-complex space was already introduced in [1]. To help the reader, it is presented anew in Appendix B in a partially renovated notation, which seems to us more practical. In Sect. 4 we introduce the theory of Dirac fermions in a MAT background, we define the relevant energy-momentum tensors (they are two, the ordinary one and its axial companion) and analyse their classical Ward identities with respect to ordinary and axial diffeormorphisms and Weyl transformations. We also define the 'square' of the Dirac operator, which is crucial for the application of the Schwinger-DeWitt method. In Sect. 5 we explain this method and compute the relevant heat kernel coefficients. In Sect. 6 we apply these results to the non-perturbative computation of the (odd) trace anomalies of the two em tensors with two different regularization, the dimensional and ζ-function ones. Then we compute the collapsing limit and show that the two anomalies collapse to a single one and take the form of the odd trace anomaly

already computed in [2,3] and [1], as expected. Section 7 is devoted to our conclusions. Appendix A is a summary of Bardeen's method. Appendix C contains a short account of fermion propagators in a MAT background.

Overview of the literature There exists a vast literature on even trace anomalies in 4d, mostly old [7–32] but also recent [33–35], denoting a renewed interest in the subject. The literature on the odd parity trace anomaly in 4d (still in a settling phase) consists of [1–3,36–40]. Textbooks on anomalies are [41–43]. Aspects of split and non-split anomalies, which are relevant to this paper, were discussed in [44,45]. A regularization, not used in this paper, but which would be interesting to explore is the one introduced in [46]. Hypercomplex analysis in physical problems was introduced and used in [47–53].

2 Axial-complex analysis

Axial-complex numbers are defined by

$$\hat{a} = a_1 + \gamma_5 a_2, \tag{1}$$

where a_1 and a_2 are real numbers. Arithmetic is defined in the obvious way. We can define a conjugation operator

$$\bar{\hat{a}} = a_1 - \gamma_5 a_2. \tag{2}$$

We will denote by $\mathcal{AC}$ the set axial-complex numbers, by $\mathcal{AR}$ the set of axial-complex numbers with $a_2 = 0$ (the axial-real numbers) and by $\mathcal{AI}$ the set of axial-complex numbers with $a_1 = 0$ (the axial-imaginary numbers). We can define a (pseudo)norm

$$(a, a) = \hat{a}\bar{\hat{a}} = a_1^2 - a_2^2. \tag{3}$$

This determines an axial-light-cone with all the related problems. In general, whenever possible, we will keep away from it by considering the case $|a_1| > |a_2|$. Alternatively we will use an axial-Wick-rotation (analogous to the Wick rotation for the Minkowski spacetime light-cone) $a_2 \to i a_2$. Whenever we resort to it explicit mention will be made.

Introducing the chiral projectors $P_\pm = \frac{1 \pm \gamma_5}{2}$, we can also write

$$\hat{a} = a_+ P_+ + a_- P_-, \qquad a_\pm = a_1 \pm a_2. \tag{4}$$

We will consider functions $\hat{f}(\hat{x})$ of the axial-complex variable

$$\hat{x} = x_1 + \gamma_5 x_2 \tag{5}$$

from $\mathcal{AC}$ to $\mathcal{AC}$, which are axial-analytic, i.e. admit a Taylor expansion, and actually identify the functions with their expansions. Using the property of the projectors it is easy to see that

$$\hat{f}(\hat{x}) = P_+ \hat{f}(x_+) + P_- \hat{f}(x_-) = \frac{1}{2} \left(\hat{f}(x_+) + \hat{f}(x_-) \right)$$
$$+ \frac{\gamma_5}{2} \left(\hat{f}(x_+) - \hat{f}(x_-) \right). \tag{6}$$

In the same way we will consider functions from $\mathcal{AC}^4$ to $\mathcal{AC}$, with analogous properties.

$$\hat{f}(\hat{x}^\mu) = P_+ \hat{f}(x_+^\mu) + P_- \hat{f}(x_-^\mu)$$
$$= \frac{1}{2} \left(\hat{f}(x_+^\mu) + \hat{f}(x_-^\mu) \right) + \frac{\gamma_5}{2} \left(\hat{f}(x_+^\mu) - \hat{f}(x_-^\mu) \right) \tag{7}$$

with $\mu = 0, 1, 2, 3$, and

$$\hat{x}^\mu = x_1^\mu + \gamma_5 x_2^\mu \tag{8}$$

are the axial-complex coordinates.

Axial-complex numbers and analysis are a particular case of pseudo-complex or hyper-complex numbers and analysis, [47,48].

Derivatives are defined in the obvious way:

$$\frac{\partial}{\partial \hat{x}^\mu} = \frac{1}{2} \left(\frac{\partial}{\partial x_1^\mu} + \gamma_5 \frac{\partial}{\partial x_2^\mu} \right),$$
$$\frac{\partial}{\partial \bar{\hat{x}}^\mu} = \frac{1}{2} \left(\frac{\partial}{\partial x_1^\mu} - \gamma_5 \frac{\partial}{\partial x_2^\mu} \right). \tag{9}$$

Notice that for axial-analytic functions

$$\frac{d}{d\hat{x}} = \frac{\partial}{\partial x_1} \equiv \frac{\partial}{\partial \hat{x}}, \tag{10}$$

whereas $\frac{\partial}{\partial \bar{\hat{x}}} \widehat{f}(\hat{x}) = 0$.

As for integrals, since we will always have to do with rapidly decreasing functions at infinity, we define

$$\int d\hat{x} \, \widehat{f}(\hat{x})$$

as the rapidly decreasing primitive $\widehat{g}(\hat{x})$ of $\widehat{f}(\hat{x})$. Therefore the property

$$\int d\hat{x} \, \frac{\partial}{\partial \hat{x}^\mu} \hat{f}(\hat{x}) = 0 \tag{11}$$

follows immediately. As a consequence of (10) it follows that, for an axial-analytic function,

$$\int d\hat{x} \, \widehat{f}(\hat{x}) = \int dx_1 \, \widehat{f}(\hat{x}) \tag{12}$$

and we can define definite integrals such as

$$\int_{\hat{a}}^{\hat{b}} d\hat{x} \, \widehat{f}(\hat{x}) = \widehat{g}(\hat{b}) - \widehat{g}(\hat{a}). \tag{13}$$

In this axial-spacetime we introduce an axial-Riemannian geometry as follows. Starting from a metric $\widehat{g}_{\mu\nu} = g_{\mu\nu} +$

$\gamma_5 f_{\mu\nu}$, the Christoffel symbols (see Appendix B) are defined by

$$\widehat{\Gamma}^\lambda_{\mu\nu} = \frac{1}{2}\widehat{g}^{\lambda\rho}\left(\frac{\partial}{\partial\widehat{x}^\mu}\widehat{g}_{\rho\nu} + \frac{\partial}{\partial\widehat{x}^\nu}\widehat{g}_{\mu\rho} - \frac{\partial}{\partial\widehat{x}^\rho}\widehat{g}_{\mu\nu}\right). \tag{14}$$

They split as follows

$$\widehat{\Gamma}^\mu_{\nu\lambda} = \Gamma^{(1)\mu}_{\nu\lambda} + \gamma_5\Gamma^{(2)\mu}_{\nu\lambda} \tag{15}$$

and are such that the metricity condition is satisfied

$$\frac{\partial}{\partial\widehat{x}^\mu}\widehat{g}_{\nu\lambda} = \widehat{\Gamma}^\rho_{\mu\nu}\,\widehat{g}_{\rho\lambda} + \widehat{\Gamma}^\rho_{\mu\lambda}\,\widehat{g}_{\nu\rho}, \tag{16}$$

which, in $\mathcal{AR}^4$, takes the form

$$\frac{\partial}{\partial\widehat{x}^\mu}g_{\nu\lambda} = \Gamma^{(1)\rho}_{\mu\nu}\,g_{\rho\lambda} + \Gamma^{(1)\rho}_{\mu\lambda}\,g_{\nu\rho}$$
$$+\Gamma^{(2)\rho}_{\mu\nu}\,f_{\rho\lambda} + \Gamma^{(2)\rho}_{\mu\lambda}\,f_{\nu\rho} \tag{17}$$
$$\frac{\partial}{\partial\widehat{x}^\mu}f_{\nu\lambda} = \Gamma^{(1)\rho}_{\mu\nu}\,f_{\rho\lambda} + \Gamma^{(1)\rho}_{\mu\lambda}\,f_{\nu\rho}$$
$$+\Gamma^{(2)\rho}_{\mu\nu}\,g_{\rho\lambda} + \Gamma^{(2)\rho}_{\mu\lambda}\,g_{\nu\rho}. \tag{18}$$

3 MAT geodesics

Let us set

$$\widehat{\Gamma}^\mu_{\nu\lambda} = \Gamma^{(1)\mu}_{\nu\lambda} + \gamma_5\Gamma^{(2)\mu}_{\nu\lambda}. \tag{19}$$

The equation for MAT geodesics is

$$\ddot{\widehat{x}}^\mu + \widehat{\Gamma}^\mu_{\nu\lambda}\dot{\widehat{x}}^\nu\dot{\widehat{x}}^\lambda = 0, \tag{20}$$

where a dot denotes derivation with respect to an axial-affine parameter $t = t_1 + \gamma_5 t_2$. For axial-real and axial-imaginary components this means

$$\ddot{x}^\mu_1 + \Gamma^{(1)\mu}_{\nu\lambda}(\dot{x}^\nu_1\dot{x}^\lambda_1 + \dot{x}^\nu_2\dot{x}^\lambda_2)$$
$$+\Gamma^{(2)\mu}_{\nu\lambda}(\dot{x}^\nu_1\dot{x}^\lambda_2 + \dot{x}^\nu_2\dot{x}^\lambda_1) = 0 \tag{21}$$
$$\ddot{x}^\mu_2 + \Gamma^{(1)\mu}_{\nu\lambda}(\dot{x}^\nu_1\dot{x}^\lambda_2 + \dot{x}^\nu_2\dot{x}^\lambda_1)$$
$$+\Gamma^{(2)\mu}_{\nu\lambda}(\dot{x}^\nu_1\dot{x}^\lambda_1 + \dot{x}^\nu_2\dot{x}^\lambda_2) = 0. \tag{22}$$

These geodesic equations can be obtained as equations of motion from the action

$$\widehat{S} = \int d\widehat{t}\sqrt{\widehat{g}_{\mu\nu}\dot{\widehat{x}}^\mu\dot{\widehat{x}}^\nu} = S_1 + \gamma_5 S_2, \tag{23}$$

where $\widehat{g}_{\mu\nu} = g_{\mu\nu} + \gamma_5 f_{\mu\nu}$.

The action takes values in $\mathcal{AC}$. For instance, setting the proper time $\widehat{\tau} = \tau_1 + \gamma_5\tau_2$,

$$\widehat{S}[\widehat{x}] = \int d\widehat{\tau}\left(\widehat{g}_{\mu\nu}\dot{\widehat{x}}^\mu\dot{\widehat{x}}^\nu\right)^{\frac{1}{2}}. \tag{24}$$

But unlike [47,48] we require the action principle to be specified by $\delta\widehat{S}[\widehat{x}] = 0$.

Taking the variation of $S[\widehat{x}]$ with respect to $\delta\widehat{x} = \delta x_1 + \gamma_5\delta x_2$, with

$$\delta\widehat{g}_{\mu\nu} = \frac{\partial\widehat{g}_{\mu\nu}}{\partial\widehat{x}^\lambda}\delta\widehat{x}^\lambda, \quad \text{i.e.}$$

$$\delta g_{\mu\nu} = \frac{1}{2}\left(\frac{\partial g_{\mu\nu}}{\partial x^\lambda_1} + \frac{\partial f_{\mu\nu}}{\partial x^\lambda_2}\right)\delta x^\lambda_1 + \left(\frac{\partial f_{\mu\nu}}{\partial x^\lambda_1} + \frac{\partial g_{\mu\nu}}{\partial x^\lambda_2}\right)\delta x^\lambda_2$$

$$= \frac{\partial g_{\mu\nu}}{\partial x^\lambda_1}\delta x^\lambda_1 + \frac{\partial f_{\mu\nu}}{\partial x^\lambda_1}\delta x^\lambda_2$$

$$\delta f_{\mu\nu} = \frac{1}{2}\left(\frac{\partial g_{\mu\nu}}{\partial x^\lambda_1} + \frac{\partial f_{\mu\nu}}{\partial x^\lambda_2}\right)\delta x^\lambda_2 + \left(\frac{\partial f_{\mu\nu}}{\partial x^\lambda_1} + \frac{\partial g_{\mu\nu}}{\partial x^\lambda_2}\right)\delta x^\lambda_1$$

$$= \frac{\partial g_{\mu\nu}}{\partial x^\lambda_1}\delta x^\lambda_2 + \frac{\partial f_{\mu\nu}}{\partial x^\lambda_1}\delta x^\lambda_1 \tag{25}$$

we get the eom

$$\widehat{g}_{\mu\rho}\ddot{\widehat{x}}^\rho + \widehat{\Gamma}^\rho_{\nu\lambda}\widehat{g}_{\mu\rho}\dot{\widehat{x}}^\mu\dot{\widehat{x}}^\nu = 0, \quad \text{i.e.} \quad \ddot{\widehat{x}}^\mu + \widehat{\Gamma}^\mu_{\nu\lambda}\dot{\widehat{x}}^\nu\dot{\widehat{x}}^\lambda = 0. \tag{26}$$

Let us rewrite

$$\sqrt{\widehat{g}_{\mu\nu}\dot{\widehat{x}}^\mu\dot{\widehat{x}}^\nu} = \sqrt{A + \gamma_5 B},$$
$$A = g_{\mu\nu}\left(\dot{x}^\mu_1\dot{x}^\nu_1 + \dot{x}^\mu_2\dot{x}^\nu_2\right) + 2f_{\mu\nu}\dot{x}^\mu_1\dot{x}^\nu_2,$$
$$B = f_{\mu\nu}\left(\dot{x}^\mu_1\dot{x}^\nu_1 + \dot{x}^\mu_2\dot{x}^\nu_2\right) + 2g_{\mu\nu}\dot{x}^\mu_1\dot{x}^\nu_2, \tag{27}$$

so that we have

$$\widehat{S}[\widehat{x}] = \int d\widehat{\tau}\sqrt{\widehat{g}_{\mu\nu}\dot{\widehat{x}}^\mu\dot{\widehat{x}}^\nu}$$
$$= \frac{1}{2}\left[\int d\tau_1\left(\sqrt{A+B} + \sqrt{A-B}\right)\right.$$
$$\left. + \int d\tau_2\left(\sqrt{A+B} - \sqrt{A-B}\right)\right]$$
$$+ \frac{\gamma_5}{2}\left[\int d\tau_1\left(\sqrt{A+B} - \sqrt{A-B}\right)\right.$$
$$\left. + \int d\tau_2\left(\sqrt{A+B} + \sqrt{A-B}\right)\right]. \tag{28}$$

Varying this action with respect to δx^λ we obtain the same eom (26). This is due to (12) and to the fact that, the action is an analytic function of $\widehat{x}$, so that the variation with respect to $\delta\widehat{x}^\lambda$ is the same as the variation of δx^λ_1.

Eventually we will set $x_2 = 0$ everywhere, but it is very convenient to keep the axial-analytic notation as far as possible.

3.1 Geodetic interval and distance

The quantity

$$\widehat{E} = E_1 + \gamma_5 E_2 = \frac{1}{2}\widehat{g}_{\mu\nu}\dot{\widehat{x}}^\mu\dot{\widehat{x}}^\nu \tag{29}$$

is conserved as a function of $\hat{t}$. Since $\widehat{g}_{\mu\nu}\overset{\star}{\hat{x}}{}^{\mu}\overset{\star}{\hat{x}}{}^{\nu}$ is constant for geodesics, we can write for the arc length parameter $\widehat{s}$

$$\frac{d\widehat{s}}{d\hat{t}} = \sqrt{\widehat{g}_{\mu\nu}\overset{\star}{\hat{x}}{}^{\mu}\overset{\star}{\hat{x}}{}^{\nu}}, \tag{30}$$

and

$$\widehat{s} - \widehat{s'} = \int_{\hat{t}'}^{\hat{t}} d\hat{t}\,\sqrt{2\widehat{E}} = \sqrt{2\widehat{E}}\,(\hat{t} - \hat{t}'). \tag{31}$$

$\widehat{s} - \widehat{s'}$ is the axial arc length along the geodesic between $\widehat{x}$ and $\widehat{x}'$. The half square of it is called the *world function* and it is denoted

$$\widehat{\sigma}(\widehat{x}, \widehat{x}') = \frac{1}{2}(\widehat{s} - \widehat{s'})^2 = \widehat{E}(\hat{t} - \hat{t}')^2 = (\hat{t} - \hat{t}')\int_{\hat{t}'}^{\hat{t}} \widehat{E} d\hat{t}. \tag{32}$$

The main properties are

$$\widehat{\sigma}_{;\mu} = \widehat{\partial}_{\mu}\widehat{\sigma} = (\hat{t} - \hat{t}')\widehat{g}_{\mu\nu}\overset{\star}{\hat{x}}{}^{\nu} \equiv -\widehat{g}_{\mu\nu}\widehat{y}^{\nu} \tag{33}$$

$\widehat{y}^{\mu}$ are the *normal coordinates* based at $\widehat{x}$. Using (32,33) one can see that

$$\frac{1}{2}\widehat{\sigma}_{;\mu}\widehat{\sigma}_{;}{}^{\mu} = \widehat{\sigma}. \tag{34}$$

The subscript $_{;\mu}$ means the covariant derivative with respect to $\widehat{x}^{\mu}$, while $_{;\mu'}$ means the covariant derivative with respect to $\widehat{x}'^{\mu'}$.

Remark 1 $\widehat{\sigma} = \sigma_1 + \gamma_5\sigma_2$, but notice that, even when we set $x_2 = 0$, we cannot infer that $\sigma_2 = 0$. This descends from Eq. (30). Looking at (28), we see that B does not vanish even when $x_2^{\nu} = 0$. As a consequence the axial-imaginary part of (27) does not vanish, so the axial-imaginary part of Eq. (30) will not automatically vanish either.

