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Multiplicative renormalization of Yang–Mills theories in the background-field formalism

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Abstract   In the paper, within the background field method, the structure of renormalizations is studied as for Yang–Mills fields interacting with a multiplet of spinor fields. By extending the Faddeev–Popov action with extra fields and parameters, one is allowed to establish the multiplicative character of the renormalizability. The renormalization of the physical parameters is shown to be gauge-independent.

1 Introduction

When quantizing non-Abelian gauge field theories [1], whose gauge transformations form a group, one is naturally based on the Faddeev–Popov method [2]. It is a characteristic property of the Faddeev–Popov gauge-fixed action that the latter is invariant under global BRST supersymmetry [3,4], which, in turn, can be expressed in the form of the Zinn-Justin equation [5] for the Faddeev–Popov action. At the quantum level, the BRST symmetry as expressed in terms of the effective action, implies the Slavnov–Taylor identities [6,7] to hold. Further generalization as to the quantization of gauge theories, including the cases of field-dependent structure coefficients, as well as open and/or reducible gauge algebra, is described by the field–antifield BV formalism [8,9]. In that formalism, the effective action is BRST invariant by construction, and thus satisfies the master equation which provides for the gauge invariance of the physical sector of the theory [8,9].

An interest to the gauge dependence problem did appear from the study of the effective potential, which appeared to be gauge-dependent in Yang–Mills theories with spontaneous breaking of the symmetry, when calculating physically-

sensible results (the energy of the ground state, the masses of the physical particles, and so on) [10,11]. In Refs. [12,13] it was established that the energy of the ground state was gauge-independent. Later, it was proved [14,15] that in Yang–Mills theories the dependence of gauge parameters in the effective action could be described in terms of gauge-invariant functional whose arguments (fields) were gauge-dependent (see also recent Refs. [16,17] devoted to that problem as resolved via the procedure of redefinition of the field variables, found in [14,15]). Notice that in the general case of gauge theories, a variation in gauge condition is described in the form of certain change of the field variables (in terms of anticanonical transformations) [18,19].

Although there are many papers devoted to various aspects of renormalizability of Yang–Mills theories, gauge dependence of renormalization constants has been studied explicitly only as for the gauge field sector [20]. In the present paper, within the background field formalism, it is studied a multiplicative renormalization procedure and gauge dependence as for Yang–Mills fields interacting with a multiplet of spinor fields. It is shown that renormalizations of physical parameters of the theory are gauge-independent.

The paper is organized as follows. In Sect. 2, it is discussed the action of Yang–Mills fields and spinor fields in the standard approach and in the background field method; it is also introduced extended action, which leads in the background field method to a multiplicative renormalizable theory of the fields considered; it is also studied the symmetry of the extended action. In Sect. 3, it is established the structure and the arbitrariness is described as for any local functional with the quantum numbers of the extended action that satisfies the same set of equations as the extended action. In Sect. 4, the equations are derived for the generating functional of vertexes (effective action), as a consequence at the quantum level, of the symmetry property of the extended action; and it is shown that the generating functional of vertexes satis-

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fies the same equations as the extended classical action. In Sect. 5, it is studied the renormalization procedure of the theory considered when using the loop expansion technique and the minimal subtraction scheme; and thus the multiplicative renormalizability of the theory is proved. In Sect. 6, the relations are found between the parameters of the renormalized action and the standard renormalization constants of fields and vertexes of the interaction, and renormalized physical parameters are shown to be gauge-independent. Concluding remarks are given in Sect. 7.

Condensed DeWitt's notations [21] are used through the paper. Functional derivatives with respect to field variables are understood as the left. Right derivatives of a quantity f with respect to the variable φ are denoted as $f \overset{\leftarrow}{\frac{\delta}{\delta\varphi}}$.

2 Extended action for Yang–Mills theories

Let us consider a gauge theory of non-abelian vector fields $A_\mu^\alpha = A_\mu^\alpha(x)$ and spinor fields $\psi_j = \psi_j(x), \overline{\psi}_j = \overline{\psi}_j(x)$ in the $D = 4$ Minkowski space-time with the action

$$\mathcal{S}_{YM}(A, \Psi) = \int dx \left(-\frac{1}{4} G_{\mu\nu}^\alpha(A) G_{\mu\nu}^\alpha(A) \right.$$
$$\left. + i \overline{\psi}_j \gamma^\mu D_{\psi\mu jk}(A) \psi_k - m \overline{\psi}_j \psi_j \right), \quad (2.1)$$

where the notations

$$G_{\mu\nu}^\alpha(A) = \partial_\mu A_\nu^\alpha - \partial_\nu A_\mu^\alpha + g f^{\alpha\beta\gamma} A_\mu^\beta A_\nu^\gamma,$$
$$D_{\psi\mu jk}(A) = \partial_\mu \delta_{jk} + g t_{jk}^\alpha A_\mu^\alpha, \quad \Psi = \{\psi, \overline{\psi}\} \quad (2.2)$$

are used. In Eqs. (2.1), (2.2) $f^{\alpha\beta\gamma}$ are structure coefficients of a simple compact gauge Lie group, $t^\alpha = \{t_{jk}^\alpha\}$ are generators of gauge transformations in sector of spinor fields satisfying the properties,

$$[t^\alpha, t^\beta] = f^{\alpha\beta\gamma} t^\gamma, \quad (t^\gamma)^+ = -t^\gamma, \quad [\gamma^\mu, t^\alpha] = 0. \quad (2.3)$$

Here γ^μ are the Dirac matrices, g and m are the coupling constant of gauge interaction and the mass parameter of spinor field, respectively. The action (2.1) is invariant under gauge transformations with gauge parameters $\omega_\alpha = \omega_\alpha(x)$, $\delta_\omega \mathcal{S}_{YM}(A, \Psi) = 0$,

$$\delta_\omega A_\mu^\alpha = \left(\partial_\mu \delta_{\alpha\beta} + g f^{\alpha\sigma\beta} A_\mu^\sigma \right) \omega_\beta = D_\mu^{\alpha\beta}(A) \omega_\beta,$$
$$\delta_\omega \psi_j = -g t_{jk}^\beta \psi_k \omega_\beta, \quad \delta_\omega \overline{\psi}_j = g \overline{\psi}_k t_{kj}^\beta \omega_\beta. \quad (2.4)$$

The corresponding Faddeev–Popov action [2] $S_{FP}^{(1)} = S_{FP}^{(1)}(A, \Psi, C, \overline{C}, B, \xi)$ in the Feynman gauge has the form

$$S_{FP}^{(1)} = \mathcal{S}_{YM}(A, \Psi) + \int dx \left(\overline{C}^\alpha \partial_\mu D_\mu^{\alpha\beta}(A) C^\beta \right.$$
$$\left. + B^\alpha \partial_\mu A_\mu^\alpha + (\xi/2) B^\alpha B^\alpha \right), \quad (2.5)$$

where ξ is a constant gauge parameter, $C^\alpha = C^\alpha(x), \overline{C}^\alpha = \overline{C}^\alpha(x)$ represent the Faddeev–Popov ghost fields, $B^\alpha = B^\alpha(x)$ are auxiliary fields introducing a gauge fixing condition. The action (2.5) is invariant under global supersymmetry (BRST symmetry) [3,4],

$$\delta_\lambda A_\mu^\alpha = D_\mu^{\alpha\beta}(A) C^\beta \lambda, \quad \delta_\lambda \psi_j = -g t_{jk}^\alpha \psi_k C^\alpha \lambda,$$
$$\delta_\lambda \overline{\psi}_j = g \overline{\psi}_k t_{kj}^\alpha C^\alpha \lambda, \quad (2.6)$$
$$\delta_\lambda C^\alpha = \frac{g}{2} f^{\alpha\beta\gamma} C^\beta C^\gamma \lambda, \quad \delta_\lambda \overline{C}^\alpha = B^\alpha \lambda, \quad \delta_\lambda B^\alpha = 0,$$

where λ is a constant anticommuting parameter.

In the background field formalism [22,23] a gauge field A_μ^α entering the classical action (2.1) is replaced by $A_\mu^\alpha + \mathcal{B}_\mu^\alpha$,

$$\mathcal{S}_{YM}(A, \Psi) \rightarrow \mathcal{S}_{YM}(A + \mathcal{B}, \Psi), \quad (2.7)$$

where $\mathcal{B}_\mu^\alpha$ is considered as an external vector field. The Faddeev–Popov action is constructed by using the modified Feynman gauge (the background gauge condition), and reads

$$S_{FP}^{(2)} = \mathcal{S}_{YM}(A + \mathcal{B}, \Psi) + \int dx \left(\overline{C}^\alpha D_\mu^{\alpha\gamma}(\mathcal{B}) D_\mu^{\gamma\beta}(A + \mathcal{B}) C^\beta \right.$$
$$\left. + B^\alpha D_\mu^{\alpha\beta}(\mathcal{B}) A_\mu^\alpha + (\xi/2) B^\alpha B^\alpha \right). \quad (2.8)$$

This action is invariant under BRST transformations of the form (2.6) with the following modification of the transformation law in the gauge field sector,

$$\delta_\lambda A_\mu^\alpha = D_\mu^{\alpha\beta}(A + \mathcal{B}) C^\beta \lambda. \quad (2.9)$$

The invariance property of Faddeev–Popov actions (2.5) and (2.8) under BRST transformations can be described in the form of non-linear functional equations for the extended action $\mathcal{S}_{ext}$ with the help of additional variables (antifields) $A_\mu^{*\alpha}, \psi_j^*, \overline{\psi}_j^*, C^{*\alpha}, \overline{C}^{*\alpha}$, being sources to the generators of BRST transformations,

$$\mathcal{S}_{ext}^{(a)} = S_{FP}^{(a)} + \int dx \left(Q^* \mathcal{R}_Q^{(a)} + \overline{C}^{*\alpha} B^\alpha \right), \quad a = 1, 2, \quad (2.10)$$

where Q means the set of the fields $\{A_\mu^\alpha, \psi_j, \overline{\psi}_j, C^\alpha\}$ and the symbol Q^* is used to indicate the set of the corresponding antifields for fields Q, wherein the BRST transformations (2.4), (2.6) are presented as $\delta_\lambda Q = \mathcal{R}_Q^{(a)} \lambda$, $a = 1, 2$. Then, as a consequence of the BRST symmetry, the actions $\mathcal{S}_{ext}^{(a)}$ satisfy the master-equation

$$\int dx \left(\mathcal{S}_{ext}^{(a)} \overset{\leftarrow}{\frac{\delta}{\delta Q}} \frac{\delta}{\delta Q^*} \mathcal{S}_{ext}^{(a)} + \mathcal{S}_{ext}^{(a)} \overset{\leftarrow}{\frac{\delta}{\delta \overline{C}}} \frac{\delta}{\delta \overline{C}^*} \mathcal{S}_{ext}^{(a)} \right) = 0, \quad a = 1, 2. \quad (2.11)$$

To study the structure of renormalizations it is convenient to extend the original set of the variables with extra fields and auxiliary quantities. An initial action, we proceed from, when studying the structure of renormalizations and dependence of renormalization constants on gauge fixing is the extended action $S_{ext} = S_{ext}(Q, Q^*, \overline{C}, B, \mathcal{B}, \xi, \theta, \chi)$,

$$
\begin{aligned}
S_{ext} = {} & S_{YM}(A + \mathcal{B}, \Psi) + \int dx \, Q^* \mathcal{R}_Q \\
& + \int dx \left(\overline{C}^\alpha D_\mu^{\alpha\gamma}(\mathcal{B}) D_\mu^{\gamma\beta}(A + \mathcal{B}) C^\beta \right. \\
& \left. + B^\alpha D_\mu^{\alpha\beta}(\mathcal{B}) A_\mu^\beta + (\xi/2) B^\alpha B^\alpha \right) \\
& + \int dx \left(\theta_\mu^\alpha [D_\mu^{\alpha\beta}(A + \mathcal{B}) \overline{C}^\beta - A_\mu^{*\alpha}] \right. \\
& + \chi [(A_\mu^{*\alpha} - D_\mu^{\alpha\beta}(\mathcal{B}) \overline{C}^\beta) A_\mu^\alpha + C^{*\alpha} C^\alpha + \psi_j^* \psi_j \\
& \left. + \overline{\psi}_j^* \overline{\psi}_j] \right),
\end{aligned}
\tag{2.12}
$$

where $\theta_\mu^\alpha = \theta_\mu^\alpha(x)$ are anticommuting extra fields and χ is a constant nilpotent parameter.[1]

The action (2.12) is invariant ($\delta S_{ext} = 0$) under the following transformations of the quantities entered,

$$
\delta A_\mu^\alpha = D_\mu^{\alpha\beta}(A + \mathcal{B}) C^\beta + \theta_\mu^\alpha = \frac{\delta}{\delta A_\mu^{*\alpha}} S_{ext} + \chi A_\mu^\alpha, \tag{2.13}
$$

$$
\begin{aligned}
\delta B^\alpha = {} & -\frac{1}{\xi} \left[D_\mu^{\alpha\gamma}(\mathcal{B}) D^{\gamma\beta}(A + \mathcal{B}) C^\beta + D_\mu^{\alpha\beta}(A + \mathcal{B}) \theta_\mu^\beta \right. \\
& \left. + \chi D_\mu^{\alpha\beta}(\mathcal{B}) A_\mu^\beta \right] - \chi B^\alpha \\
= {} & -\frac{1}{\xi} \frac{\delta}{\delta \overline{C}^\alpha} S_{ext} - \chi B^\alpha,
\end{aligned}
\tag{2.14}
$$

$$
\delta C^\alpha = \frac{g}{2} f^{\alpha\beta\gamma} C^\beta C^\gamma = \frac{\delta}{\delta C^{*\alpha}} S_{ext} - \chi C^\alpha, \tag{2.15}
$$

$$
\begin{aligned}
\delta \overline{C}^\alpha = {} & -\frac{1}{\xi} D_\mu^{\alpha\beta}(\mathcal{B}) A_\mu^\beta + \chi \overline{C}^\alpha = -\frac{1}{\xi} \frac{\delta}{\delta B^\alpha} S_{ext} + B^\alpha \\
& + \chi \overline{C}^\alpha,
\end{aligned}
\tag{2.16}
$$

$$
\delta \psi_j = -g t_{jk}^\alpha \psi_k C^\alpha = \frac{\delta}{\delta \psi_j^*} S_{ext} - \chi \psi_j, \tag{2.17}
$$

$$
\delta \overline{\psi}_j = g \overline{\psi}_k t_{kj}^\alpha C^\alpha = \frac{\delta}{\delta \overline{\psi}_j^*} S_{ext} - \chi \overline{\psi}_j, \tag{2.18}
$$

$$
\delta A_\mu^{*\alpha} = \chi A_\mu^{*\alpha}, \quad \delta C^{*\alpha} = -\chi C^{*\alpha}, \quad \delta \psi_j^* = -\chi \psi_j^*,
$$
$$
\delta \overline{\psi}_j^* = -\chi \overline{\psi}_j^*, \tag{2.19}
$$

$$
\delta \xi = 2\xi \chi, \quad \delta \mathcal{B}_\mu^\alpha = -\theta_\mu^\alpha, \quad \delta \theta_\mu^\alpha = 0, \quad \delta \chi = 0. \tag{2.20}
$$

