

The T-dual symmetries of a bosonic string

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Abstract We investigate whether the symmetry transformations of a bosonic string are connected by T-duality. We start with a standard closed string theory. We continue with a modified open string theory, modified to preserve the symmetry transformations possessed by the closed string theory. Because the string theory is conformally invariant world-sheet field theory, in order to find the transformations which preserve the physics, one has to demand the isomorphism between the conformal field theories corresponding to the initial and the transformed field configurations. We find the symmetry transformations corresponding to the similarity transformation of the energy-momentum tensor, and find that their generators are T-dual. Particularly, we find that the general coordinate and local gauge transformations are T-dual, so we conclude that T-duality in addition to the well-known exchanges, transforms symmetries of the initial and its T-dual theory into each other.

1 Introduction

One of the most important notions in theoretical physics is a symmetry. What is a symmetry of the string theory is not yet clear mainly because the theory itself is not yet formulated in a background independent way, which would enlighten its deeper principles. However, it is believed that the symmetry does exist, and that it will lead to finding the physically indistinguishable solutions to the string equations of motion and choosing the correct vacuum [1,2].

String theory revealed a T-duality, a symmetry which is a consequence of the string's extended nature. T-duality connects seemingly different string theories by exchanging, for example, the characteristics of the strings momentum and winding [3–6]. So, it should exchange the symmetries of

string theories as well. If that holds, one can say that symmetries of string theories always appear in pairs.

In this paper, we will investigate symmetry of the space-time in which the bosonic string moves, using a world-sheet formulation [1]. The formalism differs from the usual, where the symmetry is a transformation of the space-time fields which leave the classical action invariant. This concept of a symmetry does not apply now, because only the world-sheet values of the space-time fields appear in the string action. Still, a symmetry should be a change in the space-time fields which does not change the physics. Thus, suppose one considers a string theory with some chosen space-time field configuration, and a string theory with a modified space-time configuration. How does one determine if these two descriptions are physically equivalent? The string theory is a conformally invariant world-sheet field theory. The physics is determined by the conformal field theory, corresponding to the field configuration in question. The transformation on the fields will be a symmetry if the corresponding conformal field theories are isomorphic [1,2].

If one is given a conformal field theory, one will obtain a physically identical conformal field theory by performing a similarity transformation on the operators of the initial conformal field theory

$$\hat{O} \rightarrow e^{-i\hat{T}} \hat{O} e^{i\hat{T}}.$$

This transformation does not change the algebraic properties, so the new theory will be physically the same as the initial theory. However, the transformation will in general make changes to the world-sheet energy-momentum tensor. If these changes can be interpreted as changes in the space-time fields, then the latter are the symmetry transformations of the target space.

This idea was introduced in [1,7], where this automorphism of the operator algebra was seen as an analog of the change of variables in a partition function. The problem of finding symmetries was reduced to the problem of finding the operator generating the symmetry transformation. The first

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investigations were treating the string massless fields, but the method was then generalized to treat conformal deformations of conformal field theory [8, 9].

The classical analog of a similarity transformation is a transformation of a variable of interest by a Poisson bracket between a generator and a variable. In this paper, we will apply these transformations to the string sigma model energy-momentum tensor. We will start by considering a small modification of the space-time background $\Pi_{\pm\mu\nu} \rightarrow \Pi_{\pm\mu\nu} + \delta\Pi_{\pm\mu\nu}$. It will induce the transformation of the energy-momentum tensor. We will demand that the new energy-momentum tensor satisfies the Virasoro algebra as well. This means that the transformed theory is physically equivalent to the initial theory, or that these field transformations are the symmetry of space-time theory. In this way we will find a transformation of space-time fields corresponding to a similarity transformation and the generator of this symmetry.

We will consider both closed and open string theory. For the open string we will consider a modified action which different from the closed string action has an additional surface term which enables the invariance of the complete action to the general coordinate transformations and the gauge transformations, which are the symmetries of the closed string. For the open string theory, the boundary conditions can be satisfied by choosing either the Neumann or the Dirichlet boundary condition for every coordinate direction. If the choice is made, the modified action surface term is given in terms of the corresponding Neumann and the Dirichlet gauge fields. The closed string symmetries remain the symmetries of the open string theory taking the appropriate transformation of these gauge fields.

If one includes T-duality into the consideration, one can conclude that the general coordinate transformations and the local gauge transformations are not independent. Comparing their generators, using the T-dual coordinate transformation laws, one concludes that they are T-dual. Therefore, the symmetries are T-dual and the complete generator of symmetries is self-dual.