3.2 Normal coordinates

Normal coordinates can be defined based at x or at x':

$$\widehat{y}^{\mu'}(\widehat{x}', \widehat{x}) = (\hat{t} - \hat{t}')\frac{d\widehat{x}^{\mu'}}{d\hat{t}'} \tag{35}$$

and

$$\widehat{y}^{\mu}(\widehat{x}, \widehat{x}') = (\hat{t}' - \hat{t})\frac{d\widehat{x}^{\mu}}{d\hat{t}}. \tag{36}$$

The tangent vector $\frac{d\widehat{x}^{\mu}}{d\hat{t}}$ to the geodesic at $\hat{x}$ satisfies

$$\frac{D}{d\hat{t}}\frac{d\widehat{x}^{\mu}}{d\hat{t}} = \frac{d^2\widehat{x}^{\mu}}{d\hat{t}^2} + \widehat{\Gamma}^{\mu}_{\nu\lambda}\frac{d\widehat{x}^{\nu}}{d\hat{t}}\frac{d\widehat{x}^{\lambda}}{d\hat{t}} = 0 \tag{37}$$

and an analogous equation at $\hat{x}'$. Now we can write

$$\begin{aligned}
\widehat{y}^{\mu'}{}_{;\nu}(\widehat{x}', \widehat{x})\widehat{y}^{\nu}(\widehat{x}, \widehat{x}') &= (\hat{t}' - \hat{t})\widehat{y}^{\mu'}{}_{;\nu}(\widehat{x}', \widehat{x})\frac{d\widehat{x}^{\nu}(\hat{t})}{d\hat{t}} \\
&= (\hat{t}' - \hat{t})\frac{d}{d\hat{t}}\widehat{y}^{\mu'}(\widehat{x}', \widehat{x}) \\
&= (\hat{t}' - \hat{t})\frac{d\widehat{x}^{\mu'}(\hat{t}')}{d\hat{t}'} \\
&= -\widehat{y}^{\mu'}(\widehat{x}', \widehat{x}).
\end{aligned} \tag{38}$$

Dividing by $\hat{t} - \hat{t}'$ the second and fourth terms and taking the coincidence limit $\widehat{x}' \to \widehat{x}$, one gets

$$[\widehat{y}^{\mu'}{}_{;\nu}]\frac{d\widehat{x}^{\nu}}{d\hat{t}} = \frac{d\widehat{x}^{\mu}}{d\hat{t}} \quad \to \quad [\widehat{y}^{\mu'}{}_{;\nu}] = \delta^{\mu}_{\nu}, \tag{39}$$

where $[X]$ denotes the result of the coincidence limit on the quantity X. In a similar way one can prove

$$[\widehat{y}^{\mu'}{}_{;\nu'}]\frac{d\widehat{x}^{\nu}}{d\hat{t}} = -\frac{d\widehat{x}^{\mu}}{d\hat{t}} \quad \to \quad [\widehat{y}^{\mu'}{}_{;\nu'}] = -\delta^{\mu}_{\nu} \tag{40}$$

$$[\widehat{y}^{\mu}{}_{;\nu}]\frac{d\widehat{x}^{\nu}}{d\hat{t}} = -\frac{d\widehat{x}^{\mu}}{d\hat{t}} \quad \to \quad [\widehat{y}^{\mu}{}_{;\nu}] = -\delta^{\mu}_{\nu} \tag{41}$$

$$[\widehat{y}^{\mu}{}_{;\nu'}]\frac{d\widehat{x}^{\nu}}{dt} = \frac{d\widehat{x}^{\mu}}{dt} \quad \to \quad [\widehat{y}^{\mu}{}_{;\nu'}] = \delta^{\mu}_{\nu}. \tag{42}$$

From (38) we get

$$\widehat{y}^{\mu'}{}_{;\nu}\widehat{y}^{\nu} + \widehat{y}^{\mu'} = 0. \tag{43}$$

In a similar way one derives also

$$\widehat{y}^{\mu'}{}_{;\nu'}\widehat{y}^{\nu'} + \widehat{y}^{\mu'} = 0 \tag{44}$$

$$\widehat{y}^{\mu}{}_{;\nu'}\widehat{y}^{\nu'} + \widehat{y}^{\mu} = 0 \tag{45}$$

$$\widehat{y}^{\mu}{}_{;\nu}\widehat{y}^{\nu} + \widehat{y}^{\mu} = 0. \tag{46}$$

For instance, differentiating (44) with respect to $\widehat{x}^{\lambda'}$, one gets

$$\widehat{y}^{\mu'}{}_{;\nu'\lambda'}\widehat{y}^{\nu'} + \widehat{y}^{\mu'}{}_{;\nu'}\widehat{y}^{\nu'}{}_{;\lambda'} + \widehat{y}^{\mu'}{}_{;\lambda'} = 0$$

taking the coincidence limit, and using (40), one finds an identity, because $[\widehat{y}^{\mu'}] = 0$. Differentiating another time with respect to $\widehat{x}^{\rho'}$ one gets

$$[\widehat{y}^{\mu'}{}_{;\lambda'\rho'}] = 0. \tag{47}$$

Differentiating again with respect to $\widehat{x}^{\tau'}$ and using the Bianchi identity for $\widehat{R}^{\mu}{}_{\lambda\rho\tau} = R^{(1)\mu}{}_{\lambda\rho\tau} + \gamma_5 R^{(2)\mu}{}_{\lambda\rho\tau}$, one finds

$$[\widehat{y}^{\mu'}{}_{;\lambda'\rho'\tau'}] = \frac{1}{3}\left(\widehat{R}^{\mu}{}_{\rho\lambda\tau} + \widehat{R}^{\mu}{}_{\tau\lambda\rho}\right) \tag{48}$$

and, in a similar way,

$$[\widehat{y}^{\mu'}{}_{;\lambda\rho\tau}] = \frac{1}{3}\left(\widehat{R}^{\mu}{}_{\lambda\rho\tau} + \widehat{R}^{\mu}{}_{\rho\lambda\tau}\right) \tag{49}$$

and

$$[\widehat{y}^{\mu}{}_{;\lambda\rho\tau}] = \frac{1}{3}\left(\widehat{R}^{\mu}{}_{\tau\lambda\rho} + \widehat{R}^{\mu}{}_{\rho\lambda\tau}\right). \tag{50}$$

3.3 Coincidence limits of $\widehat{\sigma}$

Covariantly differentiating (34) we get

$$\widehat{\sigma}_{,v} = \widehat{\sigma}_{;\mu v}\,\widehat{\sigma}_{;}{}^{\mu}. \tag{51}$$

In the coincidence limit $[\widehat{\sigma}_{,v}] = 0$. Therefore (51) is trivial in the coincidence limit. Differentiating the first and last member of (33) we get

$$\widehat{\sigma}_{;\mu\lambda} = -\widehat{g}_{\mu v}\,\widehat{y}^{v}_{;\lambda}. \tag{52}$$

Using (41) one gets

$$[\widehat{\sigma}_{;\mu\lambda}] = \widehat{g}_{\mu\lambda}. \tag{53}$$

Similarly

$$[\widehat{\sigma}_{;\mu\lambda'}] = -\widehat{g}_{\mu\lambda}. \tag{54}$$

Differentiating (51) once more one gets

$$\widehat{\sigma}_{;v\lambda} = \widehat{\sigma}_{;\mu v\lambda}\,\widehat{\sigma}_{;}{}^{\mu} + \widehat{\sigma}_{;\mu v}\,\widehat{\sigma}_{;\lambda}^{\mu}$$

which, in the coincidence limit, using the previous results, yields an identity. Differentiating it again

$$\widehat{\sigma}_{;v\lambda\rho} = \widehat{\sigma}_{;\mu v\lambda\rho}\,\widehat{\sigma}_{;}{}^{\mu} + \widehat{\sigma}_{;\mu v\lambda}\,\widehat{\sigma}_{;\rho}^{\mu} + \widehat{\sigma}_{;\mu v\rho}\,\widehat{\sigma}_{;\lambda}^{\mu}$$
$$+\widehat{\sigma}_{;\mu v}\,\widehat{\sigma}_{;}{}^{\mu}{}_{\lambda\rho}. \tag{55}$$

In the coincidence limit this becomes

$$[\widehat{\sigma}_{;v\lambda\rho}] = [\widehat{\sigma}_{;\rho v\lambda}] + [\widehat{\sigma}_{;\lambda v\rho}] + [\widehat{\sigma}_{;v\lambda\rho}]. \tag{56}$$

Since $\widehat{\sigma}$ is a biscalar we have

$$[\widehat{\sigma}_{;v\lambda\rho}] = [\widehat{\sigma}_{;v\rho\lambda}] + \widehat{R}_{\rho\lambda v}{}^{\tau}[\widehat{\sigma}_{;\tau}] = [\widehat{\sigma}_{;\rho v\lambda}]. \tag{57}$$

Therefore

$$[\widehat{\sigma}_{;\rho v\lambda}] = [\widehat{\sigma}_{;\lambda v\rho}] = [\widehat{\sigma}_{;v\lambda\rho}] = 0. \tag{58}$$

Differentiating (55) once more and taking the coincidence limit one gets

$$[\widehat{\sigma}_{;v\lambda\rho\tau}] = -\frac{1}{3}\left(\widehat{R}_{v\tau\lambda\rho} + \widehat{R}_{v\rho\lambda\tau}\right) \equiv \widehat{S}_{v\lambda\rho\tau}, \tag{59}$$

where $\widehat{R}_{v\tau\lambda\rho} = \widehat{g}_{v\mu}\widehat{R}^{\mu}{}_{\tau\lambda\rho}$. Differentiating once more

$$[\widehat{\sigma}_{;v\lambda\rho\sigma\tau}] = \frac{3}{4}\left(\widehat{S}_{v\lambda\sigma\tau;\rho} + \widehat{S}_{v\lambda\sigma\rho;\tau} + \widehat{S}_{v\lambda\tau\rho;\sigma}\right). \tag{60}$$

We will need also the coincidence limits of tensors covariantly differentiated with respect to a primed index v'. In general

$$[t_{\mu_1\ldots\mu_k;v'}] = [t_{\mu_1\ldots\mu_k}]_{;v} - [t_{\mu_1\ldots\mu_k;v}]. \tag{61}$$

So

$$[\widehat{\sigma}_{;\mu v'}] = [\widehat{\sigma}_{;\mu}]_{;v} - [\widehat{\sigma}_{;\mu v}] = -\widehat{g}_{\mu v} \tag{62}$$

$$[\widehat{\sigma}_{;\mu v'\lambda}] = v[\widehat{\sigma}_{;\mu\lambda v'}] = [\widehat{\sigma}_{;\mu\lambda}]_{;v} - [\widehat{\sigma}_{;\mu\lambda v}] = 0 \tag{63}$$

$$[\widehat{\sigma}_{;\mu v'\lambda\rho}] = [\widehat{\sigma}_{;\mu\lambda\rho v'}] = [\widehat{\sigma}_{;\mu\lambda\rho}]_{;v} - [\widehat{\sigma}_{;\mu\lambda\rho v}]$$
$$= -[\widehat{\sigma}_{;\mu\lambda\rho v}] = -\widehat{S}_{\mu\lambda\rho v} \tag{64}$$

and

$$[\widehat{\sigma}_{;\mu v'\lambda\rho\sigma}] = [\widehat{\sigma}_{;\mu\lambda\rho\sigma v'}] = [\widehat{\sigma}_{;\mu\lambda\rho\sigma}]_{;v} - [\widehat{\sigma}_{;\mu\lambda\rho\sigma v}]$$
$$= \frac{1}{4}\widehat{S}_{\mu\lambda\rho\sigma;v} - \frac{3}{4}\left(\widehat{S}_{\mu\lambda v\rho;\sigma} + \widehat{S}_{\mu\lambda\sigma v;\rho}\right). \tag{65}$$

Similarly, one obtains

$$[\widehat{\sigma}_{;\mu}{}^{\mu}{}_{v}{}^{v}{}_{\rho}{}^{\rho}] = -\frac{8}{5}R_{;\mu}{}^{\mu} + \frac{4}{15}\widehat{R}_{\mu v}\widehat{R}^{\mu v} - \frac{4}{15}\widehat{R}_{\mu v\lambda\rho}\widehat{R}^{\mu v\lambda\rho}$$

$$[\widehat{\sigma}_{;\mu v}{}^{v}{}_{\rho}{}^{\rho\mu}]s = -[\widehat{\sigma}_{;\mu}{}^{\mu'}{}_{v}{}^{v}{}_{\rho}{}^{\rho}] = \frac{2}{5}R_{;\mu}{}^{\mu}$$
$$-\frac{1}{15}\widehat{R}_{\mu v}\widehat{R}^{\mu v} - \frac{4}{15}\widehat{R}_{\mu v\lambda\rho}\widehat{R}^{\mu v\lambda\rho}$$

3.4 Van Vleck-Morette determinant

The Van Vleck-Morette determinant in MAT is defined by

$$\widehat{D}(\widehat{x},\widehat{x}') = \det(-\widehat{\sigma}_{;\mu v'}). \tag{66}$$

$\widehat{D}(\widehat{x},\widehat{x}')$ is a bidensity of weight 1 both at $\widehat{x}$ and $\widehat{x}'$. Later on we will need a bidensity of weight 0:

$$\widehat{\Delta}(\widehat{x},\widehat{x}') = \frac{1}{\sqrt{\widehat{g}(\widehat{x})}}\widehat{D}(\widehat{x},\widehat{x}')\frac{1}{\sqrt{\widehat{g}(\widehat{x}')}}. \tag{67}$$

The VVM determinant also satisfies (for 4 dimensions)

$$(\widehat{D}(\widehat{x},\widehat{x}')\widehat{\sigma}^{;\mu})_{;\mu} = 4\widehat{D}(\widehat{x},\widehat{x}'). \tag{68}$$

In the coincidence limit

$$[\widehat{\Delta}^{\frac{1}{2}}_{;\lambda}] = [\widehat{g}^{-\frac{1}{4}}(\widehat{x})\sqrt{\widehat{D}(\widehat{x},\widehat{x}')}\frac{1}{2}\left(\widehat{\sigma}^{-1\mu v'}\widehat{\sigma}_{;\mu v'\lambda}\right)\widehat{g}^{-\frac{1}{4}}(\widehat{x}')]$$
$$= \frac{1}{2}[\widehat{\sigma}^{\mu}_{;\mu\lambda}] = 0. \tag{69}$$

We need to compute the covariant derivatives of $\widehat{\sigma}^{-1\mu v'} \equiv \{\widehat{\sigma}^{-1}_{;\mu v'}\}$. The latter is defined as

$$\widehat{\sigma}^{-1\mu v'}\widehat{\sigma}_{;v'\lambda} = \delta^{\mu}_{\lambda}. \tag{70}$$

Differentiating this relation once, twice and thrice one gets

$$[\widehat{\sigma}^{-1\mu v'}_{;\lambda}] = 0,$$
$$[\widehat{\sigma}^{-1}_{\mu\lambda';\rho\sigma}] = -[\widehat{\sigma}_{;\mu'\lambda\rho\sigma}] = [\widehat{\sigma}_{;\lambda\rho\sigma\mu}] = \widehat{S}_{\lambda\rho\sigma\mu} \tag{71}$$

and

$$[\widehat{\sigma}^{-1}_{\mu\lambda';\rho\sigma\tau}] = -[\widehat{\sigma}_{;\lambda\mu'\rho\sigma\tau}] = \frac{1}{4}\widehat{S}_{\mu\rho\sigma\tau;\lambda}$$
$$-\frac{3}{4}\left(\widehat{S}_{\mu\rho\lambda\sigma;\tau} + \widehat{S}_{\mu\rho\tau\lambda;\sigma}\right). \tag{72}$$

Differentiating once more one gets

$$[\widehat{\Delta}^{\frac{1}{2}}_{;\lambda\rho}] = \frac{1}{6}\widehat{g}^{\mu v}\left(\widehat{R}_{\mu v\lambda\rho} + \widehat{R}_{\mu\lambda v\rho}\right) = \frac{1}{6}\widehat{g}^{\mu v}\widehat{g}_{\mu\sigma}\widehat{R}^{\sigma}{}_{\lambda v\rho}$$
$$= \frac{1}{6}\left(R^{(1)}_{\lambda\rho} + \gamma_5 R^{(2)}_{\lambda\rho}\right) \tag{73}$$

and

$$[\widehat{\Delta}^{\frac{1}{2}}_{;\lambda\rho\sigma}] = \frac{1}{12}\left(\widehat{R}_{\lambda\rho;\sigma} + \widehat{R}_{\rho\sigma;\lambda} + \widehat{R}_{\sigma\lambda;\rho}\right). \tag{74}$$

Finally

$$[\widehat{\Delta}^{\frac{1}{2}}{}_{;\mu}{}^{\mu}{}_{\nu}{}^{\nu}] = +\frac{1}{5}\widehat{R}_{;\mu}{}^{\mu} + \frac{1}{36}\widehat{R}^2 - \frac{1}{30}\widehat{R}_{\mu\nu}\widehat{R}^{\mu\nu} \\ + \frac{1}{30}\widehat{R}_{\mu\nu\lambda\rho}\widehat{R}^{\mu\nu\lambda\rho}. \tag{75}$$

3.5 The geodetic parallel displacement matrix

The geodetic parallel displacement matrix $\widehat{G}^{\mu}{}_{\nu'}(\widehat{x}, \widehat{x}')$ is needed in order to parallel displace vectors from one end to the other of the geodetic interval. It is defined by

$$[\widehat{G}^{\mu}{}_{\nu'}] = \delta^{\mu}_{\nu}, \qquad \widehat{G}^{\mu}{}_{\nu';\lambda}\widehat{\sigma}^{;\lambda} = 0. \tag{76}$$

The second condition means that the covariant derivative of $\widehat{G}^{\mu}{}_{\nu'}$ vanishes in directions parallel to the geodesic. Since tangents to the geodesics are self-parallel, it follows that

$$\widehat{G}_{\mu}{}^{\nu'}\widehat{\sigma}_{;\nu'} = -\sigma_{;\mu}, \qquad \widehat{\sigma}_{;\mu}\widehat{G}^{\mu}{}_{\nu'} = -\widehat{\sigma}_{;\nu'} \\ \widehat{G}_{\mu\nu'} = \widehat{G}_{\nu'\mu}, \qquad \widehat{\sigma}_{;}{}^{\lambda'}\widehat{G}^{\mu}{}_{\nu';\lambda'} = 0 \\ \widehat{G}_{\mu}{}^{\nu'}\widehat{G}_{\nu'}{}^{\lambda} = \delta^{\lambda}_{\mu}. \tag{77}$$

The analogous parallel displacement for spinors is denoted $I(x, x')$: the object $I(x, x')\psi(x')$ is the spinor $\psi(x)$ obtained by parallel displacement of $\psi(x')$ along the geodesic from x' to x. It is a bispinor quantity satisfying

$$\widehat{\sigma}_{;}{}^{\mu}\widehat{I}_{;\mu} = 0, \qquad [\widehat{I}] = \mathbf{1} \tag{78}$$

and $\mathbf{1}$ is the identity matrix in the spinor space. Differentiating (78) once we get $[\widehat{I}_{;\mu}] = 0$. Differentiating twice we get

$$[\widehat{I}_{;(\mu\nu)}] = 0, \tag{79}$$

while

$$\widehat{I}(x, x')_{;\mu\nu} - \widehat{I}(x, x')_{;\nu\mu} = -\frac{1}{2}\left(d\widehat{\Omega} + \widehat{\Omega}\widehat{\Omega}\right)_{\mu\nu}\widehat{I}(x, x') \\ = -\frac{1}{2}\widehat{\mathcal{R}}_{\mu\nu}I(x, x'), \tag{80}$$

where $\widehat{\mathcal{R}}_{\mu\nu} = \widehat{R}_{\mu\nu}{}^{ab}\Sigma_{ab}$. So

$$[\widehat{I}(x, x')_{;[\mu,\nu]}] = [\widehat{I}(x, x')_{;\mu\nu}] = -\frac{1}{4}\widehat{\mathcal{R}}_{\mu\nu}. \tag{81}$$

Proceeding with the differentiations of (78) we find

$$[\widehat{I}_{;\nu\lambda\rho}] + [\widehat{I}_{;\lambda\nu\rho}] + [\widehat{I}_{;\rho\lambda\nu}] = 0. \tag{82}$$

