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Two fundamental constants of gravity unifying dark matter and dark energy

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Abstract The common nature of the dark sector—dark energy and dark matter—as shown in Gurzadyan (arXiv:1712.10014, 2017) follows readily from the consideration of the generalized Newtonian potential as a weak-field version of General Relativity. The generalized potential satisfying Newton's theorem on the equivalence of sphere's gravity and that of a point mass located in its center contains an additional constant, which along with the gravitational constant is able to explain quantitatively both dark energy (the cosmological constant) and dark matter. So, gravity is defined not by one but two fundamental constants. We show that the second constant is dimension-independent and matter-uncoupled and hence is even more universal than the gravitational constant, thus affecting the strategy of observational studies of dark energy and of the search of dark matter.

1 Introduction

The discovery of the dark sector as a dominant constituent of the universe is one of outstanding recent astrophysical achievements and continues to be a key puzzle for physical theories. Various modifications of the Newtonian gravity and of General Relativity (GR) are being actively considered in that context.

Among the possible approaches to modified gravity, including GR, is the one based on a theorem proved by Newton in "Principia" on the equivalence of the gravity of sphere and that of a point mass located in its center. The principal importance of that theorem was obvious, since the motion of the planets, which were spheres and not point masses, could be considered to be explained by the law of gravity only upon the proof of that theorem. Now, it appears that this theorem provides a two-step path to modified gravity theories and directly to the dark sector problem [1,2]:

1. The general function satisfying that theorem provides an additional term containing a constant and thus modifying the Newtonian gravity;
2. modified Newtonian gravity leads to a modified GR with the former as its weak-field limit.

So, modified GR has to initially include that additional constant, along with the gravitational constant. As shown in [1,2] that additional constant entering both modified Newtonian gravity and GR enables one to describe by its sign and quantitative value both dark matter and dark energy. That constant appears to be the renowned cosmological constant, which was introduced by Einstein [3] in order to have static solutions to the Einstein equations.

The two constants of the gravitational interaction are able to describe self-consistently, i.e. without postulation of additional scalar or other fields, the dark matter and dark energy. This reveals their unified, gravitational nature. Namely, the dark matter appears as a result of the pure gravitational interaction, but with a law containing the additional constant.

We analyze this approach and reveal the role of that additional universal physical constant in classical and relativistic gravities. The consequences of this approach can have a direct impact on the strategy of the observational studies of the dark energy and the search of the dark matter.

2 Newton's theorem and General Relativity

The general function to satisfy Newton's theorem that a sphere acts as a point mass located in its center has the following form for the force [2]:

$$f(r) = C_1 r^{-2} + C_2 r, \tag{1}$$

where C_1 and C_2 are constants and $f(r)$ is the solution of equation

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$$\frac{r^2}{2} f''(r) + r f'(r) - f(r) = 0. \quad (2)$$

Thus, one can conclude that according to Newton's theorem the gravitational force should contain two terms, i.e. an inversed square term and a linear one.

Considering the original formulation of gravity by Newton himself, it becomes clear that the constant C_1 is written as $C_1 = G m^2$, i.e. with the familiar gravitational constant G and mass m, the latter entering also the law of mechanics. In this sense, the Newtonian gravity can be regarded as a very special case, i.e. $C_2 = 0$ of all possible forms of gravitational fields where one can consider spherical objects as points. Furthermore, it should be noticed that Newton himself did not consider the most general form of the force before formulation of his theory of gravity, although he proved that in the context of his theory it is possible to consider spheres as points.

In this context the presence of a linear term was forgotten for hundreds of years until the formulation of GR and introduction of cosmological constant Λ in

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}. \quad (3)$$

After that it became obvious that, by considering Λ, as introduced above, the GR weak-field limit will contain an additional linear term. In this sense the metric tensor components for the sphere's gravity in the weak-field limit will be

$$g_{00} = 1 - \frac{2Gm}{rc^2} - \frac{\Lambda r^2}{3}; \quad g_{rr} = 1 + \frac{2Gm}{rc^2} + \frac{\Lambda r^2}{3}. \quad (4)$$

Thus, one can conclude that although first the existence of Λ was proposed by Einstein in the context of GR [4], it would be possible to find the full GR equations (Eq. (3)), if Newton had considered both terms and formulated his theory based on Eq. (1). In this sense, by considering Newton's principle and the most general form of the force, the cosmological term appears in Einstein's equations not by the principles of GR, but as the second linear term of Newtonian gravity.

