Eur. Phys. J. C (2018) 78:612
https://doi.org/10.1140/epjc/s10052-018-6095-3

THE EUROPEAN PHYSICAL JOURNAL C

Regular Article - Theoretical Physics

A note on thin-shell wormholes with charge in $F(R)$-gravity

S. Habib Mazharimousavi[a]

Department of Physics, Faculty of Arts and Sciences, Eastern Mediterranean University, Via Mersin 10, Famagusta, North Cyprus, Turkey

Received: 5 February 2018 / Accepted: 23 July 2018 / Published online: 31 July 2018

Abstract In their recent work (Eiroa and Aguirre in Eur Phys J C 76:132, 2016), Eiroa and Aguirre introduced thin-shell wormholes in $F(R) = R + \alpha R^2$-gravity coupled with the Maxwell electromagnetic field. Here in this note we shall address an interesting feature of their results which has been missed. It will be shown that thin-shell wormhole can not be formed in the black hole spacetime solution of this theory but instead there are rooms for making stable thin-shell wormholes in non-black hole bulk spacetime as was noted in Eiroa and Aguirre (2016). This study is not a comment on very correct results of Eiroa and Aguirre (2016) but instead it is a complementary result to their paper.

1 Introduction

Thin-shell wormhole in modified theory of gravity, namely $F(R)$-gravity, seems to be more restrictive than its former version in R-gravity due to the modified junction conditions introduced in [2]. In [1], Eiroa and Aguirre constructed thin-shell wormholes in $F(R) = R + \alpha R^2$-gravity coupled with the Maxwell's electrodynamic field in the framework of constant curvature i.e., $R = R_0 = const.$ spherically symmetric bulk spacetime. Apparently their stability analysis results in stable thin-shell wormhole against a radial perturbation. Due to the modified junction conditions, one should consider additional constraint on the radius of the throat which is assumed to be larger than the radius of the event horizon. In other words, unlike R-gravity where the standard Israel junction conditions [3–5] are applicable and there is no restriction on choosing the radius of the throat except it has to be larger than the possible event horizon, in $F(R)$ -gravity it has to satisfy Eq. (12) as well. This is because of the continuity of the trace of the extrinsic curvature in $F(R)$-gravity [2] i.e., $[K_i^i] = 0$. We shall show that upon this condition thin-shell wormhole does not exist in the black hole spacetime solution of $R + \alpha R^2$-Maxwell theory of gravity.

2 Thin-shell wormhole in $F(R) = R + \alpha R^2$-Maxwell gravity

As it is chosen in [1] the action of the bulk spacetime with constant curvature is given by

$$S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} \left(F(R) - F_{\mu\nu} F^{\mu\nu} \right) \tag{1}$$

in which

$$F(R) = R + \alpha R^2 \tag{2}$$

and

$$\mathbf{F} = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu \tag{3}$$

is the Maxwell's electromagnetic field. With constant extrinsic curvature $R = R_0$ the solution for the $F(R)$-Maxwell's field equations in spherically symmetric spacetime is found to be [6,7]

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -A(r) dt^2 + \frac{dr^2}{A(r)} + r^2 d\Omega^2 \tag{4}$$

where

$$A(r) = 1 - \frac{2M}{r} + \frac{Q^2}{F'(R_0) r^2} - \frac{R_0 r^2}{12}. \tag{5}$$

Applying the standard method of cut and paste one constructs thin-shell wormhole whose throat is located at $r = a(\tau)$ in which τ is the proper time on the throat. Hence, the extrinsic curvature tensor on the sides of the throat are found to be

