

Holevo bound of entropic uncertainty in Schwarzschild spacetime

Jin-Long Huang¹, Wen-Cong Gan^{2,3}, Yunlong Xiao⁴, Fu-Wen Shu^{2,3,a}, Man-Hong Yung^{5,6,b}

¹ Department of Physics, Southern University of Science and Technology, Shenzhen 518055, China

² Department of Physics, Nanchang University, No. 999 Xue Fu Avenue, Nanchang 330031, China

³ Center for Relativistic Astrophysics and High Energy Physics, Nanchang University, No. 999 Xue Fu Avenue, Nanchang 330031, China

⁴ Department of Mathematics and Statistics and Institute for Quantum Science and Technology, University of Calgary, Calgary, AB T2N 1N4, Canada

⁵ Institute for Quantum Science and Engineering and Department of Physics, Southern University of Science and Technology, Shenzhen 518055, China

⁶ Shenzhen Key Laboratory of Quantum Science and Engineering, Southern University of Science and Technology, Shenzhen 518055, China

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Abstract For a pair of incompatible quantum measurements, the total uncertainty can be bounded by a state-independent constant. However, such a bound can be violated if the quantum system is entangled with another quantum system (called memory); the quantum correlation between the systems can reduce the measurement uncertainty. On the other hand, in a curved spacetime, the presence of the Hawking radiation can reduce quantum correlation. The interplay of quantum correlation in the curved spacetime has become an interesting arena for studying uncertainty relations. Here we demonstrate that the bounds of the entropic uncertainty relations, in the presence of memory, can be formulated in terms of the Holevo quantity, which limits how much information can be encoded in a quantum system. Specifically, we considered examples with Dirac fields with and without spin, near the event horizon of a Schwarzschild black hole, the Holevo bound provides a better bound than the previous bound based on the mutual information. Furthermore, as the memory moves away from the black hole, the difference between the total uncertainty and the new lower bound remains a constant, not depending on any property of the black hole.

1 Introduction

Traditionally, uncertainty principle in quantum mechanics has been formulated in terms of variance [1–15], while in the context of both classical and quantum information sci-

ences, it is more natural to use entropy to quantify uncertainties. The first entropic uncertainty relation (EUR) for position and momentum was given in [16], which can be shown to be equivalent to Heisenberg's original relation [17]. Later Deutsch [18] formulated entropic uncertainty relation for any pair of observables with bounded spectrums. An improvement of Deutsch's entropic uncertainty relation was subsequently conjectured by Kraus [19] and later proved by Maassen and Uffink [20]. Recently, more improvements have been formulated [21–25]. However, if the measured system ρ_A is prepared with a quantum memory ρ_B , correlation between ρ_A and ρ_B will decrease the entropic uncertainties of ρ_A . The entropic uncertainty relation in the presence of quantum memory was proposed by Berta et al.:

$$H(M_1|B) + H(M_2|B) \geq U_1, \quad (1)$$

where $U_1 \equiv -\log c_1 + H(A) - \mathcal{I}(A : B)$ [26], with $H(\rho) \equiv -\text{tr}(\rho \ln \rho)$ is von Neumann entropy for density matrix ρ and $\mathcal{I}(A : B) = H(A) + H(B) - H(A, B)$ is the quantum mutual information.

However, as the correlation between two parties changes, is this mutual information $\mathcal{I}(A : B)$ a good measure to quantify how much the uncertainty will change? Another candidate is the well-known Holevo quantity (or Holevo bound) [27], which has a wide range of applications [28–30]. Reference [28] proves a universal bound on the quantum channel capacity for two distant systems. Holevo quantity is used to connect mutual information and an upper bound for vacuum-subtracted entropy of signal ensemble. In Ref. [29], authors adopt Holevo information to quantify distinguishability of black hole microstates by measurements performed on subregion A of a Cauchy surface. Their calculation is

Jin-Long Huang, Wen-Cong Gan and Yunlong Xiao contributed equally.

^a e-mail: shufuwen@ncu.edu.cn

^b e-mail: yung@sustc.edu.cn

based on the assumption that the vacuum conformal block dominates in the entropy calculation. Reference [30] added a correction term when this assumption fails. In these articles quantum channel is considered from a global state to a state restricted in a subregion, while in this paper quantum channel is from two parties ρ_{AB} to one party ρ_B . This channel depends on Unruh effect and measurements performed by Alice.