Now

$$[\widehat{I}_{;\nu\lambda\rho}] - [\widehat{I}_{;\nu\rho\lambda}] = \frac{1}{2}\widehat{\mathcal{R}}_{\rho\lambda}[\widehat{I}_{;\nu}] = 0 \tag{83}$$

and

$$3[\widehat{I}_{;\nu\lambda\rho}] = \frac{1}{2}\widehat{\nabla}_{\rho}\widehat{\mathcal{R}}_{\lambda\nu} + \frac{1}{2}\widehat{\nabla}_{\lambda}\widehat{\mathcal{R}}_{\rho\nu}. \tag{84}$$

In particular

$$[\widehat{I}_{;\nu}{}^{\nu}{}_{\rho}] = \frac{1}{6}\widehat{\nabla}^{\nu}\widehat{\mathcal{R}}_{\rho\nu}. \tag{85}$$

Differentiating (78) once more with respect to x^{σ}, using (59) and then contracting with $\widehat{g}^{\nu\lambda}\widehat{g}^{\sigma\rho}$ we find, after simplifying,

$$[\widehat{I}_{;\mu}{}^{\mu}{}_{\nu}{}^{\nu}] + [\widehat{I}_{;\mu\nu}{}^{\nu\mu}] = 0. \tag{86}$$

A contraction with $\widehat{g}^{\nu\sigma}\widehat{g}^{\lambda\rho}$ gives:

$$[\widehat{I}_{;\mu\nu}{}^{\nu\mu}] + 2[\widehat{I}_{;\mu\nu}{}^{\mu\nu}] + [\widehat{I}_{;\mu}{}^{\mu}{}_{\nu}{}^{\nu}] = 0. \tag{87}$$

Using (80), we get

$$[\widehat{I}_{;\sigma\rho\mu\nu}] = [\widehat{\nabla}_{\nu}\widehat{\nabla}_{\mu}(\widehat{I}_{;\sigma\rho})] = -\frac{1}{2}\widehat{\mathcal{R}}_{\sigma\rho;\mu\nu} \\ + \frac{1}{8}\widehat{\mathcal{R}}_{\sigma\rho}\widehat{\mathcal{R}}_{\mu\nu} + [\widehat{I}_{;\rho\sigma\mu\nu}]. \tag{88}$$

Contracting with $\widehat{g}^{\mu\sigma}\widehat{g}^{\nu\rho}$ gives

$$[\widehat{I}_{;\mu\nu}{}^{\mu\nu}] = 0 + \frac{1}{8}\widehat{\mathcal{R}}_{\mu\nu}\widehat{\mathcal{R}}^{\mu\nu} + [\widehat{I}_{;\mu\nu}{}^{\nu\mu}] \tag{89}$$

since by Walker's identity

$$\widehat{\nabla}_{\rho}\widehat{\nabla}_{\lambda}\widehat{\mathcal{R}}^{\rho\lambda} = 0. \tag{90}$$

Finally, by using (86), (87), one gets

$$[\widehat{I}_{;\nu}{}^{\nu}{}_{\rho}{}^{\rho}] = \frac{1}{8}\widehat{\mathcal{R}}_{\rho\lambda}\widehat{\mathcal{R}}^{\rho\lambda}. \tag{91}$$

4 Fermions in MAT background

The action of a fermion interacting with a metric and an axial tensor is

$$\widehat{S} = \int d^4\widehat{x}\left(i\overline{\psi}\sqrt{\overline{\widehat{g}}}\gamma^a\widehat{e}^{\mu}_a\left(\partial_{\mu} + \frac{1}{2}\widehat{\Omega}_{\mu}\right)\psi\right)(\widehat{x}) \\ = \int d^4\widehat{x}\left(i\overline{\psi}\sqrt{\overline{\widehat{g}}}\gamma^a(\tilde{e}^{\mu}_a + \gamma_5\tilde{c}^{\mu}_a)\right. \\ \left. \times \left(\partial_{\mu} + \frac{1}{2}\left(\Omega^{(1)}_{\mu} + \gamma_5\Omega^{(2)}_{\mu}\right)\right)\psi\right)(\widehat{x}) \\ = \int d^4\widehat{x}\left(i\overline{\psi}\sqrt{\overline{\widehat{g}}}(\tilde{e}^{\mu}_a - \gamma_5\tilde{c}^{\mu}_a)\right. \\ \left. \times \left[\frac{1}{2}\gamma^a\overset{\leftrightarrow}{\partial}_{\mu} + \frac{1}{4}\left(\gamma^a\widehat{\Omega}_{\mu} + \overline{\widehat{\Omega}}_{\mu}\gamma^a\right)\right]\psi\right)(\widehat{x}) \\ = \int d^4\widehat{x}\left(i\overline{\psi}\sqrt{\overline{\widehat{g}}}(\tilde{e}^{\mu}_a - \gamma_5\tilde{c}^{\mu}_a)\right. \\ \left. \times \left[\frac{1}{2}\gamma^a\overset{\leftrightarrow}{\partial}_{\mu}\psi + \frac{i}{4}\gamma_d\epsilon^{dabc}\widehat{\Omega}_{\mu bc}\gamma_5\right]\psi\right)(\widehat{x}). \tag{92}$$

It must be noticed that this action takes axial-real values.[1] The field $\psi(\widehat{x})$ can be understood, classically, as a series of powers of $\widehat{x}$ applied to constant spinors on their right and the symmetry transformations act on it from the left. The analogous definitions for $\psi^\dagger$ are obtained via hermitean conjugation. In the second line it is stressed that the action contains also an axial part. It is understood that $\partial_\mu = \frac{\partial}{\partial\widehat{x}^\mu}$ applies only to ψ or $\overline{\psi}$, as indicated, and $\overline{\overline{g}}$ denotes, as usual, the axial-complex conjugate of $\widehat{g}$.

A few comments are in order. As was explained in [1], the density $\sqrt{\overline{\overline{g}}}$ must be inserted between $\overline{\psi}$ and ψ, due to the presence in it of the γ_5 matrix. Moreover one has to take into account that the kinetic operator contains a γ matrix that anticommutes with γ_5. Thus, for instance, using $\widehat{D}_\lambda \widehat{g}_{\mu\nu} = 0$ and $(\widehat{D}_\lambda + \frac{1}{2}\widehat{\Omega}_\lambda)\widehat{e} = 0$, where $\widehat{D} = \partial + \widehat{\Gamma}$, one gets

$$\overline{\psi}\gamma^a \widehat{e}^\mu_a \left(\partial_\mu + \frac{1}{2}\Omega_\mu\right)\psi = \overline{\psi}(\overline{D}_\mu + \frac{1}{2}\overline{\Omega}_\mu)\gamma^a \widehat{e}^\mu_a \psi. \quad (93)$$

We recall again that a bar denotes axial-complex conjugation, i.e. a sign reversal in front of each γ_5 contained in the expression, for instance $\overline{\widehat{\Omega}}_\mu = \Omega^{(1)}_\mu - \gamma_5 \Omega^{(2)}_\mu$.

To obtain the two last lines in (92) one must use (253) and (93).

4.1 Classical Ward identities

Let us consider AE (axially extended) diffeomorphisms first, (232). It is not hard to prove that the action (92) is invariant under these transformations. Now, define the full MAT e.m. tensor by means of

$$\mathbf{T}^{\mu\nu} = \frac{2}{\sqrt{\widehat{g}}}\frac{\overleftarrow{\delta}\,\widehat{S}}{\delta\widehat{g}_{\mu\nu}}. \quad (94)$$

This formula needs a comment, since $\sqrt{\widehat{g}}$ contains γ_5. To give a meaning to it we understand that the operator $\frac{2}{\sqrt{\widehat{g}}}\frac{\overleftarrow{\delta}}{\delta\widehat{g}_{\mu\nu}}$ in the RHS acts on the operatorial expression, say $\mathcal{O}\sqrt{\widehat{g}}$, which is inside the scalar product $\overline{\psi}\mathcal{O}\sqrt{\widehat{g}}\psi$. Moreover the functional derivative acts from the right of the action. Now the conservation law under diffeomorphisms is

$$0 = \delta_{\widehat{\xi}} S = \int \overline{\psi}\frac{\overleftarrow{\delta}\,\mathcal{O}}{\delta\widehat{g}_{\mu\nu}}\delta\widehat{g}_{\mu\nu}\psi$$

$$= \int \overline{\psi}\frac{\overleftarrow{\delta}\,\mathcal{O}}{\delta\widehat{g}_{\mu\nu}}\left(\widehat{D}_\mu\widehat{\xi}_\nu + \widehat{D}_\nu\widehat{\xi}_\mu\right)\psi$$

$$= -2\int \overline{\psi}\frac{\overleftarrow{\delta}\,\mathcal{O}}{\delta\widehat{g}_{\mu\nu}}\overleftarrow{\widehat{D}}_\mu\widehat{\xi}_\nu\psi, \quad (95)$$

where $\widehat{D}$ acts (from the right) on everything except the parameter $\widehat{\xi}_\nu$. Differentiating with respect to the arbitrary parameters ξ^μ and ζ^ν we obtain two conservation laws involving the two tensors

$$T^{\mu\nu} = 2\overline{\psi}\frac{\overleftarrow{\delta}\,\mathcal{O}}{\delta\widehat{g}_{\mu\nu}}\psi \quad (96)$$

$$T^{\mu\nu}_5 = 2\overline{\psi}\frac{\overleftarrow{\delta}\,\mathcal{O}}{\delta\widehat{g}_{\mu\nu}}\gamma_5\psi. \quad (97)$$

To give a less abstract idea of these tensors, at the lowest order (flat background) and setting $x^\mu_2 = 0$, they are given by

$$T^{\mu\nu} \approx T^{\mu\nu}_{flat} = -\frac{i}{4}\left(\overline{\psi}\gamma^\mu\overleftrightarrow{\partial}^\nu\psi + \mu \leftrightarrow \nu\right), \quad (98)$$

and

$$T^{\mu\nu}_5 \approx T^{\mu\nu}_{5flat} = \frac{i}{4}\left(\overline{\psi}\gamma_5\gamma^\mu\overleftrightarrow{\partial}^\nu\psi + \mu \leftrightarrow \nu\right). \quad (99)$$

Repeating the same derivation for the axial complex Weyl transformation one can prove that, assuming for the fermion field the transformation rule

$$\psi \rightarrow e^{-\frac{3}{2}(\omega+\gamma_5\eta)}\psi, \quad (100)$$

(92) is invariant, and obtain the Ward identity

$$0 = \int \overline{\psi}\frac{\overleftarrow{\delta}\,\mathcal{O}}{\delta\widehat{g}_{\mu\nu}}\widehat{g}_{\mu\nu}(\omega + \gamma_5\eta)\psi. \quad (101)$$

One gets in this way two WI's

$$\mathcal{T}(x) \equiv T^{\mu\nu}g_{\mu\nu} + T^{\mu\nu}_5 f_{\mu\nu} = 0, \quad (102)$$

$$\mathcal{T}_5(x) \equiv T^{\mu\nu}f_{\mu\nu} + T^{\mu\nu}_5 g_{\mu\nu} = 0. \quad (103)$$

4.2 A more precise formula for the e.m. tensor

In our calculation a more explicit formula of the e.m. tensor is needed. The e.m. tensor is defined by

$$\mathbf{T}^{\mu\nu} = \frac{2}{\sqrt{\widehat{g}}}\frac{\overleftarrow{\delta}\,\widehat{S}}{\delta\widehat{g}_{\mu\nu}} = \frac{1}{2}\left(\mathbf{T}^\mu_a\widehat{e}^{a\nu} + \mathbf{T}^\nu_a\widehat{e}^{a\mu}\right), \quad (104)$$

where

$$\mathbf{T}^\mu_a = \frac{1}{\sqrt{|\widehat{g}|}}\frac{\overleftarrow{\delta}\,\widehat{S}}{\delta\widehat{e}^a_\mu}. \quad (105)$$

Let us prove first that the functional derivative of $\widehat{\Omega}_m$ does not contribute to the e.m. tensor. Consider the general variational formula

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[1] One could consider also an axial complex action, but for our purposes this is a useless complication. That is why we use the notation ψ instead of $\widehat{\psi}$.

$$\delta\widehat{\Omega}_\mu^{bc} = \frac{1}{2}\widehat{e}^{bv}\left(\widehat{\nabla}_\mu(\delta\widehat{e}_v^c) - \widehat{\nabla}_v(\delta\widehat{e}_\mu^c)\right)$$
$$-\frac{1}{2}\widehat{e}^{cv}\left(\widehat{\nabla}_\mu(\delta\widehat{e}_v^b) - \widehat{\nabla}_v(\delta\widehat{e}_\mu^b)\right)$$
$$+\frac{1}{2}\widehat{e}^{bv}\widehat{e}^{c\lambda}\left(\widehat{\nabla}_\lambda(\delta\widehat{e}_v^e) - \widehat{\nabla}_v(\delta\widehat{e}_\lambda^e)\right)\widehat{e}_{e\mu}, \tag{106}$$

where $\widehat{\nabla}$ denotes the covariant derivative such that $\widehat{\nabla}_\mu\widehat{e}_\lambda^a = 0$. After some algebra one gets

$$\gamma_d\,\epsilon^{dabc}\,\widehat{e}_a^\mu\,\delta\widehat{\Omega}_{\mu bc} = \gamma_d\,\epsilon^{dabc}\,\widehat{e}_a^\mu\widehat{e}_b^v\nabla_\mu\delta e_{cv}. \tag{107}$$

Now use this and

$$\frac{\delta\widehat{e}_\mu^a(x)}{\delta\widehat{e}_v^b(y)} = \delta_b^a\delta_\mu^v\delta(x,y)$$

and insert them into the definition (104). The relevant contribution is

$$\mathbf{T}_\Omega^{\lambda\rho} = \frac{1}{2}\left(\mathbf{T}_a^\lambda\widehat{e}^{a\rho} + \mathbf{T}_a^\rho\widehat{e}^{a\lambda}\right)_\Omega$$
$$\equiv \frac{1}{8}\int\overline{\psi}\gamma_d\epsilon^{dabc}\widehat{e}_a^\mu\left(\frac{\delta\widehat{\Omega}_{\mu bc}}{\delta\widehat{e}_\lambda^e}\widehat{e}^{e\rho} + \frac{\delta\widehat{\Omega}_{\mu bc}}{\delta\widehat{e}_\rho^e}\widehat{e}^{e\lambda}\right)\gamma_5\psi$$
$$= \frac{1}{8}\int\overline{\psi}\gamma_d\epsilon^{dabc}\widehat{e}_a^\mu\left(\widehat{e}_b^\lambda\widehat{e}_c^\rho\widehat{\nabla}_\mu\delta(x,y)\right.$$
$$\left. +\widehat{e}_b^\rho\widehat{e}_c^\lambda\widehat{\nabla}_\mu\delta(x,y)\right)\gamma_5\psi = 0. \tag{108}$$

Therefore the only contribution to the em tensor comes from the variation of the first $\widehat{e}_a^m$ factor in (92). The result is

$$\mathbf{T}^{\lambda\rho} = -\frac{i}{2}\overline{\psi}\widehat{\gamma}^\lambda\widehat{g}^{\rho\mu}\left(\partial_\mu + \frac{1}{2}\widehat{\Omega}_\mu\right) + (\lambda\leftrightarrow\rho)$$
$$= -\frac{i}{2}\overline{\psi}\widehat{\gamma}^\lambda\widehat{\nabla}^\rho\psi + (\lambda\leftrightarrow\rho), \tag{109}$$

where $\widehat{\gamma}^\lambda = \gamma^a\widehat{e}_a^\lambda$.

It is useful to write it as a trace

$$\mathbf{T}^{\lambda\rho}(x) = \frac{i}{2}\text{tr}\left(\eta\widehat{\gamma}^{(\lambda}\widehat{\nabla}^{\rho)}\psi(x)\psi^\dagger(x)\right)$$
$$= \frac{i}{4}\text{tr}\left(\eta\widehat{\gamma}^{(\lambda}[\widehat{\nabla}^{\rho)}\psi(x), \psi^\dagger(x)]\right), \tag{110}$$

where $\eta\equiv\gamma_0$, the flat gamma matrix. The commutator is interpreted as

$$[\widehat{\nabla}^\rho\psi, \psi^\dagger](x) = \frac{1}{2}\lim_{x'\to x}\left([\widehat{\nabla}^\rho\psi(x), \psi^\dagger(x')]\right.$$
$$\left. + [\widehat{\nabla}^\rho\psi(x'), \psi^\dagger(x)]\right). \tag{111}$$

Inserting (110) in the path integral it becomes

$$\langle\!\langle\mathbf{T}^{\lambda\rho}(x)\rangle\!\rangle$$
$$= \frac{i}{8}\lim_{x'\to x}\text{tr}\left(\eta\widehat{\gamma}^{(\lambda}\left(\widehat{\mathbb{S}}^{(1);\rho)}(x,x') - \widehat{\mathbb{S}}^{(1);\rho')}(x,x')\right)\right), \tag{112}$$

where $\widehat{\mathbb{S}}^{(1)}$ is the Hadamard function

$$\widehat{\mathbb{S}}^{(1)}(x,x') = \langle\!\langle[\psi(x), \psi^\dagger(x')]\rangle\!\rangle. \tag{113}$$

This leads to Christensen's method [14,15], to compute the energy-momentum tensor and related quantities, such as trace anomalies. We will not pursue this point of view here although it could be done. It is in fact strictly connected with the main approach we will follow later on, which we consider simpler. They are both based on fermion propagators such as $\widehat{\mathbb{S}}^{(1)}(x,x')$. A discussion of fermion propagators and their properties in a MAT background is presented in Appendix C.

4.3 The Dirac operator and its inverse

In the action (92) the Dirac operator is

$$\widehat{F} = i\widehat{\gamma}\cdot\widehat{\nabla} = i\widehat{\gamma}^\mu\widehat{\nabla}_\mu = i\gamma^a\widehat{e}_a^\mu\widehat{\nabla}_\mu \equiv \gamma^a\,\widehat{F}_a, \tag{114}$$

where the $\widehat{\nabla}$ operator is, schematically, $\widehat{D} + \frac{1}{2}\widehat{\Omega}$ and satisfies $\widehat{\nabla}_\mu\widehat{e}_v^a = 0$.

Under AE diffeomorphisms ψ transforms as: $\delta_{\widehat{\xi}}\psi = \widehat{\xi}\partial\psi$, while

$$\delta_{\widehat{\xi}}\left(i\widehat{\gamma}\cdot\widehat{\nabla}\psi\right) = \overline{\widehat{\xi}}\cdot\partial\left(i\widehat{\gamma}\cdot\widehat{\nabla}\psi\right). \tag{115}$$

Under AE Weyl transformation $\widehat{F}$ transform as

$$\delta_{\widehat{\omega}}\widehat{F} = -\frac{1}{2}\gamma^a\{\widehat{F}_a, \widehat{\omega}\} \tag{116}$$

and it has the following hermiticity property

$$\widehat{F}^\dagger = \eta\widehat{F}\eta, \tag{117}$$

where $\eta = \gamma_0$ and γ_0 is the nondynamical (flat) gamma matrix. To obtain (117) use $\widehat{\Omega}^\dagger = -\eta\overline{\widehat{\Omega}}^\dagger\eta$, etc.

