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Relativistic motion enhanced quantum estimation of κ-deformation of spacetime

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Abstract We probe the κ-deformation of spacetime using a two-level atom as a detector coupled to a κ-deformed massless scalar field which is invariant under a κ-Poincaré algebra and written in commutative spacetime. To address the quantum bound to the estimability of the deformation parameter κ, we perform measurements on the two-level detector and maximize the value of quantum Fisher information over all possible detector preparations. We prove that the population measurement is the optimal measurement in the estimation of the deformation parameter κ. In particular, we show that the relativistic motion of the detector affects the precision in the estimation of the parameter κ, which can effectively improve this precision comparing to that of the static detector case by many orders of magnitude.

1 Introduction

In the quantum theory of gravity, the spacetime coordinates get quantized, and the notion of spacetime, as well as the notions of its symmetry, is usually modified at the microscopic level. An interesting symmetry algebra of certain quantum gravity models is the κ-Poincaré algebra [1,2]. Such algebra is deeply connected with a "Lie algebra"-type of noncommutative spacetime named the κ-Minkowski spacetime. The κ-Poincaré algebra and the κ-deformed spacetime are also known to be related to the doubly special relativity [3], which introduces a fundamental constant of length dimension in addition to the constant velocity of light present in the special theory of relativity. Besides, it could be possible to experimentally detectable signatures of the deformed kinematics associated with such symmetries [4–6]. As a consequence of that, a lot of attentions have recently been paying on the physics involving κ-deformed spacetime [7–31].

The κ-deformation parameter does not correspond to a proper observable, and thus its value should be inferred through some indirect measurements. In this regard, let us note that any conceivable strategy aimed at evaluating the quantity of interest ultimately reduces to a parameter-estimation problem that may be properly addressed in the framework of quantum estimation theory (QET) [32–36]. With the application of QET, one can estimate the parameter of interest with a higher precision which is beyond the standard classical limits [37]. Recently, this theory has been successfully applied to a wide range of metrological problems [38], relativistic quantum field [39–45], quantum illumination [46], biology [47], the experiments with photons [48,49] and trapped ions [50,51], and so on. Besides, to enhance the relevant estimation, different ways, such as using of quantum entanglement, feedback control, and nonlinear dynamics, have been explored [52–58].

In this manuscript we study the ultimate limits of precision in the estimation of the κ-deformation parameter employing the quantum metrology [35,59]. We consider a two-level atom as a probe coupled to a κ-deformed massless scalar field which is defined in the commutative spacetime [28]. In this paper, the authors started with the κ-deformed Klein–Gordon equation, which is invariant under the action of κ-Poincaré algebra and written in the commutative spacetime itself. Since the model of detector-field is constructed in the commutative spacetime, the standard field theory techniques developed for the commutative spacetime are allowed to study the κ-deformed scalar theory. We calculate the QFI of the detector and maximize it, i.e., to find the ultimate limits of the precision, over all the possible detector preparations. By studying the Fisher information (FI) with respect to the population measurement on the detector, we find its maximal value which is equal to that of QFI, and thus the population measurement is the optimal measurement in the estimation of parameter κ. Especially, motivated by the rel-

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ativistic quantum information [60–67], we analyze how the relativistic motion of the detector affects the estimation of the parameter κ. Interestingly, our results show that the relativistic motion can effectively enhance the precision in the estimation of κ-deformation parameter, i.e., the moving detector case can exceed the static detector case by many orders of magnitude.

The paper is structured as follows. In Sect. 2, we introduce the local QET and present the expression of QFI for arbitrary qubit states in the Bloch representation. We also take a review of κ-deformed scalar theory written in commutative spacetime and presenting its propagator. This κ-deformed scalar field interacts with a detector whose dynamic evolution with an arbitrary initial state has been simply reviewed. Sections 3 and 4 are devoted to studying how to estimate the κ-deformation of spacetime and how the relativistic motion affects the relevant estimation. Finally, Sect. 5 concludes the paper.

Throughout the whole paper we employ natural units $c = \hbar = 1$. Relevant constants are restored when needed for the sake of clarity.

2 local quantum estimation theory and physical model

Now, we are going to exploit local QET to find out the quantum measurement that maximizes the quantum Fisher information (QFI) [35], which aims to evaluate the ultimate limits of precision in the estimation of the κ-deformation parameter. We explore how the precision in the estimation of the parameter κ changes when a two-level moving detector interacts with κ-deformed massless scalar field.

2.1 Local quantum estimation theory

Any inference strategy amounts to find an estimator, i.e., a mapping $\hat{\lambda} = \hat{\lambda}(x_1, x_2, \ldots, x_n)$ from the set of measurement outcomes into the space of parameters. According to the Cramér–Rao theorem, the optimal estimators are bounded by the inequality

$$\text{Var}(\lambda) \geq \frac{1}{MF(\lambda)} ,\qquad (1)$$

which sets a lower bound on the mean square error of any estimator of the parameter λ. In Eq. (1) M is the number of measurements and $F(\lambda) = \sum_x p(x \mid \lambda)[\partial_\lambda \ln p(x \mid \lambda)]^2$ is the FI. Here, $p(x \mid \lambda)$ denotes the conditional probability of obtaining the outcome x. Before collecting the outcomes, we find $p(x \mid \lambda) = \text{Tr}[\rho(\lambda)\Pi_x]$, where $\{\Pi_x\}$ are the elements of a positive operator-valued measure (POVM) and saturate to $\int_x dx\,\Pi_x = 1$. Upon maximizing the FI over all the possible quantum measurements, we have that the FI $F(\lambda)$ of any