Due to the variations (2.13)–(2.20), the invariance condition of the action rewrites

$$
\begin{aligned}
& \int dx \left(S_{ext} \frac{\overleftarrow{\delta}}{\delta Q} \frac{\delta}{\delta Q^*} S_{ext} - B \frac{\delta}{\delta \overline{C}} S_{ext} - \theta \frac{\delta}{\delta \mathcal{B}} S_{ext} \right) \\
& + \chi \int dx \left[Q \frac{\delta}{\delta Q} - Q^* \frac{\delta}{\delta Q^*} - \overline{C} \frac{\delta}{\delta \overline{C}} - B \frac{\delta}{\delta B} \right] S_{ext} \\
& + 2\chi\xi \frac{\partial}{\partial \xi} S_{ext} = 0.
\end{aligned}
\tag{2.21}
$$

Also, the action (2.12) satisfies the equation

$$
S_{ext} \overleftarrow{H^\alpha} \omega_\alpha = 0, \tag{2.22}
$$

where the notation

$$
\begin{aligned}
\overleftarrow{H^\alpha} \omega_\alpha = \int dx \Bigg\{ & \left[\frac{\overleftarrow{\delta}}{\delta \mathcal{B}_\mu^\beta} D_\mu^{\beta\alpha}(\mathcal{B}) \right. \\
& + g f^{\beta\gamma\alpha} \left(\frac{\overleftarrow{\delta}}{\delta A_\mu^\beta} A_\mu^\gamma + \frac{\overleftarrow{\delta}}{\delta B^\beta} B^\gamma \right) \\
& + g f^{\beta\gamma\alpha} \left(\frac{\overleftarrow{\delta}}{\delta C^\beta} C^\gamma + \frac{\overleftarrow{\delta}}{\delta \overline{C}^\beta} \overline{C}^\gamma + \frac{\overleftarrow{\delta}}{\delta A_\mu^{*\beta}} A_\mu^{*\gamma} \right. \\
& \left. + \frac{\overleftarrow{\delta}}{\delta C^{*\beta}} C^{*\gamma} + \frac{\overleftarrow{\delta}}{\delta \theta_\mu^\beta} \theta_\mu^\gamma \right) \\
& - g t_{jk}^\alpha \left(\frac{\overleftarrow{\delta}}{\delta \psi_j} \psi_k + \frac{\overleftarrow{\delta}}{\delta \overline{\psi}_j^*} \overline{\psi}_k^* \right) \\
& \left. + g \left(\frac{\overleftarrow{\delta}}{\delta \overline{\psi}_j} \overline{\psi}_k + \frac{\overleftarrow{\delta}}{\delta \psi_j^*} \psi_k^* \right) t_{kj}^\alpha \right] \omega_\alpha \Bigg\},
\end{aligned}
\tag{2.23}
$$

is used for the operator describing the gauge transformations of the variables $\mathcal{B}_\mu, \psi, \overline{\psi}$ and simultaneously the tensor transformation of fields and antifields $A_\mu, C, \overline{C}, B, \theta_\mu, A_\mu^*, \psi^*, \overline{\psi}^*, C^*$. Finally, we notice that the action (2.12) satisfies the two important relations linear in fields A_μ, B and also in derivatives of variables $B, \overline{C}, A_\mu^*$,

$$
\frac{\delta}{\delta B^\alpha} S_{ext} = D_\mu^{\alpha\beta}(\mathcal{B}) A_\mu^\beta + \xi B^\alpha, \tag{2.24}
$$

$$
D_\mu^{\alpha\beta}(\mathcal{B}) \frac{\delta}{\delta A_\mu^{*\beta}} S_{ext} - \frac{\delta}{\delta \overline{C}^\alpha} S_{ext} = -g f^{\alpha\beta\gamma} A_\mu^\beta \theta_\mu^\gamma. \tag{2.25}
$$

The Eq. (2.25) means that the action S_{ext} (2.12) depends on variables $A_\mu^{*\alpha}$ "Z", "p" $\overline{C}^\alpha$ in combination $A_\mu^{*\alpha} - D_\mu^{\alpha\beta}(\mathcal{B}) \overline{C}^\beta$ only when $\theta_\mu^\beta = 0$.

We give the table of "quantum" numbers of fields, antifields, auxiliary fields and constant quantities which have been used in constructing S_{ext}:

---

[1] These extra variables have been used first in Ref. [20].

Quantity	$A, \mathcal{B}$	$\psi, \overline{\psi}$	$C, \overline{C}$	B	ξ	θ	A^*	$\psi^*, \overline{\psi}^*$	C^*	dx	∂_x	χ
ε	0	1	1	0	0	1	1	0	0	0	0	1
gh	0	0	$1, -1$	0	0	1	-1	-1	-2	0	0	1
dim	1	3/2	1	2	0	2	2	3/2	2	-4	1	1
ε_f	0	$1, -1$	0	0	0	0	0	$-1, 1$	0	0	0	0

where "ε" describes the Grassmann parity, the symbol "gh" is used for the ghost number, "dim" denotes the canonical dimension and "ε_f" means the fermion number. Using this table of "quantum" numbers it is easy to establish quantum numbers of any quantities found in the text.

3 General structure of renormalized action

It is to be proved below that the renormalizable action is a local functional of field variables, carries the quantum number of the action $\mathcal{S}_{ext}$ (2.12), and satisfies the same Eqs. (2.21)–(2.25) as the action S_{ext}. In that Section we will find the general solution to the Eqs. (2.21)–(2.25) under the extra conditions mentioned.

So, let

$$P = \int dx\, P(x), \qquad (3.1)$$

where $P(x)$ is a local polynomial in all variables $Q, Q^*, \overline{C}$, $B, \mathcal{B}, \xi, \theta, \chi$ with $\dim(P(x)) = 4$. Require the functional P to satisfy the Eqs. (2.21)–(2.25) with substitution $S_{ext} \to P$, and let P be of the form

$$P = P_{00} + P^{(1)} + \chi P^{(2)}, \qquad (3.2)$$

where

$$P_{00} = \int dx \left(B^\alpha D_\mu^{\alpha\beta}(\mathcal{B}) A_\mu^\beta + \frac{\xi}{2} B^\alpha B^\alpha \right.$$
$$\left. + g\theta_\mu^\alpha f^{\alpha\beta\gamma} A_\mu^\beta \overline{C}^\gamma \right), \qquad (3.3)$$

$$\varepsilon(P^{(1)}) = 0, \quad \mathrm{gh}(P^{(1)}) = 0,$$
$$\dim(P^{(1)}) = 0, \quad \varepsilon_f((P^{(1)})) = 0, \qquad (3.4)$$
$$\varepsilon(P^{(2)}) = 1, \quad \mathrm{gh}(P^{(2)}) = -1,$$
$$\dim(P^{(2)}) = -1, \quad \varepsilon_f((P^{(2)})) = 0, \qquad (3.5)$$

and the functionals $P^{(1)}$ and $P^{(2)}$ do not depend on χ. It follows from the Eq. (2.24) for P, and representation (3.3) that $P^{(1)}$ and $P^{(2)}$ do not depend on B^α, "Z", "p",

$$P^{(k)} = P^{(k)}(Q, Q^*, \overline{C}, \mathcal{B}, \xi, \theta), \quad k = 1, 2, \qquad (3.6)$$

By introducing new variables $\mathcal{A}_\mu^{*\alpha}(x)$,

$$\mathcal{A}_\mu^{*\alpha} = A_\mu^{*\alpha} - D_\mu^{\alpha\beta}(\mathcal{B})\overline{C}^\beta, \quad \mathcal{A}_\mu^\alpha = A_\mu^\alpha, \qquad (3.7)$$

we define new functionals $\tilde{P}^{(k)}$ by the rule

$$\tilde{P}^{(k)} = \tilde{P}^{(k)}(\Omega, \Omega^*, \mathcal{B}, \overline{C}, \xi, \theta) \qquad (3.8)$$
$$= P^{(k)}(Q, Q^*, \mathcal{B}, \overline{C}, \xi, \theta)|_{A^* \to \mathcal{A}^* + D(\mathcal{B})\overline{C}}, \quad k = 1, 2,$$

to find that $\tilde{P}^{(k)}$ do not depend on the fields $\overline{C}^\alpha$,

$$\tilde{P}^{(k)} = \tilde{P}^{(k)}(\Omega, \Omega^*, \mathcal{B}, \xi, \theta). \qquad (3.9)$$

In the relations (3.8) and (3.9) the following notations

$$\Omega = \{\mathcal{A}, \psi, \overline{\psi}, C\}, \ \mathcal{A} = A, \ \Omega^* = \{\mathcal{A}^*, \psi^*, \overline{\psi}^*, C^*\} \qquad (3.10)$$

are used.

Independence of functionals $\tilde{P}^{(k)}$ of the fields $\overline{C}^\alpha$ and relations

$$P^{(k)} \int dx \left[\frac{\overleftarrow{\delta}}{\delta \mathcal{B}_\mu^\beta} D_\mu^{\beta\alpha}(\mathcal{B}) + gf^{\beta\gamma\alpha} \left(\frac{\overleftarrow{\delta}}{\delta \overline{C}^\beta} \overline{C}^\gamma + \frac{\overleftarrow{\delta}}{\delta A_\mu^{*\beta}} A_\mu^{*\gamma} \right) \right]$$
$$= \tilde{P}^{(k)} \int dx \left[\frac{\overleftarrow{\delta}}{\delta \mathcal{B}_\mu^\beta} D_\mu^{\beta\alpha}(\mathcal{B}) + gf^{\beta\gamma\alpha} \frac{\overleftarrow{\delta}}{\delta \mathcal{A}_\mu^{*\beta}} \mathcal{A}_\mu^{*\gamma} \right], \qquad (3.11)$$

allow one to write down the following set of equations as for $\tilde{P}^{(k)}$,

$$\int dx \left[\tilde{P}^{(1)} \frac{\overleftarrow{\delta}}{\delta \Omega} \frac{\delta}{\delta \Omega^*} \tilde{P}^{(1)} - \theta_\mu^\alpha \frac{\delta}{\delta \mathcal{B}_\mu^\alpha} \tilde{P}^{(1)} \right] = 0,$$

$$2\xi \frac{\partial}{\partial \xi} \tilde{P}^{(1)} = \int dx \left[\tilde{P}^{(1)} \left(\frac{\overleftarrow{\delta}}{\delta \Omega} \frac{\delta}{\delta \Omega^*} - \frac{\overleftarrow{\delta}}{\delta \Omega^*} \frac{\delta}{\delta \Omega} \right) \tilde{P}^{(2)} \right.$$
$$\left. - \theta_\mu^\alpha \frac{\delta}{\delta \mathcal{B}_\mu^\alpha} \tilde{P}^{(2)} \right]$$
$$+ \int dx \left[\left(\Omega^* \frac{\delta}{\delta \Omega^*} - \Omega \frac{\delta}{\delta \Omega} \right) \tilde{P}^{(1)} \right], \qquad (3.12)$$

$$\tilde{P}^{(k)} \overleftarrow{\tilde{h}}^\alpha \omega_\alpha = 0, \ k = 1, 2, \qquad (3.13)$$

where

$$\overleftarrow{\tilde{h}}^{\alpha} \omega_{\alpha} = \int dx \Bigg\{ \Bigg[\frac{\overleftarrow{\delta}}{\delta \mathcal{B}_{\mu}^{\beta}} D_{\mu}^{\beta\alpha}(\mathcal{B})$$

$$+ g f^{\beta\gamma\alpha} \left(\frac{\overleftarrow{\delta}}{\delta \mathcal{A}_{\mu}^{\beta}} \mathcal{A}_{m|\mu}^{\gamma} + \frac{\overleftarrow{\delta}}{\delta C^{\beta}} C^{\gamma} \right)$$

$$+ g f^{\beta\gamma\alpha} \left(\frac{\overleftarrow{\delta}}{\delta \mathcal{A}_{\mu}^{*\beta}} \mathcal{A}_{\mu}^{*\gamma} + \frac{\overleftarrow{\delta}}{\delta C^{*\beta}} C^{*\gamma} + \frac{\overleftarrow{\delta}}{\delta \theta^{\beta}} \theta_{\mu}^{\gamma} \right)$$

$$- g t_{jk}^{\alpha} \left(\frac{\overleftarrow{\delta}}{\delta \psi_j} \psi_k \right) \tag{3.14}$$

$$+ \frac{\overleftarrow{\delta}}{\delta \overline{\psi}_j^*} \overline{\psi}_k^* \Bigg) + g \left(\frac{\overleftarrow{\delta}}{\delta \overline{\psi}_j} \overline{\psi}_k + \frac{\overleftarrow{\delta}}{\delta \psi_j^*} \psi_k^* \right) t_{kj}^{\alpha} \Bigg] \omega_{\alpha} \Bigg\}.$$

When studying the structure of functionals and further investigating it appears useful a consequence of the Eq. (3.13) at $\omega_{\alpha} = \text{const}$,

$$\tilde{P}^{(k)} \overleftarrow{T}^{\alpha} = 0, \quad k = 1, 2, \tag{3.15}$$

where

$$\overleftarrow{T}^{\alpha} = \int dx \Bigg\{ f^{\beta\gamma\alpha} \left(\frac{\overleftarrow{\delta}}{\delta \mathcal{B}_{\mu}^{\beta}} \mathcal{B}_{\mu}^{\gamma} + \frac{\overleftarrow{\delta}}{\delta \mathcal{A}_{\mu}^{\beta}} \mathcal{A}_{\mu}^{\gamma} + \frac{\overleftarrow{\delta}}{\delta C^{\beta}} C^{\gamma} \right)$$

$$+ f^{\beta\gamma\alpha} \left(\frac{\overleftarrow{\delta}}{\delta \mathcal{A}_{\mu}^{*\beta}} \mathcal{A}_{\mu}^{*\gamma} + \frac{\overleftarrow{\delta}}{\delta C^{*\beta}} C^{*\gamma} + \frac{\overleftarrow{\delta}}{\delta \theta^{\beta}} \theta_{\mu}^{\gamma} \right)$$

$$- t_{jk}^{\alpha} \left(\frac{\overleftarrow{\delta}}{\delta \psi_j} \psi_k + \frac{\overleftarrow{\delta}}{\delta \overline{\psi}_j^*} \overline{\psi}_k^* \right)$$

$$+ \left(\frac{\overleftarrow{\delta}}{\delta \overline{\psi}_j} \overline{\psi}_k + \frac{\overleftarrow{\delta}}{\delta \psi_j^*} \psi_k^* \right) t_{kj}^{\alpha} \Bigg\}. \tag{3.16}$$

We refer to equations of the form (3.15) as the ones of the T-symmetry for the corresponding functional.