Using Holevo quantity, we generalized entropic uncertainty with quantum memory. Similar improvement has considered before in [31,32]. The only difference is that they considered Holevo quantity on Bob’s ensemble, which is prepared after Alice optimizes her measurement choices. Instead, our lower bound is dependent on Alice’s measurement choice. Let $\mathcal{J}(B|M_1) \equiv H(B) - \sum_j p_j^1 H(\rho_{B|u_j^1})$ be the Holevo quantity for Bob about Alice’s M_1 measurement outcomes. The new uncertainty relation says that

$$H(M_1|B) + H(M_2|B) \geq U_2, \tag{2}$$

where $U_2 \equiv -\log c_1 + H(A) - \mathcal{J}(B|M_1) - \mathcal{J}(B|M_2)$. The new entropic uncertainty relation has the property that the difference (which we denote as Δ_2 below) between entropic uncertainty (LHS of (2)) and the new uncertainty lower bound U_2 only depends on incompatible measurements M_1, M_2 and ρ_A , and independent of quantum memory,

$$\Delta_2^{M_1 M_2} = H(M_1) + H(M_2) + \log c_1 - H(A). \tag{3}$$

That is to say, when we change quantum memory ρ_B , the change of entropic uncertainty can be completely reflected by the change of lower bound U_2 . This is a remarkable result which means that the new lower bound U_2 can capture the characteristic how the entropic uncertainty would behave corresponding to different quantum memory ρ_B , since the difference between U_2 and LHS is always a constant. While for U_1 the difference between LHS and U_1 may increase or decrease as the quantum memory ρ_B changes. Thus Holevo quantity $\mathcal{J}(B|M_1) + \mathcal{J}(B|M_2)$, as a new measure of correlation, rather than previous measure $\mathcal{I}(A : B)$, describes the underlying quantity between the quantum memory and entropic uncertainty. Additionally, the terms $-\mathcal{J}(B|M_1) - \mathcal{J}(B|M_2)$ in the RHS of (2) can be further lower bounded according to enhanced Information Exclusion Relations [33].

While entropic uncertainty can be reduced by quantum correlation, quantum correlation can be affected by Hawking radiation in curved spacetime [34–39]. It has been shown in [34–39] that Hawking radiation can reduce mutual information $\mathcal{I}(A : B)$ and thus increase entropic uncertainty. It is interesting to investigate how Hawking radiation would modify the new uncertainty relation (2). To demonstrate this, we calculated the simplest black hole: Schwarzschild black hole, and the examples we considered are Dirac fields states.

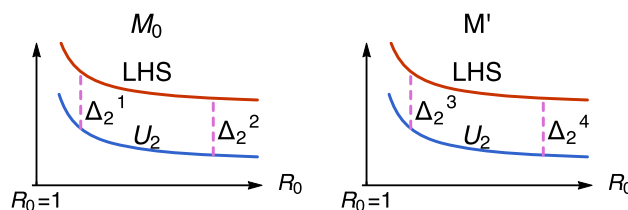


Fig. 1 Entropic uncertainty (LHS) and lower bound U_2 using accessible information for black hole with mass M_0 and M' . $\Delta_2^1 = \Delta_2^2 = \Delta_2^3 = \Delta_2^4$ where Δ_2^1 and Δ_2^2 correspond to different relative distance $R_0 = r_0/2M$ from event horizon. As shown above, at event horizon $R_0 = 1$. It is sufficient to conduct experiments near *one* Schwarzschild black hole with mass M_0 to obtain LHS. For any other Schwarzschild black holes with mass M' , we can precisely predict its LHS using LHS for M_0, Δ_2 and U_2 for M'

In this case quantum states are superposition of vacuum state and excited states of Dirac fields. We believe similar results can hold for other black holes and fields. In order to compare with results in [39], spinless field states with basis $|0\rangle$ and $|1\rangle$ are also calculated. In Schwarzschild spacetime, we consider the case in which the quantum memory ρ_B hovers near the event horizon outside the black hole and the measured system ρ_A is free falling. It has been known that the entanglement between ρ_A and ρ_B would degrade when ρ_B get closer to the event horizon due to Unruh effect on ρ_B [34], so the lower bound for entropic uncertainty would increase [39]. Because Δ_2 is independent of ρ_B , U_2 is always a constant away from LHS. In other words, when quantum memory gets closer to the event horizon, the correlation between it and measured system is decreased, and the decreased correlation is equal to increased entropic uncertainty (Fig. 1).

From an experiment point of view [40,41], a proper uncertainty game [17] can be conducted to measure Bob’s uncertainty LHS about Alice’s measurement outcomes for a particular Schwarzschild black hole with mass M_0 . U_2 and Δ_2 can be calculated for this black hole from Rindler decomposition. Since the difference Δ_2 between LHS and U_2 is independent of mass of black hole M , energy ω of quantum state and the relative distance R_0 of quantum memory from event horizon, Δ_2 for different ρ_B in different black hole backgrounds can be obtained. Fixing the energy of mode ω , for an arbitrary Schwarzschild black hole with different mass M' , we can predict Bob’s entropic uncertainty accurately without conducting any other new experiments.