quantum measurement is upper bounded by the QFI $H(\lambda)$, i.e., $F(\lambda) \leq H(\lambda) \equiv \text{Tr}[\rho(\lambda)L(\lambda)^2]$, where $L(\lambda)$ represents the symmetric logarithmic derivative satisfying the partial differential equation $\partial_\lambda \rho(\lambda) = \frac{1}{2}(L(\lambda)\rho(\lambda) + \rho(\lambda)L(\lambda))$. Therefore, we have the quantum Cramér–Rao bound

$$\text{Var}(\lambda) \geq \frac{1}{MF(\lambda)} \geq \frac{1}{MH(\lambda)} ,\qquad (2)$$

for the variance of any estimator. The quantum Cramér–Rao bound provides the ultimate bound to precision in the estimation of parameter λ for a state of the family $\rho(\lambda)$. On the other hand, the optimal quantum measurement for the estimation of parameter κ corresponds to POVM with the FI equal to the QFI.

For a two-level quantum system, the reduced density matrix of the system can be expressed in the Bloch sphere representation as

$$\rho(\tau) = \frac{1}{2}(\text{I} + \omega \cdot \sigma),\qquad (3)$$

where $\omega = (\omega_1, \omega_2, \omega_3)$ is the Bloch vector and $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ denotes the Pauli matrices. As a result, the QFI can be described as follows [68]

$$H(\lambda) = \begin{cases} |\partial_\lambda \omega|^2 + \frac{(\omega\partial_\lambda\omega)^2}{1-|\omega|^2}, & (|\omega| < 1) , \\ |\partial_\lambda \omega|^2, & (|\omega| = 1) . \end{cases} \qquad (4)$$

2.2 κ-deformed Klein–Gordon theory

Let us briefly review the κ-deformed massless scalar field which is the external environment. It is worthy noting that the κ-deformed Klein–Gordon theory can be written in the commutative spacetime [28]. Since the relevant model is constructed in the commutative spacetime, it allows us to unambiguously define the trajectory of the motional detector, that is needed for the study of the influence of the relativistic motion on the κ parameter estimation. We start with the Lie algebra-type commutation relation between the coordinates of the κ-deformed Minkowski spacetime,

$$[\hat{x}_i, \hat{x}_j] = 0 , \quad [\hat{x}_0, \hat{x}_i] = \frac{i}{\kappa}\hat{x}_i , \quad i, j = 1, 2, 3 ,\qquad (5)$$

where κ is a positive parameter donating the deformation of the spacetime. It is well known that the symmetry of κ spacetime is the κ-Poincaré algebra [1,2], whose defining relations involve the deformation parameter. In the limit of $\kappa \to \infty$, the Poincaré algebra is recovered. To construct the κ-Poincaré algebra, we require the realizations of the noncommutative coordinates $\hat{x}_\mu$ in terms of ordinary commutative coordinates x_μ and their derivatives ∂_μ, where $\partial_\mu = \frac{\partial}{\partial x_\mu}$ [24–31]. A family of realizations for noncommu-

tative coordinates $\hat{x}_\mu$ satisfying the algebra in Eq. (5) is

$$\hat{x}_i = x_i \varphi(A) , \quad \hat{x}_0 = x_0 \psi(A) + \frac{i}{\kappa} x_i \partial_i \gamma(A) . \tag{6}$$

This realization defines a unique mapping from the functions on noncommutative space to the functions on commutative space. Note that the noncommutative coordinates $\hat{x}_\mu$ in Eq. (6) contain three arbitrary parameters φ, ψ and γ which are functions of $A = -\frac{i}{\kappa} \partial_0$. However, applying the Eq. (6) to (5), we have

$$\gamma = 1 + \frac{\varphi'}{\varphi} \psi, \tag{7}$$

where $\varphi' = \frac{d\varphi}{dA}$ and these functions satisfy the boundary conditions $\varphi(0) = 1$, $\psi(0) = 1$ and $\gamma(0) = 1 + \varphi'(0)$ is finite and all are positive functions.

Then we introduce the generators $M_{\mu\nu}$ of the κ-Poincaré algebra satisfying the ordinary undeformed $so(n-1, n)$ algebra

$$[M_{\mu\nu}, M_{\lambda\rho}] = \eta_{\nu\lambda} M_{\mu\rho} - \eta_{\mu\lambda} M_{\nu\rho} - \eta_{\nu\rho} M_{\mu\lambda} + \eta_{\mu\rho} M_{\nu\lambda}, \tag{8}$$

where $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$. The generators $M_{\mu\nu}$ are required to be linear in the commutative coordinates x_λ with an infinite series in coordinates derivatives ∂_λ. Furthermore, the commutators $[M_{\mu\nu}, \hat{x}_\lambda]$ should be linear functions of $\hat{x}_\lambda$ and $M_{\mu\nu}$, antisymmetric in the indices μ and ν, and have a smooth limit $[M_{\mu\nu}, \hat{x}_\lambda] \to x_\mu \eta_{\nu\lambda} - x_\nu \eta_{\mu\lambda}$ when $\kappa \to \infty$. Therefore, one can easily see that there are only two classes of possible realization, one where $\psi = 1$ and the other one $\psi = 1 + 2A$. We restrict ourselves to the first realization $\psi = 1$, then the explicit of $M_{\mu\nu}$ are

$$M_{i0} = x_i \partial_0 \varphi \frac{e^{2A} - 1}{2A} - x_0 \partial_i \frac{1}{\varphi} + \frac{i}{\kappa} x_i \partial_k \partial_k \frac{1}{2\varphi} - \frac{i}{\kappa} x_k \partial_k \partial_i \frac{\gamma}{\varphi},$$
$$M_{ij} = x_i \partial_j - x_j \partial_i. \tag{9}$$