Integrating out the fermion field in (92) means, roughly speaking, evaluating the determinant of the Dirac operator $\widehat{F}$. This is however not what we need. First, because the log of the determinant is formally the trace of the log of $\widehat{F}$; taking this trace means integrating over spacetime and tracing over the gamma matrices: this would suppress any explicit γ_5 dependence and, thus, any axial splitting. Second, because $\widehat{F}$ is local, while, in order to exploit a coincidence limit (in order to guarantee covariance), we need a bilocal quantity. This quantity exists, it is the inverse of $\widehat{F}$: the fermion propagator. The Schwinger-DeWitt method is based on it. Let us explain this approach, adapting it to MAT.

One starts from

$$\widehat{G}(\widehat{x}, \widehat{x}') = \langle 0|\mathcal{T}\psi(\widehat{x})\psi^\dagger(\widehat{x}')|0\rangle \tag{118}$$

which satisfies

$$i\sqrt{\widehat{g}}\eta\,\widehat{\gamma}^\mu\widehat{\nabla}_\mu\widehat{G}(\widehat{x}, \widehat{x}') = -\mathbf{1}\delta(\widehat{x}, \widehat{x}'), \tag{119}$$

where $\mathbf{1}$ is the unit matrix in the spinor space. $\widehat{G}$ is not yet what we need. The Schwinger-DeWitt method requires a quadratic operator and, in addition, we must get rid of the γ matrices,

except γ_5. This is achieved with the ansatz

$$\widehat{G}(x, x') = -i\overline{\widehat{\gamma}}^\mu \overline{\widehat{\nabla}}_\mu \overline{\overline{\mathcal{G}}}(x, x')\eta^{-1}. \tag{120}$$

Remark 2 Why the ansatz (120)

In ordinary gravity, from the diff invariance of the fermion action, we can extract the transformation rule

$$\delta_\xi \left(i\gamma^\mu \nabla_\mu \psi \right) = \xi \cdot \partial \left(i\gamma \cdot \nabla \psi \right) \tag{121}$$

while $\delta_\xi \psi = \xi \cdot \partial \psi$. Therefore it makes sense to apply $\gamma \cdot \nabla$ to $\gamma \cdot \nabla \psi$, because the latter transforms as ψ. This allows us to define the square of the Dirac operator:

$$F^2 \psi = (i\gamma \cdot \nabla)^2 \psi. \tag{122}$$

It is not possible to repeat the same thing for MAT because of (115), from which we see that $\left(i\widehat{\gamma} \cdot \widehat{\nabla} \psi \right)$ does not transform like ψ, and an expression like $\left(i\widehat{\gamma} \cdot \widehat{\nabla} \right)^2 \psi$ would break general covariance. Noting that

$$\delta_{\widehat{\xi}} \left(i\overline{\widehat{\gamma}} \cdot \overline{\widehat{\nabla}} \psi \right) = \widehat{\xi} \cdot \partial \left(i\overline{\widehat{\gamma}} \cdot \overline{\widehat{\nabla}} \psi \right) \tag{123}$$

when $\delta_{\overline{\widehat{\xi}}} \psi = \overline{\widehat{\xi}} \cdot \partial \psi$, we will consider instead the *covariant* quadratic operator

$$\left(i\overline{\widehat{\gamma}} \cdot \overline{\widehat{\nabla}} \right) \left(i\widehat{\gamma} \cdot \widehat{\nabla} \right) \psi. \tag{124}$$

Let us quote next a few useful identities.

$$\overline{\widehat{\nabla}}_\mu \widehat{\gamma}_\nu - \widehat{\gamma}_\nu \overline{\widehat{\nabla}}_\mu = \gamma^a \left(\partial_\mu \widehat{e}_{a\nu} - \widehat{\Gamma}^\lambda_{\mu\nu} \widehat{e}_{a\lambda} + \frac{1}{2} \widehat{\Omega}_{\mu ab} \widehat{e}^b_\nu \right) = 0 \tag{125}$$

because of metricity, and

$$\overline{\widehat{\nabla}}_\mu \gamma^a - \gamma^a \widehat{\nabla}_\mu = 0. \tag{126}$$

The axial conjugate relation holds as well. Therefore

$$\widehat{\gamma}^\mu \widehat{\nabla}_\mu \overline{\widehat{\gamma}}^\nu \overline{\widehat{\nabla}}_\nu = \gamma^a \gamma^b \overline{\widehat{e}}^\mu_a \overline{\widehat{e}}^\nu_b \overline{\widehat{\nabla}}_\mu \overline{\widehat{\nabla}}_\nu$$
$$= \eta^{ab} \overline{\widehat{e}}^\mu_a \overline{\widehat{e}}^\nu_b \overline{\widehat{\nabla}}_\mu \overline{\widehat{\nabla}}_\nu + \Sigma^{ab} \overline{\widehat{e}}^\mu_a \overline{\widehat{e}}^\nu_b [\overline{\widehat{\nabla}}_\mu, \overline{\widehat{\nabla}}_\nu]. \tag{127}$$

On the other hand, when acting on a (bi-)spinor quantity

$$\Sigma^{ab} \overline{\widehat{e}}^\mu_a \overline{\widehat{e}}^\nu_b [\overline{\widehat{\nabla}}_\mu, \overline{\widehat{\nabla}}_\nu] = \frac{1}{8} \gamma^a \gamma^b \gamma^c \gamma^d \widehat{R}_{abcd}$$
$$= -\frac{1}{4} \widehat{R}_{\mu\nu\lambda\rho} \widehat{g}^{\mu\lambda} \widehat{g}^{\nu\rho} = -\frac{1}{4} \widehat{R}, \tag{128}$$

where use is made of

$$\widehat{R}_{abcd} = \widehat{e}^\mu_a \widehat{e}^\nu_b \widehat{e}^\lambda_c \widehat{e}^\rho_d \widehat{R}_{\mu\nu\lambda\rho}. \tag{129}$$

Now replacing (120) into (119) and using the above we get

$$\sqrt{|\overline{\widehat{g}}|} \left(\overline{\widehat{\nabla}}_\mu \overline{\widehat{g}}^{\mu\nu} \overline{\widehat{\nabla}}_\nu - \frac{1}{4} \overline{\widehat{R}} \right) \overline{\overline{\mathcal{G}}}(\widehat{x}, \widehat{x}') = -\mathbf{1}\delta(\widehat{x}, \widehat{x}'). \tag{130}$$

The differential operator acting on $\overline{\overline{\mathcal{G}}}$ will be denoted by $\overline{\overline{\mathcal{F}}}_{\widehat{g}}$. In compact operator notation

$$\overline{\overline{\mathcal{F}}}_{\widehat{g}} \overline{\overline{\mathcal{G}}}_{\widehat{g}} = -\mathbf{1}, \tag{131}$$

with $\langle \widehat{x} | \overline{\overline{\mathcal{G}}}_{\widehat{g}} | \widehat{x}' \rangle = \overline{\overline{\mathcal{G}}}_{\widehat{g}}(\widehat{x}, \widehat{x}')$.

As a consequence of (117) we have

$$\left[\sqrt{\widehat{g}} \left(\widehat{\nabla}_\mu \widehat{g}^{\mu\nu} \widehat{\nabla}_\nu - \frac{1}{4} \widehat{R} \right) \right]^\dagger$$
$$= \eta \left[\sqrt{|\widehat{g}|} \left(\widehat{\nabla}_\mu \widehat{g}^{\mu\nu} \widehat{\nabla}_\nu - \frac{1}{4} \widehat{R} \right) \right] \eta \tag{132}$$

or

$$\left(\overline{\mathcal{F}}_{\widehat{g}} \right)^\dagger = \eta \, \widehat{\mathcal{F}}_{\widehat{g}} \, \eta. \tag{133}$$

We shall refer often to the related operator

$$\widehat{\mathcal{F}} = \frac{1}{\sqrt{\widehat{g}}} \widehat{\mathcal{F}}_{\widehat{g}}, \qquad \widehat{\mathcal{F}}^\dagger = \eta \widehat{\mathcal{F}} \eta \tag{134}$$

and to its inverse $\widehat{\mathcal{G}}$: $\widehat{\mathcal{F}}\widehat{\mathcal{G}} = -\mathbf{1}$.

Remark 3 The operator $\widehat{\mathcal{F}}$ is the main intermediate result of our paper. It is natural to assume that its inverse $\widehat{\mathcal{G}}$ exists. There is no reason to believe that it does not, because, the differential operator $\widehat{\mathcal{F}}$ (after a Wick rotation) can be defined as an axial-elliptic operator, at least under reasonable conditions on the axial tensor $f_{\mu\nu}$. In fact its quadratic part can be cast in the form $-\partial_i A_{ij}(x)\partial_j$, where A_{ij} is an invertible matrix and its dominating part is symmetric and positive definite. However, no doubt, it would be desirable to have a mathematical (possibly constructive) proof of the existence of $\widehat{\mathcal{G}}$. In Appendix C we discuss this issue and, following [5], we give some arguments in this direction.

5 The Schwinger proper time method

From now on, for practical reasons, we drop the bar symbol of axial conjugation. At the end we will axially-conjugate the result.

Let us define the amplitude

$$\langle \widehat{x}, \widehat{s} | \widehat{x}', 0 \rangle = \langle \widehat{x} | e^{i\widehat{\mathcal{F}}\widehat{s}} | \widehat{x}' \rangle \tag{135}$$

which satisfies the (heat kernel) differential equation

$$i \frac{\partial}{\partial \widehat{s}} \langle \widehat{x}, \widehat{s} | \widehat{x}', 0 \rangle = -\widehat{\mathcal{F}}_{\widehat{x}} \langle \widehat{x}, \widehat{s} | \widehat{x}', 0 \rangle \equiv K(\widehat{x}, \widehat{x}', \widehat{s}), \tag{136}$$

where $\widehat{\mathcal{F}}_{\widehat{x}}$ is the differential operator

$$\widehat{\mathcal{F}}_{\widehat{x}} = \widehat{\nabla}_\mu \widehat{g}^{\mu\nu} \widehat{\nabla}_\nu - \frac{1}{4} \widehat{R}. \tag{137}$$

Then we make the ansatz

$$\langle \widehat{x}, \widehat{s} | \widehat{x}', 0 \rangle = - \lim_{m \to 0} \frac{i}{16\pi^2} \frac{\sqrt{\widehat{D}(\widehat{x}, \widehat{x}')}}{\widehat{s}^2}$$
$$\times e^{i \left(\frac{\widehat{\sigma}(\widehat{x}, \widehat{x}')}{2\widehat{s}} - m^2 \widehat{s} \right)} \widehat{\Phi}(\widehat{x}, \widehat{x}', \widehat{s}), \qquad (138)$$

where $\widehat{D}(\widehat{x}, \widehat{x}')$ is the VVM determinant and $\widehat{\sigma}$ is the world function (see above). $\widehat{\Phi}(\widehat{x}, \widehat{x}', \widehat{s})$ is a function to be determined. It is useful to introduce also the mass parameter m, which we will eventually set to zero. In the limit $\widehat{s} \to 0$ the RHS of (138) becomes the definition of a delta function multiplied by $\widehat{\Phi}$. More precisely, since it must be $\langle \widehat{x}, 0 | \widehat{x}', 0 \rangle = \delta(\widehat{x}, \widehat{x}')$, and

$$\lim_{\widehat{s} \to 0} \frac{i}{4\pi^2} \frac{\sqrt{\widehat{D}(\widehat{x}, \widehat{x}')}}{\widehat{s}^2} e^{i \left(\frac{\widehat{\sigma}(\widehat{x}, \widehat{x}')}{2\widehat{s}} - m^2 \widehat{s} \right)} = \sqrt{|\widehat{g}(x)|} \, \delta(\widehat{x}, \widehat{x}'), \qquad (139)$$

we must have

$$\lim_{\widehat{s} \to 0} \widehat{\Phi}(\widehat{x}, \widehat{x}', \widehat{s}) = \mathbf{1}. \qquad (140)$$

Equation (136) becomes an equation for $\widehat{\Phi}(\widehat{x}, \widehat{x}', \widehat{s})$. Using (34) and (68), after some algebra one gets

$$i \frac{\partial \widehat{\Phi}}{\partial \widehat{s}} + \frac{i}{\widehat{s}} \widehat{\nabla}^\mu \widehat{\Phi} \widehat{\nabla}_\mu \widehat{\sigma} + \frac{1}{\sqrt{\widehat{D}}} \widehat{\nabla}^\mu \widehat{\nabla}_\mu \left(\sqrt{\widehat{D}} \widehat{\Phi} \right)$$
$$- \left(\frac{1}{4} \widehat{R} - m^2 \right) \widehat{\Phi} = 0. \qquad (141)$$

Now we expand

$$\widehat{\Phi}(\widehat{x}, \widehat{x}', \widehat{s}) = \sum_{n=0}^{\infty} \widehat{a}_n(\widehat{x}, \widehat{x}')(i\widehat{s})^n \qquad (142)$$

with the boundary condition $[\widehat{a}_0] = 1$. The $\widehat{a}_n$ must satisfy the recursive relations:

$$(n+1)\widehat{a}_{n+1} + \widehat{\nabla}^\mu \widehat{a}_{n+1} \widehat{\nabla}_\mu \widehat{\sigma} - \frac{1}{\sqrt{\widehat{D}}} \widehat{\nabla}^\mu \widehat{\nabla}_\mu \left(\sqrt{\widehat{D}} \widehat{a}_n \right)$$
$$+ \left(\frac{1}{4} \widehat{R} - m^2 \right) \widehat{a}_n = 0. \qquad (143)$$

Using these relations and the coincidence results of Sects. 3.3, 3.4 and 3.5, it is possible to compute each coefficient a_n at the coincidence limit.

5.1 Computing $\widehat{a}_n$

In this subsection we wish to compute $[\widehat{a}_1]$ and $[\widehat{a}_2]$, which will be needed later on. We start from (143) for $n = -1$.:

$$\widehat{\nabla}^\mu \widehat{a}_0 \, \sigma_{;\mu} = 0, \quad \text{with} \quad [\widehat{a}_0] = \mathbf{1}, \qquad (144)$$

which implies that

$$\widehat{a}_0(\widehat{x}, \widehat{x}') = \widehat{I}(\widehat{x}, \widehat{x}'). \qquad (145)$$

Replacing this inside (143) for $n = 0$ one gets

$$\widehat{a}_1(\widehat{x}, \widehat{x}') + \widehat{\nabla}^\mu \widehat{\sigma} \nabla_\mu \widehat{a}_1(\widehat{x}, \widehat{x}') - \frac{1}{\sqrt{\widehat{\Delta}}} \widehat{\nabla}^\mu \widehat{\nabla}_\mu \left(\sqrt{\widehat{\Delta}} \, \widehat{I}(\widehat{x}, \widehat{x}') \right)$$
$$+ \left(\frac{1}{4} \widehat{R} - m^2 \right) \widehat{I}(\widehat{x}, \widehat{x}') = 0, \qquad (146)$$

which implies

$$[\widehat{a}_1] = \left(-\frac{1}{12} \widehat{R} + m^2 \right) \mathbf{1}. \qquad (147)$$

Moreover differentiating (146) with respect to ∇_λ and taking the coincidence limit:

$$2[\widehat{\nabla}_\lambda \widehat{a}_1] = \frac{1}{4} \widehat{R}_{;\lambda} \mathbf{1} - [\sqrt{\widehat{\Delta}}_{;\mu}{}^\mu{}_\lambda \widehat{I} + \widehat{\nabla}_\lambda \widehat{\nabla}^\mu \widehat{\nabla}_\mu \widehat{I}]$$

so

$$[\widehat{\nabla}_\lambda \widehat{a}_1] = \left(\frac{1}{12} \widehat{\mathcal{R}}_{\lambda\nu;}{}^\nu - \frac{1}{24} \widehat{R}_{;\lambda} \right) \mathbf{1}. \qquad (148)$$

Next we have

$$[\widehat{\nabla}^\lambda \widehat{\nabla}_\lambda \left(\widehat{a}_1 + \widehat{\nabla}^\mu \widehat{\sigma} \, \widehat{\nabla}_\mu \widehat{a}_1 \right)] = 3[\widehat{\nabla}^\lambda \widehat{\nabla}_\lambda \widehat{a}_1]$$

so that

$$[\widehat{\nabla}^\lambda \widehat{\nabla}_\lambda \widehat{a}_1] = \frac{1}{3}[\widehat{\nabla}^\lambda \widehat{\nabla}_\lambda \left(\frac{1}{\sqrt{\widehat{\Delta}}} \widehat{\nabla}^\mu \widehat{\nabla}_\mu \left(\sqrt{\widehat{\Delta}} \, \widehat{I} \right) \right.$$
$$\left. - \left(\frac{1}{4} \widehat{R} - m^2 \right) \widehat{I} \right)] \qquad (149)$$

$$= \frac{1}{3} \left(-\frac{1}{20} \widehat{R}_{;\mu}{}^\mu - \frac{1}{30} \widehat{R}_{\mu\nu} \widehat{R}^{\mu\nu} \right.$$
$$\left. + \frac{1}{30} \widehat{R}_{\mu\nu\lambda\rho} \widehat{R}^{\mu\nu\lambda\rho} + \frac{1}{8} \widehat{\mathcal{R}}_{\mu\nu} \widehat{\mathcal{R}}^{\mu\nu} \right). \qquad (150)$$

Finally

$$[\widehat{a}_2] = \frac{1}{2}[\widehat{\nabla}^\lambda \widehat{\nabla}_\lambda \widehat{a}_1 - \left(\frac{1}{12} \widehat{R} - m^2 \right) \widehat{a}_1]$$
$$= \frac{1}{2} m^4 - \frac{1}{12} m^2 \widehat{R} + \frac{1}{288} \widehat{R}^2 - \frac{1}{120} \widehat{R}_{;\mu}{}^\mu$$
$$- \frac{1}{180} \widehat{R}_{\mu\nu} \widehat{R}^{\mu\nu} + \frac{1}{180} \widehat{R}_{\mu\nu\lambda\rho} \widehat{R}^{\mu\nu\lambda\rho} + \frac{1}{48} \widehat{\mathcal{R}}_{\mu\nu} \widehat{\mathcal{R}}^{\mu\nu}. \qquad (151)$$

We recall that $\widehat{\mathcal{R}}_{\mu\nu} = \widehat{R}_{\mu\nu}{}^{ab} \Sigma_{ab}$.

6 The odd trace anomaly

We are now ready to compute that odd parity trace anomaly. Beside the point-splitting, which we have used above, we need a regulator to get rid of the infinities at coincident point. We will use two regularizations: the dimensional and zeta function ones.