This article is organized as follows. In Sect. 2 we propose to use Holevo χ -quantity as a part of lower bound for entropic uncertainty relation with quantum memory, and prove that the difference between two sides of this inequality is independent of quantum memory B . Then in Sect. 3 by calculating different examples in Schwarzschild spacetime with Dirac field states, we demonstrate that the new lower bound U_2 is tighter than previous bound U_1 , and more importantly, the Holevo quantity $\mathcal{J}(B|M_1) + \mathcal{J}(B|M_2)$ serves as a better

correlation measure to reveal how quantum memory would change entropic uncertainty.

2 Entropic uncertainty relation and information exclusion principle

Entropic uncertainty relation proved by Maassen and Uffink [20] is (we use base 2 log throughout this paper),

$$H(M_1) + H(M_2) \geq \log \frac{1}{c_1}, \tag{4}$$

where $M_1 = \{|u_j\rangle\}$ and $M_2 = \{|v_k\rangle\}$ are two orthonormal bases on d -dimensional Hilbert space \mathcal{H}_A , and $H(M_1) = -\sum_j p_j \log p_j$ is the Shannon entropy of the probability distribution $\{p_j = \langle u_j | \rho_A | u_j \rangle\}$ for state ρ_A of \mathcal{H}_A (similarly for $H(M_2)$ and $\{q_k = \langle v_k | \rho_A | v_k \rangle\}$). The number c_1 is the largest overlap among all $c_{jk} = |\langle u_j | v_k \rangle|^2 (\leq 1)$ between M_1 and M_2 .

When measured system is a mixed state, EUR (4) can be improved as [26]

$$H(M_1) + H(M_2) \geq \log \frac{1}{c_1} + H(A), \tag{5}$$

where $H(A)$ characterize the amount of uncertainty increased by the mixedness of A .

However, if the measured system A is prepared with a quantum memory B , then the entropic uncertainties in the presence of memory are

$$H(M_1|B) + H(M_2|B), \tag{6}$$

where $H(M_1|B) = H(\rho_{M_1B}) - H(\rho_B)$ is the conditional entropy with $\rho_{M_1B} = \sum_j (|u_j\rangle\langle u_j| \otimes I)(\rho_{AB})(|u_j\rangle\langle u_j| \otimes I)$ (similarly for $H(M_2|B)$). Then the difference between $H(M_1) + H(M_2)$ and $H(M_1|B) + H(M_2|B)$, i.e.

$$H(M_1) + H(M_2) - H(M_1|B) - H(M_2|B), \tag{7}$$

reveals the uncertainty decrease due to the correlations between measured system A and quantum memory B .

At the heart of information theory lies the mutual information, Shannon’s fundamental theorem [42, Chapter 12] states that the mutual information corresponding to a measurement is the average amount of error-free data which may be gained through the measurement of system. Information is a natural tool and concept in communications and physics, in an operation of measurement or communication, one may seek to maximize the gained information. This kind of optimization is trivial for classical systems [43].

One route to generalize mutual information is motivated by replacing the classical probability distribution by the density matrices of quantum systems., e. g.,

$$\mathcal{I}(A : B) = H(A) + H(B) - H(A, B). \tag{8}$$

Here, $H(A)$ stands for the von Neumann entropy of quantum state A and $H(A, B)$ denotes the information of combined system.

On the other hand, the quantum memory B , after the measurement corresponding to $|u_j\rangle$ (M_1) has been performed, becomes

$$\rho_{B|u_j} = \langle u_j | \rho_{AB} | u_j \rangle / \text{Tr}(\langle u_j | \rho_{AB} | u_j \rangle), \tag{9}$$

with probability $p_j = \text{Tr}(\langle u_j | \rho_{AB} | u_j \rangle)$. Similarly, we can define q_k and $\rho_{B|v_k}$ for measurement M_2 . Here $H(\rho_{B|u_j})$ is the missing information about quantum memory. The entropies $H(\rho_{B|u_j})$ with weighted probability p_j leads to a second quantum generalization of mutual information

$$\mathcal{J}(B|M_1) = H(B) - \sum_j p_j H(\rho_{B|u_j}). \tag{10}$$

This quantity reveals the information gained about the quantum memory through the measurement M_1 . The difference between $\mathcal{I}(A : B)$ and $\mathcal{J}(B|M_1)$ is related to quantum discord [44].