It should be noticed that, although Eq. (4) has been considered previously in different contexts (e.g. [4,5] and the references therein), the approach of [1,2] from the roots of Newtonian gravity/GR provides insight into the unified nature of dark matter and dark energy. Namely, the presence of Λ as an additional linear term in the Newtonian regime, i.e. Eq. (4), enables one to describe the dark matter in galaxies, as the cosmological constant in GR describes the dark energy, and as shown in [1] both values of Λ, i.e. those describing dark matter and dark energy (cosmological constant) quantitatively agree with each other. In the case of dark matter, it is of principal importance since Eq. (1) describes a non-force-free field inside a shell except for its center, while for the Newtonian law the force-free field is entirely inside the shell. This fact agrees with the observational evidence that the galactic halos determine the properties of galactic disks [6].

Table 1 Background geometries for vacuum solutions

Sign	Spacetime	Isometry Group	Curvature
$\Lambda > 0$	de Sitter (dS)	O(1, 4)	+
$\Lambda = 0$	Minkowski (M)	IO(1, 3)	0
$\Lambda < 0$	Anti de Sitter (AdS)	O(2, 3)	−

3 Group-theoretical analysis of Newton's theorem

In the previous section we have shown that it is possible to justify the existence of the second term in Eq. (1) as the weak-field limit of the GR equations written with Λ. However, as mentioned above, Λ was introduced not by Newton's theorem but according to conservation of the energy-momentum tensor and the fact that $\partial^\mu g_{\mu\nu} = 0$. So it seems quite reasonable that, to make a more powerful justification, we try to infer Newton's theorem based on the above relativistic considerations. Thus we turn to the isometry groups.

In Eq. (3), depending on Λ's sign—positive, negative or zero—one has three different vacuum solutions (three different asymptotic limits) for the field equations as shown in Table 1.

The interesting feature of all these 4D maximally symmetric Lorentzian geometries is that for all of them the stabilizer subgroup of isometry group is the Lorentz group O(1, 3). This means that at each point of all these spacetimes, one has an exact Lorentz symmetry. Since O(1, 3) is the group of orthogonal transformations, one can conclude that all above spacetimes possess spherical symmetry (in Lorentzian sense) at each point. Speaking in terms of geometry, for the above three spacetimes we have

$$dS = \frac{O(1,4)}{O(1,3)}, \quad M = \frac{IO(1,3)}{O(1,3)}, \quad AdS = \frac{O(2,3)}{O(1,3)}. \quad (5)$$

It is clear that in the non-relativistic limit the full Poincaré group IO(1, 3) is reduced to the Galilei group Gal(4) = $(\mathrm{O}(3) \times \mathrm{R}) \ltimes \mathrm{R}^6$, which is the action of $\mathrm{O}(3) \times \mathrm{R}$ (as the direct product of spatial orthogonal transformations and of time translation) on the group of boosts and spatial translations R^6. In the same way one can find the non-relativistic limit of O(1, 4) and O(2, 3) groups,

$$O(1,4) \rightarrow (O(3) \times O(1,1)) \ltimes R^6,$$
$$O(2,3) \rightarrow (O(3) \times O(2)) \ltimes R^6. \quad (6)$$

Table 2 3D background geometries with O(3) as the stabilizer

Space	Isometry group	Curvature
Spherical	O(4)	+
Euclidean	E(3)	0
Hyperbolic	$O^+(1,3)$	−

Furthermore, considering the fact that the Galilei spacetime is achieved via quotienting Gal(4) by $O(3) \ltimes R^3$ (the group generated by orthogonal transformations and boosts), one can continue the analogy and find the so-called Newton–Hooke $NH(4)^{\pm}$ spacetimes by the same quotient group, but now for the groups of Eq. (6) (see [7–9]). In this sense, depending on the sign of Λ, we cannot only find the general form of the Newtonian modified gravity (according to Sect. 2), but also the non-relativistic background geometries of the Lorentzian spacetimes in Table 1 and their symmetries.

To complete the proof, one has to check whether it is possible to apply Newton's theorem to these spacetimes or not. As stated above, to apply the gravity law to planets (spheres) Newton considered them as points. Speaking in terms of mathematics it means that at each point one should have O(3) symmetry. This statement is similar to what we showed for the 4D geometries of Table 1 and the Lorentz group $O(1, 3)$. The possible 3-geometries with such a property are listed in Table 2.