To derive the κ-deformed Klein–Gordon equation which is invariant under the κ-Poincaré algebra, it is natural to introduce the Dirac derivatives D_μ transforming like a vector under $M_{\mu\nu}$ as follows [24–28]

$$D_i = \partial_i \frac{e^{-A}}{\varphi} , \quad D_0 = \partial_0 \frac{\sinh A}{A} + \frac{i}{\kappa} \nabla^2 \left(\frac{e^{-A}}{2\varphi^2} \right). \tag{10}$$

These generators of the κ-Poincaré algebra, $M_{\mu\nu}$ and D_μ, satisfy

$$[M_{\mu\nu}, D_\lambda] = \eta_{\nu\lambda} D_\mu - \eta_{\mu\lambda} D_\nu, \quad [D_\mu, D_\nu] = 0. \tag{11}$$

Notice that the defining relations of the κ-Poincaré algebra are the same as that of the usual Poincaré algebra, but the explicit form of the generators are modified and those modifications are dependent on the deformation parameter. It is

clear that the Casimir of the κ-Poincaré algebra, $D_\mu D_\mu$, can be expressed in terms of the $\square$ operator as

$$D_\mu D_\mu = \square \left(1 - \frac{1}{4\kappa^2} \square \right), \tag{12}$$

with

$$\square = \nabla^2 \left(\frac{e^{-A}}{\varphi^2} \right) + 2\partial_0^2 \frac{1 - \cosh A}{A^2} , \tag{13}$$

where $\nabla^2 = \partial_i \partial_i$. The $\square$ operator satisfies

$$[M_{\mu\nu}, \square] = 0, \quad [\square, \hat{x}_\mu] = 2D_\mu. \tag{14}$$

Note that the Casimir, $D_\mu D_\mu$ reduces to the usual relativistic dispersion relation in the limit $\kappa \to \infty$. The arbitrary realizations of the κ spacetime coordinates in terms of commutative coordinates and their derivatives, are characterized by φ appearing in the above equation. The generalized Klein–Gordon equation is written in term of the Casimir of the κ-Poincare algebra as

$$\square \left(1 - \frac{1}{4\kappa^2} \square \right) \Phi(x) - m^2 \Phi(x) = 0 . \tag{15}$$

Note that the κ-deformed Klein–Gordon equation is invariant under the action of κ-Poincare algebra. We also note that since the generators and Casimir of the κ-Poincare algebra are expressed in terms of the commutative coordinates and their derivatives, the scalar field and the operators appearing in the above κ-deformed Klein–Gordon equation are defined in the commutative spacetime. By reexpressing the noncommutative coordinates of the κ-deformed spacetime in terms of the commutative coordinates and their derives, one can facilitate the construction of the κ-Poincaré algebra. In doing so, we can map the information of the κ spacetime to a commutative spacetime. This field operators satisfy the commutative relation which is similar to that in the commutative spacetime, but it contains the deformation parameter already. In this case, the κ-deformed Klein–Gordon theory was now constructed completely in the commutative spacetime. This allows us to use the standard tools of field theory developed for the commutative spacetime to study the κ-deformed Klein–Gordon theory.

It is worth mentioning that we consider the representations of noncommutative fields in terms of commutative fields, some characters of fields, such as nonlocal and noncausal, will appear. We can see from the deformed dispersion relation which satisfies the κ-deformed Klein–Gordon equation in the Eq. (15),

$$4\kappa^2 \sinh^2 \left(\frac{p_0}{2\kappa} \right) - p_i^2 \frac{e^{-\frac{p_0}{\kappa}}}{\varphi^2 \left(\frac{p_0}{\kappa} \right)}$$
$$- \frac{1}{4\kappa^2} \left[4\kappa^2 \sinh^2 \left(\frac{p_0}{2\kappa} \right) - p_i^2 \frac{e^{-\frac{p_0}{\kappa}}}{\varphi^2 \left(\frac{p_0}{\kappa} \right)} \right]^2 = m^2, \tag{16}$$

where $p_0 = i\partial_0$ and $p_i = -i\partial_i$. Besides, one can obtain a complicated κ-deformed Klein–Gordon equation defined in the commutative spacetime. Correspondingly, the Hamiltonian for the field is complicated and difficult to express in a compact form. Hereafter, to simplify the above problem and then get the Green function, we will made a specific choice $\varphi(\frac{p_0}{\kappa}) = e^{-\frac{p_0}{2\kappa}}$, as considered in Ref. [28]. During the concrete calculation of Green function, we just take an approximate value up to second order in $1/\kappa$.

