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Note about Hamiltonian formalism for Newton–Cartan string and *p*-brane

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Abstract We construct non-relativistic string and p-brane actions in Newton–Cartan background using the limiting procedure from the relativistic string and p-brane action in general background. We also find their Hamiltonian formulations when however we restrict ourselves to the case of the vanishing gauge field m_μ.

1 Introduction

Today it is well known that strong correlated systems in condensed matter can be sucessfully described with the help of non-relativistic holography, for review see for example [1]. This duality is based on the idea that the strongly coupled theory on the boundary can be described by string theory in the bulk. Further, when the curvature of the space-time is small we can use the classical gravity instead full string theory machinery. In case of non-relativistic holography the situation is even more interesting since we have basically two possibilities: Either we use Einstein metric with non-relativistic isometries [2–4] or we introduce non-relativivistic gravities in the bulk [5,6], like Newton–Cartan gravity [7][1] or Hořava gravity [8]. Then it is certainly very interesting to study matter coupled to non-relativistic gravity. We can either study field theories on non-relativistic background as in [19–24] or particles [25–28] or even higher dimensional objects, as for example non-relativistic strings and p-branes [18,36].

In this work we would like to focus on the canonical formulation of non-relativistic string and p-brane in Newton–Cartan background. The starting point of our analysis is the relativistic string in general background that couples to NSNS two form. Then we use the limiting procedure that was proposed in [19] and try to find corresponding string action. Note that this is different limiting procedure than in case of the non-relativistic string in flat background where the non-relativistic limit is performed on coordinates [29–31].[2] It is important to stress that if we apply this limiting procedure that leads to corank-1 spatial metric and rank one temporal metric of Newton–Cartan gravity to the case of the string action we find that there is no way how to ensure that this action is finite. In order to resolve this problem we have to select two flat target space longitudial directions exactly in the same way as in [18]. Then we propose such an ansatz for NSNS two form field that is constructed with the help of the fields that define Newton–Cartan geometry and where the divergent contribution from the coupling to NSNS two form exactly cancels the divergent contribution coming from Nambu–Goto part of the action. As a result we obtain an action for the string in Newton–Cartan background that was proposed in [18] using different procedure. As the next step we proceed to the canonical formulation of this theory. Then however we encounter an obstacle in the form that we are not able to invert relation between conjugate momenta and velocity in case of non-zero gauge field m_μ^a whose explicit definition will be given in the next section. For that reason we restrict ourselves to the case of the zero gauge field keeping in mind that the case of on-zero gauge field deserves further study. Then we find Hamiltonian for this non-relativistic string that is linear combination of two first class constraints which is the manifestation of the fact that two dimensional string action is invariant under world-sheet diffeomorphism. As the next step we generalize this analysis to the case of p-brane. We firstly determine well defined action for non-relativistic p-brane when we consider specific form of the background $p+1$ form that couples to the world-volume of p-brane. Then we introduce an equivalent form of p-brane action that allows us to consider canonical analysis of this theory. Finally we determine constraint structure of this theory and we show that there are $p+1$ first class constraints,

[1] For some recent works, see [9–17].

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[2] For recent work, see for example [32–35].

p-spatial diffeomorphism constraints and one Hamiltonian constraint. We again show that these constraints are the first class constraints.

This paper is organized as follows. In the next Sect. 2 we determine the form of non-relativistic string in Newton–Cartan background and perform its Hamiltonian analysis. Then in Sect. 3 we generalize this analysis to the case of p-brane. Finally in conclusion (4) we outline our results and suggest possible extension of this work.

2 Review of the non-relativistic limit for Nambu–Goto string

In this section we derive non-relativistic form of the string action in Newton–Cartan background using the limiting procedure developed in [15]. We start with the Nambu–Goto form of the action in the general background

$$S = -\tilde{\tau}_F \int d\tau d\sigma \sqrt{-\det(E_\mu{}^A E_\nu{}^B \eta_{AB} \partial_\alpha x^\mu \partial_\beta x^\nu)} + \tilde{\tau}_F \int d\tau d\sigma B_{\mu\nu} \partial_\tau x^\mu \partial_\sigma x^\nu, \tag{1}$$

where $E_\mu{}^A$ is d-dimensional vierbein so that the metric components have the form

$$G_{\mu\nu} = E_\mu{}^A E_\nu{}^B \eta_{AB}, \eta_{AB} = \mathrm{diag}(-1, 1, \ldots, 1) \tag{2}$$

Note that the metric inverse $G^{\mu\nu}$ is defined with the help of the inverse vierbein $E^\mu_{\ B}$ that obeys the relation

$$E_\mu{}^A E^\mu_{\ B} = \delta^A_B, \quad E_\mu{}^A E^\nu_{\ A} = \delta_\mu^\nu. \tag{3}$$

Further, $B_{\mu\nu}$ is NSNS two form field that plays the crucial role in the limiting procedure. Finally $x^\mu, \mu = 0, \ldots, d-1$ are embedding coordinates of the string where the two dimensional world-sheet is parameterised by $\sigma^\alpha \equiv (\tau, \sigma)$.

Let us now proceed to the brief description of the procedure that leads to Newton–Cartan background from general background, for more detailed discussion, see the original paper [15]. The starting point is following ansatz for d-dimensional vierbein [15]

$$E_\mu{}^0 = \omega \tau_\mu + \frac{1}{2\omega} m_\mu, \quad E_\mu{}^{a'} = e_\mu{}^{a'}, \tag{4}$$

where $a' = 1, \ldots, d-1$ and where ω is free parameter which goes to infinity in the Newton–Cartan limit. Note that in this case the metric has the form

$$G_{\mu\nu} = E_\mu{}^A E_\nu{}^B \eta_{AB} = -\omega^2 \tau_\mu \tau_\nu - \frac{1}{2} \tau_\mu m_\nu - \frac{1}{2} \tau_\nu m_\mu + h_{\mu\nu} + \frac{1}{4\omega^2} m_\mu m_\nu$$
$$= -\omega^2 \tau_\mu \tau_\nu + \bar{h}_{\mu\nu} + \frac{1}{4\omega^2} m_\mu m_\nu, \quad \bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} \tau_\mu m_\nu - \frac{1}{2} \tau_\nu m_\mu, \quad h_{\mu\nu} = e_\mu{}^{a'} e_\nu{}^{b'} \delta_{a'b'}. \tag{5}$$

Inserting this metric into the Nambu–Goto action and performing expansion with respect to ω we obtain

$$S = -\tilde{\tau}_F \omega^2 \int d\tau d\sigma \sqrt{-\det \mathbf{a}} - \frac{\tilde{\tau}_F}{2} \int d\tau d\sigma \sqrt{-\det \mathbf{a}} \mathbf{a}^{\alpha\beta} \bar{h}_{\alpha\beta}, \tag{6}$$

where we defined

$$\mathbf{a}_{\alpha\beta} = \tau_{\mu\nu} \partial_\alpha x^\mu \partial_\beta x^\nu, \quad \mathbf{a}^{\alpha\beta} \mathbf{a}_{\beta\gamma} = \delta^\alpha_\gamma,$$
$$\bar{h}_{\alpha\beta} = \bar{h}_{\mu\nu} \partial_\alpha x^\mu \partial_\beta x^\nu. \tag{7}$$