Recalling that for non-relativistic theories we have two absolute notions of space and time geometry (in contrast to relativistic theories where space and time are unified in spacetime geometry), the last step is to check whether the spatial geometry of two $NH(4)^{\pm}$ spacetimes and the Galilei spacetime are equal to one of the geometries mentioned in Table 2 or not. There are several ways to check this statement; however, the most straightforward one is to check the algebraic structure of spatial geometry. Recalling the fact that for both $NH(4)^{\pm}$ spacetimes and Galilei spacetime the spatial algebra is identical and equal to the Euclidean algebra $E(3) = R^3 \rtimes O(3)$, we can conclude that for all above spacetimes we have an exact O(3) symmetry at each point of spatial geometry. In this sense we will arrive at Newton's theorem based on a group theoretical analysis of GR equations.

4 Newton's theorem in d dimensions

To shed more light on the constant Λ we consider the higher dimensional cases, which simply means that the gravitational field defined on S^{d-1} should be equal to that defined for a single point at d-dimensional space. For the potential one has

$$\Delta_{S^{d-1}} \Phi = C_1, \qquad (7)$$

where $\Delta_{S^{d-1}}$ denotes the Laplace operator defined on S^{d-1} and the constant C_1 defines the mathematical feature of the geometrical point. Now due to spherical symmetry we can write

$$\frac{1}{r^{d-1}}\left(\frac{d}{dr} r^{d-1} \frac{d}{dr}\Phi\right) = C_1. \qquad (8)$$

So the most general form of the gravitational potential Φ of a sphere in the d-dimensional case according to Newton's theorem is

$$\Phi(r) = C_1 \frac{r^2}{2d} + \frac{C_2}{(d-2)r^{d-2}}. \qquad (9)$$

In this equation C_1 is the constant of Eq. (7) and the constant C_2 arises on solving the equation. Note that for $d = 2$ the second term becomes logarithmic but the first one remains unchanged.

The potential Φ in Eq. (9) at $d = 3$ is not only in full agreement with Eq. (1), but it also leads to further insights. One can identify C_2 in Eq. (9) with the Λ constant at the d-dimensional generalization of ordinary Newtonian gravity,

$$\Phi(r) = -\frac{G_d M}{r^{d-2}} - \frac{\Lambda c^2 r^2}{2d}, \qquad (10)$$

where G_d indicates the d-dimensional gravitational constant.

Note a remarkable fact: comparing the two constants—the gravitational constant G and Λ—one can see their essential difference. Namely, the gravitational constant G is dimension-dependent and couples to matter, while Λ is neither dimension-dependent nor matter-coupled. Such universality of Λ can be considered as fitting its vacuum content noticed by Zeldovich from completely different principles [10].

Thus, gravity has not one, but two fundamental constants—G and Λ—and the second one (the cosmological constant) is more universal (dimensional-independent) than the gravitational constant! The two constants together are able to explain quantitatively the dark energy and the dark matter [1].

Then the metric component of $d+1$-dimensional spacetime is

$$g_{00} = 1 + \frac{2\Phi}{c^2}. \qquad (11)$$

We have the d-dimensional Gauss law,

$$\Delta\Phi = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} G_d \rho - \Lambda c^2, \qquad (12)$$

where ρ is the d-dimensional density of matter. Consequently one gets the Einstein constant,

$$\kappa_d = \frac{4\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \frac{G_d}{c^4}. \tag{13}$$

This completes the generalization of Newton's theorem to arbitrary dimension and its correspondence to classical and relativistic theories of gravity.

Then for the three possible maximally symmetric $(d+1)$-dimensional spacetimes defined by the value of Λ one has the following geometries:

$$dS_{d+1} = \frac{O(1, d+1)}{O(1, d)}, \quad M_{d+1} = \frac{IO(1, d)}{O(1, d)},$$
$$AdS_{d+1} = \frac{O(2, d)}{O(1, d)}, \tag{14}$$

as the generalizations of Eq. (5); for $d = 3$ one easily recovers the 4-dimensional results. It is clear that in such a case, irrespective of which geometrical spacetime is considered, one has exact O(1,d) symmetry at each point, which in its turn indicates the existence of spherical symmetry of Lorentzian geometry for all points. Fixing the relativistic geometries and symmetries one easily finds their non-relativistic limits,

$$O(1, d) \rightarrow (O(d) \times O(1, 1)) \ltimes R^{2d},$$
$$O(2, d) \rightarrow (O(d) \times O(2)) \ltimes R^{2d},$$
$$IO(1, d) \rightarrow (O(d) \times R) \ltimes R^{2d}. \tag{15}$$