Using the properties of the functional $\tilde{P}^{(2)}$ (3.5), its locality as well as axial symmetry, Poincare- and T- symmetries we find the general representation,

$$\tilde{P}^{(2)} = \int dx \Bigg[Z_1 \mathcal{A}_{\mu}^{*\alpha} \mathcal{A}_{\mu}^{\alpha} + Z_2 C^{*\alpha} C^{\alpha} + Z_3 \psi_j^* \psi_j$$

$$+ Z_4 \overline{\psi}_j^* \overline{\psi}_j + Z_1' \mathcal{A}_{\mu}^{*\alpha} \mathcal{B}_{\mu}^{\alpha} \Bigg], \tag{3.17}$$

where Z_i, $i = 1, 2, 3, 4$, "Z", "p" Z_1' are arbitrary constants. Further, when using the Eq. (3.13) for $\tilde{P}^{(2)}$ we get that $Z_1' = 0$. The final expression for $\tilde{P}^{(2)}$ has the form

$$\tilde{P}^{(2)} = \int dx \Bigg[Z_1 \mathcal{A}_{\mu}^{*\alpha} \mathcal{A}_{\mu}^{\alpha} + Z_2 C^{*\alpha} C^{\alpha}$$

$$+ Z_3 \psi_j^* \psi_j + Z_4 \overline{\psi}_j^* \overline{\psi}_j \Bigg]. \tag{3.18}$$

Notice that the functional $\tilde{P}^{(2)}$ does not depend on the fields θ_{μ}^{α}. By taking (3.18) into account the Eq. (3.12) reduces to the following one

$$2\xi \frac{\partial}{\partial \xi} \tilde{P}^{(1)} = \int dx \Bigg[(Z_1 - 1) \left(\mathcal{A}_{\mu}^{\alpha} \frac{\delta}{\delta \mathcal{A}_{\mu}^{\alpha}} - \mathcal{A}_{\mu}^{*\alpha} \frac{\delta}{\delta \mathcal{A}_{\mu}^{*\alpha}} \right)$$

$$+ (Z_2 - 1) \left(C^{\alpha} \frac{\delta}{\delta C^{\alpha}} - C^{*\alpha} \frac{\delta}{\delta C^{*\alpha}} \right)$$

$$+ (Z_3 - 1) \left(\psi_j \frac{\delta}{\delta \psi_j} - \psi_j^* \frac{\delta}{\delta \psi_j^*} \right)$$

$$+ (Z_4 - 1) \left(\overline{\psi}_j \frac{\delta}{\delta \overline{\psi}_j} - \overline{\psi}_j^* \frac{\delta}{\delta \overline{\psi}_j^*} \right) \Bigg] \tilde{P}^{(1)}, \tag{3.19}$$

describing the dependence of renormalization constants on the gauge parameter ξ. We refer to the Eq. (3.12) as an extended master-equation and to (3.19) as a gauge dependence equation.

3.1 Solution to the extended master-equation

Now we consider a solution to the extended master-equation (3.12) for the functional $\tilde{P}^{(1)}$ as presented it in the form

$$\tilde{P}^{(1)} = \tilde{P}_{\theta}^{(1)} + \tilde{P}_{\Omega^*}^{(1)} + \tilde{P}_{\psi}^{(1)} + \tilde{P}_{\mathcal{AB}}^{(1)}. \tag{3.20}$$

The functional $\tilde{P}_{\theta}^{(1)}$ rewrites as

$$\tilde{P}_{\theta}^{(1)} = \int dx\, \theta_{\mu}^{\alpha}(x) \tilde{P}_{\mu\theta}^{\alpha}(x), \tag{3.21}$$

and the functionals $\tilde{P}_{\mathcal{AB}}^{(1)}$, $\tilde{P}_{\psi}^{(1)}$, $\tilde{P}_{\Omega^*}^{(1)}$ do not depend on the fields θ_{μ}^{α}. By taking into account the properties $\dim(\tilde{P}_{\mu\theta}^{\alpha}) = 2$, $\text{gh}(\tilde{P}_{\mu\theta}^{\alpha}) = -1$, $\varepsilon(\tilde{P}_{\mu\theta}^{\alpha}) = 1$, $\varepsilon_f(\tilde{P}_{\mu\theta}^{\alpha}) = 0$, as well as the Poincare- and T-symmetries of the functional $\tilde{P}_{\theta}^{(1)}$, we find that

$$\tilde{P}_{\mu\theta}^{\alpha} = -Z_5 \mathcal{A}_{\mu}^{*\alpha}, \quad \tilde{P}_{\theta}^{(1)} = -Z_5 \int dx\, \theta_{\mu}^{\alpha}(x) \mathcal{A}_{\mu}^{*\alpha}(x)$$

$$= -Z_5 \int dx\, \theta_{\mu}^{\alpha} \mathcal{A}_{\mu}^{*\alpha}, \tag{3.22}$$

where Z_5 is an arbitrary constant.

The functional $\tilde{P}_{\Omega^*}^{(1)}$ is linear in the antifields Ω^* (3.10), and the functionals $\tilde{P}_{\mathcal{AB}}^{(1)}$ and $\tilde{P}_{\psi}^{(1)}$ do not depend on the antifields Ω^*. The functional $\tilde{P}_{\Omega^*}^{(1)}$ can be represented in the form

$$\tilde{P}_{\Omega^*}^{(1)} = \tilde{P}_{\mathcal{A}^*}^{(1)} + \tilde{P}_{C^*}^{(1)} + \tilde{P}_{\psi^*}^{(1)} + \tilde{P}_{\overline{\psi}^*}^{(1)}. \tag{3.23}$$

By using the arguments analogous to those led us to the structure of the functional $\tilde{P}_\theta^{(1)}$ (3.22), we obtain

$$\tilde{P}_{\mathcal{A}^*}^{(1)} = \int dx \Big[Z_6 \mathcal{A}_\mu^{*\alpha} D_\mu^{\alpha\beta}(\mathcal{B}) C^\beta + g Z_{7\beta\gamma}^\alpha \mathcal{A}_\mu^{*\alpha} \mathcal{A}_\mu^\beta C^\gamma$$
$$+ g Z_{7\beta\gamma}^{'\alpha} \mathcal{A}_\mu^{*\alpha} \mathcal{B}_\mu^\beta C^\gamma \Big], \tag{3.24}$$

$$\tilde{P}_{C^*}^{(1)} = \int dx \, \frac{g}{2} Z_{8\beta\gamma}^\alpha C^{*\alpha} C^\beta C^\gamma, \tag{3.25}$$

$$\tilde{P}_{\psi^*}^{(1)} = -\int dx \, g Z_{9jk}^\alpha \psi_j^* \psi_k C^\alpha, \tag{3.26}$$

$$\tilde{P}_{\overline{\psi}^*}^{(1)} = \int dx \, g Z_{10kj}^\alpha \overline{\psi}_j^* \overline{\psi}_k C^\alpha. \tag{3.27}$$

Taking into account the gauge symmetry in the external field $\mathcal{B}$ (see the Eq. (3.13)), we find that $Z_{7\beta\gamma}^{'\alpha} = 0$. The quantities "$Z$" introduced in (3.24)–(3.27) are constants that satisfy the equations,

$$F_{\gamma\delta}^\alpha Z_{7\delta\sigma}^\beta - Z_{7\gamma\delta}^\beta F_{\delta\sigma}^\alpha = f^{\alpha\beta\lambda} Z_{7\gamma\sigma}^\lambda, \tag{3.28}$$

$$F_{\gamma\delta}^\alpha Z_{8\delta\sigma}^\beta - Z_{8\gamma\delta}^\beta F_{\delta\sigma}^\alpha = f^{\alpha\beta\delta} Z_{8\gamma\sigma}^\delta, \tag{3.29}$$

$$t_{jl}^\alpha Z_{9lk}^\beta - Z_{9jl}^\beta t_{lk}^\alpha = f^{\alpha\beta\gamma} Z_{9jk}^\gamma, \tag{3.30}$$

$$t_{kl}^\alpha Z_{10lj}^\beta - Z_{10kl}^\beta t_{lj}^\alpha = f^{\alpha\beta\gamma} Z_{10kj}^\gamma. \tag{3.31}$$

Notice that if $Z_{7\gamma\sigma}^\lambda = Z_7 f^{\gamma\lambda\sigma}$, $Z_{8\gamma\sigma}^\lambda = Z_8 f^{\gamma\lambda\sigma}$, $Z_{9jk}^\alpha = Z_9 t_{jk}^\alpha$, $Z_{10jk}^\alpha = Z_{10} t_{jk}^\alpha$, then the corresponding Eqs. (3.28)–(3.31) hold and the functionals $\tilde{P}_{\mathcal{A}^*}^{(1)}$, $\tilde{P}_{C^*}^{(1)}$, $\tilde{P}_{\psi^*}^{(1)}$, $\tilde{P}_{\overline{\psi}^*}^{(1)}$ (3.24)–(3.27) satisfy the Eq. (3.13) by themselves.

In its turn, taking into account the axial symmetry, the Poincare- and the T-invariance we determine the general structure of the functional $\tilde{P}_\psi^{(1)}$,

$$\tilde{P}_\psi^{(1)} = \int dx \Big[i Z_{11} \overline{\psi}_j \gamma^\mu D_{\psi\mu}(\mathcal{B}) \psi_j$$
$$+ i g Z_{11jk}^{'\alpha} \overline{\psi}_j \gamma^\mu \mathcal{B}_\mu^\alpha \psi_k +$$
$$+ i g Z_{12jk}^\alpha \overline{\psi}_j \gamma^\mu \mathcal{A}_\mu^\alpha \psi_k - m Z_{13} \overline{\psi}_j \psi_j \Big], \tag{3.32}$$

where constants Z_{12jk}^α satisfy the equations

$$t_{jl}^\alpha Z_{12lk}^\beta - Z_{12jl}^\beta t_{lk}^\alpha = f^{\alpha\beta\gamma} Z_{12jk}^\gamma. \tag{3.33}$$

The contribution to the $\tilde{P}_\psi^{(1)} \overleftarrow{\tilde{h}}^\alpha \omega_\alpha$, proportional to $\partial_\mu \omega^\alpha$, has the form

$$i g \overline{\psi}_j Z_{11jk}^{'\alpha} \gamma^\mu \psi_k \partial_\mu \omega^\alpha, \tag{3.34}$$

so that it follows from the Eq. (3.13) that the equalities $Z_{11jk}^{'\alpha} = 0$ and

$$\tilde{P}_\psi^{(1)} = \int dx \Big[i Z_{11} \overline{\psi}_j \gamma^\mu D_{\psi\mu}(\mathcal{B}) \psi + i g Z_{12jk}^\alpha \overline{\psi}_j \gamma^\mu \mathcal{A}_\mu^\alpha \psi_k$$
$$- m Z_{13} \overline{\psi}_j \psi_j \Big] \tag{3.35}$$

hold. Notice that in the case $Z_{12jk}^\alpha = Z_{12} t_{kj}^\alpha$ the Eq. (3.33) are fulfilled and the functional $\tilde{P}_\psi^{(1)}$ (3.35) satisfies the Eq. (3.13).

Insert the representation for the functional $\tilde{P}^{(1)}$ in the form (3.20) into the Eq. (3.12). Then, analysis of the $\theta\psi\overline{\psi}$ components in the extended master-equation (3.12) yields

$$Z_{12jk}^\alpha = Z_{12} t_{jk}^\alpha, \quad Z_{12} = Z_{11}/Z_5, \tag{3.36}$$

and the possibility to represent the functional $\tilde{P}_\psi^{(1)}$ as

$$\tilde{P}_\psi^{(1)} = \int dx \left[i Z_{11} \overline{\psi}_j \gamma^\mu D_{\psi\mu jk}(U) \psi_k - m Z_{13} \overline{\psi}_j \psi_j \right], \tag{3.37}$$

where the notation

$$U = \{U_\mu^\alpha\}, \quad U_\mu^\alpha = Z_5^{-1} \mathcal{A}_\mu^\alpha + \mathcal{B}_\mu^\alpha \tag{3.38}$$

is used. The $\theta\mathcal{A}^*C$ components in the Eq. (3.12) lead to the relations

$$Z_{7\beta\gamma}^\alpha = Z_7 f^{\alpha\beta\gamma}, \quad Z_7 = \frac{Z_6}{Z_5}, \tag{3.39}$$

and to the representation

$$\tilde{P}_{\mathcal{A}^*}^{(1)} = \int dx \, Z_6 \mathcal{A}_\mu^{*\alpha} D_\mu^{\alpha\beta}(U) C^\beta. \tag{3.40}$$

Consideration of the $\mathcal{A}^* \mathcal{A} C C$ components in the Eq. (3.12) gives the relations

$$Z_{8\beta\gamma}^\alpha = Z_8 f^{\alpha\beta\gamma}, \quad Z_8 = Z_7 = \frac{Z_6}{Z_5}, \tag{3.41}$$

and the representation for the functional $\tilde{P}_{C^*}^{(1)}$ in the form

$$\tilde{P}_{C^*}^{(1)} = \int dx \, \frac{g}{2} \frac{Z_6}{Z_5} f^{\alpha\beta\gamma} C^{*\alpha} C^\beta C^\gamma. \tag{3.42}$$

Studying the $\overline{\psi}\psi\partial C$, $\overline{\psi}\psi\mathcal{B}C$ and $m\overline{\psi}\psi C$ components in the Eq. (3.12) lead to the relations

$$Z_{10jk}^\alpha = Z_{9jk}^\alpha, \quad Z_{9jk}^\alpha = Z_9 t_{jk}^\alpha, \quad Z_9 = \frac{Z_6}{Z_5},$$

$$Z_{10jk}^\alpha = \frac{Z_6}{Z_5} t_{jk}^\alpha, \tag{3.43}$$

and, as a consequence, to the representation of the functionals $\tilde{P}^{(1)}_{\psi^*}$ and $\tilde{P}^{(1)}_{\overline{\psi}^*}$ as

$$\tilde{P}^{(1)}_{\psi^*} = -\int dx \; g \frac{Z_6}{Z_5} \; \psi_j^* t_{jk}^\alpha \psi_k C^\alpha, \tag{3.44}$$

$$\tilde{P}^{(1)}_{\overline{\psi}^*} = \int dx \; g \frac{Z_6}{Z_5} \; \overline{\psi}_j^* t_{kj}^\alpha \overline{\psi}_k C^\alpha$$

$$= \int dx \; g \frac{Z_6}{Z_5} \; \overline{\psi}_j t_{jk}^\alpha \overline{\psi}_k^* C^\alpha. \tag{3.45}$$