[a] e-mail: habib.mazhari@emu.edu.tr

$$K_j^{i\pm} = \pm diag\left(\frac{A'(a)+2\ddot{a}}{2\sqrt{A(a)+\dot{a}^2}}, \frac{1}{a}\sqrt{A(a)+\dot{a}^2}, \frac{1}{a}\sqrt{A(a)+\dot{a}^2}\right) \tag{6}$$

in which a prime and a dot stand for the derivative with respect to r and τ respectively. As it was introduced in [2] and properly applied in [1] the following junction conditions have to be satisfied. First the metric tensor, the Ricci scalar and the trace of the extrinsic curvature should be continuous across the shell i.e., $[h_{ij}] = 0$, $[R] = 0$ and $[K] = 0$ respectively. Second, due to the identical constant curvature bulks in both sides of the thin-shell, the jump of the extrinsic curvature tensor gives the surface energy-momentum tensor as

$$\kappa S_i^j = -F'(R_0)\left[K_i^j\right] \tag{7}$$

where

$$S_i^j = diag(-\sigma, p, p) \tag{8}$$

in which σ is the energy density and p is the angular pressure. $[K] = 0$ implies

$$\frac{A'(a)+2\ddot{a}}{2\sqrt{A(a)+\dot{a}^2}} + \frac{2}{a}\sqrt{A(a)+\dot{a}^2} = 0 \tag{9}$$

and (7) yields to

$$\sigma = \frac{F'\left(2\ddot{a}+A'\right)}{\kappa\sqrt{A+\dot{a}^2}}, \tag{10}$$

and

$$p = -\frac{2F'}{\kappa a}\sqrt{A+\dot{a}^2}. \tag{11}$$

Introducing the equilibrium radius $a = a_0$ [1], where $\dot{a} = \ddot{a} = 0$ one finds from (9) that a_0 has to satisfy

$$a_0 A'(a_0) + 4A(a_0) = 0 \tag{12}$$

which is the additional constraint on the radius of the thin-shell wormhole and the main concern of this note. Also (10) and (11) give

$$\sigma_0 = \frac{F'A'(a_0)}{\kappa\sqrt{A(a_0)}}, \tag{13}$$

and

$$p_0 = -\frac{2F'}{\kappa a_0}\sqrt{A(a_0)}. \tag{14}$$

Furthermore, the trace of Eq. (7) implies that the trace of S_i^j vanishes i.e., $S_i^i = 0$ which gives directly the equation of state $p = \frac{\sigma}{2}$ for the perfect fluid presented on the shell. The stability analysis of the thin-shell wormhole ends up to a one-dimensional equation of motion for the throat given by

$$\dot{a}^2 + V(a) = 0 \tag{15}$$

in which

$$V(a) = A(a) - \frac{a_0^4}{a^4}A(a_0) \tag{16}$$

such that upon taking the Eq. (12) into account $V(a_0) = V'(a_0) = 0$ while

$$V''(a_0) = A''(a_0) - \frac{20}{a_0^2}A(a_0). \tag{17}$$

Hence, at the radius of the equilibrium $a = a_0$, satisfying (12), if $V''(a_0) > 0$ the thin-shell wormhole is stable against the radial perturbation. In Fig. 2 of [1] it is clearly shown that the solid curve outside the shaded region for each individual case is the stable region.

3 The new observation

Let's recall that $a = a_0$ is the equilibrium radius of the thin-shell wormhole satisfying two critical conditions: (i) $a_0 > r_h$, in which r_h is the possible event horizon of the bulk spacetime and (ii) a_0 should satisfy Eq. (12). Now in this section we show that any possible event horizon is larger than a_0 and therefore the stable thin-shell is not possible for the black hole spacetime solution to the $R + \alpha R^2$-Maxwell gravity. This however, does not contradict the results in [1] because in Fig. 2 of [1] the bulk does not need to be a black hole. We note that from Fig. 1 in [1], $|Q| < Q_c$ corresponds to an inner and an event horizon for the bulk while $|Q| > Q_c$ presents a naked singularity for the bulk spacetime. At $|Q| = Q_c$ the two horizons coincide.