2 The bosonic string essentials

The quantization of the bosonic string theory, describing the string moving in a background consisting of a space-time metric $G_{\mu\nu}$, a Kalb–Ramond field $B_{\mu\nu}$ and a dilaton field Φ , leads to the conclusion that in order to have a conformal invariance on the quantum level the energy-momentum tensor components $\hat{T}_{\pm}(\varphi)$, with $\varphi = (G_{\mu\nu}, B_{\mu\nu}, \Phi)$, have to obey the Virasoro algebras [2, 10, 11],

$$\left[\hat{T}_{\pm}(\varphi(\sigma)), \hat{T}_{\pm}(\varphi(\bar{\sigma})) \right]$$

$$= i\hbar \left[\hat{T}_{\pm}(\varphi(\sigma)) + \hat{T}_{\pm}(\varphi(\bar{\sigma})) \right] \delta'(\sigma - \bar{\sigma}),$$

$$\left[\hat{T}_{\pm}(\varphi(\sigma)), \hat{T}_{\mp}(\varphi(\bar{\sigma})) \right] = 0. \tag{1}$$

From these conditions follow the space-time equations of motion which space-time fields $G_{\mu\nu}, B_{\mu\nu}, \Phi$ have to obey. In order to obtain the symmetries of the space-time equations of motion, one does not need to find their explicit form. It is sufficient to consider the transformations which do not change the above relations.

2.1 The conformal gauge

The action which was quantized [12] for a constant dilaton field reads

$$S[x] = \kappa \int_{\Sigma} d^2\xi \sqrt{-g} \left[\frac{1}{2} g^{\alpha\beta} G_{\mu\nu}(x) + \frac{\epsilon^{\alpha\beta}}{\sqrt{-g}} B_{\mu\nu}(x) \right] \partial_{\alpha} x^{\mu} \partial_{\beta} x^{\nu}, \quad (\epsilon^{01} = -1), \tag{2}$$

where the integration goes over a two-dimensional world sheet Σ with coordinates ξ^{α} ($\xi^0 = \tau, \xi^1 = \sigma$). $g_{\alpha\beta}$ is the intrinsic world-sheet metric and $x^{\mu}(\xi)$, $\mu = 0, 1, \dots, D-1$ are the coordinates of the D-dimensional space-time and $\kappa = \frac{1}{2\pi\alpha'}$.

Taking a conformal gauge $g_{\alpha\beta} = e^{2F} \eta_{\alpha\beta}$, the action becomes

$$S[x] = \kappa \int_{\Sigma} d^2\xi \partial_{+} x^{\mu} \Pi_{+\mu\nu}(x) \partial_{-} x^{\nu}, \tag{3}$$

with

$$\Pi_{\pm\mu\nu}(x) = B_{\mu\nu}(x) \pm \frac{1}{2} G_{\mu\nu}(x), \tag{4}$$

given in terms of the light-cone coordinates $\xi^{\pm} = \frac{1}{2}(\tau \pm \sigma)$, $\partial_{\pm} = \partial_0 \pm \partial_1$.

The momentum corresponding to x^{μ} is

$$\pi_{\mu} = \frac{\partial \mathcal{L}}{\partial \dot{x}^{\mu}} = \kappa G_{\mu\nu}(x) \dot{x}^{\nu} - 2\kappa B_{\mu\nu}(x) x'^{\nu}, \tag{5}$$

and therefore the canonical Hamiltonian for the theory (3) equals

$$\mathcal{H}_c = \frac{1}{4\kappa} (G^{-1})^{\mu\nu} \left[j_{+\mu} j_{+\nu} + j_{-\mu} j_{-\nu} \right], \tag{6}$$

where the currents $j_{\pm\mu}$ are given by

$$j_{\pm\mu} = \pi_{\mu} + 2\kappa \Pi_{\pm\mu\nu}(x) x'^{\nu}. \tag{7}$$

One can rewrite the Hamiltonian in terms of the energy-momentum tensor components

$$T_{\pm} = \mp \frac{1}{4\kappa} (G^{-1})^{\mu\nu} j_{\pm\mu} j_{\pm\nu}, \tag{8}$$

as

$$\mathcal{H}_C = T_- - T_+. \tag{9}$$

2.2 The gauge invariant approach

Let us in this subsection consider a string theory without a gauge fixing. If one takes the following parametrization of the world-sheet metric tensor $g_{\alpha\beta}$ [13]:

$$g_{\alpha\beta} = e^{2F} \hat{g}_{\alpha\beta} = \frac{1}{2} e^{2F} \begin{pmatrix} -2h^- h^+ & h^- + h^+ \\ h^- + h^+ & -2 \end{pmatrix}, \tag{10}$$

with $h^- > h^+$, the action (2) becomes

$$S = 2\kappa \int_{\Sigma} d^2\xi \sqrt{-\hat{g}} \hat{\partial}_+ x^\mu \Pi_{+\mu\nu} \hat{\partial}_- x^\nu, \tag{11}$$

where $\Pi_{\pm\mu\nu}$ is defined by (4) and the partial derivative is given by

$$\hat{\partial}_\pm = \frac{\sqrt{2}}{h^- - h^+} (\partial_0 + h^\mp \partial_1). \tag{12}$$