6.1 Schwinger-DeWitt and dimensional regularization

We start again from the Dirac operator (114). We have defined above the covariant square

$$\widehat{\mathcal{F}} = -\overline{\widehat{F}}\widehat{F}. \tag{152}$$

We identify the effective action for Dirac fermions with

$$\widehat{W} = -\frac{i}{2}\text{Tr}\left(\ln\widehat{\mathcal{F}}\right) \tag{153}$$

Tr includes also the spacetime integration. The AE Weyl variation of (153) is given by

$$\delta_{\widehat{\omega}}\widehat{W} = \frac{i}{2}\text{Tr}\left(\widehat{\mathcal{G}}\,\delta_{\widehat{\omega}}\widehat{\mathcal{F}}\right), \tag{154}$$

where

$$\widehat{\mathcal{F}}\widehat{\mathcal{G}} = -1. \tag{155}$$

So we can write

$$\delta_{\widehat{\omega}}\widehat{W} = \delta_{\widehat{\omega}}\left(-\frac{1}{2}\int_0^\infty \frac{d\widehat{s}}{i\widehat{s}}e^{i\widehat{\mathcal{F}}\widehat{s}}\right)$$
$$= -\frac{1}{2}\text{Tr}\left(\int_0^\infty d\widehat{s}\,e^{i\widehat{\mathcal{F}}\widehat{s}}\delta_{\widehat{\omega}}\widehat{\mathcal{F}}\right). \tag{156}$$

It follows that, as far as the variation with respect to axial-Weyl transform is concerned, the effective action can be represented as

$$\widehat{W} = -\frac{1}{2}\int_0^\infty \frac{d\widehat{s}}{i\widehat{s}}e^{i\widehat{\mathcal{F}}\widehat{s}} + \text{const} \equiv \widehat{L} + \text{const}, \tag{157}$$

where $\widehat{L}$ is the relevant effective action

$$\widehat{L} = \int d^d\widehat{x}\,\widehat{L}(\widehat{x}) \tag{158}$$

which can be written as

$$\widehat{L}(\widehat{x}) = -\frac{1}{2}\text{tr}\int_0^\infty \frac{d\widehat{s}}{i\widehat{s}}\widehat{K}(\widehat{x},\widehat{x}',\widehat{s}), \tag{159}$$

where the kernel $\widehat{K}$ is defined by

$$\widehat{K}(\widehat{x},\widehat{x}',\widehat{s}) = e^{i\widehat{\mathcal{F}}\widehat{s}}\delta(\widehat{x},\widehat{x}'). \tag{160}$$

Inserted in $\delta_{\widehat{\omega}}\widehat{W}$, under the symbol Tr, it means integrating over x after taking the limit $x' \to x$. So, looking at (138), in dimension d,

$$\widehat{K}(\widehat{x},\widehat{x},\widehat{s}) = \frac{i}{(4\pi i\widehat{s})^{\frac{d}{2}}}\sqrt{\widehat{g}}\,e^{-im^2\widehat{s}}[\widehat{\Phi}(\widehat{x},\widehat{x},\widehat{s})]. \tag{161}$$

A specification is in order at this point. For the heat kernel method to work a Riemannian metric is required. Therefore at this stage we Wick-rotate the metric, so that the operator $\widehat{\mathcal{F}}$ becomes axial-elliptic. This operation is understood from now on. After calculating the anomaly we will return to the Lorentz signature.

6.2 Analytic continuation in d

The purpose now is to analytically continue in d. But we can do this only for dimensionless quantities. We therefore multiply $\widehat{L}$ by μ^{-d}, where μ is a mass parameter. We have for a Dirac fermion

$$\frac{\widehat{L}(x)}{\mu^d} = -\frac{i}{2}(4\pi\mu^2)\text{tr}\int_0^\infty d\widehat{s}\,(4\pi i\mu^2\widehat{s})^{-\frac{d}{2}-1}$$
$$\times\sqrt{\widehat{g}}e^{-im^2\widehat{s}}[\widehat{\Phi}(\widehat{x},\widehat{x},\widehat{s})], \tag{162}$$

where tr denotes the trace over gamma matrices.

Now we make the assumption that

$$\lim_{s\to\infty}e^{-im^2\widehat{s}}[\widehat{\Phi}(\widehat{x},\widehat{x},\widehat{s})] = 0. \tag{163}$$

As a consequence we can integrate by parts

$$\frac{\widehat{L}(x)}{\mu^d} = \frac{i}{d}\text{tr}\int_0^\infty d\widehat{s}\frac{\partial}{\partial(i\widehat{s})}(4\pi i\mu^2\widehat{s})^{-\frac{d}{2}}\sqrt{\widehat{g}}e^{-im^2\widehat{s}}[\widehat{\Phi}(\widehat{x},\widehat{x},\widehat{s})]$$
$$= -\frac{i}{d}\text{tr}\int_0^\infty d\widehat{s}\,(4\pi i\mu^2\widehat{s})^{-\frac{d}{2}}$$
$$\times\sqrt{\widehat{g}}\frac{\partial}{\partial(i\widehat{s})}\left(e^{-im^2\widehat{s}}[\widehat{\Phi}(\widehat{x},\widehat{x},\widehat{s})]\right)$$
$$= \frac{2i}{d(2-d)4\pi\mu^2}\text{tr}\int_0^\infty d\widehat{s}\,(4\pi i\mu^2\widehat{s})^{1-\frac{d}{2}}$$
$$\times\sqrt{\widehat{g}}\frac{\partial^2}{\partial(i\widehat{s})^2}\left(e^{-im^2\widehat{s}}[\widehat{\Phi}(\widehat{x},\widehat{x},\widehat{s})]\right)$$
$$= -\frac{4i}{d(2-d)(4-d)}\frac{1}{(4\pi\mu^2)^2}\text{tr}\int_0^\infty d\widehat{s}\,(4\pi i\mu^2\widehat{s})^{2-\frac{d}{2}}$$
$$\times\sqrt{\widehat{g}}\frac{\partial^3}{\partial(i\widehat{s})^3}\left(e^{-im^2\widehat{s}}[\widehat{\Phi}(\widehat{x},\widehat{x},\widehat{s})]\right). \tag{164}$$

Next we use

$$[\widehat{\Phi}(\widehat{x},\widehat{x},\widehat{s})] = 1 + [\widehat{a}_1]i\widehat{s} + [\widehat{a}_2](i\widehat{s})^2 + \cdots \tag{165}$$

and, around $d = 2$, we use $\frac{1}{d(2-d)} = \frac{1}{2}\left(\frac{1}{d-2} - \frac{1}{d}\right)$ and in the third line of (164) we use

$$(4\pi i\mu^2 s)^{1-\frac{d}{2}} = 1 - \frac{d-2}{2}\ln(4\pi i\mu^2 s) + \cdots$$

Then we differentiate once $[\widehat{\Phi}(\widehat{x},\widehat{x},\widehat{s})]$, and the remaining derivation we get rid of by integrating by parts. Finally one gets

$$\widehat{L}(\widehat{x}) = \frac{1}{4\pi}\left(\frac{1}{d-2} - \frac{1}{2}\right)\text{tr}\left(([\widehat{a}_1] - m^2)\sqrt{\widehat{g}}\right)$$
$$- \frac{i}{8\pi}\text{tr}\int_0^\infty d\widehat{s}\,\ln(4\pi i\mu^2\widehat{s})$$
$$\times\sqrt{\widehat{g}}\frac{\partial^2}{\partial(i\widehat{s})^2}\left(e^{-im^2\widehat{s}}[\widehat{\Phi}(\widehat{x},\widehat{x},\widehat{s})]\right). \tag{166}$$

Around $d = 4$ we use $\frac{1}{d(d-2)(d-4)} \approx \frac{1}{8}\left(\frac{1}{d-4} - \frac{3}{4}\right)$. With reference to the last line of (164), we differentiate twice

$[\widehat{\Phi}(x,x,s)]$ and integrate by parts the third derivative. The result is

$$
\begin{aligned}
\widehat{L}(\widehat{x}) \approx &\frac{1}{32\pi^2}\left(\frac{1}{d-4}-\frac{3}{4}\right)\mathrm{tr}\left(m^4-2m^2[\widehat{a}_1]+2[\widehat{a}_2]\right)\sqrt{\widehat{g}} \\
&+\frac{i}{64\pi^2}\mathrm{tr}\int_0^\infty d\widehat{s}\,\ln(4\pi i\mu^2\widehat{s}) \\
&\times\sqrt{\widehat{g}}\frac{\partial^3}{\partial(i\widehat{s})^3}\left(e^{-im^2\widehat{s}}[\widehat{\Phi}(\widehat{x},\widehat{x},\widehat{s})]\right).
\end{aligned}
\tag{167}
$$

The last line depends explicitly on the parameter μ and represent a nonlocal part

6.3 The anomaly

Let us take the variation of (167) with respect to $\widehat{\omega}=\omega+\gamma_5\eta$.

Recall that

$$
\delta_{\widehat{\omega}}\sqrt{\widehat{g}}=d\,\widehat{\omega}\,\sqrt{\widehat{g}}
\tag{168}
$$

$$
\delta_{\widehat{\omega}}\widehat{R}=-2\widehat{\omega}\,\widehat{R}-2(d-1)\widehat{\Box}\widehat{\omega}
\tag{169}
$$

$$
\begin{aligned}
\delta_{\widehat{\omega}}\widehat{R}_{\mu\nu\lambda}{}^\rho=&-\delta_\nu^\rho\widehat{D}_\mu\widehat{D}_\lambda\widehat{\omega}+\delta_\mu^\rho\widehat{D}_\nu\widehat{D}_\lambda\widehat{\omega} \\
&+\widehat{D}_\mu\widehat{D}_\sigma\widehat{\omega}\,\widehat{g}^{\rho\sigma}\widehat{g}_{\nu\lambda}-\widehat{D}_\nu\widehat{D}_\sigma\widehat{\omega}\,\widehat{g}^{\rho\sigma}\widehat{g}_{\mu\lambda}.
\end{aligned}
\tag{170}
$$

From these follows, for instance,

$$
\delta_{\widehat{\omega}}\left(\sqrt{\widehat{g}}\widehat{R}^2\right)=(d-4)\sqrt{\widehat{g}}\,\widehat{\omega}\,\widehat{R}^2-4(d-1)\widehat{R}\sqrt{\widehat{g}}\widehat{\Box}\widehat{\omega}
\tag{171}
$$

$$
\begin{aligned}
\delta_{\widehat{\omega}}\left(\sqrt{\widehat{g}}\widehat{R}_{\mu\nu}\widehat{R}^{\mu\nu}\right)=&(d-4)\widehat{\omega}\sqrt{\widehat{g}}\,\widehat{R}_{\mu\nu}\widehat{R}^{\mu\nu} \\
&+2(2-d)\sqrt{\widehat{g}}\,\widehat{R}^{\mu\nu}\widehat{D}_\mu\widehat{D}_\nu\widehat{\omega}-2\sqrt{\widehat{g}}\,\widehat{R}\widehat{\Box}\widehat{\omega} \\
=&(d-4)\widehat{\omega}\sqrt{\widehat{g}}\,\widehat{R}_{\mu\nu}\widehat{R}^{\mu\nu}-d\sqrt{\widehat{g}}\,\widehat{R}\widehat{\Box}\widehat{\omega}
\end{aligned}
\tag{172}
$$

$$
\begin{aligned}
\delta_{\widehat{\omega}}\left(\sqrt{\widehat{g}}\widehat{R}_{\mu\nu\lambda\rho}\widehat{R}^{\mu\nu\lambda\rho}\right)=&(d-4)\widehat{\omega}\sqrt{\widehat{g}}\,\widehat{R}_{\mu\nu\lambda\rho}\widehat{R}^{\mu\nu\lambda\rho} \\
&-8\sqrt{\widehat{g}}\,\widehat{R}^{\mu\nu}\widehat{D}_\mu\widehat{D}_\nu\widehat{\omega} \\
=&(d-4)\widehat{\omega}\sqrt{\widehat{g}}\,\widehat{R}_{\mu\nu\lambda\rho}\widehat{R}^{\mu\nu\lambda\rho}-4\sqrt{\widehat{g}}\,\widehat{R}\widehat{\Box}\widehat{\omega}
\end{aligned}
\tag{173}
$$

$$
\begin{aligned}
\delta_{\widehat{\omega}}\left(\sqrt{\widehat{g}}\widehat{\Box}\widehat{R}\right)=&(d-4)\widehat{\omega}\sqrt{\widehat{g}}\widehat{\Box}\widehat{R}+(d-6)\sqrt{\widehat{g}}\,\partial_\mu\widehat{\omega}\,\partial^\mu\widehat{R} \\
&-2\sqrt{\widehat{g}}\,\widehat{R}\widehat{\Box}\widehat{\omega}-2(d-1)\sqrt{\widehat{g}}\widehat{\Box}^2\widehat{\omega} \\
=&0
\end{aligned}
\tag{}
$$

and

$$
\begin{aligned}
\delta_{\widehat{\omega}}\mathrm{tr}\left(\sqrt{\widehat{g}}\widehat{\mathcal{R}}_{\mu\nu}\widehat{\mathcal{R}}^{\mu\nu}\right)=&(d-4)\mathrm{tr}\left(\widehat{\omega}\sqrt{\widehat{g}}\widehat{\mathcal{R}}_{\mu\nu}\widehat{\mathcal{R}}^{\mu\nu}\right) \\
&+4\,\mathrm{tr}\left(\sqrt{\widehat{g}}\,\widehat{R}^{\mu\nu}\widehat{D}_\mu\widehat{D}_\nu\widehat{\omega}\right) \\
=&(d-4)\mathrm{tr}\left(\widehat{\omega}\sqrt{\widehat{g}}\widehat{\mathcal{R}}_{\mu\nu}\widehat{\mathcal{R}}^{\mu\nu}\right) \\
&+2\,\mathrm{tr}\left(\sqrt{\widehat{g}}\,\widehat{R}\widehat{\Box}\widehat{\omega}\right).
\end{aligned}
\tag{174}
$$

In the first line of (167) one can ignore m^2 or m^4 terms (either one sets $m=0$ or they can be subtracted because they

are trivial). The second line (167) does not contain singularities when $d\to 4$: it contains either vanishing or finite terms in this limit. Let us denote the second line by $\widehat{L}_R$.

$$
\widehat{L}=\frac{1}{16\pi^2}\left(\frac{1}{d-4}-\frac{3}{4}\right)\int d^d\widehat{x}\,\mathrm{tr}\left([\widehat{a}_2]|_{m=0}\sqrt{\widehat{g}}\right)+\widehat{L}_R.
\tag{175}
$$

We now act with $\delta_{\widehat{\omega}}=\int d^d\widehat{x}\,2\mathrm{tr}\left(\widehat{\omega}\,\widehat{g}_{\mu\nu}\frac{\delta}{\delta\widehat{g}_{\mu\nu}}\right)^2$

From (168)–(172) it follows that

$$
\begin{aligned}
\delta_{\widehat{\omega}}\mathrm{tr}\left(\sqrt{\widehat{g}}\,[\widehat{a}_2]|_{m=0}\right)=&(d-4)\mathrm{tr}\left(\sqrt{\widehat{g}}\,\widehat{\omega}\,[\widehat{a}_2]|_{m=0}\right) \\
&-\frac{d-4}{120}\mathrm{tr}\left(\sqrt{\widehat{g}}\,\widehat{R}\widehat{\Box}\widehat{\omega}\right).
\end{aligned}
\tag{176}
$$

The second piece can be canceled e.g. by a counterterm proportional to $\mathrm{tr}\left(\sqrt{\widehat{g}}\widehat{R}^2\right)$. Using the fact that the bare part of the action is Weyl invariant $\delta_{\widehat{\omega}}\widehat{L}=0$ and that the renormalised part $\widehat{L}_R$ defines the (quantum) energy momentum tensor $\frac{2}{\sqrt{\widehat{g}}}\frac{\delta}{\delta\widehat{g}_{\mu\nu}}\widehat{L}_R=\widehat{\Theta}^{\mu\nu}$ we get

$$
\begin{aligned}
&\int d^d\widehat{x}\,\mathrm{tr}\left(\widehat{\omega}\sqrt{\widehat{g}}\,\widehat{g}_{\mu\nu}\widehat{\Theta}^{\mu\nu}\right) \\
&=-\frac{1}{16\pi^2}\int d^d\widehat{x}\,\mathrm{tr}\left(\sqrt{\widehat{g}}\,\widehat{\omega}\,[\widehat{a}_2]|_{m=0}\right),
\end{aligned}
\tag{177}
$$

where the $d-4$ factor in (176) canceled the pole $\frac{1}{d-4}$ in (175).

Clearly, the odd parity anomaly can come only from the term $\widehat{\mathcal{R}}_{\mu\nu}\widehat{\mathcal{R}}^{\mu\nu}$ contained in $[\widehat{a}_2]$, with a coefficient of $\frac{1}{32\pi^2}$ (for Majorana fermions, $\times 2$ for Dirac fermions). For the odd part we have

$$
\int d^d\widehat{x}\,\mathrm{tr}\sqrt{\widehat{g}}\,\widehat{\omega}\widehat{\mathcal{T}}=-\frac{1}{768\pi^2}\int d^4x\,\mathrm{tr}\sqrt{\widehat{g}}\,\widehat{\omega}\,\widehat{\mathcal{R}}_{\mu\nu}\widehat{\mathcal{R}}^{\mu\nu}\Big|_{\mathrm{odd}},
\tag{178}
$$

where we denoted $\widehat{\mathcal{T}}=\widehat{g}_{\mu\nu}\widehat{\Theta}^{\mu\nu}=\widehat{g}_{\mu\nu}\langle\!\langle\widehat{T}^{\mu\nu}\rangle\!\rangle$.

The (odd parity) coefficient of ω defines $\mathcal{T}$ and the (odd parity) coefficient of η defines $\mathcal{T}_5$. Setting $\widehat{\mathcal{T}}=\mathcal{T}+\gamma_5\mathcal{T}_5$ one obtains in this way

$$
\begin{aligned}
\mathcal{T}=&-\frac{1}{4}\frac{1}{768\pi^2}\mathrm{tr}\left(\widehat{\mathcal{R}}_{\mu\nu}\widehat{\mathcal{R}}^{\mu\nu}\right)\Big|_{\mathrm{odd}} \\
=&-\frac{1}{4}\frac{2i}{768\pi^2}\epsilon^{\mu\nu\lambda\rho}R_{\mu\nu\alpha\beta}^{(1)}R_{\lambda\rho}^{(2)\alpha\beta}
\end{aligned}
\tag{179}
$$

and

$$
\begin{aligned}
\mathcal{T}_5=&-\frac{1}{4}\frac{1}{768\pi^2}\mathrm{tr}\left(\gamma_5\widehat{\mathcal{R}}_{\mu\nu}\widehat{\mathcal{R}}^{\mu\nu}\right)\Big|_{\mathrm{odd}} \\
=&-\frac{1}{4}\frac{i}{768\pi^2}\epsilon^{\mu\nu\lambda\rho}\left(R_{\mu\nu\alpha\beta}^{(1)}R_{\lambda\rho}^{(1)\alpha\beta}+R_{\mu\nu\alpha\beta}^{(2)}R_{\lambda\rho}^{(2)\alpha\beta}\right).
\end{aligned}
\tag{180}
$$

---

[2] In MAT case, $\widehat{g}_{\mu\nu}$ also has two spinor indices, so that $\omega\,g_{\mu\nu}\frac{\delta}{\delta g_{\mu\nu}}\to\widehat{\omega}_{AB}\,\widehat{g}_{\mu\nu\,BC}\frac{\delta}{\delta\widehat{g}_{\mu\nu\,AC}}$. Since in our case γ^5 is symmetric, we have $\widehat{a}_{AB}=\widehat{a}_{BA}$ and we can write $\delta_{\widehat{\omega}}$ as $\int d^d\widehat{x}\,2\mathrm{tr}\left(\widehat{\omega}\,\widehat{g}_{\mu\nu}\frac{\delta}{\delta\widehat{g}_{\mu\nu}}\right)$.