First let's look at the cases mentioned in [1] more closely as is shown in Fig. 1. In this figure the region where $V''(a_0) > 0$ is shown (as shaded region) in terms of the horizontal axes $x = \frac{a_0}{M}$ and vertical axes $y = \frac{|Q|}{M\sqrt{F'}}$, together with other two curves. The dot-dashed (blue) curve represents the points on the plane of xy which satisfy Eq. (12) which implicitly gives the radius of the throat (up to a coefficient $\frac{1}{M}$). Finally the long dashed (brown) curve stands for the point of the xy plane satisfying $A(r) = 0$ where the horizontal axes $x = \frac{r_h}{M}$ and therefore this curves implicitly reveals the event and Cauchy horizons of the spacetime. One observes that in all cases for the region where the bulk spacetime is a black hole the curve of a_0 lies under the curve of r_h showing that $a_0 < r_h$. A portion of a_0 curve remains inside the shaded

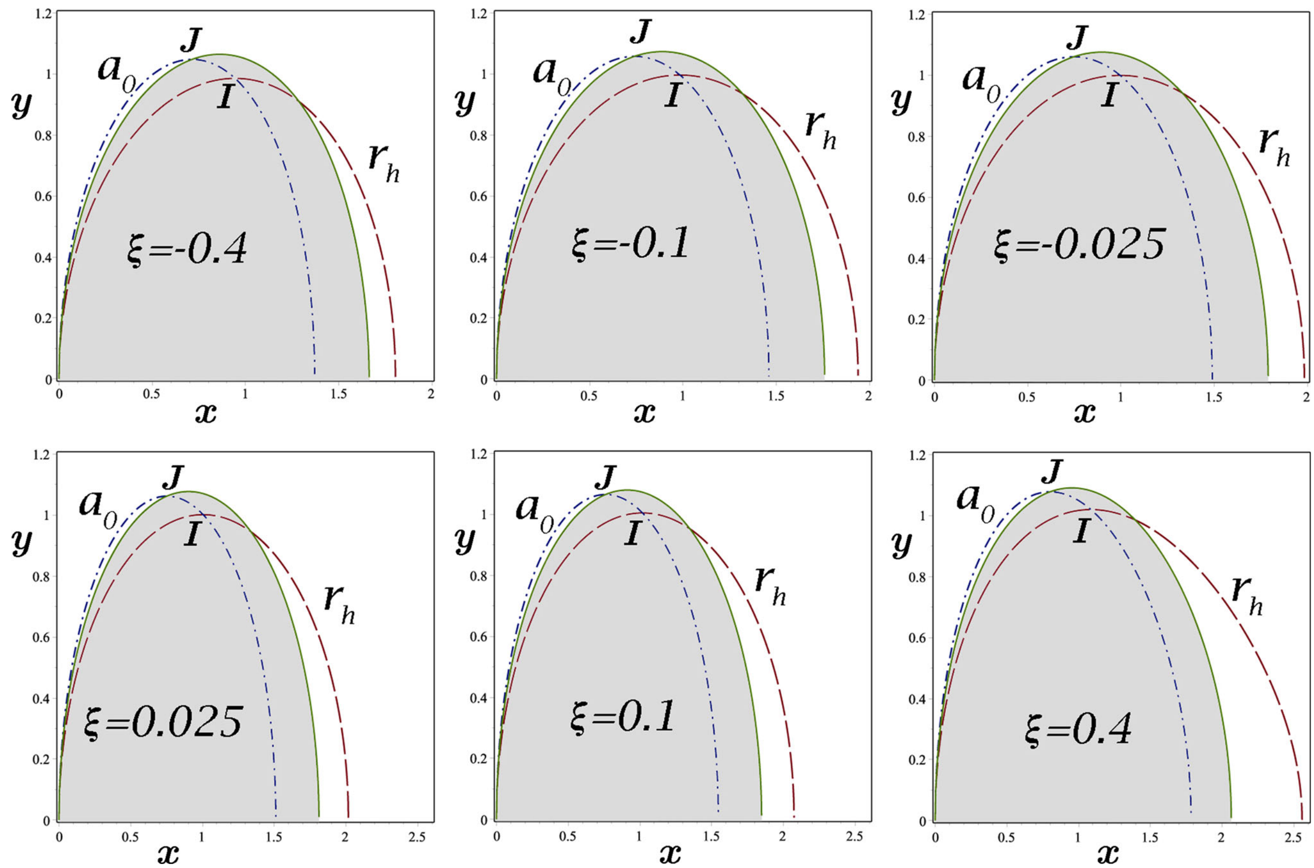

Fig. 1 A plot of $V''(a_0) > 0$ versus $x = \frac{a_0}{M}$ and $y = \frac{|Q|}{M\sqrt{F'}}$ which is shown as the shaded area. Also $\frac{a_0}{M}$ (blue dot-dashed curve) and $\frac{r_h}{M}$ (brown long-dashed curve) are plotted (horizontal axis x) versus $y = \frac{|Q|}{M\sqrt{F'}}$ satisfying $a_0 A'(a_0) + 4A(a_0) = 0$ and $A(r_h) = 0$ respectively. The value of $\xi = R_0 M^2$ are given on each individual plot. The points I and J marked on the curves are the intersections of the curves q_a vs x_a with q_h vs x_h and q_v vs x_a respectively. Definitions of these quantities are given in Sect. 3

region where the spacetime is not black hole is the stable thin-shell wormhole reported in [1].