2.3 Dynamical evolution of a two-level detector interacts with κ-deformed massless scalar field

We treat the two-level detector as an open quantum system and derive its dynamic evolution by tracing over all the degrees of freedom of the massless scalar field. Let us note that this model has been fruitfully applied in the relativistic scenario recently [43–45,58,69–73]. The total Hamiltonian of the detector-field system can be described as

$$H = H_s + H_{\Phi(x)} + H_I, \tag{17}$$

where $H_s = \frac{1}{2}\omega_0\sigma_z$ denotes the Hamiltonian of the detector, $H_{\Phi(x)}$ is the Hamiltonian of the κ-deformed massless scalar field, and H_I represents their interaction Hamiltonian. Note that ω_0 is the detector's energy-level spacing, and σ_z is the Pauli matrix. As considered in Ref. [28], the interaction Hamiltonian representing the interaction between the detector and scalar field up to first order in the parameter $1/\kappa$ can be described by the conventional Hermitian interaction Hamiltonian

$$H_I(\tau) = \mu m(\tau)\Phi(x(\tau)) = \mu[\sigma_+(\tau) + \sigma_-(\tau)]\Phi(x(\tau)), \tag{18}$$

where μ is the coupling constant, $m(\tau)$ is the monopole matrix of the detector whose spacetime coordinates are given by $x(\tau)$ with τ being the proper time, σ_+ (σ_-) is the atomic rising (lowering) operator, and $\Phi(x)$ corresponds to the scalar field operator. Notice that since the κ-deformed Klein–Gordon theory was constructed completely in the commutative space-time, this allowed us to guarantee the Hermiticity of the interaction Hamiltonian without modification. Moreover, due to the fact that the first nonvanishing $1/\kappa$ dependent modification to κ-deformed Klein–Gordon theory is in the second order in $1/\kappa$, the interaction Hamiltonian up to first order in the parameter $1/\kappa$ does not receive any modification.

As in the commutative spacetime, the field is assumed to be in Minkowski vacuum state $|0\rangle$, which is defined by $a_{\mathbf{k}}|0\rangle = 0$ for all $\mathbf{k}$. The initial state of the whole system can be written as the density matrix $\rho_{tot} = \rho(0) \otimes |0\rangle\langle 0|$, in which $\rho(0)$ is the reduced density matrix of the detector.

For the total system, its equation of motion in the interaction picture is

$$\frac{\partial \rho_{tot}(\tau)}{\partial \tau} = -i[H_I(\tau), \rho_{tot}(\tau)], \tag{19}$$

where τ is the proper time of the detector. In the limit of weak coupling between the detector and the field, the master function of the detector, by tracing over the field degrees of freedom, can be written in the Kossakowski-Lindblad form [74–76]

$$\frac{\partial \rho(\tau)}{\partial \tau} = -i[H_{eff}, \rho(\tau)] + \mathcal{L}[\rho(\tau)], \tag{20}$$

where

$$H_{eff} = \frac{1}{2}\Omega\sigma_z = \frac{1}{2}\{\omega_0 + \mu^2 \text{Im}(\Gamma_+ + \Gamma_-)\}\sigma_z, \tag{21}$$

is the effective Hamiltonian in which $\mu^2 \text{Im}(\Gamma_+ + \Gamma_-)$ can be neglected because it is far less than ω_0, and

$$\mathcal{L}[\rho(\tau)] = \sum_{j=1}^{3}[2L_j\rho L_j^\dagger - L_j^\dagger L_j\rho - \rho L_j^\dagger L_j], \tag{22}$$

is the dissipator resulting from the scalar field that the detector couples to. Here, $\Gamma_\pm = \int_0^\infty e^{i\omega_0\triangle\tau}G^+(s \pm i\epsilon)d\triangle\tau$, $L_1 = \sqrt{\frac{\gamma_-}{2}}\sigma_-$, $L_2 = \sqrt{\frac{\gamma_+}{2}}\sigma_+$, $L_3 = \sqrt{\frac{\gamma_z}{2}}\sigma_z$, $\gamma_z = 0$, $\gamma_\pm = \mu^2 \int_{-\infty}^{\infty} e^{\mp i\omega_0\triangle\tau}G^+(\triangle\tau - i\epsilon)d\triangle\tau$ and $\triangle\tau = \tau - \tau'$. It was shown in Ref. [28] that the correlation function $G^+(x-x') = \langle 0|\Phi(x)\Phi(x')|0\rangle = \frac{1}{4\pi^2(\triangle x^2 - \triangle t^2)}\left[1 - \frac{1}{4\kappa^2}\frac{\triangle x^2 + 3\triangle t^2}{(\triangle x^2 - \triangle t^2)^2} - \frac{1}{\kappa^2}\frac{(\triangle x^2 + \triangle t^2)\triangle t^2}{(\triangle x^2 - \triangle t^2)^3}\right]$ corresponding to the massless κ-deformed Klein–Gordon operator in Eq. (15) terms up to second order in $1/\kappa$, which satisfies $G^+(x-x') = G^-(-(x-x'))$, where $\triangle x = |x - x'|$ and $\triangle t = t - t'$. However, if one includes higher order terms in parameter $1/\kappa$ this may not be true, as the exact correlation function should exhibit the violation of Lorentz invariance due to the κ deformation.