Varying the action over x^μ , one obtains the equations of motion

$$\hat{\nabla}_\pm \hat{\partial}_\mp x^\mu + \Gamma_{\mp\nu\rho}^\mu \partial_\pm x^\nu \partial_\mp x^\rho = 0, \tag{13}$$

with $\hat{\nabla}_\pm$ being the covariant derivative [13], defined by

$$\hat{\nabla}_\pm x_n = (\hat{\partial}_\pm + n \hat{\omega}_\pm) x_n, \quad \omega_\pm = \mp \frac{\sqrt{2} h^\mp}{h^- - h^+}, \tag{14}$$

where x_n is a scalar, vector or tensor and n is the sum of its world-sheet indices, taking 1 for plus and -1 for minus. The generalized connection is defined by

$$\Gamma_{\pm\nu\rho}^\mu = \Gamma_{\nu\rho}^\mu \pm B_{\nu\rho}^\mu, \tag{15}$$

given in terms of the Christoffel symbol by

$$\Gamma_{\nu\rho}^\mu = \frac{1}{2} (G^{-1})^{\mu\sigma} (\partial_\nu G_{\rho\sigma} + \partial_\rho G_{\sigma\nu} - \partial_\sigma G_{\nu\rho}),$$

and the field strength of the field $B_{\mu\nu}$,

$$B_{\nu\rho}^\mu = (G^{-1})^{\mu\sigma} B_{\sigma\nu\rho} = (G^{-1})^{\mu\sigma} (\partial_\sigma B_{\nu\rho} + \partial_\nu B_{\rho\sigma} + \partial_\rho B_{\sigma\nu}).$$

The momentum corresponding to x^μ is

$$\pi_\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = \frac{\kappa G_{\mu\nu}(x)}{h^- - h^+} (2\dot{x}^\nu + (h^+ + h^-) x'^\nu) - 2\kappa B_{\mu\nu}(x) x'^\nu. \tag{16}$$

One can extract \dot{x}^μ from the last equation, to obtain

$$\dot{x}^\mu = \frac{(G^{-1})^{\mu\nu}}{2\kappa} (h^- (\pi_\nu + 2\kappa \Pi_{-\nu\rho} x'^\rho))$$

$$-h^+ (\pi_\nu + 2\kappa \Pi_{+\nu\rho} x'^\rho)). \tag{17}$$

Using the currents (7), the coordinate derivatives over world-sheet parameters become

$$\dot{x}^\mu = \frac{(G^{-1})^{\mu\nu}}{2\kappa} (h^- j_{-\nu} - h^+ j_{+\nu}), \tag{18}$$

$$x'^\mu = \frac{(G^{-1})^{\mu\nu}}{2\kappa} (j_{+\nu} - j_{-\nu}). \tag{19}$$

The canonical Hamiltonian density $\mathcal{H}_C = \pi_\mu \dot{x}^\mu - \mathcal{L}$ is

$$\mathcal{H}_C = -h^- T_+ - h^+ T_-, \tag{20}$$

with T_\pm defined in (8). For $h^\pm = \mp 1$, one returns to the conformal gauge.

3 The symmetries of space-time

In this section, we will search for the symmetries of the space-time in which the closed and the open strings propagate, and for their generators. We will investigate the change of the world-sheet energy-momentum tensor caused by the change in the space-time fields. We will demand that the transformed energy-momentum tensor $T_\pm + \delta T_\pm$ still obeys the classical analog of the Virasoro algebra (1).

The energy-momentum tensor components T_\pm satisfy two independent copies of the Virasoro algebra. To find a symmetry of the equations of motion, one should conclude what kind of transformation of fields $\varphi \rightarrow \varphi + \delta\varphi$, and consequently of the energy-momentum tensor

$$\begin{aligned} \hat{T}_\pm(\varphi + \delta\varphi) &= \hat{T}_\pm(\varphi) + \delta\hat{T}_\pm(\varphi), \\ \delta\hat{T}_\pm(\varphi) &= {}_G\hat{T}_\pm^{\mu\nu} \delta G_{\mu\nu} + {}_B\hat{T}_\pm^{\mu\nu} \delta B_{\mu\nu} + \phi \hat{T}_\pm \delta\Phi, \end{aligned} \tag{21}$$

conserves the Virasoro algebra. One does not need to know the explicit form of the space-time equations of motion to find its symmetry transformations. In order to have a conserved Virasoro algebra, one should find transformations for which the following conditions are fulfilled:

$$\begin{aligned} & \left[\hat{T}_\pm(\varphi(\sigma)), \delta\hat{T}_\pm(\varphi(\bar{\sigma})) \right] + \left[\delta\hat{T}_\pm(\varphi(\sigma)), \hat{T}_\pm(\varphi(\bar{\sigma})) \right] \\ &= i\hbar \left[\delta\hat{T}_\pm(\varphi(\sigma)) + \delta\hat{T}_\pm(\varphi(\bar{\sigma})) \right] \delta'(\sigma - \bar{\sigma}), \\ & \left[\delta\hat{T}_\pm(\varphi(\sigma)), \hat{T}_\mp(\varphi(\bar{\sigma})) \right] + \left[\hat{T}_\pm(\varphi(\sigma)), \delta\hat{T}_\mp(\varphi(\bar{\sigma})) \right] = 0. \end{aligned} \tag{22}$$