We also used the fact that $\mathbf{a}_{\alpha\beta}$ is 2×2 matrix that is non-singular. Apparently we see from (7) that there is a term proportional to ω^2 that cannot be made finite by rescaling of $\tilde{\tau}_F$. In case of the string in the flat non-relativistic background such a term can be canceled with the suitable form of the background $NSNS$ two form. Further, , this two form field should be build from the fields that define Newton–Cartan theory as m_μ, τ_ν. However it turns out that it is not possible to find such a NSNS two form due to the fact that it has to be antisymmetryc in space-time indicies. In order to find solution of this problem we implement the generalization of the Newton–Cartan gravity that was introduced in [18]. Explicitly, we split the target-space indices A into $A = (a', a)$ where now $a = 0, 1$ label longitudial and $a' = 2, \ldots, d-1$ correspond to transverse directions. Then we introduce $\tau_\mu{}^a$ so that we write

$$\tau_{\mu\nu} = \tau_\mu{}^a \tau_\nu{}^b \eta_{ab}, \quad a, b = 0, 1, \quad \eta_{ab} = \mathrm{diag}(-1, 1). \tag{8}$$

In the same way we introduce vierbein $e_\mu{}^{a'}, a' = 2, \ldots, d-1$ and also we generalize m_μ into $m_\mu{}^a$. The $\tau_\mu{}^a$ can be interpreted as the gauge fields of the longitudinal translations while $e_\mu{}^{a'}$ as the gauge fields of the transverse translations [18]. Then we can also introduce their inverses with respect to their longitudinal and transverse subspaces

$$e_\mu{}^{a'} e^\mu_{\ b'} = \delta^{a'}_{b'}, \quad e_\mu{}^{a'} e^\nu_{\ a'} = \delta^\nu_\mu - \tau_\mu{}^a \tau^\nu_{\ a},$$
$$\tau^\mu_{\ a} \tau_\mu{}^b = \delta^b_a, \quad \tau^\mu_{\ a} e_\mu{}^{a'} = 0, \quad \tau_\mu{}^a e^\mu_{\ a'} = 0. \tag{9}$$

Performing this generalization implies following form of the vierbein

$$E_\mu{}^a = \omega \tau_\mu{}^a + \frac{1}{2\omega} m_\mu{}^a, \quad E_\mu{}^{a'} = e_\mu{}^{a'} \tag{10}$$

so that we find following form of the metric

$$G_{\mu\nu} = E_\mu{}^a E_\nu{}^b \eta_{ab} + E_\mu{}^{a'} E_\nu{}^{b'} \delta_{a'b'}$$

$$= \omega^2\tau_{\mu\nu} + h_{\mu\nu} + \frac{1}{2}\tau_\mu{}^a m_\nu{}^b\eta_{ab} + \frac{1}{2}m_\mu{}^a\tau_\nu{}^b\eta_{ab} + \frac{1}{4\omega^2}m_\mu{}^a m_\nu{}^b\eta_{ab}. \quad (11)$$

It was shown in [15] that in order to find the right form of the particle action in Newton–Cartan background we should consider following ansatz for the background gauge field $A_\mu = \omega\tau_\mu - \frac{1}{2\omega}m_\mu$. In order to find correct form of the action for the string in Newton–Cartan background we propose analogue form of the NSNS two form[3]

$$\begin{aligned} B_{\mu\nu} &= \left(\omega\tau_\mu{}^a - \frac{1}{2\omega}m_\mu{}^a\right)\left(\omega\tau_\nu{}^b - \frac{1}{2\omega}m_\nu{}^b\right)\epsilon_{ab} \\ &= \omega^2\tau_\mu{}^a\tau_\nu{}^b\epsilon_{ab} - \frac{1}{2}(m_\mu{}^a\tau_\nu{}^b + \tau_\mu{}^a m_\nu{}^b)\epsilon_{ab} \\ &\quad + \frac{1}{4\omega^2}m_\mu{}^a m_\mu{}^b\epsilon_{ab}, \quad \epsilon_{ab} = -\epsilon_{ba}, \quad \epsilon_{01} = 1. \end{aligned} \quad (12)$$

It is important that terms proportional to ω^{-2} and ω^{-4} vanish in the limit $\omega \to \infty$ while the divergent contribution cancel the divergence coming from Nambu–Goto part of the action since

$$\begin{aligned} &-\tilde{\tau}_F\omega^2\int d^2\sigma\sqrt{-\det\mathbf{a}} + \frac{\tilde{\tau}_F}{2}\int d^2\sigma\epsilon^{\alpha\beta}B_{\mu\nu}\partial_\alpha x^\mu\partial_\beta x^\nu \\ &= -\tilde{\tau}_F\omega^2\int d^2\sigma \det\tau_\alpha{}^a \\ &\quad + \omega^2\frac{\tilde{\tau}_F}{2}\int d^2\sigma\epsilon^{\alpha\beta}\epsilon_{ab}\tau_\alpha{}^a\tau_\beta{}^b = 0, \end{aligned} \quad (13)$$

where we introduced 2×2 matrix $\tau_\alpha{}^a \equiv \tau_\mu{}^a\partial_\alpha x^\mu$ and where we used the fact that $\det\tau_\alpha{}^a = \frac{1}{2}\epsilon^{\alpha\beta}\epsilon_{ab}\tau_\alpha{}^a\tau_\beta{}^b$ where $\epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha}, \epsilon^{01} = 1$ is antisymmetric symbol with upper indices.

In summary we obtain the action for non-relativistic string in Newton–Cartan background in the form

$$\begin{aligned} S &= -\frac{\tau_F}{2}\int d^2\sigma\sqrt{-\det\mathbf{a}}\mathbf{a}^{\alpha\beta}\bar{h}_{\alpha\beta} \\ &\quad + \frac{\tau_F}{2}\int d\tau d\sigma(m_\mu{}^a\tau_\nu{}^b + \tau_\mu{}^a m_\nu{}^b)\epsilon_{ab}\partial_\tau x^\mu\partial_\sigma x^\nu, \end{aligned} \quad (14)$$

where $\tilde{\tau}_F = \tau_F$.