To complete our note let's find above observation analytically. The two equations i.e., $A(r_h) = 0$, and $a_0 A'(a_0) + 4A(a_0) = 0$ after change of variable as $x_a = \frac{a_0}{M}$, $x_h = \frac{r_h}{M}$ and $q = \frac{|Q|}{M\sqrt{F'}}$ become

$$1 - \frac{\xi}{12}x_h^2 - \frac{2}{x_h} + \frac{q^2}{x_h^2} = 0 \tag{18}$$

and

$$4 - \frac{\xi}{2}x_a^2 - \frac{6}{x_a} + \frac{2q^2}{x_a^2} = 0, \tag{19}$$

respectively, in which $\xi = R_0 M^2$. Solving both equations (18) and (19) for q, reveals

$$q_h = \sqrt{\frac{x_h\left(-12x_h + 24 + \xi x_h^3\right)}{12}} \tag{20}$$

and

$$q_a = \sqrt{\frac{x_a\left(-8x_a + 12 + \xi x_a^3\right)}{4}} \tag{21}$$

in which a sub h/a stands for the horizon/throat and refers to the solutions of the Eq. (18)/(19). In Fig. 1 the brown long-dashed and the blue dot-dashed curves are $q_h(= y)$ versus $x_h(= x)$ and $q_a(= y)$ versus $x_a(= x)$, respectively, for different values of ξ. Next we find the intersection point between two curves. From Fig. 1 we see that there are two points of intersections between two curves, one at the origin which is trivially seen from (20) and (21) and the second point is the maximum point of the curve q_h versus x_h i.e., point I shown on the curve, which is not trivial. Here we show that irrespective of the value of ξ the second intersection point is actually the maximum of q_h. To do so we find the extremum/maximum of q_h by finding its first derivative and equate it to zero i.e., $\frac{dq_h}{dx_h} = 0$ which gives

$$\frac{1}{3}\xi\tilde{x}_h^3 - 2\tilde{x}_h + 2 = 0 \tag{22}$$

in which $\tilde{x}_h$ is the horizontal location of the extremum point of q_h. Solving (22) for ξ and inserting it in (20) one finds the extremum of $q_h = \tilde{q}_h$ which is given by

$$\tilde{q}_h = \sqrt{\frac{\tilde{x}_h\left(3 - \tilde{x}_h\right)}{2}}. \tag{23}$$