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In the last step we have Wick-rotated back the result: this is the origin of the i in the anomaly coefficient. At this point we can safely set $x_2^\mu = 0$ everywhere.

6.4 ζ-function regularization

Given a differential operator A in analogy with the Riemann ζ function, the expression A^{-z}, for complex z, is called ζ function regularization of A:

$$\zeta(z, A) = A^{-z} = \frac{1}{\Gamma(z)} \int_0^\infty dt\, t^{z-1}\, e^{-tA}. \tag{181}$$

We will apply this representation to the operator $\widehat{\mathcal{F}}(\widehat{x}, \widehat{x})$, :

$$(\widehat{\mathcal{F}}(\widehat{x}))^{-z} = \frac{1}{\Gamma(z)} \int_0^\infty dt\, t^{z-1}\, \langle \widehat{x} | e^{-t\widehat{\mathcal{F}}} | \widehat{x} \rangle, \tag{182}$$

where $\langle \widehat{x} | e^{-t\widehat{\mathcal{F}}} | \widehat{x} \rangle$ means the coincidence limit of $\langle \widehat{x} | e^{-t\widehat{\mathcal{F}}} | \widehat{x}' \rangle$. Equation (182) is not quite correct because only dimensionless quantities can be raised to an arbitrary power. Moreover the object of interest will be $\widehat{\mathcal{G}}$, rather than $\widehat{\mathcal{F}}$. Thus we introduce again the mass parameter μ and shift from t to $i\widehat{s}\mu$.

$$\zeta(\widehat{x}, z) \equiv (\mu^2 \widehat{\mathcal{G}}(\widehat{x}, \widehat{x}))^z$$
$$= \frac{1}{\Gamma(z)} \int_0^\infty (i\mu^2) d\widehat{s}\, (i\widehat{s}\mu^2)^{z-1} \langle x | e^{i\widehat{s}\widehat{\mathcal{F}}} | \widehat{x} \rangle. \tag{183}$$

Finally we replace $\langle \widehat{x} | e^{i\widehat{s}\widehat{\mathcal{F}}} | \widehat{x} \rangle$ with $\widehat{K}(\widehat{x}, \widehat{x}, \widehat{s})$ in Eq. (161). The result is

$$\zeta(\widehat{x}, z) = (\mu^2 \widehat{\mathcal{G}}(\widehat{x}, \widehat{x}))^z = \frac{i}{\Gamma(z)} \frac{\mu^d}{(4\pi)^{\frac{d}{2}}}$$
$$\times \sqrt{\widehat{g}} \int_0^\infty (i\mu^2) d\widehat{s}\, (i\widehat{s}\mu^2)^{z-1-\frac{d}{2}}\, e^{-im^2\widehat{s}} [\widehat{\Phi}(\widehat{x}, \widehat{x}, \widehat{s})] \tag{184}$$

which can be rewritten as

$$\zeta(\widehat{x}, z) = (\mu^2 \widehat{\mathcal{G}}(\widehat{x}, \widehat{x}))^z$$
$$= -\frac{i}{\Gamma(z)} \frac{\mu^{d-4}}{(4\pi)^{\frac{d}{2}}} \frac{\sqrt{\widehat{g}}}{(z-\frac{d}{2})(z-\frac{d}{2}+1)(z-\frac{d}{2}+2)}$$
$$\times \int_0^\infty d(i\widehat{s})\, (i\widehat{s}\mu^2)^{z-\frac{d}{2}+2} \frac{\partial^3}{\partial(i\widehat{s})^3} \left(e^{-im^2\widehat{s}} [\widehat{\Phi}(\widehat{x}, \widehat{x}, \widehat{s})] \right). \tag{185}$$

This is well defined for $d = 4$ at $z = 0$.

$$\zeta(\widehat{x}, 0) = \frac{i\sqrt{\widehat{g}}}{2(4\pi)^2} \left[\frac{\partial^2}{\partial(i\widehat{s})^2} \left(e^{-im^2\widehat{s}} [\widehat{\Phi}(\widehat{x}, \widehat{x}, \widehat{s})] \right) \right]_{\widehat{s}=0}. \tag{186}$$

Now, differentiating (181) with respect to z and evaluating at $z = 0$, we get formally

$$\frac{d}{dz} \zeta(z, A)|_{z=0} = -\text{Tr} \ln A. \tag{187}$$

This suggest the procedure to regularize $\widehat{W}$ (which is the trace of a log). More precisely

$$\widehat{W} \to \widehat{W}_\zeta = -\frac{i}{2} \zeta'(0), \quad \text{where}$$
$$\zeta(z) = \int \text{tr}\, \zeta(\widehat{x}, z) d^d \widehat{x}. \tag{188}$$

As a consequence for $d = 4$:

$$\widehat{L}_\zeta(x) = \frac{1}{64\pi^2} (\gamma + \frac{3}{2} - \ln(4\pi))$$
$$\times \sqrt{\widehat{g}}\, \text{tr} \left(2[\widehat{a}_2(x)] - 2m^2[\widehat{a}_1(\widehat{x})] + m^4 \right)$$
$$- \frac{i}{64\pi^2} \sqrt{\widehat{g}} \int_0^\infty d\widehat{s}\, \ln(4\pi i\mu^2\widehat{s})$$
$$\times \frac{\partial^3}{\partial(i\widehat{s})^3} \left(e^{-im^2\widehat{s}} [\widehat{\Phi}(\widehat{x}, \widehat{x}, \widehat{s})] \right). \tag{189}$$

Now, suppose that the operator A, under a symmetry transformation with parameter ϵ, transforms as

$$\delta_\epsilon A = \{A, \epsilon\}. \tag{190}$$

Then

$$\delta_\epsilon \text{Tr} A^{-z} = -2z \text{Tr} \left(A^{-z} \epsilon \right) = -2z \text{Tr} \left(\zeta(z, A) \epsilon \right). \tag{191}$$

Since the relevant result is obtained by differentiating with respect to z and setting $z = 0$, once the functional is regularized, the anomalous part of the effective action is extremely easy to derive:

$$\widehat{L}_\mathcal{A} = -2\text{Tr} \left(\zeta(0, A) \epsilon \right). \tag{192}$$

Let us return to the our problem. The operator to be regulated is $\widehat{\mathcal{F}} = \widehat{\mathcal{F}}_{\widehat{x}}$. Its AE Weyl transformation is

$$\delta_{\widehat{\omega}} \widehat{\mathcal{F}} = -2\widehat{\omega} \widehat{\mathcal{F}} + \left(\overline{\widehat{\gamma}}^\mu \widehat{\gamma}^\nu + \widehat{g}^{\mu\nu} \right) \partial_\nu \widehat{\omega} \widehat{\nabla}_\mu + \frac{3}{2} \widehat{\Box} \widehat{\omega}$$
$$= -2\widehat{\omega} \widehat{\mathcal{F}} + \widehat{\mathcal{F}} \left[\frac{1}{\widehat{\mathcal{F}}} \left(\left(\overline{\widehat{\gamma}}^\mu \widehat{\gamma}^\nu + \widehat{g}^{\mu\nu} \right) \partial_\nu \widehat{\omega} \widehat{\nabla}_\mu + \frac{3}{2} \widehat{\Box} \widehat{\omega} \right) \right]$$

$\widehat{\mathcal{G}}(\widehat{x}, \widehat{x})$ is the inverse of $\widehat{\mathcal{F}}$ and its transformation is similar:

$$\delta_{\widehat{\omega}} \widehat{\mathcal{G}} = 2 \widehat{\mathcal{G}} \widehat{\omega} + \widehat{\mathcal{G}} \left[\left(\left(\overline{\widehat{\gamma}}^\mu \widehat{\gamma}^\nu + \widehat{g}^{\mu\nu} \right) \partial_\nu \widehat{\omega} \widehat{\nabla}_\mu + \frac{3}{2} \widehat{\Box} \widehat{\omega} \right) \widehat{\mathcal{G}} \right]$$

The first piece in the RHS reproduces exactly the mechanism in (191). The second is a nonlocal term of the effective action; it does not concern us here and we drop it. As noticed above this procedure does not lead directly to the anomaly.

It rather gives the anomalous part of the effective action, i.e. the anomaly integrated with the insertion of $\sqrt{g}$:

$$
\begin{aligned}
\widehat{L}_{\mathcal{A}}(\widehat{\omega}) &= -i \operatorname{Tr}(\widehat{\omega}\,\zeta(\widehat{x}, 0)) \\
&= i \operatorname{Tr}\left(\frac{\sqrt{\widehat{g}}}{2(4\pi)^2}\left[\frac{\partial^2}{\partial(i\widehat{s})^2}\left(e^{-im^2\widehat{s}}[\widehat{\Phi}(\widehat{x}, \widehat{x}, \widehat{s})]\right)\right]_{s=0}\widehat{\omega}\right) \\
&= i \operatorname{Tr}\left(\frac{\sqrt{\widehat{g}}}{2(4\pi)^2}\left(2[\widehat{a}_2(\widehat{x})] - 2m^2[\widehat{a}_1(\widehat{x})] + m^4\right)\widehat{\omega}\right).
\end{aligned}
\tag{193}
$$

Now, proceeding as before, we differentiate with respect to $\widehat{\omega}$ and strip off $\sqrt{\widehat{g}}$, multiply back $\widehat{\omega}$ and obtain the true integrated anomaly. This leads to the same results as above.

6.5 The collapsing limit

After computing the trace anomalies (179) and (180) of a Dirac fermion coupled to a metric and an axial symmetric tensor, we are now interested in returning to the original problem, that is the trace anomaly of a Weyl tensor in an chiral fermion theory coupled to ordinary gravity. To this end we take the collapsing limit. In [1] the latter was defined as $h_{\mu\nu} \to \frac{h_{\mu\nu}}{2}$, $k_{\mu\nu} \to \frac{h_{\mu\nu}}{2}$, with $h_{\mu\nu}$ and $k_{\mu\nu}$ both infinitesimal. Here we do not put such a limitation. The collapsing limit is defined by making the replacements

$$
g_{\mu\nu} \to \eta_{\mu\nu} + \frac{h_{\mu\nu}}{2}, \qquad f_{\mu\nu} \to \frac{h_{\mu\nu}}{2}.
\tag{194}
$$

in the previous formulas, with finite $h_{\mu\nu}$. With this choice one has

$$
\begin{aligned}
\widehat{g}_{\mu\nu} &= \frac{1}{2}(1 - \gamma^5)\,\eta_{\mu\nu} + \frac{1}{2}(1 + \gamma^5)\,G_{\mu\nu}, \\
G_{\mu\nu} &\equiv \eta_{\mu\nu} + h_{\mu\nu}.
\end{aligned}
\tag{195}
$$

From this we see that the left-handed part couples to the flat metric, while the right-handed part couples to the (generic) metric $G_{\mu\nu}$. As a consequence we have also

$$
\begin{aligned}
\widehat{e}_m^a &\to \delta_m^a \frac{1 - \gamma_5}{2} + e_m^a \frac{1 + \gamma_5}{2}, \\
\widehat{e}_a^m &\to \delta_a^m \frac{1 - \gamma_5}{2} + e_a^m \frac{1 + \gamma_5}{2},
\end{aligned}
\tag{196}
$$

as well as

$$
\sqrt{\widehat{g}} \to \frac{1 - \gamma_5}{2} + \frac{1 + \gamma_5}{2}\sqrt{G}.
\tag{197}
$$

Similarly for the Christoffel symbols

$$
\Gamma_{\mu\nu}^{(1)\lambda} \to \frac{1}{2}\Gamma_{\mu\nu}^\lambda, \qquad \Gamma_{\mu\nu}^{(2)\lambda} \to \frac{1}{2}\Gamma_{\mu\nu}^\lambda,
\tag{198}
$$

for the spin connections

$$
\Omega_\mu^{(1)ab} \to \frac{1}{2}\omega_\mu^{ab}, \qquad \Omega_\mu^{(2)ab} \to \frac{1}{2}\omega_\mu^{ab},
\tag{199}
$$

and for the curvatures

$$
R_{\mu\nu\lambda}^{(1)}{}^{\rho} \to \frac{1}{2}R_{\mu\nu\lambda}{}^{\rho}, \qquad R_{\mu\nu\lambda}^{(2)}{}^{\rho} \to \frac{1}{2}R_{\mu\nu\lambda}{}^{\rho},
\tag{200}
$$

where all the quantities on the RHS of these limits are built with the metric $G_{\mu\nu}$.

As a consequence, the action (92) becomes

$$
\begin{aligned}
\widehat{S} \longrightarrow S' = &\int d^4x\left[i\overline{\psi}\gamma^a\frac{1 - \gamma_5}{2}\partial_a\psi \right. \\
&\left. + \int d^4x\sqrt{G}\,i\overline{\psi}\gamma^a e_a^\mu\left(\partial_\mu + \frac{1}{2}\omega_\mu\right)\frac{1 + \gamma_5}{2}\psi\right],
\end{aligned}
\tag{201}
$$

where γ^a is the flat (non-dynamical) gamma matrix while the vierbein e_a^μ and the connection ω_μ are compatible with the metric $G_{\mu\nu}$. Up to the term that represents a decoupled left-handed fermion in the flat spacetime, the action S' is the action of a right-handed Weyl fermion coupled to the ordinary gravity.

In the collapsing limit we have

$$
\mathcal{T}(x) = \mathcal{T}_5(x) = -\frac{1}{16}\frac{2i}{768\pi^2}\epsilon^{\mu\nu\lambda\rho}R_{\mu\nu\alpha\beta}R_{\lambda\rho}{}^{\alpha\beta}
\tag{202}
$$

The integrated anomaly (178) corresponding to $\widehat{S}$ thus becomes

$$
\begin{aligned}
\int d^d\widehat{x}\,\operatorname{tr}\sqrt{\widehat{g}}\,\widehat{\omega}\,\widehat{\mathcal{T}} &= \int d^dx\sqrt{G}\,(\omega + \eta)\,(\mathcal{T} + \mathcal{T}_5)\operatorname{tr}P_+ \\
&\quad + \int d^dx\,(\omega - \eta)\,(\mathcal{T} - \mathcal{T}_5)\operatorname{tr}P_- \\
&= 4\int d^dx\sqrt{G}\,\omega_+\,\mathcal{T},
\end{aligned}
\tag{203}
$$

where we used $\operatorname{tr}P_+ = 2$, $\mathcal{T} - \mathcal{T}_5 = 0$ and set $\omega_+ = \omega + \eta$. Notice that due to (195) the transformation property of $G_{\mu\nu}$ is $G_{\mu\nu} \to e^{2\omega_+}G_{\mu\nu}$. To extract an anomaly of the right fermion of the effective action corresponding to (201) we take its Weyl variation with respect to the metric $G_{\mu\nu}$

$$
\int d^dx\sqrt{G}\,\omega_+\,\mathcal{T}',
\tag{204}
$$

where we denoted $\mathcal{T}' = G_{\mu\nu}\Theta'^{\mu\nu} = G_{\mu\nu}\langle\!\langle T'^{\mu\nu}\rangle\!\rangle$.

Comparing (203) and (204) we get

$$
\mathcal{T}'(x) = -\frac{i}{1536\pi^2}\epsilon^{\mu\nu\lambda\rho}R_{\mu\nu\alpha\beta}R_{\lambda\rho}{}^{\alpha\beta}
\tag{205}
$$

If we instead of (194) take the following collapsing limit

$$
g_{\mu\nu} \to \eta_{\mu\nu} + \frac{h_{\mu\nu}}{2}, \qquad f_{\mu\nu} \to -\frac{h_{\mu\nu}}{2}
\tag{206}
$$

then one obtains

$$
\begin{aligned}
\widehat{g}_{\mu\nu} &= \frac{1}{2}(1 - \gamma^5)\,G_{\mu\nu} + \frac{1}{2}(1 + \gamma^5)\,\eta_{\mu\nu}, \\
G_{\mu\nu} &\equiv \eta_{\mu\nu} + h_{\mu\nu}
\end{aligned}
\tag{207}
$$

Now the right handed part is coupled to the flat metric and left handed part to generic curved metric. We can now repeat the arguments from above and obtain the Pontryagin Weyl anomaly for left-hended Weyl fermion

$$\mathcal{T}'(x) = \frac{i}{1536\pi^2} \epsilon^{\mu\nu\lambda\rho} R_{\mu\nu\alpha\beta} R_{\lambda\rho}{}^{\alpha\beta}. \tag{208}$$

The relative minus sign with respect to right-handed case is because of the opposite sign in front of γ_5 matrix in the defining relation for projectors $P_\pm$.

7 Conclusion

In [2] the odd parity (Pontryagin) trace anomaly was calculated using a Feynman diagram approach coupled to dimensional regularization. Only the lowest order diagrams were computed, they allowed to identify the lowest order term of the anomaly. The full anomaly was then reconstructed by covariantization, which is correct if the diffeomorphisms are preserved by the regularization procedure. This turned out to be the case, as was shown in [3]. After these two papers a negative result was obtained in [37]. Using a heat kernel method with a Pauli–Villars regularization the authors found a vanishing odd parity trace anomaly in 4d. At this point it was imperative to find the culprit. In [2,3] the approach may appear too simple-minded, because only two Feynman three-legged diagrams were considered, the triangle and the bubble diagram. As was shown in the first part of [1] there are several additional diagrams that may affect the final result. But, in fact, the accurate analysis carried out in [1] showed that such additional diagrams cannot change the result as far as the odd parity trace anomaly is concerned. It must be admitted however that for such a delicate calculation an approach based solely on Feynman diagrams may not be satisfactory. The reason is the preservation of chirality throughout the anomaly computation.

It may appear obvious that if one wants to compute the anomaly of a left-handed fermion coupled to gravity one has to respect its left-handedness and avoid mixing different chiralities in the course of the computation. But this is not as easy to do as to claim. As pointed out many times, the trouble arises with the path integral measure, which is hard if not impossible to define for Weyl fermions. If one uses a Fujikawa or heat kernel method (they are relatives) the problem is transferred to the 'square' of the Dirac operator, that is an (Euclidean) elliptic operator that is used in these methods to define the fermion determinant. The problem is: is there a quadratic operator that preserves the same handedness as the linear Weyl operator? As was pointed out in Ref. [1] one such operator could be $\slashed{D}_L^\dagger \slashed{D}_L$, where $\slashed{D}_L = \slashed{D} P_L$ with $\slashed{D}$ the ordinary Dirac operator and P_L the chiral projector, but, with this choice, a phase would remain completely undeter-

mined. We do not know if it is possible to solve this problem, but we are sure the solution is not the choice made in [37], because the operator chosen by the authors there includes both chiralities. Of course, with this choice, the result for the odd trace anomaly cannot be but 0.