For any quantum systems, the quantity $H(M_1) + H(M_2) - H(M_1|B) - H(M_2|B)$ describes the uncertainty decrease according to the extra quantum memory, while on the same time $\mathcal{J}(B|M_1) + \mathcal{J}(B|M_2)$ is the increase of information content of observables due to quantum memory. What is the relation between the values of uncertainty decrease and information increase in the presence of quantum memory? Actually, we can rewrite $\rho_{M_1B} = \sum_j (|u_j\rangle\langle u_j| \otimes I)(\rho_{AB})(|u_j\rangle\langle u_j| \otimes I)$ as $\rho_{M_1B} = \sum_j p_j |u_j\rangle\langle u_j| \otimes \rho_B^j$, where ρ_B^j is density matrix for B if Alice measurement outcome is j . According to joint entropy theorem [45], we have $H(\rho_{M_1B}) = H(\rho_{M_1B}) - H(\rho_B) = H(M_1) - \mathcal{J}(B|M_1)$, thus

$$\begin{aligned} H(M_1) + H(M_2) - H(M_1|B) - H(M_2|B) \\ = \mathcal{J}(B|M_1) + \mathcal{J}(B|M_2), \end{aligned} \tag{11}$$

for incompatible observables M_1 and M_2 . Through this unified equation, we have shown that the increase of information content of quantum observables in the presence of quantum memory equals to the decrease of quantum uncertainties due to the extra quantum memory. Now these two fundamental concepts in quantum theory and information theory have been unified.

The entropic uncertainty relation now reads that $H(M_1|B) + H(M_2|B) \geq U_2$, where $U_2 \equiv -\log c_1 + H(A) - \mathcal{J}(B|M_1) - \mathcal{J}(B|M_2)$. The left hand side (LHS) minus right hand side (RHS) equals to $\Delta_2^{M_1 M_2} = H(M_1) + H(M_2) + \log c_1 - H(A)$ which is independent of quantum memory B . Actually, Δ_2 is the difference between LHS and RHS of (5). Put another way, in the presence of quantum memory, the amount of uncertainty decrease equals to the amount of decrease of the lower bound U_2 . This fact has been revealed in (11). In next section, we will apply this Holevo quantity generalized entropic uncertainty relation to the cases with Dirac field states in Schwarzschild spacetime.

Note that the quantity $\mathcal{J}(B|M_i)$ ($i = 1, 2$) is related to the optimal bound of the Holevo–Schumacher–Westmoreland (HSW) channel capacity [45–49], i.e.,

$$C_{\text{HSW}} = \max_{\sum_j p_j \rho_j = \rho} \left\{ H \left(\sum_j p_j \varepsilon(\rho_j) \right) - \sum_j p_j H(\varepsilon(\rho_j)) \right\}, \tag{12}$$

here ε denotes a quantum channel, and $\{p_j, \rho_j\}$ is a ensemble decomposition for the density matrix ρ . When we set $\varepsilon = \text{id}$, the HSW channel capacity degenerates to the Holevo quantity. However, if the particle (quantum memory) is prepared to be entangled with a measuring system ρ_A , then the HSW channel capacity C_{HSW} can be generalized to

$$\begin{aligned} C_{\text{GHSW}}^{\max}(M_1) &= \max_{\rho_{AB}} \left\{ H \left(\sum_j p_j \varepsilon(\rho_{B|u_j}) \right) - \sum_j p_j H(\varepsilon(\rho_{B|u_j})) \right\}, \\ C_{\text{GHSW}}^{\min}(M_1) &= \min_{\rho_{AB}} \left\{ H \left(\sum_j p_j \varepsilon(\rho_{B|u_j}) \right) - \sum_j p_j H(\varepsilon(\rho_{B|u_j})) \right\}, \end{aligned} \tag{13}$$

with measurement M_1 perform on measured system A and all $\rho_{B|u_j}$ satisfy the condition $\sum_j p_j \rho_{B|u_j} = \rho_B$. Similarly, we define the generalized HSW (GHSW) quantities for measurement M_2 , i.e. $C_{\text{GHSW}}^{\max}(M_2)$ and $C_{\text{GHSW}}^{\min}(M_2)$. The maximal value of GHSW is related with the asymptotic rate at which classical information can be transmitted over a quantum channel ε per channel use in the presence of quantum memory [45–48]. On the other hand, the minimal value of GHSW plays an important role in generalized uncertainty relations, especially the sum form $C_{\text{GHSW}}^{\min}(M_1, M_2)$,