It is known [14] that a similarity transformation applied to \hat{T}_\pm ,

$$\hat{T}_\pm \rightarrow e^{-i\hat{G}} \hat{T}_\pm e^{i\hat{G}},$$

ensures the physical equivalence of the corresponding theories, and it makes the change in \hat{T}_\pm , which corresponds to a change in the space-time fields, without changing the

physics. This kind of change in the space-time fields is therefore a symmetry transformation. The similarity transformation implies that the change of \hat{T}_\pm is just

$$\delta\hat{T}_\pm(\varphi) = -i\left[\hat{T}, \hat{T}(\varphi)\right]. \tag{23}$$

One can confirm that the last relation solves the conditions for the Virasoro algebra conservation (22).

In the subsequent sections, we will be interested in finding the change in the space-time fields, which transform T_\pm in a way which preserves the classical version of the Virasoro algebra. We will search for a generator Γ_Λ (where Λ is some parameter) such that its Poisson bracket with energy-momentum components T_\pm produces the variation $\delta T_\pm = \{\Gamma, T_\pm\}$, equal to the change of energy-momentum tensor caused by the variation of fields $\varphi \rightarrow \varphi + \delta\varphi$. If such a generator exists, then the previous variation is a symmetry transformation of the space-time.

3.1 T-duality of the closed string symmetry generators

Our goal in this and the subsequent sections is to find the generators of the general symmetry transformations corresponding to the similarity transformation. Thus, let us suppose the background fields undergo a small change in value $\Pi_{\pm\mu\nu} \rightarrow \Pi_{\pm\mu\nu} + \delta\Pi_{\pm\mu\nu}$. Let us find the generators of the symmetries Γ , for this transformation of the background fields. The currents change by

$$\delta j_{\pm\mu} = 2\kappa\delta\Pi_{\pm\mu\nu}(x)x'^\nu, \tag{24}$$

and therefore

$$\delta T_\pm = \frac{1}{2\kappa}\delta\Pi_{\pm\mu\nu}j_\pm^\mu j_\mp^\nu. \tag{25}$$

Let us determine the algebra of the currents (7). Using the standard Poisson brackets between the coordinates and the momenta

$$\{x^\mu(\sigma), \pi_\nu(\bar{\sigma})\} = \delta_\nu^\mu\delta(\sigma - \bar{\sigma}), \tag{26}$$

one obtains

$$\begin{aligned} \{j_{\pm\mu}(\sigma), j_{\pm\nu}(\bar{\sigma})\} &= \\ &\pm 2\kappa\Gamma_{\mp\mu, \nu\rho}x'^\rho(\sigma)\delta(\sigma - \bar{\sigma}) \\ &\pm 2\kappa G_{\mu\nu}(x(\sigma))\delta'(\sigma - \bar{\sigma}), \\ \{j_{\pm\mu}(\sigma), j_{\mp\nu}(\bar{\sigma})\} &= \\ &\pm 2\kappa\Gamma_{\mp\rho, \mu\nu}x'^\rho(\sigma)\delta(\sigma - \bar{\sigma}), \end{aligned} \tag{27}$$

where the generalized connection is defined by (15). Consequently, the Poisson brackets between T_\pm , defined by (8),

and currents are

$$\begin{aligned} \{T_\pm(\sigma), j_{\pm\mu}(\bar{\sigma})\} &= \pm\frac{1}{2\kappa}\Gamma_{\mp\nu, \mu\rho}j_\pm^\nu j_\mp^\rho\delta(\sigma - \bar{\sigma}) - j_{\pm\mu}(\sigma)\delta'(\sigma - \bar{\sigma}), \\ \{T_\pm(\sigma), j_{\mp\mu}(\bar{\sigma})\} &= \pm\frac{1}{2\kappa}\Gamma_{\mp\rho, \nu\mu}j_\pm^\nu j_\mp^\rho\delta(\sigma - \bar{\sigma}). \end{aligned} \tag{28}$$

Finally, we obtain the Virasoro algebra

$$\begin{aligned} \{T_\pm(\sigma), T_\pm(\bar{\sigma})\} &= -\left[T_\pm(\sigma) + T_\pm(\bar{\sigma})\right]\delta'(\sigma - \bar{\sigma}), \\ \{T_\pm(\sigma), T_\mp(\bar{\sigma})\} &= 0, \end{aligned} \tag{29}$$

in agreement with the condition (1).