If we choose the initial state of the detector as $|\psi(0)\rangle = \sin\frac{\theta}{2}|0\rangle + e^{-i\phi}\cos\frac{\theta}{2}|1\rangle$, substituting Eq. (3) into (20), the Bloch vector with respect to the proper time τ can be expressed as

$$\omega_1 = \sin\theta\cos(\Omega\tau + \phi)e^{-\frac{1}{2}A\tau},$$
$$\omega_2 = \sin\theta\sin(\Omega\tau + \phi)e^{-\frac{1}{2}A\tau},$$
$$\omega_3 = \cos\theta e^{-A\tau} + \frac{B}{A}(1 - e^{-A\tau}), \tag{23}$$

where $A = \gamma_+ + \gamma_-$ and $B = \gamma_+ - \gamma_-$.

Let us note that A, and B are related to the field correlation function that depends on the property of spacetime. As a consequence of the interaction between the detector and field, the information of κ-deformation of spacetime could

be encoded into the quantum state of the detector. By performing the measurements on the detector, we can infer the κ-deformation parameter through the relevant outcomes.

3 The detector of uniform motion

To analyze how the relativistic motion of the detector affects the estimation of the deformation parameter κ, we consider a relativistic motion detector whose spacetime coordinates are given by

$$t(\tau) = \gamma\tau , \quad x(\tau) = x_0 + v\gamma\tau , \quad y(\tau) = y_0,$$
$$z(\tau) = z_0 , \tag{24}$$

where τ denotes the proper time of the detector, v is the velocity of the detector and $\gamma = 1/\sqrt{1-v^2}$ is the usual Lorentz factor. According to the κ-deformed massless scalar field correlation function and (24), γ_+ and γ_- should be

$$\gamma_+ = 0 , \quad \gamma_- = \frac{\mu^2\omega_0}{2\pi}\left[1 - \frac{\omega_0^2}{24\kappa^2} - \frac{\omega_0^2}{3\kappa^2}\frac{v^2}{(1-v^2)^2}\right]. \tag{25}$$

As a result, we have

$$A = \gamma_+ + \gamma_- = \frac{\mu^2\omega_0}{2\pi}\left[1 - \frac{\omega_0^2}{24\kappa^2} - \frac{\omega_0^2}{3\kappa^2}\frac{v^2}{(1-v^2)^2}\right],$$
$$B = \gamma_+ - \gamma_- = -\frac{\mu^2\omega_0}{2\pi}\left[1 - \frac{\omega_0^2}{24\kappa^2} - \frac{\omega_0^2}{3\kappa^2}\frac{v^2}{(1-v^2)^2}\right]. \tag{26}$$

For simplicity, in this section we will work with dimensionless quantities by rescaling time τ and deformation parameter κ

$$\tilde{\tau} \equiv \gamma_0\tau , \quad \tilde{\kappa} \equiv \frac{\kappa}{\omega_0} , \tag{27}$$

where $\gamma_0 = \frac{\mu^2\omega_0}{2\pi}$ is the spontaneous emission rate of the detector.

In generally, an estimation scheme can be performed in three steps: (i) the preparation of a probe state, (ii) its interaction with the dynamic system in which the estimated parameters are involved, (iii) measure the probe after the evolution [77]. In our system, the state of two-level atom which is coupled to the κ-deformed massless scalar field acts as a probe state. For different positive operator-valued measure (POVM) Π_x, the different values of Fisher information can be obtained in terms of the classical probabilities $p(x \mid \lambda)$. The aim of this paper is finding the highest precision for the estimation, which means that we should carry out the optimization over measurement processes, i.e., maximizing the FI over all possible quantum measurements on the quantum system to obtain the QFI. According to the Eq. (4), we can evaluate the QFI of the deformation parameters as follows
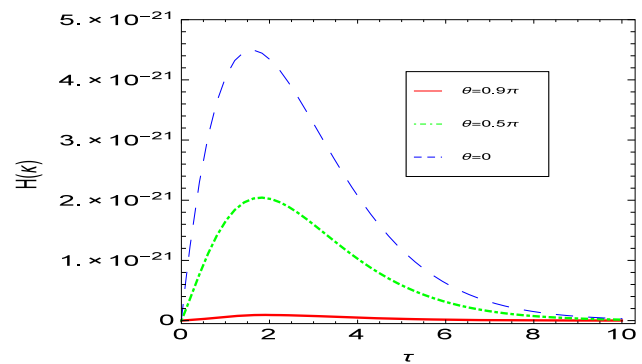

Fig. 1 QFI in the estimation of the deformation parameter as a function of the effective time τ with fixed values of θ. The effective deformation parameter is fixed as $\kappa = 1000$ and the velocity of detector is given by $v = 0$. The different initial state parameter $\theta = 0, 0.5\pi, 0.9\pi$, respectively

$$H(\tilde{\kappa}) = \frac{M[1 + 8\frac{v^2}{(1-v^2)^2}]^2[(1-\cos\theta)M - 2]\tilde{\tau}^2\cos\frac{\theta}{2}}{288(M-1)\tilde{\kappa}^6} , \tag{28}$$

where $M = \text{Exp}[[1 + 8\frac{v^2}{(1-v^2)^2} - 24\tilde{\kappa}^2]\tilde{\tau}/24\tilde{\kappa}^2]$. It is interesting to note that the QFI is independent of quantum phase ϕ, and only depends on the parameter $\theta, v, \tilde{\tau}$. In the following, we want to find out the optimal initial detector preparation and whether the relativistic motion of the detector can enhance the precision in the estimation of deformation parameter. Hereafter, for the sake of simplicity we term $\tilde{\tau}$ and $\tilde{\kappa}$ as τ and κ respectively.