The functional $\tilde{P}^{(1)}_{\mathcal{AB}}$ depends on the fields $\mathcal{A}$ and $\mathcal{B}$ only. The $\theta \mathcal{AB}$ components in the Eq. (3.12) allow us to conclude that the functional $\tilde{P}^{(1)}_{\mathcal{AB}}$ depends on the fields $\mathcal{A}$ and $\mathcal{B}$ only in combination (3.38),

$$\tilde{P}^{(1)}_{\mathcal{AB}}(\mathcal{A}, \mathcal{B}) = X(U). \tag{3.46}$$

Finally, consideration of the $\mathcal{ABC}$ components in the Eq. (3.12) leads to equations for the functional $X(U)$ (3.46)

$$D_\mu^{\alpha\beta}(U) \frac{\delta}{\delta U_\mu^\beta(x)} X(U) = 0. \tag{3.47}$$

The required solution to the Eq. (3.47) can be written in the form

$$\tilde{P}^{(1)}_{\mathcal{AB}}(\mathcal{A}, \mathcal{B}) = X(U) = -\int dx \; \frac{1}{4} Z_{14} \; G_{\mu\nu}^\alpha(U) G_{\mu\nu}^\alpha(U), \tag{3.48}$$

where

$$G_{\mu\nu}^\alpha(U) = \partial_\mu U_\nu^\alpha - \partial_\nu U_\mu^\alpha + g f^{\alpha\beta\gamma} U_\mu^\beta U_\nu^\gamma. \tag{3.49}$$

Thus the general solution to the extended master-equation, $\tilde{P}^{(1)}$, is constructed. It is defined by fifth independent arbitrary constants $Z_5, Z_6, Z_{11}, Z_{13}, Z_{14}$ and has the form

$$\tilde{P}^{(1)} = \int dx \left[-\frac{1}{4} Z_{14} \; G_{\mu\nu}^\alpha(U) G_{\mu\nu}^\alpha(U) \right.$$

$$+ i Z_{11} \overline{\psi}_j \gamma^\mu D_{\psi\mu jk}(U) \psi_k - m Z_{13} \overline{\psi}_j \psi_j$$

$$- Z_5 \theta_\mu^\alpha \mathcal{A}_\mu^{*\alpha} + Z_6 \mathcal{A}_\mu^{*\alpha} D_\mu^{\alpha\beta}(U) C^\beta$$

$$+ g \frac{Z_6}{Z_5} \left(f^{\alpha\beta\gamma} C^{*\alpha} C^\beta C^\gamma + \overline{\psi}_j^* t_{kj}^\alpha \overline{\psi}_k C^\alpha \right.$$

$$\left. \left. - \psi_j^* t_{jk}^\alpha \psi_k C^\alpha \right) \right]. \tag{3.50}$$

Notice that at $Z_1 = Z_2 = Z_3 = Z_4 = Z_5 = Z_6 = Z_{11} = Z_{13} = Z_{14} = 1$, the equality (the initial condition),

$$P_{Z=1} = S_{ext}, \tag{3.51}$$

holds where S_{ext} is given by the formula (2.12).

3.2 Solution to the gauge dependence equation

Consider now a solution to the Eq. (3.19) describing the gauge dependence of the constants entering the general solution constructed, $\tilde{P}^{(1)}$, to the extended master-equation (3.50). By studying the $\mathcal{A}^*\theta$, $\mathcal{A}D(U)G(U)$, $\mathcal{A}^* f \mathcal{A}C$ and $\overline{\psi}\gamma t \mathcal{A}\psi$ structures in the Eq. (3.19), we derive the following relation

$$2\xi \dot{Z}_5 = -(Z_1 - 1)Z_5 \; \Rightarrow \; Z_1 = 1 - 2\xi \frac{\dot{Z}_5}{Z_5}. \tag{3.52}$$

Henceforth we use the notation

$$\dot{I} \equiv \frac{\partial}{\partial \xi} I, \tag{3.53}$$

for any quantity $I = I(\xi, \ldots)$ depending on the gauge parameter ξ.

Analysis of the $\mathcal{A}^* D(U) C$ components in the Eq. (3.19) gives the relation

$$2\xi \dot{Z}_6 = (Z_2 - Z_1)Z_6 \; \Rightarrow \; Z_2 = 1 + 2\xi \left(\frac{\dot{Z}_6}{Z_6} - \frac{\dot{Z}_5}{Z_5} \right). \tag{3.54}$$

Considering the $\overline{\psi}\gamma D_\psi(U)\psi$ components in the Eq. (3.19), we obtain

$$2\xi \dot{Z}_{11} = (Z_3 + Z_4 - 2)Z_{11}. \tag{3.55}$$

Analyzing the $m\overline{\psi}\psi$ components in the Eq. (3.19), we find

$$2\xi \dot{Z}_{13} = (Z_3 + Z_4 - 2)Z_{13}. \tag{3.56}$$

By making use of the change of constants "Z"

$$Z_{13} = Z_{11}Z_{15}, \quad Z_3 - Z_4 = 2Z_{16}, \tag{3.57}$$

the Eqs. (3.55), (3.56) rewrite in the form

$$\dot{Z}_{15} = 0, \quad Z_3 = 1 + \xi \frac{\dot{Z}_{11}}{Z_{11}} + Z_{16}, \quad Z_4 = 1 + \xi \frac{\dot{Z}_{11}}{Z_{11}} - Z_{16}. \tag{3.58}$$

Finally, consideration of the $G(U)G(U)$ components in the Eq. (3.19) leads to the important statement that,

$$\dot{Z}_{14} = 0. \tag{3.59}$$

Analysis of the $\psi^* t \psi C$, $\overline{\psi}^* t^t \overline{\psi} C$ and $C^* f C C$ components in the Eq. (3.19) gives no new information.

Below, in Sect. 5 find that all constants "Z" can be interpreted as renormalization constants which are uniquely defined from the conditions of divergence elimination.

Let us formulate the results obtained in that Section in the form of a lemma.

Lemma *Let*

$$P = \int dx \, P(Q, Q^*, \overline{C}, B, \mathcal{B}, \xi, \theta, \chi), \tag{3.60}$$

be a local functional of all variables, obey the quantum numbers of the action S_{ext} and satisfy all Eqs. (2.21)–(2.25) as well as extra symmetries (Poincare-invariance and so on) which have been used in solving the Eqs. (2.21)–(2.25) with substitution $S_{ext} \to P$.

Then the functional P has the form

$$P = P_{00} + P^{(1)} + \chi P^{(2)}, \tag{3.61}$$

where P_{00} is given by the formula (3.3), $P^{(1)}$ and $P^{(2)}$ do not depend on B^α and χ and are functionals of arguments Ω, Ω^*, $\mathcal{B}, \overline{C}, \xi, \theta$,

$$P^{(k)} = P^{(k)}(Q, Q^*, \overline{C}, \mathcal{B}, \xi, \theta) = \tilde{P}^{(k)}$$
$$= \tilde{P}^{(k)}(\Omega, \Omega^*, \mathcal{B}, \overline{C}, \xi, \theta), \; k = 1, 2, \tag{3.62}$$

$$\Omega = \{\mathcal{A}, \psi, \overline{\psi}, C\}, \; \Omega^* = \{\mathcal{A}^*, \psi^*, \overline{\psi}^*, C^*\}, \tag{3.63}$$

$$\mathcal{A}_\mu^{*\alpha} = A_\mu^{*\alpha} - D_\mu^{\alpha\beta}(\mathcal{B})\overline{C}^\beta, \; \mathcal{A}_\mu^\alpha = A_\mu^\alpha. \tag{3.64}$$

The functionals $\tilde{P}^{(k)}$ read

$$\tilde{P}^{(1)} = \int dx \Big[-\frac{1}{4} Z_{14} \, G_{\mu\nu}^\alpha(U) G_{\mu\nu}^\alpha(U)$$
$$+ i Z_{11} \overline{\psi}_j \gamma^\mu D_{\psi\mu jk}(U) \psi_k - m Z_{13} \overline{\psi}_j \psi_j$$
$$- Z_5 \theta_\mu^\alpha \mathcal{A}_\mu^{*\alpha} + + Z_6 \mathcal{A}_\mu^{*\alpha} D_\mu^{\alpha\beta}(U) C^\beta$$
$$+ g \frac{Z_6}{Z_5} \Big(f^{\alpha\beta\gamma} C^{*\alpha} C^\beta C^\gamma + \overline{\psi}_j^* t_{kj}^\alpha \overline{\psi}_k C^\alpha$$
$$- \psi_j^* t_{jk}^\alpha \psi_k C^\alpha \Big) \Big], \tag{3.65}$$

$$\tilde{P}^{(2)} = \int dx \Big[Z_1 \mathcal{A}_\mu^{*\alpha} \mathcal{A}_\mu^\alpha + Z_2 C^{*\alpha} C^\alpha + Z_3 \psi_j^* \psi_j$$
$$+ Z_4 \overline{\psi}_j^* \overline{\psi}_j \Big]. \tag{3.66}$$

$$U = \{U_\mu^\alpha\}, \quad U_\mu^\alpha = Z_5^{-1} \mathcal{A}_\mu^\alpha + \mathcal{B}_\mu^\alpha, \tag{3.67}$$

$$Z_1 = 1 - 2\xi \frac{\dot{Z}_5}{Z_5}, \quad Z_2 = 1 + 2\xi \Big(\frac{\dot{Z}_6}{Z_6} - \frac{\dot{Z}_5}{Z_5} \Big),$$

$$Z_3 = 1 + \xi \frac{\dot{Z}_{11}}{Z_{11}} + Z_{16}, \quad Z_4 = 1 + \xi \frac{\dot{Z}_{11}}{Z_{11}} - Z_{16}, \tag{3.68}$$

$$Z_{13} = Z_{11} Z_{15}, \quad \dot{Z}_{14} = 0, \quad \dot{Z}_{15} = 0, \tag{3.69}$$

where Z_5, Z_6, Z_{11}, Z_{16} are arbitrary constants depending perhaps on ξ, and Z_{14}, Z_{15} are arbitrary constants not depending on ξ.

The inverse statement, being perhaps trivial but nevertheless important, is true: if the functional P has the form (3.61), (3.62), (3.65), (3.66) and the relations (3.67), (3.68), (3.69) are fulfilled then this functional satisfies the Eqs. (2.21)–(2.25).

4 Generating functional of vertex functions

It is convenient to define the generating functional of Green functions by making use of the action functional P constructed in the previous Section as the action yields then a finite theory certainly. In what follows we re-denote the functional P, $P \equiv S_R$, and, respectively, $P^{(k)} \equiv S_R^{(k)}$, $\tilde{P}^{(k)} \equiv \tilde{S}_R^{(k)}$, $k = 1, 2$.

The generating functional of Green functions is given by the functional integral,

$$Z(J_\Phi, L) = \int d\Phi \exp \Big(\frac{i}{\eta} \Big[S_R + J_\Phi \Phi \Big] \Big)$$
$$= \exp \Big\{ \frac{i}{\eta} W(J_\Phi, L) \Big\}, \tag{4.1}$$

with η standing for a parameter of a loop expansion as to the expression in the exponential in (4.1), $W(J_\Phi, L)$ that is the generating functional of connected Green functions, and the notations are introduced $\Phi = \{Q, \overline{C}, B\}$ "Z", "p" $L = \{L^A\} = \{\mathcal{B}, Q^*, \xi, \theta, \chi\}$, and J_Φ as for the sources to the fields Φ. Also, we assume that all the constants "Z" are functions of η, "Z"="Z"(η), expandable in Taylor power series, $Z_i(0) = 1$, $\dot{Z}_i = O(\eta)$, $i = 5, 6, 11, 14, 15$, $Z_{16} = O(\eta)$. In that case the functional S_R becomes a function of η,

$$S_R = S_R(\eta) = \sum_{l=0}^\infty \eta^l S_{R,l}, \quad S_R^{[k]} = \sum_{l=0}^k \eta^l S_{R,l}, \tag{4.2}$$

so that all the functionals $S_{R,l}$ are linear combination of a single set of monomials,

$$S_{R,l} = \sum_{i=1}^I a_{l,i} S_i, \tag{4.3}$$

where $\{S_i, i = 1, \ldots, I\}$ is a sub-set of monomials which the action S_{ext} is expanded in, and $a_{l,i}$ are constant coefficients for l-loop order.

The generating functional of vertex Green functions (effective action) is defined by the Legendre transformation

$$\Gamma(\Phi_{m|}, L) = W(J_\Phi, L) - J_\Phi \Phi_{m|}, \quad \Phi_{m|} = \frac{\delta}{\delta J_\Phi} W(J_\Phi, L), \tag{4.4}$$

has the quantum numbers $\varepsilon(\Gamma) = 0$, $gh(\Gamma) = 0$, $\dim(\Gamma) = 0$, $\varepsilon_f(\Gamma) = 0$, and satisfies the relations

$$\Gamma(\Phi_{m|}, L) \frac{\overleftarrow{\delta}}{\delta \Phi_{m|}} = -J_\Phi(\Phi_{m|}, L),$$

$$\Gamma(\Phi_{m|}, L) \frac{\overleftarrow{\delta}}{\delta L^A} = W(J_\Phi, L) \frac{\overleftarrow{\delta}}{\delta L^A}. \tag{4.5}$$

Functional average of the Eqs. (2.21)–(2.25) with substitution $S_{ext} \rightarrow S_R$ yields the corresponding equations for the functional $\Gamma = \Gamma(\Phi_{m|}, L)$, copying the equations for S_R,

$$\int dx \left(\Gamma \frac{\overleftarrow{\delta}}{\delta Q_{m|}} \frac{\delta}{\delta Q^*} \Gamma - B_{m|} \frac{\delta}{\delta \overline{C}_{m|}} \Gamma - \theta \frac{\delta}{\delta \mathcal{B}} \Gamma \right)$$

$$+ 2\chi \xi \frac{\partial}{\partial \xi} \Gamma + \chi \int dx \left[\left(Q_{m|} \frac{\delta}{\delta Q_{m|}} - Q^* \frac{\delta}{\delta Q^*} \right. \right.$$

$$\left. \left. - \overline{C}_{m|} \frac{\delta}{\delta \overline{C}_{m|}} - B_{m|} \frac{\delta}{\delta B_{m|}} \right) \Gamma \right] = 0, \tag{4.6}$$

$$\Gamma \overleftarrow{H^\alpha_{m|}} \omega_\alpha = 0, \tag{4.7}$$

where $\overleftarrow{H^\alpha_{m|}} \omega_\alpha$ is given by the expression (2.23) with the replacement $\Phi \rightarrow \Phi_{m|}$,

$$\frac{\delta}{\delta B^\alpha_{m|}} \Gamma = D^{\alpha\beta}_\mu(\mathcal{B}) A^\beta_{m|\mu} + \xi B^\alpha_{m|}, \tag{4.8}$$

$$D^{\alpha\beta}_\mu(\mathcal{B}) \frac{\delta}{\delta A^{*\beta}_\mu} \Gamma - \frac{\delta}{\delta \overline{C}^\alpha_{m|}} \Gamma = -g f^{\alpha\beta\gamma} A^\beta_{m|\mu} \theta^\gamma_\mu \tag{4.9}$$