A way out is provided by Bardeen's method, which we have used in this paper. This method bypasses the difficulty mentioned above because it utilizes Dirac fermions, and so it is not hard to define a 'square' Dirac operator, $\widehat{\mathcal{F}}$ (see Eq. (130)) which respects the (axially extended) diffeomorphisms (and, of course, can avoid the formidable obstacle of being chiral). The desired handedness is obtained by taking the collapsing limit $h_{\mu\nu} \rightarrow \frac{h_{\mu\nu}}{2}$, $f_{\mu\nu} \rightarrow \frac{h_{\mu\nu}}{2}$ (or $h_{\mu\nu} \rightarrow \frac{h_{\mu\nu}}{2}$, $f_{\mu\nu} \rightarrow -\frac{h_{\mu\nu}}{2}$ for the opposite handedness). This limit is smooth: we have not found any evidence of singularity in it. This method admits different possible regularizations. We have utilized two: the dimensional and the ζ-function regularization, with identical results. The latter absolutely agree with the perturbative results previously obtained in [1–3].

On the basis of the evidence collected so far, with no convincing counterevidence, we conclude that not only does the parity odd trace anomaly exist, but all the procedures used in [1–3] and the present paper are in accord.[3] It is reassuring in particular that there are different ways of doing the same calculations while preserving chirality.

Next let us comment on/recall some characteristics and possible consequences of the odd trace anomaly. Although we have done the calculation in 4d it is easy to see that a parity odd trace anomaly may appear only in dimensions multiple of 4. Therefore, in particular, they do not affect critical (super)string theories. Moreover, as was already pointed out in [2], the Pontryagin density vanish for a number of background metrics, among which the FRW one. But let us see the possible consequences of the instances in which such anomaly does not vanish. In this regard we cannot but repeat what was pointed out in the conclusion of [2]. The parity odd trace anomaly in Lorentzian metric has an imaginary coefficient, which means in particular that the hamiltonian may be complex. This may not be a problem as long as the fermion model is used in an effective field theory context. A problem certainly arises when gravity is itself quantized, because the lack of reality (hermiticity) of the em tensor might propagate in the internal lines. Using this anomaly as a selective criterion in the same way as chiral consistent gauge anomalies were used in the past, we should conclude that theories of massless Weyl fermions interacting with gravity, with a

---

[3] We think the doubts raised in [1] in regard to the Pauli–Villars regularization, as being unable to produce the same results, are worth a very detailed scrutiny. Unfortunately, we are unable to say a final word on this issue due to the exceeding complexity of the calculation (at least in this particular case) and we have to postpone it to another occasion.

definite imbalance of chiralities (an explicit example, the old fashioned standard model, is shown in [2]), should be excluded from the realm of good theories, or at least very critically considered, because they may turn out to be non-unitary.[4] Even though, as we just saw, critical (super)string theory is unaffected by the parity odd trace anomaly, any 4d theory which has is UV completion in a superstring theory should be completely anomaly free (and unitary) at any intermediate energy regime from Planck all the way to low energy. Finally, speaking of unitarity, we cannot refrain from a comment on a claim which is sometimes met in the literature: unitary theories cannot have such kind of anomalies as the odd parity trace anomaly. Although we believe the connection between unitary theories and absence of such anomalies is true, we think the logical order should be reversed. One cannot impose unitarity on a theory; unitarity must be the outcome of quantization. We think a more sensible claim is: there are classical theories which are potentially unitary (because they are based, say, on self-adjoint operators), but one has to verify that unitarity persists after quantization; in this sense the absence of the Pontryagin trace anomaly in a theory is a basic building block of its unitarity.

Appendices

A Bardeen's method

This appendix is a short account of Bardeen's method to derive gauge anomalies [4].

We consider a theory of Dirac fermions coupled to two non-Abelian (vector V_μ and axial A_μ) gauge potentials, both valued in a Lie algebra with anti-hermitean generators T^a, with $[T^a, T^b] = f^{abc} T^c$. The action is

---

$$S[V, A] = i \int d^4x \, \overline{\psi} \left(\slashed{\partial} + \slashed{V} + \gamma_5 \slashed{A} \right) \psi. \tag{209}$$

It is invariant under two sets of gauge transformations

$$\begin{cases} V_\mu \longrightarrow V_\mu + D_{V\mu}\alpha \\ A_\mu \longrightarrow A_\mu + [A_\mu, \alpha], \\ \quad \psi \longrightarrow (1-\alpha)\psi \end{cases} \quad \begin{cases} V_\mu \longrightarrow V_\mu + [A_\mu, \beta] \\ A_\mu \longrightarrow A_\mu + D_{V\mu}\beta \\ \quad \psi \longrightarrow (1+\gamma_5\beta)\psi \end{cases} \tag{210}$$

where $D_{V\mu} = \partial_\mu + [V_\mu, \ \cdot \ ]$ and $\alpha = \alpha^a(x)T^a, \beta = \beta^a(x)T^a$.

As a consequence there are two covariantly conserved currents, $j_\mu = j_\mu^a T^a$ and $j_{5\mu} = j_{5\mu}^a T^a$, where

$$j_\mu^a = \overline{\psi}\gamma_\mu T^a \psi, \qquad j_{5\mu}^a = \overline{\psi}\gamma_\mu\gamma_5 T^a \psi. \tag{211}$$

In the one-loop quantum theory it is impossible to preserve both conservations. The most one can do is to preserve, for instance, the vector one

$$[D_V^\mu j_\mu]^a + [A^\mu, j_{5\mu}]^a = 0 \tag{212}$$

while the axial conservation becomes anomalous:

$$\begin{aligned} [D_V^\mu &j_{5\mu}]^a + [A^\mu, j_\mu]^a \\ &= \frac{1}{4\pi^2}\varepsilon_{\mu\nu\lambda\rho}\text{tr}\Bigg[T^a \Bigg(\frac{1}{4}F_V^{\mu\nu}F_V^{\lambda\rho} + \frac{1}{12}F_A^{\mu\nu}F_A^{\lambda\rho} \\ &\quad - \frac{1}{6}F_V^{\mu\nu}A^\lambda A^\rho - \frac{1}{6}A^\mu A^\nu F_V^{\lambda\rho} - \frac{2}{3}A^\mu F_A^{\nu\lambda}A^\rho \\ &\quad - \frac{1}{3}A^\mu A^\nu A^\lambda A^\rho \Bigg) \Bigg], \end{aligned} \tag{213}$$

where $F_V^{\mu\nu} = \partial^\mu V^\nu - \partial^\nu V^\mu + [V^\mu, V^\nu] + [A^\mu, A^\nu]$, and $F_A^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu + [V^\mu, A^\nu] + [A^\mu, V^\nu]$.

From this expression we can derive two results in particular. Setting $A_\mu = 0$ we get the covariant anomaly

$$[D_V^\mu j_{5\mu}]^a = \frac{1}{16\pi^2}\varepsilon_{\mu\nu\lambda\rho}\text{tr}\left(T^a F_V^{\mu\nu} F_V^{\lambda\rho} \right). \tag{214}$$

Taking the collapsing limit $V \to \frac{V}{2}, A \to \frac{V}{2}$, and adding (212) to (213) we get

$$[D_{V\mu} j_L^\mu]^a = \frac{1}{24\pi^2}\varepsilon_{\mu\nu\lambda\rho}\text{tr}\left[T^a \partial^\mu \left(V^\nu \partial^\lambda V^\rho + \frac{1}{2}V^\nu V^\lambda V^\rho \right) \right], \tag{215}$$

where $j_{L\mu} = \overline{\psi}_L \gamma_\mu \psi_L$, here $\psi_L = \frac{1+\gamma_5}{2}\psi$, which is the consistent non-Abelian gauge anomaly.

B The axial-Riemannian geometry

In this Appendix we collect the formulas, relevant to this paper, of axial-Riemannian geometry. Such formulas have already appeared in [1], although in a somewhat different notation. An important difference with [1] is that, there, all

the quantities where functions of x^μ. In this appendix, and throughout the paper they are functions of $\hat{x}^\mu$ unless otherwise specified.

The main changes in notation are

$$
\begin{aligned}
&G_{\mu\nu} \longrightarrow \widehat{g}_{\mu\nu}, \quad \hat{G}^{\mu\nu} \longrightarrow \widehat{g}^{\mu\nu}, \quad \hat{g} \longrightarrow \tilde{g}, \quad \hat{f} \longrightarrow \tilde{f} \\
&E_\mu^a \longrightarrow \widehat{e}_\mu^a, \quad \hat{E}_a^\mu \longrightarrow \widehat{e}_a^\mu, \quad \hat{e}_a^\mu \longrightarrow \tilde{e}_a^\mu, \quad \hat{c}_a^\mu \longrightarrow \tilde{c}_a^\mu \\
&\gamma_{\mu\nu}^\lambda \longrightarrow \Gamma_{\mu\nu}^\lambda, \quad \Gamma_{\mu\nu}^\lambda \longrightarrow \widehat{\Gamma}_{\mu\nu}^\lambda, \quad \Omega_\mu^{ab} \longrightarrow \widehat{\Omega}_\mu^{ab}, \quad \Xi^\mu \longrightarrow \widehat{\xi}^\mu \\
&\mathcal{R} \longrightarrow \hat{R}, \quad \mathcal{R}^{(1,2)} \longrightarrow \hat{R}^{(1,2)}
\end{aligned}
$$

B.1 Axial metric

We use the symbols $g_{\mu\nu}$, $g^{\mu\nu}$ and e_μ^a, e_a^μ in the usual sense of metric and vierbein and their inverses, except for the fact that they are functions of $\hat{x}^\mu$. Then we introduce the MAT metric

$$\widehat{g}_{\mu\nu} = g_{\mu\nu} + \gamma_5 f_{\mu\nu}, \tag{216}$$

where f is a symmetric tensor. Their background values are $\eta_{\mu\nu}$ and 0, respectively. So, we write as usual $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$.

In matrix notation the inverse of $\widehat{g}$, $\widehat{g}^{-1}$, is defined by

$$\widehat{g}^{-1} = \tilde{g} + \gamma_5 \tilde{f}, \quad \widehat{g}^{-1}\widehat{g} = 1, \quad \widehat{g}^{\mu\lambda}\widehat{g}_{\lambda\nu} = \delta_\nu^\mu \tag{217}$$

which implies

$$\tilde{g}f + \tilde{f}g = 0, \quad \tilde{g}g + \tilde{f}f = 1. \tag{218}$$

So

$$
\begin{aligned}
\tilde{g} &= (1 - g^{-1}fg^{-1}f)^{-1}g^{-1}, \\
\tilde{f} &= -(1 - g^{-1}f\,g^{-1}f)^{-1}g^{-1}f\,g^{-1}.
\end{aligned}
\tag{219}
$$

B.2 MAT vierbein

Likewise for the vierbein one writes

$$\widehat{e}_\mu^a = e_\mu^a + \gamma_5 c_\mu^a, \quad \widehat{e}_a^\mu = \tilde{e}_a^\mu + \gamma_5 \tilde{c}_a^\mu. \tag{220}$$

This implies

$$\eta_{ab}\left(e_\mu^a e_\nu^b + c_\mu^a c_\nu^b\right) = g_{\mu\nu}, \quad \eta_{ab}\left(e_\mu^a c_\nu^b + e_\nu^a c_\mu^b\right) = f_{\mu\nu}. \tag{221}$$

Moreover, from $\widehat{e}_a^\mu \widehat{e}_\nu^a = \delta_\nu^\mu$,

$$\tilde{e}_a^\mu c_\nu^a + \tilde{c}_a^\mu e_\nu^a = 0, \quad \tilde{e}_a^\mu e_\nu^a + \tilde{c}_a^\mu c_\nu^a = \delta_\nu^\mu, \tag{222}$$

one gets

$$\tilde{e}_a^\mu = \left(\frac{1}{1 - e^{-1}c\,e^{-1}c}e^{-1}\right)_a^\mu \tag{223}$$

and

$$\tilde{c}_a^\mu = -\left(e^{-1}c\frac{1}{1 - e^{-1}c\,e^{-1}c}e^{-1}ce^{-1}\right)_a^\mu. \tag{224}$$

B.3 Christoffel and Riemann

The ordinary Christoffel symbols are

$$\Gamma_{\mu\nu}^\lambda = \frac{1}{2}g^{\lambda\rho}\left(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right). \tag{225}$$

The MAT Christoffel symbols are defined in a similar way

$$
\begin{aligned}
\widehat{\Gamma}_{\mu\nu}^\lambda &= \frac{1}{2}\widehat{g}^{\lambda\rho}\left(\partial_\mu \widehat{g}_{\rho\nu} + \partial_\nu \widehat{g}_{\rho\mu} - \partial_\rho \widehat{g}_{\mu\nu}\right) \\
&= \frac{1}{2}\left(\tilde{g}^{\lambda\rho}\left(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right)\right. \\
&\quad \left. + \tilde{f}^{\lambda\rho}\left(\partial_\mu f_{\rho\nu} + \partial_\nu f_{\rho\mu} - \partial_\rho f_{\mu\nu}\right)\right) \\
&\quad + \frac{1}{2}\gamma_5\left(\tilde{g}^{\lambda\rho}\left(\partial_\mu f_{\rho\nu} + \partial_\nu f_{\rho\mu} - \partial_\rho f_{\mu\nu}\right)\right. \\
&\quad \left. + \tilde{f}^{\lambda\rho}\left(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right)\right) \\
&\equiv \Gamma_{\mu\nu}^{(1)\lambda} + \gamma_5 \Gamma_{\mu\nu}^{(2)\lambda},
\end{aligned}
\tag{226}
$$

where it is understood that $\partial_\mu = \frac{\partial}{\partial \hat{x}^\mu}$, etc.

Proceeding the same way one can define the MAT Riemann tensor via $\widehat{R}_{\mu\nu\lambda}{}^\rho$:

$$
\begin{aligned}
\widehat{R}_{\mu\nu\lambda}{}^\rho &= -\partial_\mu \widehat{\Gamma}_{\nu\lambda}^\rho + \partial_\nu \widehat{\Gamma}_{\mu\lambda}^\rho - \widehat{\Gamma}_{\mu\sigma}^\rho \widehat{\Gamma}_{\nu\lambda}^\sigma + \widehat{\Gamma}_{\nu\sigma}^\rho \widehat{\Gamma}_{\mu\lambda}^\sigma \\
&= -\partial_\mu \Gamma_{\nu\lambda}^{(1)\rho} + \partial_\nu \Gamma_{\mu\lambda}^{(1)\rho} - \Gamma_{\mu\sigma}^{(1)\rho}\Gamma_{\nu\lambda}^{(1)\sigma} \\
&\quad + \Gamma_{\nu\sigma}^{(1)\rho}\Gamma_{\mu\lambda}^{(1)\sigma} - \Gamma_{\mu\sigma}^{(2)\rho}\Gamma_{\nu\lambda}^{(2)\sigma} + \Gamma_{\nu\sigma}^{(2)\rho}\Gamma_{\mu\lambda}^{(2)\sigma} \\
&\quad + \gamma_5\Big(-\partial_\mu \Gamma_{\nu\lambda}^{(2)\rho} + \partial_\nu \Gamma_{\mu\lambda}^{(2)\rho} - \Gamma_{\mu\sigma}^{(1)\rho}\Gamma_{\nu\lambda}^{(2)\sigma} \\
&\quad + \Gamma_{\nu\sigma}^{(1)\rho}\Gamma_{\mu\lambda}^{(2)\sigma} - \Gamma_{\mu\sigma}^{(2)\rho}\Gamma_{\nu\lambda}^{(1)\sigma} + \Gamma_{\nu\sigma}^{(2)\rho}\Gamma_{\mu\lambda}^{(1)\sigma}\Big) \\
&\equiv \widehat{R}_{\mu\nu\lambda}^{(1)}{}^\rho + \gamma_5 \widehat{R}_{\mu\nu\lambda}^{(2)}{}^\rho.
\end{aligned}
\tag{227}
$$

The MAT spin connection is introduced in analogy

$$\widehat{\Omega}_\mu^{ab} = \widehat{e}_\nu^a\left(\partial_\mu \widehat{e}^{\nu b} + \widehat{e}^{\sigma b}\widehat{\Gamma}_{\sigma\mu}^\nu\right) = \Omega_\mu^{(1)ab} + \gamma_5 \Omega_\mu^{(2)ab}, \tag{228}$$

where

$$
\begin{aligned}
\Omega_\mu^{(1)ab} &= e_\nu^a\left(\partial_\mu \tilde{e}^{\nu b} + \tilde{e}^{\sigma b}\Gamma_{\sigma\mu}^{(1)\nu} + \tilde{c}^{b\sigma}\Gamma_{\sigma\mu}^{(2)\nu}\right) \\
&\quad + c_\nu^a\left(\partial_\mu \tilde{c}^{\nu b} + \tilde{e}^{\sigma b}\Gamma_{\sigma\mu}^{(2)\nu} + \tilde{c}^{b\sigma}\Gamma_{\sigma\mu}^{(1)\nu}\right)
\end{aligned}
\tag{229}
$$

$$
\begin{aligned}
\Omega_\mu^{(2)ab} &= e_\nu^a\left(\partial_\mu \tilde{c}^{\nu b} + \tilde{e}^{\sigma b}\Gamma_{\sigma\mu}^{(2)\nu} + \tilde{c}^{b\sigma}\Gamma_{\sigma\mu}^{(1)\nu}\right) \\
&\quad + c_\nu^a\left(\partial_\mu \tilde{e}^{\nu b} + \tilde{e}^{\sigma b}\Gamma_{\sigma\mu}^{(1)\nu} + \tilde{c}^{b\sigma}\Gamma_{\sigma\mu}^{(2)\nu}\right).
\end{aligned}
\tag{230}
$$

B.4 Transformations: diffeomorphisms

We recall that under a diffeomorphism, $\delta x^\mu = \xi^\mu$, the ordinary Christoffel symbols transform as tensors except for one non-covariant piece

$$\delta_\xi^{(n.c.)}\Gamma_{\mu\nu}^\lambda = \partial_\mu\partial_\nu\xi^\lambda. \tag{231}$$

In the MAT context it is more opportune to introduces also axially-extended (AE) diffeomorphisms. They are defined by

$$\widehat{x}^\mu \rightarrow \widehat{x}^\mu + \widehat{\xi}^\mu(\widehat{x}^\mu), \qquad \widehat{\xi}^\mu = \xi^\mu + \gamma_5 \zeta^\mu. \tag{232}$$

Since operationally these transformations act in the same way as the usual diffeomorphisms, it is easy to obtain for the non-covariant part

$$\delta^{(n.c.)}\widehat{\Gamma}^\lambda_{\mu\nu} = \partial_\mu \partial_\nu \widehat{\xi}^\lambda, \tag{233}$$

where the derivatives are understood with respect to $\widehat{x}^\mu$ and $\widehat{x}^\nu$. This means in particular that $\Gamma^{(2)\lambda}_{\mu\nu}$ is a tensor.

We have also

$$\delta_{\widehat{\xi}}\widehat{g}_{\mu\nu} = \widehat{D}_\mu \widehat{\xi}_\nu + \widehat{D}_\nu \widehat{\xi}_\mu, \tag{234}$$

where $\widehat{\xi}_\mu = \widehat{g}_{\mu\nu}\widehat{\xi}^\nu$ and $\widehat{D}_\mu$ is the covariant derivative with respect to $\widehat{\Gamma}$.