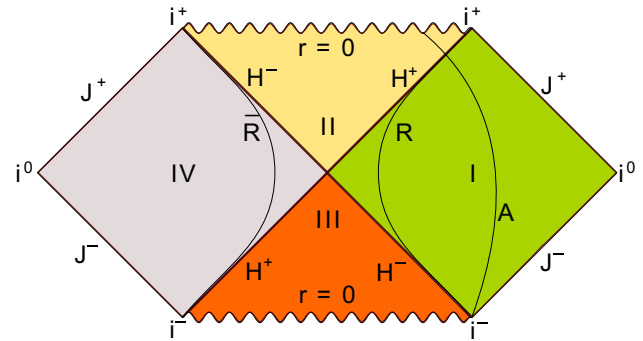


Fig. 2 Penrose diagram for maximally extended black hole which shows the world-line of Alice, Rob and Anti-Rob. i^0 denotes the spatial infinities, i^- (i^+) denotes timelike past (future) infinity. J^- (J^+) denotes lightlike past (future) infinity. H^\pm denote the event horizons of the black hole

$$\begin{aligned} C_{\text{GHSW}}^{\min}(M_1, M_2) &= \min_{\rho_{AB}} \left\{ H \left(\sum_j p_j \varepsilon(\rho_{B|u_j}) \right) - \sum_j p_j H(\varepsilon(\rho_{B|u_j})) \right. \\ &\quad \left. + H \left(\sum_k q_k \varepsilon(\rho_{B|v_k}) \right) - \sum_k q_k H(\varepsilon(\rho_{B|v_k})) \right\}. \end{aligned} \tag{14}$$

Based on $C_{\text{GHSW}}^{\min}(M_1, M_2)$, we obtain the following relation:

$$\begin{aligned} \max_{\rho_{AB}} U_2 &= \max_{\rho_{AB}} \{-\log c_1 + H(A) - \mathcal{J}(B|M_1) - \mathcal{J}(B|M_2)\} \\ &= -\log c_1 + H(A) - C_{\text{GHSW}}^{\min}(M_1, M_2)|_{\varepsilon=\text{id}}. \end{aligned} \tag{15}$$

3 Generalized entropic uncertainty relations in Schwarzschild spacetime

3.1 Dirac field in Schwarzschild spacetime

We first review the definition of proper accelerated observer’s vacuum states in Schwarzschild spacetime. A Schwarzschild black hole in Schwarzschild coordinates is given by

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{16}$$

where M is the mass of black hole. Near the event horizon, the metric has similar structure as Rindler horizon in flat spacetime [37]. The Penrose diagram of the Schwarzschild spacetime is plotted in Fig. 2.

To compare two lower bound in uncertainty relation, the field states can be considered as bosonic field states or Dirac field states. In this thesis we choose Dirac field states. However, to make comparison with results in [39], spinless field states with basis $|0\rangle$ and $|1\rangle$ are also calculated.

Dirac field state is considered here instead of bosonic state because there is at most one particle for each spin in one mode due to Pauli’s exclusion principle [37]:

$$\begin{aligned}
 |\sigma_{\omega_i}\rangle_I &= c_{1,\omega_i,\sigma}^\dagger |0\rangle_I, \\
 |\sigma_{\omega_i}\rangle_{IV} &= d_{IV,\omega_i,\sigma}^\dagger |0\rangle_{IV}, \\
 |p_{\omega_i}\rangle_I &= c_{1,\omega_i,\uparrow}^\dagger c_{1,\omega_i,\downarrow}^\dagger |0\rangle_I = -c_{1,\omega_i,\downarrow}^\dagger c_{1,\omega_i,\uparrow}^\dagger |0\rangle_I, \\
 |p_{\omega_i}\rangle_{IV} &= d_{IV,\omega_i,\uparrow}^\dagger d_{IV,\omega_i,\downarrow}^\dagger |0\rangle_{IV} = -d_{IV,\omega_i,\downarrow}^\dagger d_{IV,\omega_i,\uparrow}^\dagger |0\rangle_{IV},
 \end{aligned}
 \tag{17}$$

where p_{ω_i} represents a pair of spin states in the mode with frequency ω_i , $\sigma = \uparrow$ or \downarrow , and $c_{1,\omega_i,\sigma}^\dagger, d_{IV,\omega_i,\sigma}^\dagger$ are create operators for particle and anti-particle, respectively. Thus, for each mode, a Dirac particle has four basis states: $|0\rangle, |\uparrow\rangle, |\downarrow\rangle, |p\rangle$.