The static case. The corresponding behavior of the QFI is shown in Figs. 1 and 2, which is the function of the effective time τ (initial state parameter θ) with different initial state parameter θ (effective time τ) and the velocity of the detector $v = 0$, i.e., the Lorentz factor $\gamma = 1$. It is shown that for $\theta = 0$ the QFI is larger than the other cases all the time. In addition, from Fig. 1 we see that the QFI always obtain the maximum value when the detector evolves for a limited time under the same conditions. Thus, we arrive at the conclusion that the initial exited state of the detector is the optimal state, i.e., $\theta = 0$, and the maximum sensitivity in the predictions for the deformation parameter κ can be obtained when the detector evolves for a limited time.

Moreover, the ultimate bound on the precision of the estimator is determined by the QFI, but it is hard to obtain the optimal measurement to achieve the ultimate bound because the QFI is independent on any measurement. However, we can obtain it by maximizing the FI over all possible quantum measurements on the quantum system. Therefore, to find out which the measurement is optimal to estimate the deformation parameter κ, our task is to calculate the FI for the population measurement and compare it with the QFI to determine whether the population measurement is optimal according to the condition that POVM with a FI equal to the QFI. Thus,
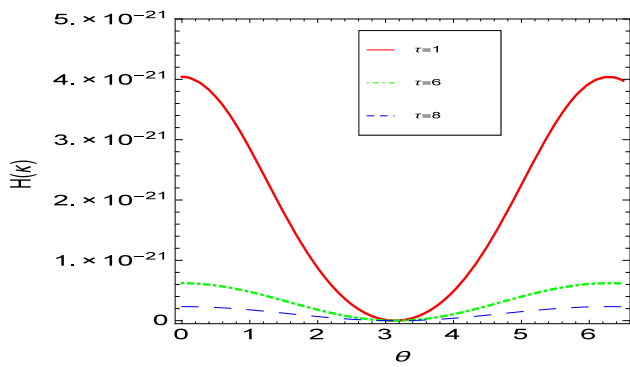
Fig. 2 QFI in the estimation of the deformation parameter as a function of the initial state parameter θ with fixed effective time τ. The effective deformation parameter is fixed as $\kappa = 1000$ and the velocity of detector is given by $v = 0$. The different effective time $\tau = 1$, 6, 8, respectively

Fig. 3 QFI in the estimation of the deformation parameter as a function of the effective time τ with $v = 0.9992$, 0.9994, respectively. By preparing the detector initially in the exited state $\theta = 0$ and the effective deformation parameter $\kappa = 1000$

according to Eq. (23), we can calculate the FI for the population measurement as follow

$$F(\kappa) = \frac{(\partial_\kappa \rho_{11})^2}{\rho_{11}} + \frac{(\partial_\kappa \rho_{22})^2}{\rho_{22}}$$

$$= \frac{e^{\frac{\tau[1+8(1-v^2)^{-2}]}{24\kappa^2}}[1 + 8\frac{v^2}{(1-v^2)^2}]^2 \tau^2 (1 + \cos\theta)}{114\kappa^6[2e^{\tau(1+\frac{1}{3\kappa^2(1-v^2)})} - e^{\frac{\tau[1+8(1-v^2)^{-2}]}{24\kappa^2}}(1 + \cos\theta)]},$$

$$(29)$$

where ρ_{11} and ρ_{22} are the diagonal elements of the quantum state of the detector respectively. Note that for $\theta = 0$, the FI is larger than the other cases, which means that the initial exited state of the detector is the optimal probe state. Therefore, the corresponding FI can be evaluated as

$$F(\kappa) = \frac{(\partial_\kappa \lambda_+)^2}{\lambda_+} + \frac{(\partial_\kappa \lambda_-)^2}{\lambda_-}, \qquad (30)$$

where

$$\lambda_+ = \frac{1}{2}\left[1 + e^{-A\tau} + \frac{A}{B}(1 - e^{-A\tau})\right],$$

$$\lambda_- = \frac{1}{2}\left[1 - e^{-A\tau} - \frac{A}{B}(1 - e^{-A\tau})\right], \qquad (31)$$

are two eigenvalues of the evolution state of the probe in Eq. (23). It is interesting to note that this choice gives $F(\kappa)$ (FI) equal to $H(\kappa)$ (QFI) in Eq. (28). From Fig. 1, we find that both the FI and QFI obtain the maximum value, when the static detector ($v = 0$) evolves for a limited time and is prepared in the exited state initially. Because the FI of any measurement process is upper bounded by the QFI, it means that the quantum estimation process is the optimization over measurement processes now. We easily find that the maximized FI is equal to the maximized QFI, which means that the estimation of parameter κ via the population measurement is optimal and the above is the ultimate bound to precision of estimation of the deformation parameter κ.