Our goal is to find Hamiltonian formulation of this theory. To do this we rewrite the Lagrangian density introduced in (14) into the form

$$\begin{aligned} \mathcal{L} &= \frac{1}{4\lambda^\tau}(\bar{h}_{\tau\tau} - 2\lambda^\sigma\bar{h}_{\tau\sigma} + (\lambda^\sigma)^2\bar{h}_{\sigma\sigma}) - \lambda^\tau\tau_F^2\bar{h}_{\sigma\sigma} \\ &\quad + B^\tau\left(\lambda^\tau - \frac{\sqrt{-\det\mathbf{a}}}{2\tau_F\mathbf{a}_{\sigma\sigma}}\right) + B^\sigma\left(\lambda^\sigma - \frac{\mathbf{a}_{\tau\sigma}}{\mathbf{a}_{\sigma\sigma}}\right) \\ &\quad + \frac{\tau_F}{2}(m_\mu{}^a\tau_\nu{}^b + \tau_\mu{}^a m_\nu{}^b)\epsilon_{ab}\partial_\tau x^\mu\partial_\sigma x^\nu. \end{aligned} \quad (15)$$

It is easy to see an equivalence of these two Lagrangians since the equations of motion for B^τ and B^σ give

$$\lambda^\tau = \frac{\sqrt{-\det\mathbf{a}}}{2\tau_F\mathbf{a}_{\sigma\sigma}}, \quad \lambda^\sigma = \frac{\mathbf{a}_{\tau\sigma}}{\mathbf{a}_{\sigma\sigma}}. \quad (16)$$

Inserting this result into (15) and using the fact that

$$\begin{aligned} &\frac{1}{\lambda^\tau} = -2\tau_F\mathbf{a}^{\sigma\sigma}\sqrt{-\det\mathbf{a}}, \quad \frac{\lambda^\sigma}{\lambda^\tau} = 2\tau_F\mathbf{a}^{\tau\sigma}\sqrt{-\det\mathbf{a}}, \\ &\frac{1}{4\lambda^\tau}((\lambda^\sigma)^2 - 4\tau_F^2(\lambda^\tau)^2) = -\frac{\tau_F}{2}\mathbf{a}^{\sigma\sigma}\sqrt{-\det\mathbf{a}} \end{aligned} \quad (17)$$

we find that (15) reduces into (14). Then from (15) we obtain conjugate momenta

$$\begin{aligned} p_\mu &= \frac{1}{2\lambda^\tau}\bar{h}_{\mu\nu}\partial_\tau x^\nu - \frac{\lambda^\sigma}{2\lambda^\tau}\bar{h}_{\mu\nu}\partial_\sigma x^\nu \\ &\quad - B^\tau\frac{1}{2\tau_F\mathbf{a}_{\sigma\sigma}}\tau_{\mu\nu}\partial_\alpha x^\nu\mathbf{a}^{\alpha\tau}\sqrt{-\det\mathbf{a}} - \frac{B^\sigma}{\mathbf{a}_{\sigma\sigma}}\tau_{\mu\nu}\partial_\sigma x^\nu \\ &\quad + \frac{\tau_F}{2}(m_\mu{}^a\tau_\nu{}^b + \tau_\mu{}^a m_\nu{}^b)\epsilon_{ab}\partial_\sigma x^\nu, \\ P_B^\tau &= \frac{\partial L}{\partial\partial_\tau B^\tau} \approx 0, \quad P_B^\sigma = \frac{\partial L}{\partial\partial_\tau B^\sigma} \approx 0, \\ P_\lambda^\tau &= \frac{\partial L}{\partial\partial_\tau\lambda^\tau} \approx 0, \quad P_\lambda^\sigma = \frac{\partial L}{\partial\partial_\tau\lambda^\sigma} \approx 0 \end{aligned} \quad (18)$$

Now we come to the most important problem in our analysis which is an imposibility to invert the relation between p_μ and $\partial_\tau x^\mu$ in order to express $\partial_\tau x^\mu$ using the canonical variables. The reason why we are not able to do is in the presence of the vector field $m_\mu{}^a$ so that the contraction of the metric $\bar{h}_{\mu\nu}$ with τ^μ is non-zero and carries also longitudinal index a and hence further manipulation is non-trivial and it is not clear for us how to proceed further. For that reason we restrict to the simpler case when $m_\mu{}^a = 0$.[4] Despite of this simplification

[3] It is clear that the background fields have to obey equations of motion of the effective string theory. Certainly if we insert proposed ansatz for metric and NSNS two form into these equations of motion we get set of differential equations for Newton–Cartan background fields. Since we are interested in the dynamics of probe string in Newton–Cartan background it is not necessary to address this question and we postpone this analysis for future work.

[4] It is instructive compare this problem with the Hamiltonian analysis of particle in NC background. Recall that the particle action in Newton–Cartan background has the form [27]

$$S = \frac{m}{2}\int d\tau\frac{\dot{x}^\mu\bar{h}_{\mu\nu}\dot{x}^\nu}{\dot{x}^\rho\tau_\rho}, \quad \bar{h}_{\mu\nu} = h_{\mu\nu} - m_\mu\tau_\nu - \tau_\mu m_\nu, \quad \dot{x}^\mu \equiv \frac{dx^\mu}{d\tau}. \quad (19)$$

From this action we obtain conjugate momenta

$$p_\mu = m\frac{\bar{h}_{\mu\nu}\dot{x}^\nu}{\dot{x}^\rho\tau_\rho} - \frac{m}{2}\frac{\dot{x}^\rho\bar{h}_{\rho\sigma}\dot{x}^\sigma}{(\dot{x}^\rho\tau_\rho)^2}\tau_\mu. \quad (20)$$

Firstly we multiply this relation with τ^μ and we obtain

$$\tau^\mu p_\mu = -\tau^\mu m_\mu - \frac{m}{2}\frac{\dot{x}^\rho h_{\rho\sigma}\dot{x}^\sigma}{(\dot{x}^\omega\tau_\omega)^2}. \quad (21)$$

we will see that even in this case the Hamiltonian formulation of the non-relativistic string in Newton–Cartan background is non-trivial task. In case when $m_\mu^{\ a}=0$ we have $\bar{h}_{\mu\nu}=h_{\mu\nu}$ and $\bar{h}_{\mu\nu}h^{\nu\rho}\bar{h}_{\rho\sigma}=h_{\rho\sigma}$ and hence from (18) we obtain

$$\left(p_\mu+\frac{\lambda^\sigma}{2\lambda^\tau}h_{\mu\rho}\partial_\sigma x^\rho\right)h^{\mu\nu}\left(p_\nu+\frac{\lambda^\sigma}{2\lambda^\tau}h_{\nu\sigma}\partial_\sigma x^\sigma\right)=\frac{1}{4(\lambda^\tau)^2}\partial_\tau x^\mu h_{\mu\nu}\partial_\tau x^\nu. \quad (25)$$