The vacuum corresponding to free falling observer is called Hartle–Hawking vacuum $|0\rangle_H$, which is analogous to Minkowski vacuum. The vacuum corresponding to proper accelerated observer is called Boulware vacuum $|0\rangle_R$, which is analogous to the Rindler vacuum. There is another Boulware vacuum $|0\rangle_{\bar{R}}$ in region IV. Vacuum is made of different frequency modes $|0_H\rangle \equiv \otimes_i |0_{\omega_i}\rangle_H$ and similarly for first excitation $|1_H\rangle \equiv \otimes_i |1_{\omega_i}\rangle_H$. The relation between different notation is

$$\begin{aligned}
 |0\rangle_R &\leftrightarrow |0\rangle_I, \\
 |0\rangle_{\bar{R}} &\leftrightarrow |0\rangle_{IV}, \\
 |0\rangle_{A,B} &\leftrightarrow |0\rangle_H.
 \end{aligned}
 \tag{18}$$

Just like the case in Rindler spacetime, vacuum and excited state for different observer are related by Bogoliubov transformation [37,38,50]

$$\begin{aligned}
 |0_{\omega_i}\rangle_H &= (\cos q_{d,i})^2 |0_{\omega_i}\rangle_R |0_{\omega_i}\rangle_{\bar{R}} \\
 &\quad + \sin q_{d,i} \cos q_{d,i} (|\uparrow_{\omega_i}\rangle_R |\downarrow_{\omega_i}\rangle_{\bar{R}} + |\downarrow_{\omega_i}\rangle_R |\uparrow_{\omega_i}\rangle_{\bar{R}}) \\
 &\quad + (\sin q_{d,i})^2 |p_{\omega_i}\rangle_R |p_{\omega_i}\rangle_{\bar{R}},
 \end{aligned}
 \tag{19}$$

and for one particle state of Hartle–Hawking vacuum

$$\begin{aligned}
 |\uparrow_{\omega_i}\rangle_H &= \cos q_{d,i} |\uparrow_{\omega_i}\rangle_R |0_{\omega_i}\rangle_{\bar{R}} + \sin q_{d,i} |p_{\omega_i}\rangle_R |\uparrow_{\omega_i}\rangle_{\bar{R}}, \\
 |\downarrow_{\omega_i}\rangle_H &= \cos q_{d,i} |\downarrow_{\omega_i}\rangle_R |0_{\omega_i}\rangle_{\bar{R}} - \sin q_{d,i} |p_{\omega_i}\rangle_R |\downarrow_{\omega_i}\rangle_{\bar{R}},
 \end{aligned}
 \tag{20}$$

with

$$\tan q_{d,i} = \exp\left(-\frac{\Omega}{2}\sqrt{1-1/R_0}\right),
 \tag{21}$$

where $R_0 = r_0/R_H = r_0/2M$, $\Omega = \omega/T_H = 8\pi\omega M$ and ω is the mode frequency measured by Bob, just the same as above. Rindler approximation is only valid in vicinity of event horizon as mentioned above, i.e. $R_0 - 1 \ll 1$ [37].

For spinless state spanned by $\{|0\rangle, |1\rangle\}$, the Hartle–Hawking vacuum $|0\rangle$ and its first excitation $|1\rangle$ can be expressed in Rindler basis as [37,39]

$$|0_{\omega_i}\rangle_H = [1 + \exp(-\Omega\sqrt{1-1/R_0})]^{-\frac{1}{2}} |0_{\omega_i}\rangle_I |0_{\omega_i}\rangle_{IV}
 \tag{22}$$

$$+ [1 + \exp(\Omega\sqrt{1-1/R_0})]^{-\frac{1}{2}} |1_{\omega_i}\rangle_I |1_{\omega_i}\rangle_{IV}
 \tag{23}$$

$$|1_{\omega_i}\rangle_H = |1_{\omega_i}\rangle_I |1_{\omega_i}\rangle_{IV}
 \tag{24}$$

where $R_0 = r_0/R_H = r_0/2M$, $\Omega = \omega/T_H = 8\pi\omega M$ and ω is the mode frequency measured by Bob.

3.2 Incompatible measurements

For spinless field state, normal 2-dimensional Pauli matrices are used. $\sigma_x = |0\rangle\langle 1| + |1\rangle\langle 0|$, $\sigma_y = -i|0\rangle\langle 1| + i|1\rangle\langle 0|$, $\sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1|$.

For Dirac field state, 4-dimensional Pauli matrices are utilized for measurements [51]

$$\sigma_x \equiv \frac{1}{2} \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix},
 \tag{25}$$

$$\sigma_y \equiv \frac{1}{2} \begin{pmatrix} 0 & -i\sqrt{3} & 0 & 0 \\ i\sqrt{3} & 0 & -2i & 0 \\ 0 & 2i & 0 & -i\sqrt{3} \\ 0 & 0 & i\sqrt{3} & 0 \end{pmatrix},
 \tag{26}$$

$$\sigma_z \equiv \frac{1}{2} \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}.
 \tag{27}$$

For each pair of them, after calculating their eigenvectors, we can find the incompatible term $-\log c_1 = -\log \max_{i_1, i_2} |\langle u_{i_1}^{M_1} | u_{i_2}^{M_2} \rangle|^2$ is always $\log \frac{8}{3}$. Thus in the following discussion, without loss of generality, we choose σ_x and σ_y only.