Represent the functional Γ in the following form

$$\Gamma = \Gamma_{00} + \Gamma^{(1)} + \chi \Gamma^{(2)}, \tag{4.10}$$

where

$$\Gamma_{00} = \int dx \left(B_{m|} D(\mathcal{B}) A_{m|} + \frac{\xi}{2} B^2_{m|} + g \theta^\alpha_\mu f^{\alpha\beta\gamma} A^\beta_{m|\mu} \overline{C}^\gamma_{m|} \right), \tag{4.11}$$

and the functionals $\Gamma^{(1)}$ and $\Gamma^{(2)}$ do not depend on the parameter χ. Due to the structure chosen for the functional (4.11) it follows from the Eqs. (4.8) and (4.9) that the functionals $\Gamma^{(1)}$ and $\Gamma^{(2)}$ do not depend on the fields $B^\alpha_{m|}$,

$$\frac{\delta}{\delta B_{m|}} \Gamma^{(k)} = 0, \ \Gamma^{(k)} = \Gamma^{(k)}(Q_{m|}, \overline{C}_{m|}, \mathcal{B}, Q^*, \xi, \theta),$$

$$k = 1, 2, \tag{4.12}$$

and satisfy the equations

$$\left(D^{\alpha\beta}_\mu(\mathcal{B}) \frac{\delta}{\delta A^{*\beta}_\mu} - \frac{\delta}{\delta \overline{C}^\alpha_{m|}} \right) \Gamma^{(k)} = 0, \quad k = 1, 2, \tag{4.13}$$

In its turn, the Eq. (4.6) splits in the two, one of which is closed as for the functional $\Gamma^{(1)}$,

$$\int dx \left[\Gamma^{(1)} \frac{\overleftarrow{\delta}}{\delta Q_{m|}} \frac{\delta}{\delta Q^*} \Gamma^{(1)} - g \theta^\alpha_\mu f^{\alpha\beta\gamma} \overline{C}^\beta_{m|} \frac{\delta}{\delta A^{*\gamma}_\mu} \Gamma^{(1)} \right.$$

$$\left. - \theta^\alpha_\mu \frac{\delta}{\delta \mathcal{B}^\alpha_\mu} \Gamma^{(1)} \right] = 0, \tag{4.14}$$

and the second includes both the functionals and describes their dependence on the gauge parameter ξ,

$$2\xi \frac{\partial}{\partial \xi} \Gamma^{(1)} = \int dx \left[\Gamma^{(1)} \left(\frac{\overleftarrow{\delta}}{\delta Q_{m|}} \frac{\delta}{\delta Q^*} - \frac{\overleftarrow{\delta}}{\delta Q^*} \frac{\delta}{\delta Q_{m|}} \right) \Gamma^{(2)} \right.$$

$$- \left(g \theta^\alpha_\mu f^{\alpha\beta\gamma} \overline{C}^\beta_{m|} \frac{\delta}{\delta A^{*\gamma}_\mu} + \theta^\alpha_\mu \frac{\delta}{\delta \mathcal{B}^\alpha_\mu} \right) \Gamma^{(2)} \bigg]$$

$$+ \int dx \left[\left(\overline{C}_{m|} \frac{\delta}{\delta \overline{C}_{m|}} - Q_{m|} \frac{\delta}{\delta Q_{m|}} \right. \right.$$

$$\left. \left. + Q^* \frac{\delta}{\delta Q^*} \right) \Gamma^{(1)} \right]. \tag{4.15}$$

The Eq. (4.7) rewrites now in the form of the two equations as for the functionals $\Gamma^{(1)}$ and $\Gamma^{(2)}$,

$$\Gamma^{(k)} \overleftarrow{h^\alpha_{m|}} \omega_\alpha = 0, \quad k = 1, 2, \tag{4.16}$$

where

$$\overleftarrow{h^\alpha_{m|}} \omega_\alpha = \int dx \left\{ \left[\frac{\overleftarrow{\delta}}{\delta \mathcal{B}^\beta_\mu} D^{\beta\alpha}_\mu(\mathcal{B}) \right. \right.$$

$$+ g f^{\beta\gamma\alpha} \left(\frac{\overleftarrow{\delta}}{\delta A^\beta_{m|\mu}} A^\gamma_{m|\mu} + \frac{\overleftarrow{\delta}}{\delta C^\beta_{m|}} C^\gamma_{m|} \right)$$

$$+ g f^{\beta\gamma\alpha} \left(\frac{\overleftarrow{\delta}}{\delta \overline{C}^\beta_{m|}} \overline{C}^\gamma_{m|} + \frac{\overleftarrow{\delta}}{\delta A^{*\beta}_\mu} A^{*\gamma}_\mu + \frac{\overleftarrow{\delta}}{\delta C^{*\beta}} C^{*\gamma} + \frac{\overleftarrow{\delta}}{\delta \theta^\beta_\mu} \theta^\gamma_\mu \right)$$

$$- g t^\alpha_{jk} \left(\frac{\overleftarrow{\delta}}{\delta \psi_{m|j}} \psi_{m|k} + \frac{\overleftarrow{\delta}}{\delta \overline{\psi}^*_j} \overline{\psi}^*_k \right)$$

$$+ g \left(\frac{\overleftarrow{\delta}}{\delta \overline{\psi}_{m|j}} \overline{\psi}_{m|k} + \frac{\overleftarrow{\delta}}{\delta \psi^*_j} \psi^*_k \right) t^\alpha_{kj} \right] \omega_\alpha \bigg\}. \tag{4.17}$$

As for the Eq. (4.13), it is convenient to introduce the variables $\mathcal{A}^{*\alpha}_\mu = \mathcal{A}^{*\alpha}_\mu(x)$, $\mathcal{A}^{*\alpha}_{m|\mu} = \mathcal{A}^{*\alpha}_{m|\mu}(x)$

$$\mathcal{A}^{*\alpha}_\mu = A^{*\alpha}_\mu - D^{\alpha\beta}_\mu(\mathcal{B}) \overline{C}^\beta, \quad \mathcal{A}^{*\alpha}_{m|\mu} = A^{*\alpha}_\mu - D^{\alpha\beta}_\mu(\mathcal{B}) \overline{C}^\beta_{m|}, \tag{4.18}$$

and to use the following convention

$$\mathcal{A}^\alpha_\mu = A^\alpha_\mu, \tag{4.19}$$

as for the sake of uniformity. Also, introduce the new functionals $\tilde{\Gamma}^{(k)}$ by the rule,

$$\tilde{\Gamma}^{(k)}(\mathcal{B}, \overline{C}_{m|}, \mathcal{A}^*_{m|}, \Lambda_{m|})$$

$$= \Gamma^{(k)}(\mathcal{B}, \overline{C}_{m|}, A^*, \Lambda_{m|}) \big|_{A^* \rightarrow \mathcal{A}^*_{m|} + D^{\alpha\beta}_\mu(\mathcal{B}) \overline{C}^\beta_{m|}}, \tag{4.20}$$

where the notation

$$\Lambda = \{Q, \psi^*, \overline{\psi}^*, C^*, \xi, \theta\},$$

$$\Lambda_{m|} = \{Q_{m|}, \psi^*, \overline{\psi}^*, C^*, \xi, \theta\} \tag{4.21}$$

is used. With the definitions (4.18)–(4.20) taken into account, we have

$$\frac{\delta}{\delta A_\mu^{*\alpha}} \Gamma^{(k)} = \frac{\delta}{\delta \mathcal{A}_{m|\mu}^{*\alpha}} \tilde{\Gamma}^{(k)}, \tag{4.22}$$

$$\frac{\delta}{\delta \overline{C}_{m|}^\alpha} \Gamma^{(k)} = \frac{\delta}{\delta \overline{C}_{m|}^\alpha} \tilde{\Gamma}^{(k)} + D_\mu^{\alpha\beta}(\mathcal{B}) \frac{\delta}{\delta \mathcal{A}_{m|\mu}^{*\beta}} \tilde{\Gamma}^{(k)}, \tag{4.23}$$

$$\frac{\delta}{\delta \mathcal{B}_\mu^\alpha} \Gamma^{(k)} = \frac{\delta}{\delta \mathcal{B}_\mu^\alpha} \tilde{\Gamma}^{(k)} - g f^{\alpha\beta\gamma} \overline{C}_{m|}^\beta \frac{\delta}{\delta \mathcal{A}_{m|\mu}^{*\gamma}} \tilde{\Gamma}^{(k)},$$
$$k = 1, 2. \tag{4.24}$$

Then, we find from the Eqs. (4.13), (4.20), (4.22), (4.23) that

$$\frac{\delta}{\delta \overline{C}_{m|}^\alpha} \tilde{\Gamma}^{(k)} = 0, \tag{4.25}$$

the functionals $\tilde{\Gamma}^{(k)}$, $k = 1, 2$, do not depend on the fields $\overline{C}_{m|}^\alpha$,

$$\tilde{\Gamma}^{(k)} = \tilde{\Gamma}^{(k)}(\Omega_{m|}, \Omega_{m|}^*, \mathcal{B}, \xi, \theta). \tag{4.26}$$

Henceforth we use the notations

$$\Omega_{m|} = \{\mathcal{A}_{m|}, \psi_{m|}, \overline{\psi}_{m|}, C_{m|}\}, \quad \mathcal{A} = A,$$
$$\Omega_{m|}^* = \{\mathcal{A}_{m|}^*, \psi^*, \overline{\psi}^*, C^*\}, \quad k = 1, 2. \tag{4.27}$$

Now, with (4.20), (4.26), (4.27) taken into account, the ones (4.14), (4.15) rewrite in the form

$$\frac{1}{2}(\tilde{\Gamma}^{(1)}, \tilde{\Gamma}^{(1)}) - \int dx \left(\theta \frac{\delta}{\delta \mathcal{B}} \right) \tilde{\Gamma}^{(1)} = 0, \tag{4.28}$$

$$2\xi \frac{\partial}{\partial \xi} \tilde{\Gamma}^{(1)} = (\tilde{\Gamma}^{(1)}, \tilde{\Gamma}^{(2)})$$
$$+ \int dx \left(\Omega_{m|}^* \frac{\delta}{\delta \Omega_{m|}^*} - \Omega_{m|} \frac{\delta}{\delta \Omega_{m|}} \right) \tilde{\Gamma}^{(1)}$$
$$- \int dx \left(\theta \frac{\delta}{\delta \mathcal{B}} \right) \tilde{\Gamma}^{(2)}, \tag{4.29}$$

where the notation for the antibracket [8,9] is used,

$$(F, G) = \frac{1}{2} F \int dx \left(\frac{\overleftarrow{\delta}}{\delta \Omega_{m|}} \frac{\delta}{\delta \Omega_{m|}^*} - \frac{\overleftarrow{\delta}}{\delta \Omega_{m|}^*} \frac{\delta}{\delta \Omega_{m|}} \right) G. \tag{4.30}$$

Further, with the relations (4.20), (4.26) and

$$\Gamma^{(k)} \int dx \left[\frac{\overleftarrow{\delta}}{\delta \mathcal{B}_\mu^\beta} D_\mu^{\beta\alpha}(\mathcal{B}) \right.$$
$$\left. + g f^{\beta\gamma\alpha} \left(\frac{\overleftarrow{\delta}}{\delta \overline{C}_{m|}^\beta} \overline{C}_{m|}^\gamma + \frac{\overleftarrow{\delta}}{\delta A_\mu^{*\beta}} A_\mu^{*\gamma} \right) \right] \tag{4.31}$$
$$= \tilde{\Gamma}^{(k)} \int dx \left[\frac{\overleftarrow{\delta}}{\delta \mathcal{B}_\mu^\beta} D_\mu^{\beta\alpha}(\mathcal{B}) + g f^{\beta\gamma\alpha} \frac{\overleftarrow{\delta}}{\delta \mathcal{A}_{m|\mu}^{*\beta}} \mathcal{A}_{m|\mu}^{*\gamma} \right]$$

we find that

$$\Gamma^{(k)} \overleftarrow{\tilde{h}_{m|}^\alpha} \omega_\alpha = \tilde{\Gamma}^{(k)} \overleftarrow{\tilde{h}_{m|}^\alpha} \omega_\alpha = 0, \quad k = 1, 2, \tag{4.32}$$

where the operator $\overleftarrow{\tilde{h}_{m|}^\alpha} \omega_\alpha$ is defined in the equality (3.14) with the replacement $\Omega \to \Omega_{m|}$, $\Omega^* \to \Omega_{m|}^*$.

Then, when studying the tensor structure of divergence parts of the generating functional of vertexes, it is convenient to use a consequence of the Eq. (4.32) in particular case $\omega_\alpha(x) = \text{const}$, i.e. as to a global $T_{m|}$-symmetry:

$$\tilde{\Gamma}^{(k)} \overleftarrow{T_{m|}^\alpha} = 0, \ k = 1, 2, \tag{4.33}$$

where the operators $\overleftarrow{T_{m|}^\alpha}$ are defined by the Eq. (3.16) with the replacement $\Omega \to \Omega_{m|}$, $\Omega^* \to \Omega_{m|}^*$.

5 Renormalization

In that section we study the structure of renormalizations, and show the multiplicative character of the renormalizabiliuty of the model considered. The main role in that study is given to resolving the extended master-equation (3.12) and the one (3.19) describing the gauge dependence. We show that the renormalized quantum action and the effective action satisfy exactly their master equations to each subsequent order in loops. In this resolving, the structure of the renormalized quantum action is determined by the same monomials in fields and antifields as it does for the non-renormalized quantum action with constants determined by the divergencies of the effective action. For the sake of notational simplicity, we omit lower case $m|$ of any arguments of any functionals.