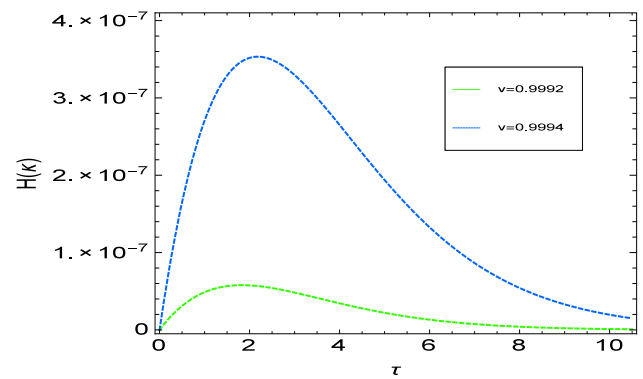*The motion case.* Let us consider that the detector is in uniform motion with constant velocity v and assume that the effective deformation parameter has the value $\kappa = 1000$. It is similar to the case of the static detector, we also can find out the optimal measurement to achieve the ultimate bound on the precision of the estimator for the case of the uniform motion detector. With the help of the Eq. (28), we obtain that for any time the maximal QFI is always obtained by taking $\theta = 0$, i.e., by preparing the detector in the exited state. It means that the optimal initial state of the detector is the exited state. Similarly, we find that for a limited time both the FI and QFI take the same maximum value when the detector in its exited state. Therefore, it is natural to achieve the ultimate bound to the precision by performing a population measurement on the uniform motion detector.

To analyze the relativistic motion of the detector how to effect the precision in the estimation of deformation parameter κ, we plot the QFI for different fixed velocity v of the detector as a function of the effective time τ in Fig. 3. We find that the higher the velocity, the bigger the QFI is, i.e., it is easier to achieve a given precision in the estimation of deformation parameter and the order of magnitude of the QFI is as high as 10^{-7}. As a result, we argue that the relativistic motion can effectively improve the QFI, which can exceed the QFI for the static detector case by the orders of magnitude 10^{14}. Therefore, we emphasize that the precision in the estimation of the deformation parameter κ can be effectively improved by the relativity motion of the detector.

4 The detector of uniform circular motion

In this section, we want to explore the case for the detector follows a uniform circular path with centripetal acceleration $a = \frac{\gamma^2 v^2}{R}$, whose coordinates are described as
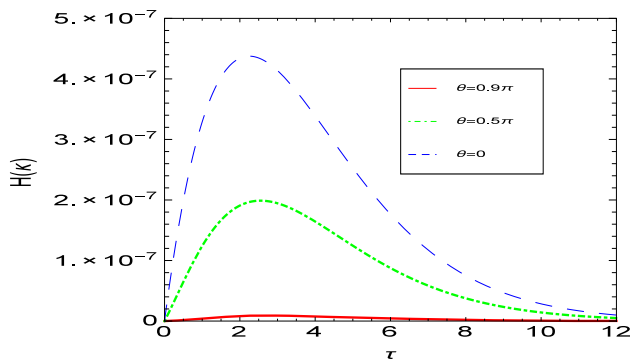
Fig. 4 QFI in the estimation of the deformation parameter as a function of the effective time τ with fixed values of θ. The effective deformation parameter is fixed as $\kappa = 1000$, the velocity of detector is given by $v = 0.9994$ and the effective centripetal acceleration $a = 0.2$. The different initial state parameter $\theta = 0,\ 0.5\pi,\ 0.9\pi$, respectively

$$t(\tau) = \gamma\tau\ , \quad x(\tau) = R\cos\frac{\gamma v\tau}{R}\ ,$$
$$y(\tau) = R\sin\frac{\gamma v\tau}{R}\ , \quad z(\tau) = z_0\ , \tag{32}$$

where R denotes the radius of the orbit. Applying the trajectory of the uniform circular detector to the κ-deformed massless scalar field correlation function, we easily obtain the γ_+ and γ_- under the ultra-relativistic limit $\gamma \gg 1$ [78,79]. Therefore, we can calculate A and B for the uniform circular motion case straightly.

We evaluate the performance of the quantum measurement in the estimation of the deformation parameter through the calculation of the QFI for the uniform circular detector. Similarly, we will work with dimensionless quantities by rescaling time τ, centripetal acceleration a and deformation parameter κ

$$\tilde{\tau} \equiv \gamma_0\tau\ , \quad \tilde{a} \equiv \frac{a}{\omega_0}\ , \quad \tilde{\kappa} \equiv \frac{\kappa}{\omega_0}\ . \tag{33}$$

With the help of Eqs. (4), (23) and (32), it is easy to obtain the formula of the QFI and to find that the QFI only depends on the parameters θ, v, $\tilde{\tau}$, and $\tilde{a}$, but is independent of quantum phase ϕ.