We will suppose the generator of the symmetries in the following form:

$$\mathcal{G} = \mathcal{G}_+ + \mathcal{G}_-, \quad \mathcal{G}_\pm = \int d\sigma \Lambda_\pm^\mu(x(\sigma))j_{\pm\mu}(\sigma). \tag{30}$$

Using (28), one obtains the Poisson brackets between T_\pm and the generators,

$$\begin{aligned} \{T_\pm(\sigma), \mathcal{G}_\pm(\bar{\sigma})\} &= \pm\frac{1}{2\kappa}\left[\Gamma_{\mp\nu\rho}^\mu\Lambda_\pm^\rho + \partial_\nu\Lambda_\pm^\mu\right]j_{\pm\mu}j_\mp^\nu, \\ \{T_\pm(\sigma), \mathcal{G}_\mp(\bar{\sigma})\} &= \pm\frac{1}{2\kappa}\left[\Gamma_{\mp\mu\rho}^\nu\Lambda_\mp^\rho + \partial_\mu\Lambda_\mp^\nu\right]j_\pm^\mu j_{\mp\nu}. \end{aligned} \tag{31}$$

If one defines the generalized covariant derivatives by

$$D_{\pm\mu}\Lambda^\nu = \partial_\mu\Lambda^\nu + \Gamma_{\pm\rho\mu}^\nu\Lambda^\rho = D_\mu\Lambda^\nu \pm B_{\rho\mu}^\nu\Lambda^\rho, \tag{32}$$

one rewrites (31) as

$$\begin{aligned} \{T_\pm(\sigma), \mathcal{G}_\pm(\bar{\sigma})\} &= \pm\frac{1}{2\kappa}\left(D_{\mp\nu}\Lambda_\pm^\mu\right)j_{\pm\mu}j_\mp^\nu, \\ \{T_\pm(\sigma), \mathcal{G}_\mp(\bar{\sigma})\} &= \pm\frac{1}{2\kappa}\left(D_{\pm\mu}\Lambda_\mp^\nu\right)j_\pm^\mu j_{\mp\nu}. \end{aligned} \tag{33}$$

We know that T_\pm transforms as (25), therefore we search for a generator $\mathcal{G} = \mathcal{G}_+ + \mathcal{G}_-$ such that

$$\delta T_\pm = \{\mathcal{G}, T_\pm\} = \frac{1}{2\kappa}\delta\Pi_{\pm\mu\nu}j_\pm^\mu j_\mp^\nu, \tag{34}$$

which implies

$$\delta\Pi_{\pm\mu\nu} = \mp\left(D_{\mp\nu}\Lambda_{\pm\mu} + D_{\pm\mu}\Lambda_{\mp\nu}\right). \tag{35}$$

Taking

$$\Lambda_{\pm\mu} = \xi_\mu \pm \Lambda_\mu, \tag{36}$$

one obtains

$$\begin{aligned} \delta G_{\mu\nu} &= -2(D_\mu\xi_\nu + D_\nu\xi_\mu), \\ \delta B_{\mu\nu} &= D_\mu\Lambda_\nu - D_\nu\Lambda_\mu - 2B_{\mu\nu}^\rho\xi_\rho. \end{aligned} \tag{37}$$

Using the currents (7) and the gauge parameter (36), we rewrite the generator \mathcal{G} as

$$\mathcal{G} = \int d\sigma \left[2\xi^\mu \pi_\mu + 2\kappa(2\xi^\mu B_{\mu\nu} + \Lambda^\mu G_{\mu\nu})x'^\nu \right]. \tag{38}$$

To simplify the last expression, one can define another gauge parameter

$$\tilde{\Lambda}_\nu = 2\xi^\mu B_{\mu\nu} + \Lambda^\mu G_{\mu\nu} = \Lambda_\nu - 2B_{\nu\mu}\xi^\mu, \tag{39}$$

so that

$$\mathcal{G} = 2 \int d\sigma \left[\xi^\mu \pi_\mu + \tilde{\Lambda}_\mu \kappa x'^\mu \right]. \tag{40}$$

In terms of the new parameter the Kalb–Ramond field transforms as

$$\delta B_{\mu\nu} = D_\mu \tilde{\Lambda}_\nu - D_\nu \tilde{\Lambda}_\mu + 2 \left[D_\mu (B_{\nu\rho}\xi^\rho) - D_\nu (B_{\mu\rho}\xi^\rho) - \xi^\rho B_{\rho\mu\nu} \right], \tag{41}$$

and the generator (40) is rewritten as

$$\mathcal{G} = \mathcal{G}_\xi + \mathcal{G}_{\tilde{\Lambda}}. \tag{42}$$

Therefore, the closed string described by (11) is invariant under the general coordinate transformations