As in Sect. 3, one can find the non-relativistic background geometries for each case by quotienting $\mathrm{O(d)} \ltimes \mathrm{R}^d$ for all three symmetric groups. The resulting spacetimes are Gal(d+1), $\mathrm{NH}^+(\mathrm{d}+1)$, $\mathrm{NH}^-(\mathrm{d}+1)$, and clearly at $d = 3$ one obtains the classical spacetimes. As we have mentioned earlier, the interesting feature of these non-relativistic geometries is the fact that, in contrast to the relativistic case, they are not metric geometries because they do not admit single metric structure and their properties can be studied via the corresponding affine connection. Furthermore, from the geometrical point of view, for all these three cases the spatial geometry seems to be Euclidean and the pure spatial algebra is equal to the Euclidean algebra E(d). Then, since $\mathrm{E(d)} = \mathrm{R}^d \rtimes \mathrm{O(d)}$, one easily concludes that in the spatial geometry the O(d) is the stabilizer group, which in its turn means that all points can be considered as d-dimensional spheres, S^{d-1}. This proves that for all these three geometries Newton's theorem holds. However, as mentioned above, the spatial parts of all three geometries are equal to each other, and the question is how Λ affects these geometries. The answer becomes clear if one considers the temporal parts of Eq. (15). Indeed, the sign of Λ indicates that we are living either in an oscillating $\mathrm{NH}(\mathrm{d}+1)^-$, a flat Gal(d+1) or an expanding $\mathrm{NH}(\mathrm{d}+1)^+$ universe. One can also check that, for all these cases, depending on the sign and the value of Λ, the affine connection can be flat, for the Gal(d+1) case, and either positive or negative for $\mathrm{NH}(\mathrm{d}+1)^+$ and $\mathrm{NH}(\mathrm{d}+1)^-$, respectively.

To conclude this brief but principal discussion, we write down the d-dimensional ($d \neq 2$) Schwarzschild metric for non-zero Λ,

$$ds^2 = \left(1 - \frac{2G_d M}{r^{d-2}c^2} - \frac{\Lambda r^2}{3}\right) c^2 dt^2 - \left(1 - \frac{2G_d M}{r^{d-2}c^2} - \frac{\Lambda r^2}{3}\right)^{-1} dr^2 - r^2 d\Omega^2_{d-1}. \tag{16}$$

Although d-dimensional cases have been considered before, our approach to GR and its weak-field limit justifies the consideration of point-like dynamics for higher-dimensional spheres based on Newton's original theorem.

5 Conclusions

Thus, according to our analysis:

1. Gravity has not one but two fundamental constants, the gravitational constant G and an additional one, Λ, which appears readily in General Relativity with the weak-field limit as modified Newtonian gravity. Moreover, the Λ constant (the cosmological constant) is dimension-independent and matter-uncoupled and hence can be considered as even more universal than the gravitational constant G.
2. The Λ constant of gravity emerges from Newton's theorem on the identity of the sphere's gravity and that of the point mass located in its center.
3. Both constants, G and Λ, jointly are able to explain quantitatively dark energy and dark matter [1], which hence appear as gravity effects.

Also, the AdS spacetime of AdS/QFT emerges here readily from the genuine structure of classical and relativistic gravities. A positive Λ constant is an essential condition in conformal cyclic cosmology [11,12].

The accuracies of the current tests of GR (e.g. [13]) are still far from enabling one to probe modified gravity as discussed above; however, for example, the astronomical observations of galactic halos [14] can be efficient in testing the predictions regarding the nature of dark matter.

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References

1. V.G. Gurzadyan, arXiv:1712.10014 (2017)
2. V.G. Gurzadyan, Observatory **105**, 42 (1985)
3. A. Einstein, Königlich-Preuische Akad. Wiss. **142**, 235–237 (1917)
4. C. O'Raifeartaigh, M. O'Keeffe, W. Nahm, S. Mitton, Eur. Phys. J. H **43**, 73 (2018)
5. S. Bahamonde et al, arXiv:1712.03107 (2017)
6. A.V. Kravtsov, Astrophys. J. Lett. **764**, L31 (2013)
7. H. Bacry, J.-M. Levy-Leblond, J. Math. Phys. **9**, 1605 (1967)
8. R. Aldrovandi, A.L. Barbosa, L.C.B. Crispino, J.G. Pereira, Class. Quantum Gravity **16**, 495 (1999)
9. G.W. Gibbons, C.E. Patricot, Class. Quantum Gravity **20**, 5225 (2003)
10. Y.B. Zeldovich, JETP Lett. **6**, 316 (1967)
11. R. Penrose, *Cycles of Time: An Extraordinary New View of the Universe* (Bodley Head, London, 2010)
12. V.G. Gurzadyan, R. Penrose, Eur. Phys. J. Plus **128**, 22 (2013)
13. I. Ciufolini et al., Eur. Phys. J. C **76**, 120 (2016)
14. V.G. Gurzadyan et al., A & A **609**, A131 (2018)