5.1 Tree approximation ($\eta = 0$)

Consider the tree approximation for the functional Γ, $\Gamma_0 = S_{\text{ext}}$, written in new variables as

$$\Gamma_0 = \Gamma_{00} + \Gamma_0^{(1)} + \chi \Gamma_0^{(2)}, \tag{5.1}$$
$$\Gamma_0^{(1)} = \tilde{\Gamma}_0^{(1)}, \quad \Gamma_0^{(2)} = \tilde{\Gamma}_0^{(2)}, \tag{5.2}$$

where

$$\Gamma_{00} = \int dx \left(B^\alpha D_\mu^{\alpha\beta}(\mathcal{B}) A_\mu^\beta + \frac{\xi}{2} B^\alpha B^\alpha + g \theta_\mu^\alpha f^{\alpha\beta\gamma} A_\mu^\beta \overline{C}^\gamma \right). \tag{5.3}$$

Represent the functional $\tilde{\Gamma}_0^{(1)}$ in the form

$$\tilde{\Gamma}_0^{(1)} = \Gamma_{0\theta} + \Gamma_{0\Omega^*} + \Gamma_{0\psi} + \Gamma_{0\mathcal{AB}},$$
$$\Gamma_{0\psi} = \Gamma_{0\psi|1} + \Gamma_{0\psi|2}, \quad \Gamma_{0\Omega^*} = \Gamma_{0\mathcal{A}^*}$$
$$+ \Gamma_{0C^*} + \Gamma_{0\psi^*} + \Gamma_{0\overline{\psi}^*}, \tag{5.4}$$

where the following notations being further useful

$$\Gamma_{0\theta} = \int dx \, \mathcal{A}_\mu^{*\alpha} \theta_\mu^\alpha, \tag{5.5}$$

$$\Gamma_{0\mathcal{A}^*} = \int dx \, \mathcal{A}_\mu^{*\alpha} D_\mu^{\alpha\beta}(\mathcal{A} + \mathcal{B})C^\beta, \tag{5.6}$$

$$\Gamma_{0C^*} = \int dx \, \frac{g}{2} f^{\alpha\beta\gamma} C^{*\alpha} C^\beta C^\gamma, \tag{5.7}$$

$$\Gamma_{0\psi^*} = -\int dx \, g \psi_j^* t_{jk}^\alpha \psi_k C^\alpha,$$

$$\Gamma_{0\overline{\psi}^*} = \int dx \, g \overline{\psi}_j^* t_{kj}^\alpha \overline{\psi}_k C^\alpha, \tag{5.8}$$

$$\Gamma_{0\psi|1} = \int dx \left[i \overline{\psi} \gamma^\mu D_{\psi\mu}(\mathcal{A} + \mathcal{B}) \psi \right],$$

$$\Gamma_{0\psi|2} = -m \int dx \, \overline{\psi} \, \psi, \tag{5.9}$$

$$\Gamma_{0\mathcal{AB}} = -\int dx \, \frac{1}{4} G_{\mu\nu}^\alpha(\mathcal{A} + \mathcal{B}) G_{\mu\nu}^\alpha(\mathcal{A} + \mathcal{B}), \tag{5.10}$$

are introduced. In its turn, the functional $\tilde{\Gamma}_0^{(2)}$ has the form

$$\tilde{\Gamma}_0^{(2)} = \int dx \left[\mathcal{A}_\mu^{*\alpha} \mathcal{A}_\mu^\alpha + C^{*\alpha} C^\alpha + \psi_j^* \psi_j + \overline{\psi}_j^* \overline{\psi}_j \right]. \tag{5.11}$$

Remind that the functional Γ_0 satisfies the Eqs. (2.21)–(2.25).

5.2 (l+1)-loop approximation

We carry out the proof of the multiplicative renormalizability via the mathematical induction method in the framework of loop expansion of the effective action with the use of the minimal subtraction scheme. To this end we suppose that we managed to find such parameters $Z_i^{[l]}$,

$$Z_i^{[l]} = \sum_{n=0}^l \eta^n z_{i,n}, \quad i = 5, 6, 11, 14, 15, 16,$$

$$\dot{Z}_{14}^{[l]} = \dot{Z}_{15}^{[l]} = 0,$$

$$1 - 2\xi \frac{\dot{Z}_5^{[l]}}{Z_5^{[l]}} = Z_1^{[l]} + O(\eta^{l+1}),$$

$$1 + 2\xi \left(\frac{\dot{Z}_6^{[l]}}{Z_6^{[l]}} - \frac{\dot{Z}_5^{[l]}}{Z_5^{[l]}} \right) = Z_2^{[l]} + O(\eta^{l+1}),$$

$$1 + \xi \frac{\dot{Z}_{11}^{[l]}}{Z_{11}^{[l]}} + Z_{16}^{[l]} = Z_3^{[l]} + O(\eta^{l+1}),$$

$$1 + \xi \frac{\dot{Z}_{11}^{[l]}}{Z_{11}^{[l]}} - Z_{16}^{[l]} = Z_4^{[l]} + O(\eta^{l+1}), \tag{5.12}$$

that the l-loop approximation for Γ, $\Gamma^{[l]} = \sum_{n=0}^l \eta^n \Gamma_n$, is a finite functional. We are to show that it is possible to pick up

the $l + 1$-loop approximation for Z_i,

$$Z_i = Z_i^{[l]} + z_{i,l+1} + O(\eta^{l+2}), \quad i = 5, 6, 11, 14, 15, 16,$$

$$\dot{z}_{14,l+1} = \dot{z}_{15,l+1} = 0, \tag{5.13}$$

which does compensate the divergences of $l+1$-loop approximation for the functional Γ.

Represent the action S_R in the form

$$S_R = S_R^{[l]} + \eta^{l+1} s_{l+1} + O(\eta^{l+2}), \tag{5.14}$$

where $S_R^{[l]}$ is the action S_R with independent parameters Z_i replaced by $Z_i^{[l]}$, and satisfying the Eqs. (2.21)–(2.25), and the functional s_{l+1} reads

$$s_{l+1} = s_{l+1}^{(1)} + \chi s_{l+1}^{(2)}. \tag{5.15}$$

For the functional $s_{l+1}^{(1)}$ we use the representation

$$s_{l+1}^{(1)} = s_{\theta,l+1} + s_{\Omega^*,l+1} + s_{\psi,l+1} + s_{\mathcal{AB},l+1}, \tag{5.16}$$

where

$$s_{\theta,l+1} = z_{5,l+1} \Gamma_{0\theta}, \tag{5.17}$$

$$s_{\mathcal{A}^*,l+1} = z_{6,l+1} \Gamma_{0\mathcal{A}^*} - z_{5,l+1} \mathcal{A} \partial_\mathcal{A} \Gamma_{0\mathcal{A}^*}, \tag{5.18}$$

$$s_{C^*,l+1} = (z_{6,l+1} - z_{5,l+1}) \Gamma_{0C^*}, \tag{5.19}$$

$$s_{\psi^*,l+1} = (z_{6,l+1} - z_{5,l+1}) \Gamma_{0\psi^*},$$

$$s_{\overline{\psi}^*,l+1} = (z_{6,l+1} - z_{5,l+1}) \Gamma_{0\overline{\psi}^*}, \tag{5.20}$$

$$s_{\psi,l+1} = z_{11,l+1} \Gamma_{0\psi|1} - z_{5,l+1} \mathcal{A} \partial_\mathcal{A} \Gamma_{0\psi|1}$$
$$+ (z_{11,l+1} + z_{15,l+1}) \Gamma_{0\psi|2}, \tag{5.21}$$

$$s_{\mathcal{AB},l+1} = z_{14,l+1} \Gamma_{0\mathcal{AB}} - z_{5,l+1} \mathcal{A} \partial_\mathcal{A} \Gamma_{0\mathcal{AB}}. \tag{5.22}$$

In its turn, the functional $s_{l+1}^{(2)}$ has the form

$$s_{l+1}^{(2)} = \int dx \Big[2\xi \dot{z}_{5,l+1} \mathcal{A}^* \mathcal{A} + 2\xi (\dot{z}_{6,l+1} - \dot{z}_{5,l+1}) C^* C$$
$$+ \xi (\dot{z}_{11,l+1} + z_{16,l+1}) \psi^* \psi +$$
$$+ \xi (\dot{z}_{11,l+1} - z_{16,l+1}) \overline{\psi}^* \overline{\psi} \Big]. \tag{5.23}$$

Here (and below in this section) we use the abbreviation to denote the variational derivatives of the kind

$$\frac{\delta}{\delta \mathcal{A}} \to \partial_\mathcal{A}, \quad \mathcal{A} \partial_\mathcal{A} = \int dx \, \mathcal{A} \frac{\delta}{\delta \mathcal{A}}, \tag{5.24}$$

when it does not cause an ambiguity.

Let us study the structure of the functional Γ with the accuracy including the $(l+1)$-loop approximation. It is described by the diagrams with vertexes from the action S_R with parameters $z_{i,n}$, $i = 5, 6, 11, 14, 15, 16, 0 \le n \le l+1$, or, in other words, by vertexes from the action $S_R^{[l]}$ and from the summand s_{l+1}. As we are interested in diagrams of the loop order not higher than $l+1$, the vertexes from s_{l+1} cannot appear in loop diagrams, i.e. vertexes from s_{l+1} give the "tree" contribution to Γ, equal to $\eta^{l+1} s_{m|,l+1}$. Other diagrams are generated by

the action $S_R^{[l]}$. Let $\Gamma(S_R^{[l]})$ be the contribution of those diagrams into the functional Γ, i.e.

$$\Gamma = \Gamma(S_R^{[l]}) + \eta^{l+1} s_{m|,l+1} + O(\eta^{l+2}). \tag{5.25}$$

As the functional $S_R^{[l]}$ satisfies the Eqs. (2.12)–(2.15), the functional $\Gamma(S_R^{[l]})$ satisfies the same equations with the replacement $Q, \overline{C}, B \to Q_{m|}, \overline{C}_{m|}, B_{m|}$.

Represent the functional $\Gamma(S_R^{[l]})$ in the form

$$\Gamma(S_R^{[l]}) = \Gamma_{00} + \Gamma^{(1)}(S_R^{[l]}) + \chi \Gamma^{(2)}(S_R^{[l]}), \tag{5.26}$$

By repeating the calculations of Sect. 3 we find that

$$\Gamma^{(k)}(S_R^{[l]}|Q_{m|}, \overline{C}_{m|}, \mathcal{B}, Q^*, B_{m|}, \xi, \theta) = \tilde{\Gamma}^{(k)}$$
$$(S_R^{[l]}|\Omega_{m|}, \Omega_{m|}^*, \mathcal{B}, \xi, \theta), \quad k = 1, 2, \tag{5.27}$$

and the functionals $\tilde{\Gamma}^{(k)}(S_R^{[l]})$ satisfy the Eqs. (4.28), (4.29) and (4.32).

Represent the functionals $\tilde{\Gamma}^{(k)}(S_R^{[l]})$ in the form of sums of divergent and finite (after removing a regularization) parts. Taking into account that the functionals $\tilde{\Gamma}^{(k)}(S_R^{[l]})$ are, by assumption, finite to the n-loop approximations, $0 \le n \le l$, we obtain

$$\tilde{\Gamma}^{(k)}(S_R^{[l]}) = \tilde{\Gamma}^{(k)}(S_R^{[l]})_{\mathrm{fin}} + \eta^{l+1} \tilde{\Gamma}^{(k)}(S_R^{[l]})_{l+1,\mathrm{div}}$$
$$+ O(\eta^{l+2}), \tag{5.28}$$
$$\Gamma = \Gamma(S_R^{[l]})_{\mathrm{fin}} + \eta^{l+1} \left[\Gamma(S_R^{[l]})_{l+1,\mathrm{div}} + s_{l+1} \right]$$
$$+ O(\eta^{l+2}), \tag{5.29}$$
$$\Gamma(S_R^{[l]})_{l+1,\mathrm{div}} = \tilde{\Gamma}^{(1)}(S_R^{[l]})_{l+1,\mathrm{div}}$$
$$+ \chi \tilde{\Gamma}^{(2)}(S_R^{[l]})_{l+1,\mathrm{div}}, \tag{5.30}$$

so that the functionals $\tilde{\Gamma}^{(k)}(S_R^{[l]})_{l+1,\mathrm{div}}$ are local ones of arguments with the quantum numbers of the action S_{ext} and contain divergent terms only (the minimal subtraction scheme). Then, as a consequence of the Eqs. (4.28), (4.29) (4.32), they satisfy the following equations,

$$(\tilde{\Gamma}_0, \tilde{\Gamma}(S_R^{[l]})_{l+1,\mathrm{div}}^{(1)})$$
$$- \int dx \left(\theta \frac{\delta}{\delta \mathcal{B}} \right) \tilde{\Gamma}(S_R^{[l]})_{l+1,\mathrm{div}}^{(1)} = 0, \tag{5.31}$$
$$2\xi \frac{\partial}{\partial \xi} \tilde{\Gamma}(S_R^{[l]})_{l+1,\mathrm{div}}^{(1)} = (\tilde{\Gamma}_0^{(1)}, \tilde{\Gamma}(S_R^{[l]})_{l+1,\mathrm{div}}^{(2)})$$
$$- (\tilde{\Gamma}_0^{(2)}, \tilde{\Gamma}(S_R^{[l]})_{l+1,\mathrm{div}}^{(1)}) \tag{5.32}$$
$$- \int dx \left(\theta \frac{\delta}{\delta \mathcal{B}} \right) \tilde{\Gamma}(S_R^{[l]})_{l+1,\mathrm{div}}^{(2)}$$
$$+ \int dx \left(\Omega_{m|}^* \frac{\delta}{\delta \Omega_{m|}^*} - \Omega_{m|} \frac{\delta}{\delta \Omega_{m|}} \right) \tilde{\Gamma}(S_R^{[l]})_{l+1,\mathrm{div}}^{(1)},$$
$$\tilde{\Gamma}(S_R^{[l]})_{l+1,\mathrm{div}}^{(k)} \overleftarrow{h^\alpha} \omega_\alpha = 0, \tag{5.33}$$
$$\tilde{\Gamma}(S_R^{[l]})_{l+1,\mathrm{div}}^{(k)} \overleftarrow{T^\alpha} = 0, \quad k = 1, 2. \tag{5.34}$$

Notice that the form of the Eqs. (5.31)–(5.34) does not depend on the label l.

By taking into account the quantum numbers, axial-, Poincare-, T-symmetries, the general expression for local functional $\tilde{\Gamma}(S_R^{[l]})_{l+1,\mathrm{div}}^{(2)}$, reads

$$\tilde{\Gamma}(S_R^{[l]})_{l+1,\mathrm{div}}^{(2)} = \int dx \Big(q_{1,l+1} \mathcal{A}^* \mathcal{A} + q_{2,l+1} C^* C$$
$$+ q_{3,l+1} \psi^* \psi + q_{4,l+1} \overline{\psi}^* \overline{\psi} + q_{1,l+1}' \mathcal{A}^* \mathcal{B} \Big), \tag{5.35}$$

where $q_{i,l+1}, i = 1, 2, 3, 4,$ "Z", "p" $q_{1,l+1}'$ are arbitrary constants. Then, by using the Eq. (5.33) for $\tilde{\Gamma}(S_R^{[l]})_{l+1,\mathrm{div}}^{(2)}$, we find that $q_{1,l+1}' = 0$. The final expression for $\tilde{\Gamma}(S_R^{[l]})_{l+1,\mathrm{div}}^{(2)}$ has the form

$$\tilde{\Gamma}(S_R^{[l]})_{l+1,\mathrm{div}}^{(2)} = \int dx \Big(q_{1,l+1} \mathcal{A}^* \mathcal{A} + q_{2,l+1} C^* C$$
$$+ q_{3,l+1} \psi^* \psi + q_{4,l+1} \overline{\psi}^* \overline{\psi} \Big), \tag{5.36}$$

Notice that the functional $\tilde{\Gamma}(S_R^{[l]})_{l+1,\mathrm{div}}^{(2)}$ does not depend on the fields θ and $\mathcal{B}$.