$$\begin{aligned} \delta_\xi G_{\mu\nu} &= -2(D_\mu \xi_\nu + D_\nu \xi_\mu), \\ \delta_\xi B_{\mu\nu} &= -2\xi^\rho B_{\rho\mu\nu} + 2(\partial_\mu b_\nu - \partial_\nu b_\mu), \quad b_\mu = B_{\mu\nu}\xi^\nu, \end{aligned} \tag{43}$$

with $D_\mu \xi_\nu = \partial_\mu \xi_\nu - \Gamma_{\mu\nu}^\rho \xi_\rho$, and the local gauge transformations

$$\begin{aligned} \delta_\Lambda G_{\mu\nu} &= 0, \\ \delta_\Lambda B_{\mu\nu} &= \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu, \end{aligned} \tag{44}$$

where we omit the tilde on Λ_μ .

If one keeps in mind T-duality, the present form of the generator offers interesting conclusions. T-duality connects physically equivalent string sigma models, and the connection between T-dual string backgrounds and their variables is simplest in the constant background case. In that case, the well-known T-duality relation [15, 16]

$$\pi_\mu \cong \kappa x'^\mu$$

stands. So, T-duality interchanges the sigma derivative of the coordinates with the momenta. The consequence of the above relation for the generator of symmetry (42) is that its constituents turn out to be T-dual as well,

$$\mathcal{G}_\xi \cong \mathcal{G}_{\tilde{\Lambda}},$$

which makes the complete generator \mathcal{G} self-dual. Because of this, the local gauge transformations and the general coordinate transformations are T-dual also. The same relation, however, does not hold in more complicated backgrounds, when T-duality is performed along the nonisometry directions or in backgrounds without the global shift symmetry. These backgrounds were discussed in [17–19] where the generalized T-dualization procedure, applicable along an arbitrary space-time direction was presented and elaborated.

4 The open string and its symmetries

The open string described by the same action as the closed string is not invariant to the above symmetries. The change in the action caused by the general coordinate transformations is

$$\delta_\xi S = 2\sqrt{2}\kappa \int d\tau \xi^\nu \left[-h^- \Pi_{+\nu\mu} \hat{\partial}_- x^\mu + h^+ \Pi_{-\nu\mu} \hat{\partial}_+ x^\mu \right] \Big|_{\sigma=0}^{\sigma=\pi}, \tag{45}$$

for the equation of motion (13) and the change by the local gauge transformations is

$$\delta_\Lambda S = 2\kappa \int d\tau \Lambda_\mu \dot{x}^\mu \Big|_{\sigma=0}^{\sigma=\pi}. \tag{46}$$

The first expression can be rewritten as

$$\delta S = -2 \int d\tau \xi^\mu \gamma_\mu^{(0)} \Big|_{\sigma=0}^{\sigma=\pi},$$

where

$$\begin{aligned} \gamma_\mu^{(0)} &= -\sqrt{2}\kappa \left[-h^- \Pi_{+\nu\mu} \hat{\partial}_- x^\mu + h^+ \Pi_{-\nu\mu} \hat{\partial}_+ x^\mu \right] \\ &= \frac{\kappa G_{\mu\nu}(x)}{h^- - h^+} \left[(h^- + h^+) \dot{x}^\nu + 2h^- h^+ x'^\nu \right] \\ &\quad + 2\kappa B_{\mu\nu}(x) \dot{x}^\nu. \end{aligned} \tag{47}$$

The boundary conditions of the open string are given in terms of this variable,

$$\gamma_\mu^{(0)} \delta x^\mu \Big|_{\sigma=0}^{\sigma=\pi} = 0. \tag{48}$$

In Ref. [20] the way to gain invariance to the transformation (44) was shown, and in Refs. [21, 22] the open string action invariant under both (43) and (44) was presented, which different from the standard action has an additional surface term,

$$S_{\text{bon}} = 2 \int d\tau \left[\kappa A_\mu(x) \dot{x}^\mu - \bar{A}_\mu(x) (G^{-1})^{\mu\nu} \gamma_\nu^{(0)} \right] \Big|_{\sigma=0}^{\sigma=\pi}. \tag{49}$$

This term makes the open string theory invariant under both general coordinate and local gauge transformations, if the introduced vector fields A_μ and \bar{A}_μ transform as

$$\begin{aligned} \delta_\Lambda A_\mu &= -\Lambda_\mu, \\ \delta_\xi \bar{A}_\mu &= -\xi_\mu. \end{aligned} \tag{50}$$

For each of the coordinates one can fulfill the boundary conditions (48), by choosing either the Neumann or the Dirichlet boundary condition. If we mark the coordinates with the Neumann condition by x^a , $a = 0, 1, \dots, p$ and the coordinates with the Dirichlet condition by x^i , $i = p + 1, \dots, D - 1$, the surface term (49) reduces to

$$S_{\text{bon}} = 2 \int d\tau \left[\kappa A_a^N(x) \dot{x}^a - A_i^D(x) (G^{-1})^{ij} \gamma_j^{(0)} \right] \Big|_{\sigma=0}^{\sigma=\pi}, \tag{51}$$

where A_a^N and A_i^D are $(p + 1)$ - and $(D - p - 1)$ -dimensional vector gauge fields, the former living on the Dp -brane and the latter orthogonal to the Dp -brane. The Neumann vector field is as usual coupled to the coordinate time parameter derivative and the Dirichlet vector field is coupled to the variable $\gamma_\mu^{(0)}$ related to the boundary condition, depending on both world-sheet parameter derivatives of the coordinates.