3.3 Setup

In this section we detail the uncertainty game between Alice and Bob. Firstly Bob sends Alice a quantum state A, entangled with his quantum memory B. In this stage, both of them free falling towards the black hole. Then Alice remains free falling into the black hole. But Bob locates at a fixed distance r_0 outside the event horizon. At this stage Alice measures her quantum system with measurement either M_1 or M_2 , then sends her measurement choice to Bob through a classical communication channel. The goal of this game is for Bob to reduce his uncertainty about Alice’s measurement outcomes. We assume that Alice has a detector which only detects mode with frequency k and Bob has a detector sensitive to mode

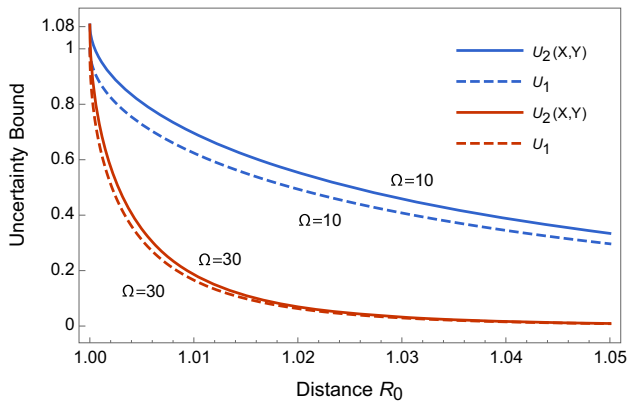


Fig. 3 Given $\Omega = \omega/T_H = 10$ or 30 , U_2^{xy} is always better than U_1

ω . Therefore, the states corresponding to mode ω must be specified in Boulware basis. Since the static observer cannot access the modes beyond the horizon, the lost information reduces the entanglement between A and B, therefore modifies the uncertainty bound.

4 Results

4.1 Spinless field states

4.1.1 Bell state

A Bell state in Hilbert space spanned by $\{|0\rangle, |1\rangle\}$ can be expressed as

$$|\Psi\rangle_H = \frac{1}{\sqrt{2}}(|0\rangle_A |0\rangle_B + |1\rangle_A |1\rangle_B). \tag{28}$$

Figure 3 depicts EUR lower bound for (28) with both U_1 and U_2^{xy} . The figure depicts the uncertainty bound U_1 and U_2^{xy} .

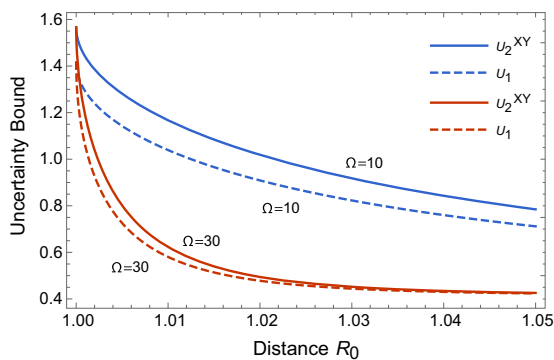


Fig. 4 The left figure depicts the uncertainty bound U_1 and U_2^{xy} with respect to $R_0 = r_0/2M$ when $\Omega = \omega/T_H = 10, 30$. The right figure shows the gap between left side and right side: $\Delta_1 = H(X|R) + H(Y|R) - U_1$ and $\Delta_2^{xy} = H(X|R) + H(Y|R) - U_2^{xy}$.

with respect to $R_0 = r_0/2M$ when $\Omega = \omega/T_H = 10, 30$. In the following calculation, T_H is Hawking temperature and ω is the frequency of the mode. The relative distance of Rob to event horizon $R_0 \leq 1.05$ is assumed thus Rindler approximation can be hold. Our calculation for U_1 agrees with bound in [39].