In components one easily finds

$$\delta_\xi g_{\mu\nu} = \xi^\lambda \partial_\lambda g_{\mu\nu} + \partial_\mu \xi^\lambda g_{\lambda\nu} + \partial_\nu \xi^\lambda g_{\lambda\mu}$$
$$\delta_\xi f_{\mu\nu} = \xi^\lambda \partial_\lambda f_{\mu\nu} + \partial_\mu \xi^\lambda f_{\lambda\nu} + \partial_\nu \xi^\lambda f_{\lambda\mu} \tag{235}$$
$$\delta_\zeta g_{\mu\nu} = \zeta^\lambda \partial_\lambda f_{\mu\nu} + \partial_\mu \zeta^\lambda f_{\lambda\nu} + \partial_\nu \zeta^\lambda f_{\lambda\mu}$$
$$\delta_\zeta f_{\mu\nu} = \zeta^\lambda \partial_\lambda g_{\mu\nu} + \partial_\mu \zeta^\lambda g_{\lambda\nu} + \partial_\nu \zeta^\lambda g_{\lambda\mu}. \tag{236}$$

Summarizing

$$\delta_\xi^{(n.c.)}\Gamma^{(1)\lambda}_{\mu\nu} = \partial_\mu \partial_\nu \xi^\lambda, \qquad \delta_\xi^{(n.c.)}\Gamma^{(2)\lambda}_{\mu\nu} = 0$$
$$\delta_\zeta^{(n.c.)}\Gamma^{(1)\lambda}_{\mu\nu} = 0, \qquad \delta_\zeta^{(n.c.)}\Gamma^{(2)\lambda}_{\mu\nu} = \partial_\mu \partial_\nu \zeta^\lambda \tag{237}$$

and the overall Riemann and Ricci tensors are tensor, and the Ricci scalar $\widehat{R}$ is a scalar. But also $\widehat{R}^{(1)}$ and $\widehat{R}^{(2)}$, separately, have the same tensorial properties.

B.5 Transformations: Weyl transformations

There are two types of Weyl transformations. The first is the obvious one

$$\widehat{g}_{\mu\nu} \longrightarrow e^{2\omega}\widehat{g}_{\mu\nu}, \qquad \widehat{g}^{\mu\nu} \rightarrow e^{-2\omega}\widehat{g}^{\mu\nu} \tag{238}$$

and

$$\widehat{e}^a_\mu \longrightarrow e^\omega \widehat{e}^a_\mu, \qquad \widehat{e}^\mu_a \rightarrow e^{-\omega}\widehat{e}^\mu_a. \tag{239}$$

This leads to the usual relations

$$\widehat{\Gamma}^\lambda_{\mu\nu} \longrightarrow \widehat{\Gamma}^\lambda_{\mu\nu} + \partial_\mu \omega \, \delta^\lambda_\nu + \partial_\nu \omega \, \delta^\lambda_\mu - \partial_\rho \omega \, \widehat{g}^{\lambda\rho}\widehat{g}_{\mu\nu} \tag{240}$$

and

$$\widehat{\Omega}^{ab}_\mu \longrightarrow \widehat{\Omega}^{ab}_\mu + \left(\widehat{e}^a_\mu \widehat{e}^{\sigma b} - \widehat{e}^b_\mu \widehat{e}^{\sigma a}\right) \partial_\sigma \omega. \tag{241}$$

The second type of Weyl transformation is the axial one

$$\widehat{g}_{\mu\nu} \longrightarrow e^{2\gamma_5\eta}\widehat{g}_{\mu\nu}, \qquad \widehat{g}^{\mu\nu} \longrightarrow e^{-2\gamma_5\eta}\widehat{g}^{\mu\nu} \tag{242}$$

and

$$\widehat{e}^a_\mu \longrightarrow e^{\gamma_5\eta}\widehat{e}^a_\mu, \qquad \widehat{e}^\mu_a \rightarrow e^{-\gamma_5\eta}\widehat{e}^\mu_a. \tag{243}$$

This leads to

$$\widehat{\Gamma}^\lambda_{\mu\nu} \longrightarrow \widehat{\Gamma}^\lambda_{\mu\nu} + \gamma_5 \left(\partial_\mu \eta \, \delta^\lambda_\nu + \partial_\nu \eta \, \delta^\lambda_\mu - \partial_\rho \eta \, \widehat{g}^{\lambda\rho}\widehat{g}_{\mu\nu}\right) \tag{244}$$

and

$$\widehat{\Omega}^{ab}_\mu \longrightarrow \widehat{\Omega}^{ab}_\mu + \gamma_5 \left(\widehat{e}^a_\mu \widehat{e}^{\sigma b} - \widehat{e}^b_\mu \widehat{e}^{\sigma a}\right) \partial_\sigma \eta. \tag{245}$$

Equation (242) implies

$$g_{\mu\nu} \longrightarrow \cosh(2\eta) \, g_{\mu\nu} + \sinh(2\eta) \, f_{\mu\nu},$$
$$f_{\mu\nu} \longrightarrow \cosh(2\eta) \, f_{\mu\nu} + \sinh(2\eta) \, g_{\mu\nu}. \tag{246}$$

We can write the axially-extended (AE) Weyl transformation in compact form using the parameter $\widehat{\omega} = \omega + \gamma_5\eta$

$$\widehat{g}_{\mu\nu} \longrightarrow e^{2\widehat{\omega}}\widehat{g}_{\mu\nu}, \tag{247}$$

etc.

B.6 Volume density

The ordinary density $\sqrt{g}$ is replaced by

$$\sqrt{\widehat{g}} = \sqrt{\det(\widehat{g})} = \sqrt{\det(g + \gamma_5 f)}. \tag{248}$$

The expression in the RHS has to be understood as a formal Taylor expansion in terms of the axial-complex variable $g + \gamma_5 f$. This means

$$\mathrm{tr}\ln(g + \gamma_5 f) = \frac{1 + \gamma_5}{2} \, \mathrm{tr}\ln(g + f)$$
$$+ \frac{1 - \gamma_5}{2} \, \mathrm{tr}\ln(g - f). \tag{249}$$

It follows that

$$\sqrt{\widehat{g}} = \frac{1}{2}\left(\sqrt{\det(g + f)} + \sqrt{\det(g - f)}\right)$$
$$+ \frac{\gamma_5}{2}\left(\sqrt{\det(g + f)} - \sqrt{\det(g - f)}\right). \tag{250}$$

$\sqrt{\widehat{g}}$ has the basic property that, under AE diffeomorphisms,

$$\delta_{\widehat{\xi}}\sqrt{\widehat{g}} = \widehat{\xi}^\lambda \partial_\lambda \sqrt{\widehat{g}} + \sqrt{\widehat{g}} \, \partial_\lambda \widehat{\xi}^\lambda. \tag{251}$$

This is a volume density, and has the following properties

$$\sqrt{\widehat{g}} \rightarrow e^{4\widehat{\omega}}\sqrt{\widehat{g}}, \tag{252}$$

under an axial-Weyl transformations. Moreover

$$\frac{1}{\sqrt{\widehat{g}}}\partial_\nu \sqrt{\widehat{g}} = \frac{1}{2}\widehat{g}^{\mu\lambda}\partial_\nu \widehat{g}_{\mu\lambda} = \widehat{\Gamma}^\mu_{\mu\nu}. \tag{253}$$

C Green's functions

In the text we have assumed the existence of the propagator $\widehat{\mathcal{G}}$, the inverse of $\widehat{\mathcal{F}}$. In this Appendix we discuss this question by comparing it with the ordinary case, as discussed in [5]. First we review the approach of [5] in the ordinary gravity case. Then we explain the modifications required in the MAT case.

We consider the case of a stationary metric and axial-metric background. We will assume eventually that the results hold also for nonstationary case, provided the background varies mildly in time.

In this Appendix the flat gamma matrices are understood to be the Majorana ones, that is, they are purely imaginary, together with γ_5: $\gamma_0 \equiv \eta$ and γ_5 are antisymmetric, while γ_i, $i = 1, 2, 3$ are symmetric.

C.1 A summary of Green's functions

Let us give first a short review of ordinary fermionic propagators, see [5,6,14,15]. We start from

$$G(x, x') = \langle 0|\mathcal{T}\psi(x)\psi^\dagger(x')|0\rangle. \tag{254}$$

This is not the standard Feynman Green function

$$S_F(x, x') = \langle 0|\mathcal{T}\psi(x)\bar{\psi}(x')|0\rangle. \tag{255}$$

The two are related by $S_F(x, x') = G(x, x')\eta$

Other Green functions are the advanced, $G^+(x, x')$, and retarded, $G^-(x, x')$; the positive and negative frequency Green functions, $G^{(+)}(x, x')$ and $G^{(-)}(x, x')$, respectively; and the principal value Green function $\bar{G}(x, x') = \frac{1}{2}\left(G^+(x, x') + G^-(x, x')\right)$. The definitions depends only on the contour of integration of p^0 in the momentum space representation, while for the rest they are the same. The important relation in this context is

$$G(x, x') = \bar{G}(x, x') + \frac{i}{2}G^{(1)}(x, x'),$$
$$G^{(1)} = i\left(G^{(+)} - G^{(-)}\right). \tag{256}$$

For real fermions $\bar{G}(x, x')$ and $G^{(1)}(x, x')$ are real. So they represent the real and imaginary part of $G(x, x')$. $G^{(1)}(x, x')$ can be represented as

$$G^{(1)}(x, x') = \langle 0|[\psi(x), \psi^\dagger(x')]|0\rangle \equiv \mathcal{S}^{(1)}(x, x'). \tag{257}$$

The Feynman propagator satisfies the equation

$$i\sqrt{g}\eta\left(\gamma^\mu\nabla_\mu + m\right)G(x, x') = -\mathbf{1}\delta(x, x') \tag{258}$$

and $\mathbf{1}$ is the identity matrix in the spinor space. Both sides of (258) transform as a bispinor density, i.e. like $\sqrt{g}\gamma_0\psi(x)$ at x and as $\psi^\dagger(x')$ at x'. Instead

$$i\sqrt{g}\eta\left(\gamma^\mu\nabla_\mu + m\right)G^{(1)}(x, x') = 0. \tag{259}$$

The approach of [14,15] is based essentially on $G^{(1)}$.

Now let us make the ansatz

$$G(x, x') = -i\left(\gamma^\mu\nabla_\mu - m\right)\mathcal{G}(x, x')\eta^{-1}. \tag{260}$$

Inserting this into (258) one gets

$$\sqrt{g}\left(\nabla_\mu g^{\mu\nu}\nabla_\nu - \left(m^2 + \frac{1}{4}R\right)\right)\mathcal{G}(x, x') = -\mathbf{1}\delta(x, x'). \tag{261}$$

Now we represent (261) as

$$\int dx'' \mathcal{F}(x, x'')\mathcal{G}(x'', x') = -\mathbf{1}\delta(x, x') \tag{262}$$

or, in operator form,

$$\mathcal{F}\mathcal{G} = -1 \tag{263}$$

(understanding $\langle x|\mathcal{G}|x'\rangle = \mathcal{G}(x, x')$, etc.), where

$$\mathcal{F}(x, x') = \sqrt{g}\left(\nabla_\mu g^{\mu\nu}\nabla_\nu - \left(m^2 + \frac{1}{4}R\right)\right)\mathbf{1}\delta(x, x') \tag{264}$$

and the function and derivatives in the RHS are understood to be evaluated at x. Alternatively we represent (261) as

$$\mathcal{F}_x\mathcal{G}(x, x') = -\mathbf{1}\delta(x, x'), \tag{265}$$

where $\mathcal{F}_x$ is the differential operator acting on $\mathbf{1}\delta(x, x')$ in the RHS of (264).

C.2 Properties of $\mathcal{F}$

The operator $\mathcal{F}$ in (261) is not selfadjoint. In fact

$$\mathcal{F}^\dagger = \gamma_0\mathcal{F}\gamma_0. \tag{266}$$

This implies that the construction of a Green's function is not straightforward. In a stationary background a propagator is constructed out of modes which are stationary eigenfunctions (plane waves, at least asymptotically) with real frequencies. Given the Dirac equation

$$i(\gamma^\mu\nabla_\mu + m)u = 0 \tag{267}$$

by suitably fixing the gauge for diffeomorphisms, one can always define a complete set of eigenfunctions with real frequencies, symbolically $u_+ = \chi e^{-i\omega t}, u_- = \lambda e^{i\omega t}$, so that (understanding the indices and integration over the space momenta)

$$\psi = u_+ a + u_- a^\dagger, \tag{268}$$

where $a, a^\dagger$ are annihilation, creation operators (see chapter 19 of [6]).

In the same way one can infer the existence of an analogous complete set of solutions, say v_+, v_- of

$$i(\gamma^\mu\nabla_\mu - m)v = 0. \tag{269}$$

Now, even if $\mathcal{F}$ is not self-adjoint, we can construct the following operator

$$\mathcal{F} = \begin{pmatrix} 0 & \mathcal{F} \\ \mathcal{F}^\dagger & 0 \end{pmatrix} \tag{270}$$

which is self-adjoint, and whose inverse is

$$\mathcal{G} = \begin{pmatrix} 0 & \mathcal{G}^\dagger \\ \mathcal{G} & 0 \end{pmatrix}. \tag{271}$$

The mode solutions of $\mathcal{F}$ are

$$\begin{pmatrix} 0 \\ u_+ \end{pmatrix}, \quad \begin{pmatrix} 0 \\ u_- \end{pmatrix}, \quad \begin{pmatrix} \gamma_0 v_+ \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \gamma_0 v_- \\ 0 \end{pmatrix} \tag{272}$$

which have all real frequencies. It follows that we can construct the Feynman propagator of $\mathcal{F}$. Following the argument of [6], end of chapter 20, it has the form

$$\mathcal{F}^{-1} = \begin{pmatrix} 0 & -\frac{i}{\mathcal{F}^\dagger + i\epsilon} \\ -\frac{1}{\mathcal{F} + i\epsilon} & 0 \end{pmatrix}. \tag{273}$$

Comparing with (271) we get

$$\mathcal{G} = -\frac{1}{\mathcal{F} + i\epsilon}. \tag{274}$$

C.3 Existence of mode functions

The existence of mode functions, i.e. solutions of the Dirac equation (269) of the type $u = \chi e^{i\omega t}$ with real ω, in a stationary background, is the basis for the existence of propagators. In [6] the problem is discussed as follows. One shows that one can cast (269) in the form

$$Fu = 0, \qquad F = \frac{1}{2}\left\{ B^\mu, \frac{\partial}{\partial x^\mu} \right\} - C, \tag{275}$$

where

$$B^\mu = i\eta\gamma^\mu, \qquad C = -\frac{i}{4}\eta\{\gamma^\mu, \omega_\mu\}. \tag{276}$$

The important thing is that, in the Majorana representation of the γ matrices, B^μ is a symmetric matrix, while C is antisymmetric, and they are both purely imaginary. By choosing the gauge $e_0^0 = 1$, $e_0^i = 0$ for the vierbein e, the operator F becomes

$$F = \frac{1}{2}\left\{ B, \frac{\partial}{\partial t} \right\} - \mathcal{C}, \tag{277}$$

where

$$B = i, \qquad \mathcal{C} = C - \frac{1}{2}\left\{ B^i, \frac{\partial}{\partial x^i} \right\}. \tag{278}$$

Again while B is symmetric imaginary with $-iB$ being positive definite, $\mathcal{C}$ is antisymmetric imaginary. Plugging the ansatz $u_A = \chi_A e^{-i\omega_A t}$ into $Fu = 0$ one gets the eigenvalue equation

$$(\mathcal{C} + i\omega_A B)\chi_A = 0. \tag{279}$$

Due to the abovementioned propertis of B and $\mathcal{C}$, one can find eigenvalues and eigenvectors. The eigenvalues ω_A can be taken real and positive.

C.4 What changes when the background is MAT

In this case the analog of (266) is

$$\widehat{\mathcal{F}}^\dagger = \eta \widehat{\mathcal{F}} \eta. \tag{280}$$

But as above we can proceed to construct the operator

$$\widehat{\mathcal{F}} = \begin{pmatrix} 0 & \widehat{\mathcal{F}} \\ \widehat{\mathcal{F}}^\dagger & 0 \end{pmatrix} \tag{281}$$

which is self-adjoint, and whose inverse is

$$\widehat{\mathcal{G}} = \begin{pmatrix} 0 & \widehat{\mathcal{G}}^\dagger \\ \widehat{\overline{\mathcal{G}}} & 0 \end{pmatrix}. \tag{282}$$

Using the same argument as above we can conclude that

$$\widehat{\mathcal{G}} = -\frac{1}{\widehat{\mathcal{F}} + i\epsilon}. \tag{283}$$

The only delicate point in reaching this conclusion is the solutions of

$$i\widehat{\gamma}^\mu \widehat{\nabla}_\mu u = 0. \tag{284}$$

Equation (269) is real, since the gamma matrices are purely imaginary. But, in (284), the presence of γ_5 poses a problem. In a representation in which the gamma matrices are purely imaginary, the γ_5 is also imaginary, thus Eq. (284) is complex, and, based on the analogy with the previous subsection, one cannot be sure a priori that there are real frequency solutions. However we notice that the operator $\eta\widehat{F}$ is self-adjoint. This remark lends us a way out.

Another crucial point is the gauge fixing, so that one can end up with something analog to (278), in which $-iB$ is positive definite. As we saw above, this is obtained by choosing in particular $e_0^0 = 1$, $e_0^i = 0$. In MAT the coefficient of γ^0 is $\widehat{e}_0^\mu$, which contains also $\gamma_5 c_0^\mu$. We shall choose $c_0^\mu = 0$. As a consequence the analog of $Fu = 0$ is $\widehat{F}\widehat{u} = 0$ where

$$\hat{F} = \frac{1}{2}\left\{ \widehat{B}, \frac{\partial}{\partial t} \right\} - \widehat{\mathcal{C}}, \tag{285}$$

where $\widehat{B} = B$, i.e. symmetric and such that $-iB$ is positive definite. As for $\widehat{\mathcal{C}}$, it can be written as

$$\widehat{\mathcal{C}} = \widehat{\mathcal{C}}_a + \widehat{\mathcal{C}}_s, \tag{286}$$

where $\widehat{\mathcal{C}}_a$ is imaginary antisymmetric and does not contain γ_5, while $\widehat{\mathcal{C}}_s$ is real, linear in γ_5 and symmetric. However altogether it is self-adjoint.

Plugging the ansatz $\widehat{u}_A = \widehat{\chi}_A e^{-i\omega_A t}$ into $\eta\widehat{F}\widehat{u} = 0$ one gets the equation

$$(\widehat{\mathcal{C}} - \omega_A)\widehat{\chi}_A = 0 \tag{287}$$

which is an eigenvalue equation for $\widehat{\mathcal{C}}$. Since the latter is self-adjoint we know there exists a complete set of eigenfunctions. This is what we need.

So the remaining question is: is the choice $c_0^\mu = 0$ permitted? In order to see this one has to check that the defining equations (220,221) for the axial-complex vierbein and the like in Appendix B are still valid. Now, suppose the ordinary gauge fixed vierbein satisfies such defining equation (which they do in [5]). Then we can set the axial-imaginary vierbein

c and c^{-1} to 0, while preserving the defining relations. In other words, there is a large gauge freedom, and in particular we can choose $c_0^\mu = 0$.

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