On the other hand let us multiply both sides of expression (18) with $\tau^\mu_a\eta^{ab}\epsilon_{bc}\tau_\rho^{\ c}\partial_\sigma x^\rho$ and we obtain

$$\begin{aligned}
&p_\mu\tau^\mu_a\eta^{ab}\epsilon_{bc}\tau_\rho^{\ c}\partial_\sigma x^\rho\\
&=-B^\tau\frac{1}{2\tau_F\mathbf{a}_{\sigma\sigma}}\partial_\alpha x^\mu\tau_\mu^{\ a}\epsilon_{ab}\tau_\rho^{\ b}\partial_\sigma x^\rho\mathbf{a}^{\alpha\tau}\sqrt{-\det\mathbf{a}}\\
&\quad-B^\sigma\frac{1}{\mathbf{a}_{\sigma\sigma}}\partial_\sigma x^\mu\tau_\mu^{\ a}\epsilon_{ab}\tau_\nu^{\ b}\partial_\sigma x^\nu\\
&=-B^\tau\frac{1}{2\tau_F\mathbf{a}_{\sigma\sigma}}\partial_\tau x^\mu\tau_\mu^{\ a}\epsilon_{ab}\tau_\rho^{\ b}\partial_\sigma x^\rho\frac{\mathbf{a}_{\sigma\sigma}}{\det\mathbf{a}}\sqrt{-\det\mathbf{a}}\\
&=\frac{1}{2\tau_F}B^\tau
\end{aligned} \quad (26)$$

using

$$\tau^\mu_a\tau_{\mu\nu}=\tau_\nu^{\ b}\eta_{ba} \quad (27)$$

and also we used the fact that

$$\sqrt{-\det\mathbf{a}}=\det\tau_\alpha^{\ a}=\tau_\tau^a\tau_\sigma^b\epsilon_{ab}, \quad (28)$$

where $\tau_\alpha^{\ a}=\tau_\mu^{\ a}\partial_\alpha x^\mu$. Then the relation (26) implies following primary constraint

$$\Gamma^\tau\equiv 2\tau_F p_\mu\tau^\mu_a\eta^{ab}\epsilon_{bc}\tau_\rho^{\ c}\partial_\sigma x^\rho-B^\tau\approx 0. \quad (29)$$

On the other hand if we multiply the relation (18) with $\tau^{\mu\nu}\tau_{\nu\rho}\partial_\sigma x^\rho$ we obtain

$$p_\mu\tau^{\mu\nu}\tau_{\nu\rho}\partial_\sigma x^\rho=-B^\tau\frac{1}{2\tau_F\mathbf{a}_{\sigma\sigma}}\mathbf{a}_{\sigma\alpha}\mathbf{a}^{\alpha\tau}\sqrt{-\det\mathbf{a}}-B^\sigma=-B^\sigma \quad (30)$$

and hence we obtain second primary constraint

$$\Gamma^\sigma\equiv p_\mu\tau^{\mu\nu}\tau_{\nu\rho}\partial_\sigma x^\rho+B^\sigma\approx 0. \quad (31)$$

As a result the extended Hamiltonian with all primary constraints included has the form

$$\begin{aligned}
H_E=\int d\sigma\Big(&\lambda^\tau(p_\mu h^{\mu\nu}p_\nu+\tau_F^2h_{\sigma\sigma})\\
&-B^\tau\lambda^\tau-B^\sigma\lambda^\sigma+\lambda^\sigma p_\mu h^{\mu\nu}h_{\nu\rho}\partial_\sigma x^\rho\\
&+U_\tau\Gamma^\tau+U_\sigma\Gamma^\sigma+v_\tau^B P_B^\tau\\
&+v_\sigma^B P_B^\sigma+v_\tau^\lambda P_\lambda^\tau+v_\sigma^\lambda P_\lambda^\sigma\Big).
\end{aligned} \quad (32)$$

Let us now analyze the requirement of the preservation of the primary constraints $P_\lambda^\tau\approx 0,\ P_\lambda^\sigma\approx 0$. In case of P_λ^τ we obtain

$$\begin{aligned}
\partial_\tau P_\lambda^\tau&=\left\{P_\lambda^\tau,H_E\right\}=-p_\mu h^{\mu\nu}p_\nu-\tau_F^2h_{\sigma\sigma}+B^\tau\\
&=-p_\mu h^{\mu\nu}p_\nu-\tau_F^2h_{\sigma\sigma}\\
&\quad+2\tau_F p_\mu\tau^\mu_a\eta^{ab}\epsilon_{bc}\tau_\rho^{\ c}\partial_\sigma x^\rho-2\tau_F\Gamma^\tau\\
&\approx-p_\mu h^{\mu\nu}p_\nu-\tau_F^2h_{\sigma\sigma}\\
&\quad+2\tau_F p_\mu\tau^\mu_a\eta^{ab}\epsilon_{bc}\tau_\rho^{\ c}\partial_\sigma x^\rho\equiv-\mathcal{H}_\tau\approx 0
\end{aligned} \quad (33)$$

and also

$$\begin{aligned}
\partial_\tau P_\lambda^\sigma&=\left\{P_\lambda^\sigma,H\right\}=-p_\mu\partial_\sigma x^\mu+p_\mu\tau^{\mu\nu}\tau_{\nu\rho}\partial_\sigma x^\rho+B^\sigma\\
&=-p_\mu\partial_\sigma x^\mu+\Gamma^\sigma\approx-p_\mu\partial_\sigma x^\mu\equiv-\mathcal{H}_\sigma\approx 0.
\end{aligned} \quad (34)$$

We see that the requirement of the preservation of the primary constraints $P_\lambda^\tau\approx 0,\ P_\lambda^\sigma\approx 0$ implies an existence of two secondary constraints:

$$\begin{aligned}
&\mathcal{H}_\sigma=p_\mu\partial_\sigma x^\mu\approx 0,\quad \mathcal{H}_\tau=p_\mu h^{\mu\nu}p_\nu+\tau_F^2h_{\sigma\sigma}\\
&\quad-2\tau_F p_\mu\tau^\mu_a\eta^{ab}e_{bc}\tau_\rho^{\ c}\partial_\sigma x^\rho\approx 0.
\end{aligned} \quad (35)$$

Further, since $\left\{P_B^\tau(\sigma),\Gamma^\tau(\sigma')\right\}=\delta(\sigma-\sigma'),\left\{P_B^\sigma(\sigma),\Gamma^\sigma(\sigma')\right\}=-\delta(\sigma-\sigma')$ we see that these constraints are the second class constraints that can be explicitly solved for B^τ and B^σ. Then these constraints vanish strongly and hence the Hamiltonian is linear combination of the constraints

$$\mathcal{H}_E=\lambda^\tau\mathcal{H}_\tau+\lambda^\sigma\mathcal{H}_\sigma+v_\tau^\lambda P_\lambda^\tau+v_\sigma^\lambda P_\lambda^\sigma. \quad (36)$$

As the next step we should check that $\mathcal{H}_\tau\approx 0$, $\mathcal{H}_\sigma\approx 0$ are the first class constraints. To do this we introduce the smeared forms of these constraints

$$\mathbf{T}_\tau(N)=\int d\sigma N\mathcal{H}_\tau,\ \mathbf{T}_\sigma(N^\sigma)=\int d\sigma N^\sigma\mathcal{H}_\sigma. \quad (37)$$

First of all we easily find

$$\left\{\mathbf{T}_\sigma(N^\sigma),\mathbf{T}_\sigma(M^\sigma)\right\}=\mathbf{T}_\sigma(N^\sigma\partial_\sigma M^\sigma-N^\sigma\partial_\sigma M^\sigma). \quad (38)$$

Footnote 4 continued
On the other hand let us multiply (20) with $h^{\mu\nu}$ and we get

$$h^{\mu\nu}p_\nu=m\frac{h^{\mu\nu}h_{\nu\rho}\dot{x}^\rho}{\dot{x}^\rho\tau_\rho}-mh^{\mu\nu}m_\nu \quad (22)$$

and hence we find

$$(p_\mu+m_\mu)h^{\mu\nu}(p_\nu+m_\nu)=m^2\frac{\dot{x}^\mu h_{\mu\nu}\dot{x}^\nu}{(\dot{x}^\rho\tau_\rho)^2}. \quad (23)$$

Finally if we combine this result with (21) we obtain following Hamiltonian constraint [28]

$$\mathcal{H}_\tau=(p_\mu+m_\mu)h^{\mu\nu}(p_\nu+m_\nu)+2m(p_\mu+m_\mu)\tau^\mu\approx 0. \quad (24)$$

The reason, why the particle case is much simpler than string like case is, that τ_μ and m_μ do not carry tangent space indices a,b and hence the manipulations with them is much simpler.