In the following, we also continue to replace τ, a and κ with $\tilde{\tau}$, $\tilde{a}$ and $\tilde{\kappa}$, respectively. We fix the velocity of detector $v = 0.9994$ and the effective deformation parameter κ by assuming $\kappa = 1000$. To clarify what value of θ could allow better estimation, we plot the QFI as a function of the effective time τ (initial state parameter θ) with different initial state parameter θ (effective time τ) in Figs. 4 and 5. Here, we take the effective centripetal acceleration $a = 0.2$. Obviously, as we demonstrated in the uniform motion detector, the maximal value of the QFI are obtained for the uniform circular detector in its exited state initially, i.e., $\theta = 0$, and the QFI always achieves the maximum when the detector evolves for a limited time whatever the initial state is prepared in. Therefore, we arrive a conclusion that the maximum sensitivity
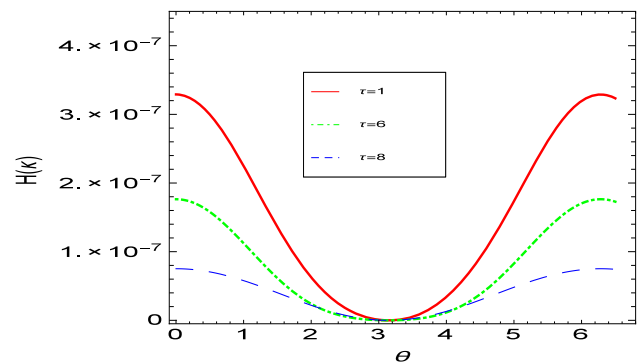

Fig. 5 QFI in the estimation of the deformation parameter as a function of the initial state parameter θ with fixed effective time τ. The effective deformation parameter is fixed as $\kappa = 1000$, the velocity of detector is given by $v = 0.9994$ and the effective centripetal acceleration is $a = 0.2$. The different effective time $\tau = 1,\ 6,\ 8$, respectively
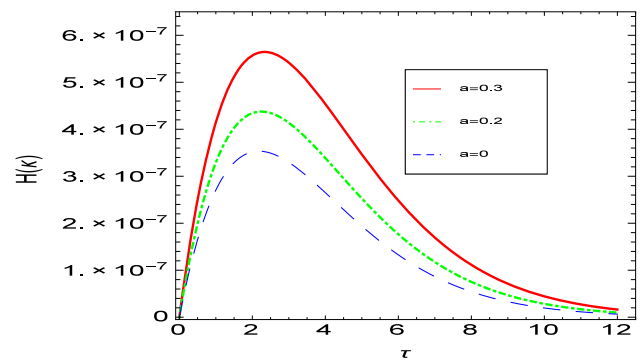

Fig. 6 QFI in the estimation of the deformation parameter as a function of the effective time τ with the effective centripetal acceleration a. The initial state parameter is fixed as $\theta = 0$, the effective deformation parameter is given by $\kappa = 1000$, and the velocity of detector is $v = 0.9994$. The different effective centripetal acceleration $a = 0,\ 0.2$,respectively

in the predictions for the deformation parameter κ can be obtained by initially preparing the detector in its exited state, which means that the exited state is the best probe state.

In Fig. 6, we plot the QFI of the probe state in the Eq. (23) as functions of the effective time τ with different effective centripetal acceleration a at fixed $v = 0.9994$ for $\theta = 0$. We find that the QFI always increase as the growth of the centripetal acceleration a, which indicates that the highest precision in the estimation of κ-deformation parameter can be obtained for a larger centripetal acceleration. It is worthy noting that the QFI of κ-deformation parameter in the uniform circular case is always larger than that in the uniform motion case from Fig. 6. Thus, we argue that relativistic motion will enhance the quantum estimation of κ-deformation of spacetime, which means that a high centripetal acceleration of the detector can provide us a better precision.

Furthermore, as the analysis for static case, we also calculate the FI for the population measurement in the case of the uniform circular detector and find that the maximal FI can

be obtained by taking $\theta = 0$. Moreover, the FI has the same configurations with the QFI when we prepare the detector in its exited state, i.e., $\theta = 0$. And we can find that both the FI and QFI not only achieve the maximum but also are equal to each other when the detector evolves for a limited time. It indicates that the estimation of deformation parameter κ via the population measurement is optimal and we can get the ultimate bound to precision of estimation. Due to the population measurement is optimal and the population of estimation measurement is allowed by the current technology [80–86], the ultimate bound to precision of estimation of the deformation parameter κ can be achieved by quantum mechanics in the capability of current technology for the optimal population measurement.

5 Conclusions

We have focused on explore the κ-deformation of spacetime using a two-level detector coupled to the κ-deformed massless scalar field, and with the quantum metrology technologies address the quantum bound to the estimability of the κ-deformation by performing measurements on the two-level detector. In particular, by preparing the proper probe state and adjusting the interaction parameters, we can obtain the optimal strategy for the estimation of deformation parameter in the estimation process.

We found that the initial probe state preparation and the motion of the detector have great effects on the precision in the estimation of κ-deformed parameter. Both the cases of the uniform motion detector and the uniform circular detector, when the initial state is prepared in its exited state, the value of QFI which corresponds to the ultimate limit of the precision in the estimation of the deformation parameter κ is always maximum. Under the conditions that when the detector is prepared in the initial exited state and evolves for a limited time, the maximum FI is equal to the maximum QFI, which indicates that the optimal measurement for the estimation of the deformation parameter κ corresponding to the population measurement. To be specific, the ultimate bound to the precision in the estimation of the deformation parameter κ can be allowed by performing a population measurement on the detector and can be achieved under the current technology imposed by quantum mechanics during the estimation process. It is worthy noting that our results demonstrated that the precision in the estimation of the parameter κ can be influenced by the relativistic motion of the detector, which effectively enhance the precision comparing with the static detector case by many orders of magnitude.

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