With the expression (5.36) for $\tilde{\Gamma}(S_R^{[l]})_{l+1,\mathrm{div}}^{(2)}$, the Eq. (5.32) reduces to the following one,

$$2\xi \frac{\partial}{\partial \xi} \tilde{\Gamma}(S_R^{[l]})_{l+1,\mathrm{div}}^{(1)} - \int dx \left[q_{1,l+1} \left(\mathcal{A} \frac{\delta}{\delta \mathcal{A}} - \mathcal{A}^* \frac{\delta}{\delta \mathcal{A}^*} \right) \right.$$
$$+ q_{2,l+1} \left(C \frac{\delta}{\delta C} - C^* \frac{\delta}{\delta C^*} \right) + q_{3,l+1} \left(\psi \frac{\delta}{\delta \psi} - \psi^* \frac{\delta}{\delta \psi^*} \right)$$
$$\left. + q_{4,l+1} \left(\overline{\psi} \frac{\delta}{\delta \overline{\psi}} - \overline{\psi}^* \frac{\delta}{\delta \overline{\psi}^*} \right) \right] \tilde{\Gamma}_0^{(1)} = 0. \tag{5.37}$$

5.2.1 Solution to equation (5.31) for $\tilde{\Gamma}(S_R^{[l]})_{l+1,\mathrm{div}}^{(1)}$

Consider a solution to the Eq. (5.31) for the functional $\tilde{\Gamma}_{l+1,\mathrm{div}}^{(1)} = \tilde{\Gamma}(S_R^{[l]})_{l+1,\mathrm{div}}^{(1)}$, represented in the form,

$$\tilde{\Gamma}_{l+1,\mathrm{div}}^{(1)} = M_{\theta,l+1} + M_{\Omega^*,l+1} + M_{\psi,l+1} + M_{\mathcal{AB},l+1},$$
$$M_{\Omega^*,l+1} = M_{\mathcal{A}^*,l+1} + M_{C^*,l+1} + M_{\psi^*,l+1}$$
$$+ M_{\overline{\psi}^*,l+1}. \tag{5.38}$$

With this aim, we find first the general form of the functional $\tilde{\Gamma}_{l+1,\mathrm{div}}^{(1)}$, using the locality, the quantum numbers, axial-, Poincare-, T-symmetries and partially the gauge symmetry in the external field $\mathcal{B}$. In fact, all required calculations do copy ones performed in Sect. 3 when constructing the general form of the functional $\tilde{P}^{(1)}$ [see formulas (3.21)–(3.35) with the obvious replacements like $\tilde{P}_\theta^{(1)} \to M_\theta$]. Here, we reproduce the final results only. The functional $M_{\theta,l+1}$ has the form

$$M_{\theta,l+1} = q_{5,l+1} \int dx \, \mathcal{A}_\mu^{*\alpha}(x)\theta_\mu^\alpha(x) = q_{5,l+1} \int dx \, \mathcal{A}^*\theta. \tag{5.39}$$

For the functionals linear in antifields we find

$$M_{\mathcal{A}^*,l+1} = \int dx \Big[q_{6,l+1} \mathcal{A}_\mu^{*\alpha} D_\mu^{\alpha\beta}(\mathcal{B}) C^\beta$$

$$+ g q_{7,l+1\beta\gamma}^\alpha \mathcal{A}_\mu^{*\alpha} \mathcal{A}_\mu^\beta C^\gamma \Big], \tag{5.40}$$

$$M_{C^*,l+1} = \int dx \frac{g}{2} q_{8,l+1\beta\gamma}^\alpha C^{*\alpha} C^\beta C^\gamma, \tag{5.41}$$

$$M_{\psi^*,l+1} = -\int dx g q_{9,l+1jk}^\alpha \psi_j^* \psi_k C^\alpha, $$

$$M_{\overline\psi^*,l+1} = \int dx g q_{10,l+1kj}^\alpha \overline\psi_j^* \overline\psi_k C^\alpha, \tag{5.42}$$

where the constants "q" satisfy the equations (we omit the lower case $l+1$)

$$F_{\gamma\delta}^\alpha q_{7\delta\sigma}^\beta - q_{7\gamma\delta}^\beta F_{\delta\sigma}^\alpha = f^{\alpha\beta\lambda} q_{7\gamma\sigma}^\lambda, \tag{5.43}$$

$$F_{\gamma\delta}^\alpha q_{8\delta\sigma}^\beta - q_{8\gamma\delta}^\beta F_{\delta\sigma}^\alpha = f^{\alpha\beta\delta} q_{8\gamma\sigma}^\delta, \tag{5.44}$$

$$t_{jl}^\alpha q_{9lk}^\beta - q_{9jl}^\beta t_{lk}^\alpha = f^{\alpha\beta\gamma} q_{9jk}^\gamma, \tag{5.45}$$

$$t_{kl}^\alpha q_{10lj}^\beta - q_{10kl}^\beta t_{lj}^\alpha = f^{\alpha\beta\gamma} q_{10kj}^\gamma. \tag{5.46}$$

For the functionals $M_{\psi,l+1}$ we obtain

$$M_{\psi,l+1} = \int dx \Big[i q_{11,l+1} \overline\psi \gamma^\mu D_{\psi\mu}(\mathcal{B})\psi$$

$$+ i g q_{12,l+1,jk}^\alpha \overline\psi_j \gamma^\mu \mathcal{A}_\mu^\alpha \psi_k - m q_{13,l+1} \overline\psi\psi \Big], \tag{5.47}$$

$$t_{jr}^\alpha q_{12,l+1,rk}^\beta - q_{12,l+1,rl}^\beta t_{rk}^\alpha = f^{\alpha\beta\gamma} q_{12,l+1,jk}^\gamma. \tag{5.48}$$

As the coefficient of the θ_μ^α term should be zero, it follows that the equation

$$q_{5,l+1}\partial_\mathcal{A}(\Gamma_{0\mathcal{A}^*} + \Gamma_{0\psi} + \Gamma_{0\mathcal{A}\mathcal{B}}) + \partial_\mathcal{A}(M_{\mathcal{A}^*,l+1} + M_{\psi,l+1}$$

$$+ M_{\mathcal{A}\mathcal{B},l+1})$$

$$- \partial_\mathcal{B}(M_{\mathcal{A}^*,l+1} + M_{\psi,l+1} + M_{\mathcal{A}\mathcal{B},l+1}) = 0, \tag{5.49}$$

holds. As the coefficient of the $\overline\psi\psi$ vertex of the Eq. (5.49) should be zero, it follows that

$$q_{12,l+1,jk}^\alpha = q_{12,l+1} t_{jk}^\alpha, \quad q_{12,l+1} = q_{11,l+1} - q_{5,l+1}. \tag{5.50}$$

In turn, as the coefficient of the $\mathcal{A}^* C$ vertex of the Eq. (5.49) should be zero, it follows that

$$q_{7,l+1,\beta\gamma}^\alpha = q_{7,l+1} f^{\alpha\beta\gamma}, \quad q_{7,l+1} = q_{6,l+1} - q_{5,l+1}. \tag{5.51}$$

When inserting the expressions found for the $q_{12,l+1,jk}^\alpha$ and $q_{7,l+1,\beta\gamma}^\alpha$ coefficients into the Eq. (5.49), it reduces to the following one

$$q_{5,l+1}\partial_{\mathcal{A}_\mu^\alpha}\Gamma_{0\mathcal{A}\mathcal{B}} + \partial_{\mathcal{A}_\mu^\alpha} M_{\mathcal{A}\mathcal{B},l+1} - \partial_{\mathcal{B}_\mu^\alpha} M_{\mathcal{A}\mathcal{B},l+1} = 0. \tag{5.52}$$

The general solution to the Eq. (5.52) reads

$$M_{\mathcal{A}\mathcal{B},l+1} = -q_{5,l+1}\mathcal{A}\partial_\mathcal{A}\Gamma_{0\mathcal{A}\mathcal{B}}(V) + M_{1,l+1}(V), \quad V = \mathcal{A} + \mathcal{B}, \tag{5.53}$$

where $M_{1,l+1}(V)$ stands for an arbitrary functional of V, at the moment.

In addition, insertion of the expression (5.51) for the $q_{7,l+1,\beta\gamma}^\alpha$ coefficient into the relation (5.40) yields

$$M_{\mathcal{A}^*,l+1} = q_{6,l+1}\Gamma_{0\mathcal{A}^*}(V) - q_{5,l+1}\mathcal{A}\partial_\mathcal{A}\Gamma_{0\mathcal{A}^*}(V), \tag{5.54}$$

and insertion (5.50) for the $q_{12,l+1,jk}^\alpha$ coefficient into (5.47) gives the expression for $M_{\psi,l+1}$,

$$M_{\psi,l+1} = q_{11,l+1}\Gamma_{0\psi|1} - q_{5,l+1}\mathcal{A}\partial_\mathcal{A}\Gamma_{0\psi|1} + q_{13,l+1}\Gamma_{0\psi|2}. \tag{5.55}$$

At $\theta = 0$ the Eq. (5.31) is reduced to the one

$$\int dx (\Gamma_{0\Omega^*} + \Gamma_{0\psi} + \Gamma_{0\mathcal{A}\mathcal{B}}) \left(\overleftarrow\partial_\Omega \partial_{\Omega^*} - \overleftarrow\partial_{\Omega^*} \partial_\Omega \right)$$

$$(M_{\Omega^*,l+1} + M_{\psi,l+1} + M_{\mathcal{A}\mathcal{B},l+1}) = 0, \tag{5.56}$$

which is not more than linear in antifields.

As the coefficient of the $\mathcal{A}^* D(\mathcal{B})CC$ vertex of the Eq. (5.56) should be zero, it follows that

$$q_{8,l+1,\beta\gamma}^\alpha = q_{8,l+1} f^{\alpha\beta\gamma}, \quad q_{8,l+1} = q_{7,l+1} = q_{6,l+1} - q_{5,l+1}. \tag{5.57}$$

Next, we consider the equations which follow from (5.56) for zero-valued antifields. They split into the two sets of equations. In the first set of equations,

$$\Gamma_{0\psi} \overleftarrow\partial_\Omega \partial_{\Omega^*} M_{\Omega^*,l+1} - \Gamma_{0\Omega^*} \overleftarrow\partial_{\Omega^*} \partial_\Omega M_{\psi,l+1} = 0, \tag{5.58}$$

all vertexes contain the spinor fields. In the second ones,

$$\Gamma_{0\mathcal{A}\mathcal{B}} \overleftarrow\partial_\mathcal{A} \partial_{\mathcal{A}^*} M_{\mathcal{A}^*,l+1} - \Gamma_{0\mathcal{A}^*} \overleftarrow\partial_{\mathcal{A}^*} \partial_\mathcal{A} M_{\mathcal{A}\mathcal{B},l+1} = 0, \tag{5.59}$$

vertexes are constructed of the fields $\mathcal{A}$, $\mathcal{B}$ and their coordinate-derivatives only.

As the coefficient of the $\overline\psi\psi \, \partial_\mu C$ vertex of the Eq. (5.58) should be zero, it follows

$$q_{9,l+1,jk}^\alpha = q_{9,l+1} t_{jk}^\alpha, \quad q_{9,l+1} = q_{6,l+1} - q_{5,l+1}. \tag{5.60}$$

As the coefficient of the $\overline\psi\psi C$ vertex of the Eq. (5.58) should be zero, it follows the relation,

$$q_{10,l+1,jk}^\alpha = q_{9,l+1,jk}^\alpha = q_{9,l+1} t_{jk}^\alpha. \tag{5.61}$$

When inserting the expressions (5.53) and (5.54) for $M_{\mathcal{AB},l+1}$ and $M_{\mathcal{A}^*,l+1}$, respectively, into the Eq. (5.59), it reduces to the following equation,

$$D(V)\partial_V M_{1,j+1}(V) = 0 \Rightarrow M_{1,j+1}(V) = q_{14,l+1}\Gamma_{0\mathcal{AB}}. \tag{5.62}$$

Thus, the functional $\Gamma(S_R^{[l]})_{l+1,\mathrm{div}}$ describing the "Z", "p" $(l+1)$-loop divergences of the functional $\Gamma(S_R^{[l]})$, has the form

$$\Gamma(S_R^{[l]})_{l+1,\mathrm{div}} = \tilde{\Gamma}^{(1)}(S_R^{[l]})_{l+1,\mathrm{div}} + \chi\tilde{\Gamma}^{(2)}(S_R^{[l]})_{l+1,\mathrm{div}}, \tag{5.63}$$

where the functional $\tilde{\Gamma}^{(2)}(S_R^{[l]})_{l+1,\mathrm{div}}$ is given by the expression (5.36). As for the functional $\tilde{\Gamma}^{(1)}(S_R^{[l]})_{l+1,\mathrm{div}}$ we use the representation

$$\tilde{\Gamma}(S_R^{[l]})_{l+1,\mathrm{div}}^{(1)} = M_{\theta,l+1} + M_{\Omega^*,l+1} + M_{\psi,l+1} + M_{\mathcal{AB},l+1}, \tag{5.64}$$

where the functionals "M",

$$M_{\theta,l+1} = q_{5,l+1}\Gamma_{0\theta}, \tag{5.65}$$
$$M_{\mathcal{A}^*,l+1} = q_{6,l+1}\Gamma_{0\mathcal{A}^*}(V) - q_{5,l+1}\mathcal{A}\partial_\mathcal{A}\Gamma_{0\mathcal{A}^*}(V), \tag{5.66}$$
$$M_{C^*,l+1} = (q_{6,l+1} - q_{5,l+1})\Gamma_{0C^*}, \tag{5.67}$$
$$M_{\psi^*,l+1} = (q_{6,l+1} - q_{5,l+1})\Gamma_{0\psi^*}, \tag{5.68}$$
$$M_{\overline{\psi}^*,l+1} = (q_{6,l+1} - q_{5,l+1})\Gamma_{0\overline{\psi}^*}, \tag{5.69}$$
$$\begin{aligned}M_{\psi,l+1} &= q_{11,l+1}\Gamma_{0\psi|1} - q_{5,l+1}\mathcal{A}\partial_\mathcal{A}\Gamma_{0\psi|1} \\ &\quad + q_{13,l+1}\Gamma_{0\psi|2}, \end{aligned} \tag{5.70}$$
$$M_{\mathcal{AB},l+1} = q_{14,l+1}\Gamma_{0\mathcal{AB}} - q_{5,l+1}\mathcal{A}\partial_\mathcal{A}\Gamma_{0\mathcal{AB}}, \tag{5.71}$$

are represented in terms of the tree loop functionals "Γ_0" (5.5)–(5.10).