In case of the Hamiltonian constraints the situation is more involved since the explicit calculation gives

$$
\begin{aligned}
&\{\mathbf{T}_\tau(N),\mathbf{T}_\tau(M)\}\\
&\quad=\int d\sigma(N\partial_\sigma M-M\partial_\sigma N)4\tau_F^2 p_\mu h^{\mu\nu}h_{\nu\rho}\partial_\sigma x^\rho\\
&\qquad+2\int d\sigma\tau_F(N\partial_\sigma M-M\partial_\sigma N)p_\mu V^\mu_\nu h^{\nu\omega}p_\omega\\
&\qquad+\int d\sigma(N\partial_\sigma M-M\partial_\sigma N)p_\rho V^\rho_\sigma V^\sigma_\omega\partial_\sigma x^\omega\\
&\qquad+\tau_F^2\int d\sigma(N\partial_\sigma M-M\partial_\sigma N)4\tau_F^2 V^\mu_\nu\partial_\sigma x^\nu h_{\mu\rho}\partial_\sigma x^\rho,
\end{aligned}
\tag{39}
$$

where we defined

$$
V^\mu_\nu=-2\tau_F\tau^\mu_a\eta^{ab}\epsilon_{bc}\tau_\nu{}^c. \tag{40}
$$

To proceed further we calculate

$$
\begin{aligned}
4p_\mu h^{\mu\nu}h_{\nu\rho}\partial_\sigma x^\rho&=4\tau_F^2 p_\mu\partial_\sigma x^\mu\\
&\quad-4\tau_F^2 p_\mu\tau^{\mu\rho}\tau_{\rho\nu}\partial_\sigma x^\nu,\\
p_\rho V^\rho_\mu V^\mu_\nu\partial_\sigma x^\nu&=4\tau_F^2 p_\mu\tau^\mu_a\tau^a_\nu\partial_\sigma x^\nu\\
&=4\tau_F^2 p_\mu\tau^{\mu\rho}\tau_{\rho\nu}\partial_\sigma x^\nu,\\
V^\mu_\nu\partial_\sigma x^\nu h_{\mu\rho}\partial_\sigma x^\rho&=0,\quad p_\mu V^\mu_\nu h^{\nu\omega}p_\omega=0.
\end{aligned}
\tag{41}
$$

Collecting all these results together we finally obtain

$$
\{\mathbf{T}_\tau(N),\mathbf{T}_\tau(M)\}=\mathbf{T}_\sigma((N\partial_\sigma M-M\partial_\sigma N)4\tau_F^2). \tag{42}
$$

Finally we also calculate Poisson bracket between $\mathbf{T}_\sigma(N^\sigma)$ and $\mathbf{T}_\tau(M)$ and we find

$$
\left\{\mathbf{T}_\sigma(N^\sigma),\mathbf{T}_\tau(M)\right\}=\mathbf{T}_\tau(\partial_\sigma MN^\sigma-M\partial_\sigma N^\sigma). \tag{43}
$$

These results show that $\mathcal{H}_\tau\approx 0,\mathcal{H}_\sigma\approx 0$ are the first class constraints and the non-relativistic string is well defined system from the Hamiltonian point of view.

Finally we also say few words about the gauge fixed theory. We showed above that the Hamiltonian and spatial diffeomorphism constraints are the first class. Standard way how to deal with such a theory is to gauge fix them. For example, we can impose the static gauge when we introduce two gauge fixing functions

$$
\mathcal{G}^0=x^0-\tau\approx 0,\quad \mathcal{G}^1=x^1-\sigma\approx 0. \tag{44}
$$

It is easy to see that $\mathcal{G}^a\approx 0$ are the second class constraints together with $\mathcal{H}_\tau\approx 0,\mathcal{H}_\sigma\approx 0$. Since then these constraints vanish strongly we can identify the Hamiltonian density on the reduced phase space from the action principle

$$
S=\int d\tau d\sigma(p_\mu\partial_\tau x^\mu-H)=\int d\tau d\sigma(p_i\partial_\tau x^i+p_0) \tag{45}
$$

so that it is natural to identify $-p_0$ as the Hamiltonian on the reduced phase space $H_{red}=-p_0$. The explicit form of the Hamiltonian follows from the Hamiltonian constraint that can be solved for p_0. Note also that $\mathcal{H}_\sigma$ can be solved for p_1 as $p_1=-p_I\partial_\sigma x^I$, $I=2,\ldots,d-1$.

3 Generalization: non-relativistic p-brane

In this section we perform generalization of the analysis presented previously to the case of non-relativistic p-brane. As the first step we determine an action for non-relativistic p-brane in Newton–Cartan backgorund in the same way as in the string case. Explicitly, we start with the relativistic p-brane action coupled to C^{p+1} form whose action has the form

$$
\begin{aligned}
&S=-\tilde{\tau}_p\int d^{p+1}\xi\sqrt{-\det\mathbf{A}_{\alpha\beta}}+\tilde{\tau}_p\int C^{(p+1)},\\
&\mathbf{A}_{\alpha\beta}=G_{\mu\nu}\partial_\alpha x^\mu\partial_\beta x^\nu,
\end{aligned}
\tag{46}
$$

where $\xi^\alpha,\alpha=0,\ldots,p$ label world-volume of p-brane and where

$$
\begin{aligned}
C^{(p+1)}=&C_{\mu_1\ldots\mu_{p+1}}dx^{\mu_1}\wedge\ldots dx^{\mu_{p+1}}\\
=&\frac{1}{(p+1)!}\epsilon^{\alpha_1\ldots\alpha_{p+1}}C_{\mu_1\ldots\mu_{p+1}}\partial_{\alpha_1}x^{\mu_1}\ldots\partial_{\alpha_{p+1}}x^{\mu_{p+1}},
\end{aligned}
\tag{47}
$$

where again $\epsilon^{\alpha_1\ldots\alpha_{p+1}}$ is totally antisymmetric symbol.