4.1 Field strengths

It is well known that, in the bosonic string action, the surface term can be rewritten in the form of the Kalb–Ramond term. If all the boundary conditions are Neumann, then the action on the boundary,

$$S_{\text{bon}} = 2\kappa \int d\tau A_\mu^N(x) \dot{x}^\mu \Big|_{\sigma=0}^{\sigma=\pi}, \tag{52}$$

can be rewritten as

$$S_{\text{bon}} = \kappa \int d^2\xi \mathcal{F}_{\mu\nu}^N \varepsilon^{\alpha\beta} \partial_\alpha x^\mu \partial_\beta x^\nu, \tag{53}$$

with

$$\mathcal{F}_{\mu\nu}^N = \partial_\mu A_\nu^N(x) - \partial_\nu A_\mu^N(x). \tag{54}$$

For arbitrary choice of the boundary conditions the action on the boundary is given by (51). Let us restrict our investigation to the following metric:

$$G_{\mu\nu} = \begin{bmatrix} G_{ab} & 0 \\ 0 & G_{ij} \end{bmatrix}. \tag{55}$$

In that case, (51) can be rewritten using (48) as

$$S_{\text{bon}} = 2\kappa \int d\tau \left[\mathcal{A}_\mu^{(0)}(x) \dot{x}^\mu + \mathcal{A}_\mu^{(1)} x'^\mu \right] \Big|_{\sigma=0}^{\sigma=\pi}, \tag{56}$$

with

$$\begin{aligned} \mathcal{A}_a^{(0)} &= A_a^N, \\ \mathcal{A}_i^{(0)} &= -\frac{h^- + h^+}{h^- - h^+} A_i^D + 2(BG^{-1})_i^j A_j^D, \\ \mathcal{A}_a^{(1)} &= 0, \\ \mathcal{A}_i^{(1)} &= -2\frac{h^- h^+}{h^- - h^+} A_i^D. \end{aligned} \tag{57}$$

In terms of the field strengths, the surface term becomes

$$S_{\text{bon}} = 2\kappa \int_\Sigma d^2\xi \sqrt{-\hat{g}} \hat{\partial}_+ x^\mu \mathcal{F}_{\mu\nu} \hat{\partial}_- x^\nu, \tag{58}$$

$$\mathcal{F}_{\mu\nu} = \mathcal{F}_{\mu\nu}^{(a)} + \frac{1}{2} \mathcal{F}_{\mu\nu}^{(s)},$$

with

$$\begin{aligned} \mathcal{F}_{ab}^{(a)} &= \partial_a A_b^N - \partial_b A_a^N, \\ \mathcal{F}_{ij}^{(a)} &= -\frac{h^- + h^+}{h^- - h^+} (\partial_i A_j^D - \partial_j A_i^D) + 2\partial_i ((BG^{-1})^k A_k^D) \\ &\quad - 2\partial_j ((BG^{-1})^l A_l^D), \\ \mathcal{F}_{ab}^{(s)} &= 0, \\ \mathcal{F}_{ij}^{(s)} &= -\frac{2}{h^-} \partial_i A_j^{(1)} + \frac{2}{h^+} \partial_j A_i^{(1)}. \end{aligned} \tag{59}$$

The above calculations are done for an open string moving in constant background fields. If a coordinate dependent background is assumed, then the field strengths $\mathcal{F}_{ij}^{(s)}$ and $\mathcal{F}_{ij}^{(a)}$ will have additional terms, coming from $\frac{h^+ - h^-}{h^+ h^-} A_i^{(1)} \Gamma_{\mp\nu\rho}^i$.