4.2 Dirac field states

4.2.1 A bell-like state

We consider a Bell-like state

$$|\Psi\rangle_H = \frac{1}{\sqrt{2}}(|0\rangle_A |0\rangle_B + |\uparrow\rangle_A |\downarrow\rangle_B). \tag{29}$$

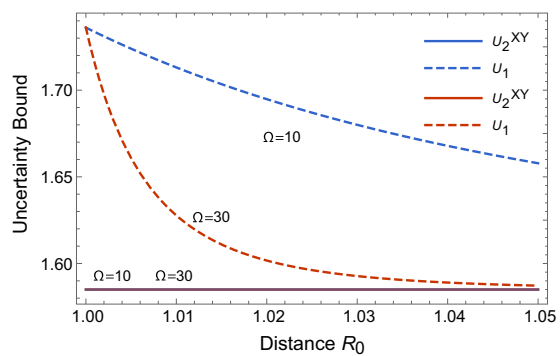
We depict its EUR lower bound for both U_1 and U_2^{xy} and the difference between $H(M_1|B) + H(M_2|B)$ in Fig. 4.

4.2.2 W state

Consider the case when Alice, Bob and Charlie initially shared a W state from perspective of inertial frame,

$$W = \frac{1}{\sqrt{3}}(|00\uparrow\rangle + |0\uparrow0\rangle + |\uparrow00\rangle) \tag{30}$$

The entropy uncertainty game is only between Alice and Bob, so Charlie has been traced. We depict EUR lower bound for Alice and Bob when Alice free falls into the black hole and Bob hovers near the event horizon. Both U_1 and U_2^{xy} and their difference with $H(M_1|B) + H(M_2|B)$ are shown in Fig. 5.



Given $\Omega = \omega/T_H = 10$ or 30 , U_2^{xy} is always better than U_1 . For different Ω and R_0 , Δ_2 is constant while Δ_1 decreases as Ω and R_0 increase

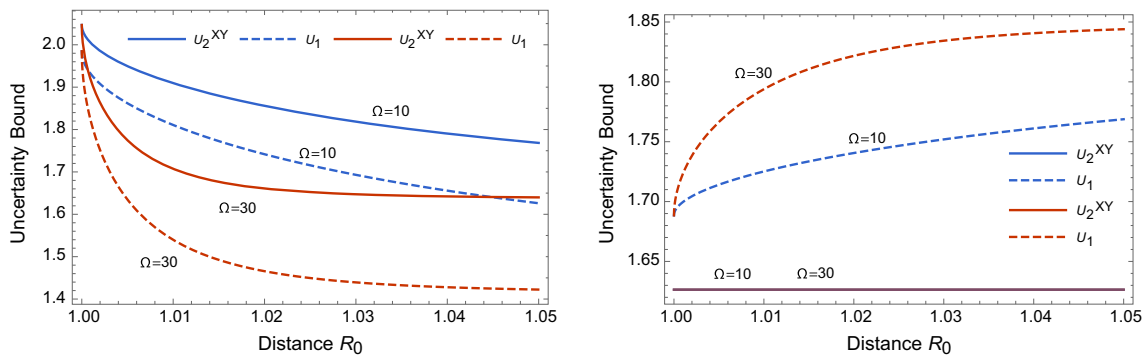


Fig. 5 Given $\Omega = \omega/T_H = 10$ or 30 , U_2^{xy} is always better than U_1 . For different Ω and R_0 , Δ_2 is constant while Δ_1 increases as Ω and R_0 increase

4.2.3 Comparison

In all examples we calculated, U_2 is tighter than U_1 . When Bob gets closer to the horizon, his uncertainty about Alice’s state gradually increases for both U_1 and U_2^{xy} . In addition, the figures shows that for a particular bound U_1 or U_2 , when $\Omega = \omega/T_H$ is larger, the uncertainty bound is lower. This is evident since fixing the mode energy ω , the larger Ω is, the lower Hawking temperature T_H is, which results in more correlation which can reduce the uncertainty. Besides, there is no surprise that $\Delta_2^{xy} = H(M_1) + H(M_2) - (-\log c_1)$ is constant as it is only influenced by the choice of measurements M_1, M_2 and measured system ρ_A , not by the quantum memory ρ_B . We can see from these figures that Δ_1 is not always constant but can decrease or increase when R_0 increase. This fact indicates that U_2 is better than U_1 in the sense that, for U_2 when the correlation decreases, the amount of increased uncertainty always equals to the amount of decreased correlation.

5 Conclusion

In this article, we calculated examples with spinless field states and Dirac field states in Schwarzschild spacetime, demonstrating that uncertainty relation generalized by Holevo quantity not only has a tighter lower bound, but reveals how the quantum memory would influence the entropic uncertainty as well. The second result has implications in experiments. It is sufficient to conduct experiments near *one* Schwarzschild black hole with mass M_0 to obtain LHS. For any other Schwarzschild black holes with mass M' , we do not need experiments and can precisely predict its LHS by only using LHS for M_0, Δ_2 and U_2 for M' .

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