With the help of the action (46) we can proceed to the definition of non-relativistic p-brane in Newton–Cartan background. As we have seen in case of the non-relativistic string the requirement that the action for non-relativistic string should be finite select two longitudial directions. Then we can deduce that in case of non-relativistic p-brane we should select $p+1$ longitudial directions. Explicitly, we presume that in case of the probe p-brane the background metric has the form

$$
\begin{aligned}
G_{\mu\nu}&=E_\mu{}^aE_\nu{}^b\eta_{ab}+E_\mu{}^{a'}E_\nu{}^{b'}\delta_{a'b'}\\
&=\omega^2\tau_{\mu\nu}+h_{\mu\nu}+\frac{1}{2}\tau_\mu{}^am_\nu{}^b\eta_{ab}+\frac{1}{2}m_\mu{}^a\tau_\nu{}^b\eta_{ab}\\
&\quad+\frac{1}{4\omega^2}m_\mu{}^am_\nu{}^b\eta_{ab},
\end{aligned}
\tag{48}
$$

where now $a,b=0,\ldots,p$ and $a',b'\cdots=(p+1,\ldots,d-1)$. Further, $\tau_{\mu\nu}$ and $h_{\mu\nu}$ are defined as

$$
\begin{aligned}
\tau_{\mu\nu}=&\tau_\mu{}^a\tau_\nu{}^b\eta_{ab},\quad \eta_{ab}=\mathrm{diag}(-1,\ldots,1),\\
h_{\mu\nu}=&e_\mu{}^{a'}e_\nu{}^{b'}\delta_{a'b'}.
\end{aligned}
\tag{49}
$$

We also introduce their inverses with respect to their longitudinal and transverse dimensions

$$e_\mu^{\ a'} e^\mu_{\ b'} = \delta^{a'}_{b'}, \quad e_\mu^{\ a'} e^\nu_{\ a'} = \delta^\nu_\mu - \tau_\mu^{\ a} \tau^\nu_{\ a}, \quad \tau^\mu_{\ a} \tau_\mu^{\ b} = \delta^b_a,$$
$$\tau^\mu_{\ a} e_\mu^{\ a'} = 0, \quad \tau_\mu^{\ a} e^\mu_{\ a'} = 0. \tag{50}$$

In case of $p+1$-form $C^{(p+1)}$ we presume, with the analogy with the string case, that it has the form

$$C_{\mu_1 \dots \mu_{p+1}} = \left(\omega \tau_{\mu_1}^{\ a_1} - \frac{1}{2\omega} m_{\mu_1}^{\ a_1}\right) \times \cdots \times \left(\tau_{\mu_{p+1}}^{\ a_{p+1}} - \frac{1}{2\omega} m_{\mu_{p+1}}^{\ a_{p+1}}\right) \epsilon_{a_1 \dots a_{p+1}}, \tag{51}$$

where $\epsilon_{a_1 \dots a_{p+1}}$ is totally antisymmetric symbol. Now we are ready to define non-relativistic limit of p-brane action. We start with the kinetic term and we obtain

$$S_{DBI} = -\tilde{\tau}_p \omega^{p+1} \int d^{p+1}\xi \sqrt{-\det \mathbf{a}} - \frac{\tilde{\tau}_p}{2} \omega^{p-1} \int d^{p+1}\xi \sqrt{-\det \mathbf{a}} \tilde{\mathbf{a}}^{\alpha\beta} \bar{h}_{\alpha\beta}, \tag{52}$$

where $\tilde{\mathbf{a}}^{\alpha\beta}$ is inverse to $\mathbf{a}_{\alpha\beta}$. In fact, it is reasonable to presume that $\mathbf{a}_{\alpha\beta} = \partial_\alpha x^\mu \eta_{ab} \partial_\beta x^\nu = \tau_\alpha^{\ a} \eta_{ab} \tau_\beta^{\ b}$ since $\tau_\alpha^{\ a}$ and η_{ab} are $(p+1) \times (p+1)$ non-singular matrices. From the requirement that the kinetic term is finite we have to perform following rescaling

$$\tilde{\tau}_p \omega^{p-1} = \tau_p. \tag{53}$$

Further, the divergent term can be written as

$$\tilde{\tau}_p \omega^{p+1} \int d^{p+1}\xi \sqrt{-\det \mathbf{a}} = \tau_p \omega^2 \int d^{p+1}\xi \det \tau_\alpha^{\ b}, \quad \tau_\alpha^{\ a} = \tau_\mu^{\ a} \partial_\alpha x^\mu. \tag{54}$$

Let us now concentrate on the second term in the action (46). If we express $\tilde{\tau}_p$ using τ_p as $\tilde{\tau}_p = \frac{1}{\omega^{p-1}} \tau_p$ we find that the only non-zero contribution comes from the product of $\tau_\mu^{\ a}$'s while remaining terms vanish in the limit $\omega \to \infty$. Then we obtain

$$\begin{aligned} S_{WZ} &= \frac{1}{\omega^{p-1}} \tau_p \int d^{p+1} \epsilon^{\alpha_1 \dots \alpha_{p+1}} \omega \tau_{\mu_1}^{\ a_1} \partial_{\alpha_1} x^{\mu_1} \dots \omega \tau_{\mu_{p+1}}^{\ a_{p+1}} \partial_{\alpha_{p+1}} x^{\mu_{p+1}} \epsilon_{a_1 \dots a_{p+1}} \\ &= \omega^2 \tau_p \int d^{p+1}\xi \frac{1}{p!} \epsilon^{\alpha_1 \dots \alpha_{p+1}} \tau_{\alpha_1}^{\ a_1} \dots \tau_{\alpha_{p+1}}^{\ a_{p+1}} \\ &= \omega^2 \tau_p \int d^{p+1}\xi \det \tau_\alpha^{\ a} \end{aligned} \tag{55}$$

and we again see that these two divergent contributions cancel. As a result we obtain well defined action for non-relativistic p-brane in Newton–Cartan background

$$S = -\frac{\tau_p}{2} \int d^{p+1}\xi \sqrt{-\det \mathbf{a}} \tilde{\mathbf{a}}^{\alpha\beta} \bar{h}_{\mu\nu} \partial_\alpha x^\mu \partial_\beta x^\nu. \tag{56}$$

Now we proceed to the Hamiltonian formulation of this theory. With the analogy with the string case we write the action as

$$\begin{aligned} S = \int d^{p+1}\xi \Big(&\frac{1}{4\lambda^\tau} (\partial_0 x^\mu - \lambda^i \partial_i x^\mu) h_{\mu\nu} (\partial_0 x^\nu - \lambda^j \partial_j x^\nu) \\ &- \lambda^\tau \tau_p^2 \det \mathbf{a}_{ij} \mathbf{a}^{ij} h_{ij} \\ &+ B^0 \left(\lambda^0 - \frac{\sqrt{-\det \mathbf{a}}}{2\tau_p \det \mathbf{a}_{ij}}\right) + B^i (\lambda_i - \mathbf{a}_{i0}) \Big), \end{aligned} \tag{57}$$

where

$$\lambda^i = \mathbf{a}^{ij} \mathbf{a}_{j0}, \quad \mathbf{a}_{ij} \mathbf{a}^{jk} = \delta_i^k. \tag{58}$$