Comparing the boundary actions (58) with the action (11), we conclude that the addition of the surface term has changed the background fields by

$$\begin{aligned} G_{\mu\nu} &\rightarrow G_{\mu\nu} + \mathcal{F}_{\mu\nu}^{(s)} \equiv \mathcal{G}_{\mu\nu}, \\ B_{\mu\nu} &\rightarrow B_{\mu\nu} + \mathcal{F}_{\mu\nu}^{(a)} \equiv \mathcal{B}_{\mu\nu}. \end{aligned} \tag{60}$$

4.2 The symmetry generators for the open string

The metric and the Kalb–Ramond field are changed in comparison with the closed string case to $G_{\mu\nu} \rightarrow \mathcal{G}_{\mu\nu} = G_{\mu\nu} + \mathcal{F}_{\mu\nu}^{(s)}$ and $B_{\mu\nu} \rightarrow \mathcal{B}_{\mu\nu} = B_{\mu\nu} + \mathcal{F}_{\mu\nu}^{(a)}$. Instead of the transformations (37), the open string symmetry transformations, for a theory with mixed boundary conditions, are

$$\begin{aligned} \delta \mathcal{G}_{\mu\nu} &= -2(\mathcal{D}_\mu \xi_\nu + \mathcal{D}_\nu \xi_\mu), \\ \delta \mathcal{B}_{\mu\nu} &= \mathcal{D}_\mu \Lambda_\nu - \mathcal{D}_\nu \Lambda_\mu - 2\mathcal{B}_{\mu\nu}^\rho \xi_\rho, \end{aligned} \tag{61}$$

where

$$\mathcal{D}_\mu \Lambda^\nu = \partial_\mu \Lambda^\nu + \Gamma_{\rho\mu}^\nu(\mathcal{G}) \Lambda^\rho$$

is a covariant derivative corresponding to the metric $\mathcal{G}_{\mu\nu}$. By $\mathcal{B}_{\mu\nu}^\rho$ we denoted the field strength of the Kalb–Ramond field. The field itself is changed; however, the field strength of the additional part is zero and therefore $\mathcal{B}_{\mu\nu\rho} = B_{\mu\nu\rho}$.

The generator of the transformation is

$$\mathcal{G} = \int d\sigma \left[2\xi^\mu \pi_\mu + 2\kappa(2\xi^\mu \mathcal{B}_{\mu\nu} + \Lambda^\mu \mathcal{G}_{\mu\nu})x'^\nu \right], \quad (62)$$

with

$$\pi_\mu = \kappa \mathcal{G}_{\mu\nu}(x) \dot{x}^\nu - 2\kappa \mathcal{B}_{\mu\nu}(x) x'^\nu. \quad (63)$$

Using the explicit form of the transformed metric, we obtain the connection

$$\Gamma_{\mu,\nu\rho}(\mathcal{G}) = \Gamma_{\mu,\nu\rho} - 2\partial_\nu \partial_\rho A_\mu^D. \quad (64)$$

If one chooses only Neumann boundary conditions, the metric remains the same as $\mathcal{F}_{\mu\nu}^{(s)} = 0$, so that only the Kalb–Ramond field changes, but not its field strength.

5 Conclusion

We considered the general coordinate and the local gauge transformations of the bosonic string, and we showed that they are T-dual to each other. We started with the bosonic string theory in a conformal gauge and in a gauge invariant form. One of the purposes of the latter was to find the separation of the Hamiltonian into two energy-momentum tensors satisfying two copies of the Virasoro algebras. These tensors are represented as products of currents, which are used for defining the generators of symmetries that were investigated. The generators were defined as the integrals over the spatial world-sheet parameter of the weighted currents.

Given the form of the generator, we investigated how it affects the variables, whose transformation is defined by a Poisson bracket between the generator of symmetry and the corresponding variable. We were interested in finding the explicit form of the change in the energy-momentum tensor caused by such a transformation. This form of transformation is used as a classical analog of the quantum transformation, known to preserve the Virasoro algebra between the energy-momentum tensor components. So, both the initial and the transformed energy-momentum tensor describe the same physics. Because of that, we were interested in finding the small variations of background fields such that the change in energy-momentum tensor they cause is exactly the considered transformation. In fact, if there exists a generator such that the described equality is possible, then the obtained transformations of space-time fields represent the symmetry of the theory.

We considered the standard closed string theory, and a modified open string theory which in comparison to the standard open string theory has an additional surface term. This term was chosen in such a way as to cancel the obstacle for

the closed string symmetries to be the open string symmetries as well. Introduction of the Neumann vector fields on the boundary is equivalent to a change of Kalb–Ramond field. In the case considered here, however, the vector fields are not coupled only to the time derivative of the coordinates but also to the functions which define the boundary conditions on the string endpoints. Therefore, both metric and Kalb–Ramond field are changed by the surface term.

We found the transformations of the background fields, which transform the energy-momentum tensor in such a way that the Virasoro algebra is preserved. It turned out that the generators of the symmetries (40) can be separated in such a way that one part of the generator, linear in x'^μ , represents the generator of the well-known local gauge transformation of the Kalb–Ramond field and the other part, proportional to π_μ , generates the general coordinate transformations. Since these quantities are T-dual $\kappa x'^\mu \cong \pi_\mu$, we conclude that these symmetries are T-dual. Thus, we showed that T-duality has an additional feature: it interchanges the symmetries of a theory. The generator of general coordinate transformations and local gauge transformations of the initial theory is T-dual to the generator of local gauge transformations and the general coordinate transformations of the T-dual theory.

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