Next, we find the value of q_a at $x_a = \tilde{x}_h$ with the same value of ξ as

$$q_a\left(x_a = \tilde{x}_h\right) = \sqrt{\frac{\tilde{x}_h\left(3 - \tilde{x}_h\right)}{2}} \tag{24}$$

which is equal to $\tilde{q}_h$. This is the end of the proof. Hence we see that only at the location of the extremal black hole when the event horizon and the Cauchy horizon coincide (point I) the value of a_0 can be equal to the radius of the horizon while for any other black hole case $r_c < a_0 < r_h$.

A similar calculation shows that, the boundary curve of the region $V''(a_0) > 0$ which is given by $V''(a_0) = 0$ reduces to

$$\frac{3\xi}{2} - \frac{20}{x_a^2} + \frac{36}{x_a^3} - \frac{14q^2}{x_a^4} = 0. \tag{25}$$

The solution of (25) for q is found to be

$$q_v = \sqrt{\frac{x_a\left(3\xi x_a^3 - 40x_a + 72\right)}{28}} \tag{26}$$

in which a sub v refers to the solution of Eq. (25). In Fig. 1 q_v versus x_a is shown with Green-Solid curve. This curve intersects the curve of q_a versus x_a at the origin and its extremum point J. The proof is similar to the case of point I which we have worked out earlier. Hence one concludes that the possible stable thin-shell wormhole is located on the curve of q_a versus x_a between the two points I and J marked on Fig. 1.

4 Conclusion

In this note the results of the recent work of Eiroa and Aguirre on stability of thin-shell wormhole in $F(R) = R + \alpha R^2$-Maxwell theory of gravity have been reconsidered. It was shown that for the black hole bulk solution in this theory there is no possible stable thin-shell wormhole. No need to mention that, the non-black hole solution to the $R + \alpha R^2$-Maxwell theory with constant curvature is naked singular and the radius of the throat of the thin-shell wormhole is located to the left of the local minimum of $A(r)$ where $A'(r) = 0$. Also to keep the bulk spacetime a non-black hole solution one must consider $|Q| > Q_c$ in which Q_c is a minimum value for the charge (see Fig. 1 in Ref. [1]). Restriction on $|Q|$ affects the physical properties of the constructed TSW, for instance the amount of the exotic matter. Therefore this study is not trivial. Once more we would like to add that our results are in agreement with the original work of Eiroa and Aguirre [1]. As our final remark we would like to look at the condition (12) once more. In this equation $A(a_0) \neq 0$ and therefore $A'(a_0) \neq 0$, in other words the point of the throat and the point where derivative of $A(r)$ is zero do not coincide. In [8] it is shown that wormhole solutions with this property are asymmetric wormhole. In our case although the condition used in [8] is satisfied but our original bulk metric is not a wormhole solution. Hence, the thin-shell wormhole considered in this study remains symmetric thin-shell wormhole.

Funded by SCOAP^3.

References

1. E.F. Eiroa, G.F. Aguirre, Eur. Phys. J. C **76**, 132 (2016)
2. J.M.M. Senovilla, Phys. Rev. D **88**, 064015 (2012)
3. G. Darmois, *Mèmorial des Sciences Màthematiques, Fascicule XXV*, Chap. V (Gauthier-Villars, Paris, 1927)
4. W. Israel, Nuovo Cim. B **44**, 1 (1966)
5. W. Israel, Nuovo Cim. B **48**, 463(E) (1967)
6. A. de la Cruz-Dombriz, A. Dobado, A.L. Maroto, Phys. Rev. D **80**, 124011 (2009)
7. A. de la Cruz-Dombriz, A. Dobado, A.L. Maroto, Phys. Rev. D **83(E)**, 029903 (2011)
8. S. Bahamonde, D. Benisty, E.I. Guendelman (2018). arXiv:1801.08334