In order to see an equivalence between the action (57) and (56) we note that the inverse matrix $\tilde{\mathbf{a}}^{\alpha\beta}$ to the matrix $\mathbf{a}_{\alpha\beta}$ has the form

$$\begin{aligned} &\tilde{\mathbf{a}}^{00} = \frac{\det \mathbf{a}_{ij}}{\det \mathbf{a}}, \quad \tilde{\mathbf{a}}^{0i} = -\mathbf{a}_{0k} \mathbf{a}^{kj} \frac{\det \mathbf{a}_{ij}}{\det \mathbf{a}}, \\ &\tilde{\mathbf{a}}^{i0} = -\mathbf{a}^{ik} \mathbf{a}_{k0} \frac{\det \mathbf{a}_{ij}}{\det \mathbf{a}}, \quad \tilde{\mathbf{a}}^{ij} = \mathbf{a}^{ij} + \frac{\det \mathbf{a}_{ij}}{\det \mathbf{a}} \mathbf{a}^{ik} \mathbf{a}_{k0} \mathbf{a}_{0l} \mathbf{a}^{lj}, \end{aligned} \tag{59}$$

where $\mathbf{a}^{ij} \mathbf{a}_{jk} = \delta^i_k$. Then the equation of motion for B^0 and B^i imply

$$\lambda^0 = -\frac{\sqrt{-\det \mathbf{a}}}{2\tau_p \det \mathbf{a}_{ij}}, \quad \lambda_i = \mathbf{a}_{0i}. \tag{60}$$

Inserting this result into (57) we obtain that it is equal to the action (56). Let us now return to the action (57) and determine conjugate momenta from it

$$\begin{aligned} p_\mu &= \frac{\partial \mathcal{L}}{\partial(\partial_0 x^\mu)} \\ &= \frac{1}{2\lambda^\tau} \bar{h}_{\mu\nu} \partial_0 x^\nu - \frac{\lambda^i}{2\lambda^\tau} \bar{h}_{\mu\nu} \partial_i x^\nu \\ &\quad - B^\tau \frac{1}{2\tau_p \det \mathbf{a}_{ij}} \tau_{\mu\nu} \partial_\alpha x^\nu (\mathbf{a}^{-1})^{\alpha 0} \sqrt{-\det \mathbf{a}} - B^i \tau_{\mu\nu} \partial_i x^\nu, \\ P_0 &= \frac{\partial \mathcal{L}}{\partial(\partial_0 \lambda^0)} \approx 0, \quad P_i = \frac{\partial \mathcal{L}}{\partial(\partial_0 \lambda^i)} \approx 0, \\ P_0^B &= \frac{\partial \mathcal{L}}{\partial(\partial_0 B^0)} \approx 0, \quad P_i^B = \frac{\partial \mathcal{L}}{\partial(\partial_0 B^i)} \approx 0. \end{aligned} \tag{61}$$

From the same reason as in case of the fundamental string we have to restrict to the case $m_\mu^{\ a} = 0$ so that $\bar{h}_{\mu\nu} = h_{\mu\nu}$. Then if we multiply (61) with $h^{\mu\nu}$ we obtain

$$\left(p_\mu + \frac{\lambda^i}{2\lambda^\tau} h_{\mu\rho} \partial_i x^\rho\right) h^{\mu\nu} \left(p_\nu + \frac{\lambda^j}{2\lambda^\tau} h_{\nu\sigma} \partial_j x^\sigma\right) = \frac{1}{4(\lambda^\tau)^2} \partial_0 x^\mu h_{\mu\nu} \partial_0 x^\nu. \tag{62}$$

On the other hand let us multiply both sides of (61) with $\tau^{\mu\nu}\tau_{\nu\rho}\partial_i x^\rho$ and we obtain

$$p_\mu\tau^{\mu\nu}\tau_{\nu\rho}\partial_i x^\rho = -B^\tau\frac{1}{2\tau_p\det\mathbf{a}_{ij}}\partial_i x^\mu\tau_{\mu\nu}\partial_\alpha x^\nu\mathbf{a}^{\alpha 0}\sqrt{-\det\mathbf{a}} -\partial_i x^\mu\tau_{\mu\nu}\partial_j x^\nu B^j = -\mathbf{a}_{ij}B^j \tag{63}$$

that imlies p-primary constraints

$$\Sigma^i \equiv p_\mu\tau^{\mu\nu}\tau_{\nu\sigma}\partial_j x^\sigma\mathbf{a}^{ji} + B^i \approx 0. \tag{64}$$

On the other hand let us multiply (61) with following expression

$$\frac{1}{p!}\tau^\mu_a\eta^{aa_1}\epsilon_{a_1\ldots a_{p+1}}\tau^{a_2}_{\nu_2}\ldots\tau^{a_{p+1}}_{\nu_{p+1}}\epsilon^{j_2\ldots j_{p+1}}\partial_{j_2}x^{\nu_2}\ldots\partial_{j_{p+1}}x^{\nu_{p+1}}. \tag{65}$$

Then using the fact that

$$\begin{aligned}
&\partial_i x^\nu\tau_{\nu\mu}\tau^\mu_a\eta^{aa_1}\epsilon_{a_1\ldots a_{p+1}}\tau^{a_2}_{\nu_2}\ldots\tau^{a_{p+1}}_{\nu_{p+1}}\epsilon^{j_2\ldots j_{p+1}}\partial_{j_2}x^{\nu_2}\\
&\quad\ldots\partial_{j_{p+1}}x^{\nu_{p+1}} = 0,\\
&\frac{1}{p!}\mathbf{a}^{0\alpha}\partial_\alpha x^\nu\tau_{\nu\mu}\tau^\mu_a\eta^{aa_1}\epsilon_{a_1\ldots a_{p+1}}\tau^{a_2}_{\nu_2}\ldots\tau^{a_{p+1}}_{\nu_{p+1}}\epsilon^{j_2\ldots j_{p+1}}\partial_{j_2}x^{\nu_2}\\
&\quad\ldots\partial_{j_{p+1}}x^{\nu_{p+1}}\frac{\sqrt{-\det\mathbf{a}}}{\det\mathbf{a}_{ij}}\\
&= -\frac{1}{(p+1)!}\epsilon_{a_1\ldots a_{p+1}}\epsilon^{j_1\ldots j_{p+1}}\tau^{a_1}_{\nu_1}\partial_{\alpha_1}x^{\nu_1}\\
&\quad\ldots\tau^{a_{p+1}}_{\nu_{p+1}}\partial_{\alpha_{p+1}}x^{\nu_{p+1}}\frac{1}{\det\tau_\alpha^{\ a}} = -1
\end{aligned} \tag{66}$$

we obtain second primary constraint

$$\Sigma^0 \equiv 2\tau_p p_\mu\frac{1}{p!}\tau^\mu_a\eta^{aa_1}\epsilon_{a_1\ldots a_{p+1}}\tau^{a_2}_{\nu_2}\ldots\tau^{a_{p+1}}_{\nu_{p+1}}\epsilon^{j_2\ldots j_{p+1}}\partial_{j_2}x^{\nu_2}\ldots\partial_{j_{p+1}}x^{\nu_{p+1}} - B^0 \approx 0. \tag{67}$$