5.2.2 Solution to equation (5.37) for $\tilde{\Gamma}(S_R^{[l]})_{l+1,\mathrm{div}}^{(1)}$

When inserting the representation for the functional $\tilde{\Gamma}^{(1)}(S_R^{[l]})_{l+1,\mathrm{div}}$ given by (5.64)–(5.71) into the Eq. (5.37), it takes the form of zero value for some linear combinations of structures appeared in the right-hand side of formulas (5.65)–(5.71).

As the coefficient of the θ_μ^α term should be zero, it follows that

$$q_{1,l+1} = -2\xi\dot{q}_{5,l+1}. \tag{5.72}$$

Then, as the coefficient of any antifield term should be zero, it follows

$$q_{2,l+1} = 2\xi(\dot{q}_{6,l+1} - \dot{q}_{5,l+1}). \tag{5.73}$$

When the relations (5.72) and (5.73) hold, then the Eq. (5.37) reduces to the two equations having obvious solutions

$$2\xi\dot{q}_{14,l+1}\Gamma_{0\mathcal{AB}} = 0 \Rightarrow \dot{q}_{14,l+1} = 0, \tag{5.74}$$
$$\begin{aligned}&2\xi\dot{q}_{11,l+1}\Gamma_{0\psi|1} + 2\xi\dot{q}_{11,l+1}\Gamma_{0\psi|2} \\ &\quad - (q_{3,l+1} + q_{4,l+1})(\Gamma_{0\psi|1} + \Gamma_{0\psi|2}) = 0 \Rightarrow \end{aligned} \tag{5.75}$$
$$2\xi\dot{q}_{11,l+1} = 2\xi\dot{q}_{13,l+1} = q_{3,l+1} + q_{4,l+1}. \tag{5.76}$$

It is convenient to introduce new parameters $q_{15,l+1}$ and $q_{16,l+1}$,

$$q_{15,l+1} = q_{13,l+1} - q_{11,l+1}, \quad 2q_{16,l+1} = q_{3,l+1} - q_{4,l+1}, \tag{5.77}$$

in terms of which the Eq. (5.76) rewrites as

$$\begin{aligned}&\dot{q}_{15,l+1} = 0, \quad q_{3,l+1} = \xi\dot{q}_{11,l+1} + q_{16,l+1}, \\ &q_{4,l+1} = \xi\dot{q}_{11,l+1} - q_{16,l+1}. \end{aligned} \tag{5.78}$$

5.3 Finiteness of Γ to (l+1)-loop approximation

Now, let us prove that one can chose the renormalization constants in such a way as to make the effective action finite to the $(l+1)$-loop approximation. To this end, we consider the divergent part of the effective action Γ, $\Gamma_{l+1,\mathrm{div}}$, described by the Eq. (5.29),

$$\Gamma_{l+1,\mathrm{div}} = \Gamma(S_R^{[l]})_{l+1,\mathrm{div}} + s_{l+1} = \Gamma_{l+1,\mathrm{div}}^{(1)} + \chi\Gamma_{l+1,\mathrm{div}}^{(2)}. \tag{5.79}$$

For the functionals $\Gamma_{l+1,\mathrm{div}}^{(1)}$, $\Gamma_{l+1,\mathrm{div}}^{(2)}$ we have the representations

$$\begin{aligned}\Gamma_{l+1,\mathrm{div}}^{(1)} &= \Gamma(S_R^{[l]})_{l+1,\mathrm{div}}^{(1)} + s_{l+1}^{(1)} \\ &= \Gamma_{\theta,l+1,\mathrm{div}} + \Gamma_{\Omega^*,l+1,\mathrm{div}} \\ &\quad + \Gamma_{\psi,l+1,\mathrm{div}} + \Gamma_{\mathcal{AB},l+1,\mathrm{div}}, \end{aligned} \tag{5.80}$$
$$\Gamma_{\theta,l+1,\mathrm{div}} = (q_{5,l+1} + z_{5,l+1})\Gamma_{0\theta}, \tag{5.81}$$
$$\begin{aligned}\Gamma_{\mathcal{A}^*\theta,l+1,\mathrm{div}} &= (q_{6,l+1} + z_{6,l+1})\Gamma_{0\mathcal{A}^*}(V) \\ &\quad - (q_{5,l+1} + z_{5,l+1})\mathcal{A}\partial_\mathcal{A}\Gamma_{0\mathcal{A}^*}(V), \end{aligned} \tag{5.82}$$
$$\Gamma_{C^*,l+1,\mathrm{div}} = (q_{6,l+1} + z_{6,l+1} - q_{5,l+1} - z_{5,l+1})\Gamma_{0C^*}, \tag{5.83}$$
$$\Gamma_{\psi^*,l+1,\mathrm{div}} = q_{6,l+1} + z_{6,l+1} - q_{5,l+1} - z_{5,l+1})\Gamma_{0\psi^*}, \tag{5.84}$$
$$\Gamma_{\overline{\psi}^*,l+1,\mathrm{div}} = q_{6,l+1} + z_{6,l+1} - q_{5,l+1} - z_{5,l+1})\Gamma_{0\overline{\psi}^*}, \tag{5.85}$$
$$\begin{aligned}\Gamma_{\psi,l+1,\mathrm{div}} &= (q_{11,l+1} + z_{11,l+1})\Gamma_{0\psi|1} \\ &\quad - (q_{5,l+1} + z_{5,l+1})\mathcal{A}\partial_\mathcal{A}\Gamma_{0\psi|1} + \\ &\quad + (q_{11,l+1} + z_{11,l+1} \\ &\quad + q_{15,l+1} + z_{15,l+1})\Gamma_{0\psi|2}, \end{aligned} \tag{5.86}$$
$$\begin{aligned}\Gamma_{\mathcal{AB},l+1,\mathrm{div}} &= (q_{14,l+1} + z_{14,l+1})\Gamma_{0\mathcal{AB}} \\ &\quad - (q_{5,l+1} + z_{5,l+1})\mathcal{A}\partial_\mathcal{A}\Gamma_{0\mathcal{AB}}, \end{aligned} \tag{5.87}$$
$$\begin{aligned}\Gamma_{l+1,\mathrm{div}}^{(2)} &= \Gamma(S_R^{[l]})_{l+1,\mathrm{div}}^{(2)} + s_{l+1}^{(2)} \\ &= \int dx\Big[2\xi(\dot{q}_{5,l+1} + \dot{z}_{5,l+1})\mathcal{A}^*\mathcal{A} + \end{aligned}$$

$$+2\xi(\dot{q}_{6,l+1} + \dot{z}_{6,l+1} - \dot{q}_{5,l+1} - \dot{z}_{5,l+1})C^*C$$
$$+(\xi\dot{q}_{11,l+1} + \xi\dot{z}_{11,l+1} + q_{16,l+1} +$$
$$+z_{16,l+1})\psi^*\psi + (\xi\dot{q}_{11,l+1} + \xi\dot{z}_{11,l+1}$$
$$-q_{16,l+1} - z_{16,l+1})\overline{\psi}^*\overline{\psi}\Big]. \tag{5.88}$$

It follows from the formulas (5.79)–(5.88), that the choice of the parameters $z_{i,l+1}$ in the form "Z", "p"

$$z_{i,l+1} = z_{i,l+1,\text{fin}}, \quad z_{i,l+1,\text{fin}} = -q_{i,l+1},$$
$$i = 5, 6, 11, 14, 15, 16, \tag{5.89}$$

provides for zero-valued coefficients to the $(l+1)$-loop divergences,

$$\Gamma_{l+1,\text{div}}\Big|_{z_{i,l+1}=z_{i,l+1,\text{fin}}} = 0,$$
$$\Gamma_{l+1}\Big|_{z_{i,l+1}=z_{i,l+1,\text{fin}}} = \Gamma_{l+1,\text{fin}}. \tag{5.90}$$

Notice that the choice of parameters $z_{i,l+1,\text{fin}}$ is unique within the minimal subtraction scheme.

5.4 $(l+2)$-loop approximation

The renormalization of S_R to the $(l + 1)$-loop approximation allows one to construct the effective action Γ, finite to that approximation; however it does not satisfy exactly the extended master-equation and the gauge dependence equation, by itself. We show the possibility to complete the renormalization constants of the action S_R with the help of the $(l + 2)$-loop approximation, so that it will satisfy the equations mentioned to the $(l + 1)$-loop approximation and, in its turn, the corresponding effective action, finite to the $(l + 1)$-loop approximation, will satisfy the set of Eqs. (2.21)–(2.25) to that approximation.

Indeed, we represent the action S_R as

$$S_R = S_R^{[l+1]} + \eta^{l+2}s_{l+2} + O(\eta^{l+3}), \tag{5.91}$$

where $S_R^{[l+1]}$ is the action S_R with independent parameters "Z", "p" Z_i replaced by $Z_i^{[l+1]}$, and s_{l+2} is equal to

$$s_{l+2} = s_{l+2}^{(1)} + \chi s_{l+2}^{(2)}, \tag{5.92}$$

where

$$s_{l+2}^{(1)} = s_{\theta,l+2} + s_{\Omega^*,l+2} + s_{\psi,l+2} + s_{\mathcal{AB},l+2}, \tag{5.93}$$
$$s_{\theta,l+2} = z_{5,l+2}\Gamma_{0\theta}, \tag{5.94}$$
$$s_{\mathcal{A}^*,l+2} = z_{6,l+2}\Gamma_{0\mathcal{A}^*} - z_{5,l+2}\mathcal{A}\partial_{\mathcal{A}}\Gamma_{0\mathcal{A}^*}, \tag{5.95}$$
$$s_{C^*,l+2} = (z_{6,l+1} - z_{5,l+2})\Gamma_{0C^*}, \tag{5.96}$$
$$s_{\psi^*,l+2} = (z_{6,l+2} - z_{5,l+2})\Gamma_{0\psi^*}, \tag{5.97}$$
$$s_{\overline{\psi}^*,l+2} = (z_{6,l+2} - z_{5,l+2})\Gamma_{0\overline{\psi}^*}, \tag{5.97}$$
$$s_{\psi,l+2} = z_{11,l+2}\Gamma_{0\psi|1} - z_{5,l+2}\mathcal{A}\partial_{\mathcal{A}}\Gamma_{0\psi|1}$$
$$+(z_{11,l+2} + z_{15,l+2})\Gamma_{0\psi|2}, \tag{5.98}$$
$$s_{\mathcal{AB},l+2} = z_{14,l+2}\Gamma_{0\mathcal{AB}} - z_{5,l+2}\mathcal{A}\partial_{\mathcal{A}}\Gamma_{0\mathcal{AB}}, \tag{5.99}$$

$$s_{l+2}^{(2)} = \int dx\Big[2\xi\dot{z}_{5,l+2}\mathcal{A}^*\mathcal{A}$$
$$+2\xi(\dot{z}_{6,l+2} - \dot{z}_{5,l+2})C^*C +$$
$$+\xi(\dot{z}_{11,l+2} + z_{16,l+2})\psi^*\psi +$$
$$+\xi(\dot{z}_{11,l+2} - z_{16,l+2})\overline{\psi}^*\overline{\psi}\Big], \tag{5.100}$$
$$\dot{z}_{14,l+2} = \dot{z}_{15,l+2} = 0. \tag{5.101}$$

Notice that the action $S_R^{[l+1]}$ satisfies the Eqs. (2.21)–(2.25).

Further calculations and consequences from them do copy exactly the results of the previous subsection with the natural replacement $l + 1 \rightarrow l + 2$.

Also, it is obvious that the procedure of divergence compensations discussed can be applied to the case $l = 0$ so that by using the loop induction method in Feynman diagrams for the functional Γ, we arrive at the following statement: for the l-loop approximation $\Gamma^{[l]}$, where l is arbitrary positive integer,

$$\Gamma^{[l]} = \sum_{n=0}^{l} \eta^n \Gamma_n, \tag{5.102}$$

of the functional Γ defined by the relations (4.1), (4.4), there exists the uniquely defined parameters $Z_i^{[l]}$, $i = 5, 6, 11, 14, 15, 16$,

$$\dot{Z}_{14}^{[l]} = 0, \quad \dot{Z}_{15}^{[l]} = 0, \ \forall l \geq 0, \tag{5.103}$$

such that the functional $\Gamma^{[l]}$ does not contain divergences and Γ satisfies the Eqs. (4.6)–(4.9).

6 Relations between parameters of S_R and standard renormalization constants

In that section we find relations between some parameters of the action S_R and the standard renormalization constants. Within the expression for S_R, we restrict ourselves only by desired vertexes in symbolic notation

$$S_R = \int dx \left(Z_{14}Z_5^{-2}\partial A\partial A + gZ_{14}Z_5^{-3}A^2\partial A + Z_{11}\overline{\psi}\partial\psi\right.$$
$$\left. + mZ_{13}\overline{\psi}\psi + \ldots\right), \tag{6.1}$$

where the ellipsis means the rest vertexes. As the propagators of fields A and ψ are finite, they should be considered as renormalized fields. Then, we find:

$$Z_A = Z_{14}^{1/2}Z_5^{-1}, \quad Z_\psi = Z_{11}^{1/2}, \tag{6.2}$$

where Z_A and Z_ψ are the renormalization constants of the bare fields A_0 and ψ_0. The coefficient of the second vertex in the expression (6.1) gives the renormalization for vertex A^3,

$$Z_{A^3} = Z_{14}Z_5^{-3} \ \Rightarrow \ g_0 = Z_g g, \ Z_g = Z_{A^3}Z_A^{-3} = Z_{14}^{-1/2}. \tag{6.3}$$

The coefficient of the forth vertex in the expression (6.1) gives the renormalization for vertex $\overline{\psi}\psi$,

$$Z_{\overline{\psi}\psi} = Z_{13} \;\Rightarrow\; m_0 = Z_m m,$$
$$Z_m = Z_{\overline{\psi}\psi} Z_\psi^{-2} = Z_{13}/Z_{11} = Z_{15}. \qquad (6.4)$$

It follows from the Eq. (5.103) that the renormalization constants of physical parameters g and m do not depend on gauge,

$$\partial_\xi Z_g = 0, \quad \partial_\xi Z_m = 0. \qquad (6.5)$$

7 Summary

In the present paper, within the background field formalism, it is studied the renormalization procedure and the gauge dependence of the theory of Yang–Mills fields interacting with a multiplet of massive spinor fields. It is shown that the extension of the Faddeev–Popov action with extra fields and parameters allows one to establish the multiplicative character of the renormalizability. The proofs given above are based on the possibility to expand the effective action in loops, as well as to use the minimal subtraction scheme as to eliminate divergences. It is a new and important result that the renormalization constant of the mass parameter is shown to be gauge-independent.

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