Using all these results we determine extended Hamiltonian with all primary constraints included in the form

$$\begin{aligned}
H_E = \int d^p\xi(&\lambda^0 p_\mu h^{\mu\nu}p_\nu + \lambda^i p_\mu h^{\mu\nu}h_{\nu\sigma}\partial_i x^\sigma\\
&+\lambda^\tau\tau_p^2\det\mathbf{a}_{ij}\mathbf{a}^{ij}h_{ij}\\
&-B^0\lambda^\tau - B^i\lambda_i + v^0P_0 + v^iP_i + v^0_BP^B_0\\
&+v^B_iP^i_B + \Psi_0\Sigma^0 + \Psi_i\Sigma^i).
\end{aligned} \tag{68}$$

Since $\left\{P^B_0(\xi),\Sigma^0(\xi')\right\} = \delta(\xi-\xi'), \left\{P^i_B(\xi),\Sigma^j(\xi')\right\} = -\delta^{ij}\delta(\xi-\xi')$ we see that that P^B_0 together with Ψ^0 are the couple of $p+1$ second class constraints. Then we can solve these constraints with respect to B^0, B^i and we we obtain the Hamiltonian in the form

$$H_B = \int d^p\xi(\lambda^0\mathcal{H}_0 + \lambda^i\mathcal{H}_i + v^0P_0 + v^iP_i) \tag{69}$$

where

$$\begin{aligned}
\mathcal{H}_0 &= p_\mu h^{\mu\nu}p_\nu + \tau_p^2\det\mathbf{a}_{ij}\mathbf{a}^{ij}h_{ij}\\
&\quad -2\tau_p p_\mu\frac{1}{p!}\tau^\mu_a\eta^{aa_1}\epsilon_{a_1\ldots a_{p+1}}\tau^{a_2}_{\nu_2}\\
&\quad\ldots\tau^{a_{p+1}}_{\nu_{p+1}}\epsilon^{j_2\ldots j_{p+1}}\partial_{j_2}x^{\nu_2}\ldots\partial_{j_{p+1}}x^{\nu_{p+1}} \approx 0,\\
\mathcal{H}_i &= p_\mu\partial_i x^\mu \approx 0.
\end{aligned} \tag{70}$$

Then the requirement of the preservation of the constraint $P_0 \approx 0$, $P_i \approx 0$ implies $p+1$ secondary constraints

$$\mathcal{H}_0 \approx 0, \quad \mathcal{H}_i \approx 0. \tag{71}$$

Now we have to check that these constraints are the first class constraints. We introduce their smeared form

$$\mathbf{T}_T(N) = \int d^p\xi N\mathcal{H}_0, \quad \mathbf{T}_S(N^i) = \int d^p\xi N^i\mathcal{H}_i \tag{72}$$

and calculate corresponding Poisson brackets. First of all we have

$$\left\{\mathbf{T}_S(N^i),\mathbf{T}_S(M^j)\right\} = \mathbf{T}_S(N^j\partial_j M^i - M^j\partial_j N^i). \tag{73}$$

In case of the calculation of the Poisson brackets between $\mathbf{T}_S(N^i)$ and $\mathbf{T}_T(M)$ we have to be more careful. First of all we have

$$\left\{\mathbf{T}_S(N^i),\tau_i^{\ a}\right\} = -N^k\partial_k\tau_i^{\ a} - \partial_i N^j\tau_j^{\ a}, \quad \tau_i^{\ a} \equiv \partial_i x^\mu\tau_\mu^{\ a}. \tag{74}$$

Then we obtain

$$\begin{aligned}
\left\{\mathbf{T}_S(N^i),\mathbf{a}_{ij}\right\} &= -N^k\partial_k\mathbf{a}_{ij} - \partial_i N^k\mathbf{a}_{kj} - \mathbf{a}_{ik}\partial_j N^k,\\
\left\{\mathbf{T}_S(N^i),\mathbf{a}^{ij}\right\} &= -N^k\partial_k\mathbf{a}^{ij} + \partial_k N^i\mathbf{a}^{kj} + \mathbf{a}^{ik}\partial_k N^j,\\
\left\{\mathbf{T}_S(N^i),\det\mathbf{a}_{ij}\right\} &= -N^k\partial_k(\det\mathbf{a}_{ij}) - 2\partial_i N^i\det\mathbf{a}_{ij}.
\end{aligned} \tag{75}$$

Using also the fact that

$$\begin{aligned}
\left\{\mathbf{T}_S(N^i),\partial_i x^\mu\right\} &= -N^k\partial_k(\partial_i x^\mu) - \partial_i N^k\partial_k x^\mu,\\
\left\{\mathbf{T}_S(N^i),h_{ij}\right\} &= -N^k\partial_k h_{ij} - \partial_i N^k h_{kj} - h_{ik}\partial_j N^k
\end{aligned} \tag{76}$$

we finally obtain

$$\left\{\mathbf{T}_S(N^i),\det\mathbf{a}_{ij}\mathbf{a}^{kl}h_{kl}\right\} = -N^k\partial_k(\det\mathbf{a}_{ij}\mathbf{a}^{kl}h_{kl}) -2\partial_i N^i\det\mathbf{a}_{ij}\mathbf{a}^{kl}h_{kl}. \tag{77}$$

Let us introduce following vector

$$V^\mu = -2\tau_p\frac{1}{p!}\tau^\mu_a\eta^{aa_1}\epsilon_{a_1\ldots a_{p+1}}\tau^{a_2}_{\nu_2}\ldots\tau^{a_{p+1}}_{\nu_{p+1}}\epsilon^{j_2\ldots j_{p+1}}\partial_{j_2}x^{\nu_2}\ldots\partial_{j_{p+1}}x^{\nu_{p+1}}. \tag{78}$$

Then after some algebra we obtain

$$\left\{\mathbf{T}_S(N^i), V^\mu\right\} = -N^k\partial_k V^\mu - 2\partial_k N^k V^\mu. \tag{79}$$

Collecting all these results together we finally find

$$\left\{\mathbf{T}_S(N^i), \mathbf{T}_S(M)\right\} = \mathbf{T}_T(N^i\partial_i M - \partial_i N^i M). \tag{80}$$

Finally we calculate Poisson brackets of the smeared forms of Hamiltonian constraints and we obtain

$$\begin{aligned}
&\{\mathbf{T}_T(N), \mathbf{T}_T(M)\} \\
&= \int d^p\xi(N\partial_i M - M\partial_i N)4\tau_p^2 \det \mathbf{a}_{ij}\mathbf{a}^{ij} p_\mu h^{\mu\nu} h_{\nu\sigma}\partial_j x^\sigma \\
&\quad +2\tau_p \int d^p\xi(N\partial_i M \\
&\quad -M\partial_i N)\frac{1}{(p-1)!} p_\nu \tau_a^\nu \eta^{aa_1}\epsilon_{a_1\dots a_{p+1}}\tau_\mu^{a_2}\tau_{\nu_3}^{a_3}\dots\tau_{\nu_{p+1}}^{a_{p+1}} \\
&\quad \times\epsilon^{ij_3\dots j_{p+1}}\partial_{j_3}x^{\nu_3}\dots\partial_{j_{p+1}}x^{\nu_{p+1}}V^\mu.
\end{aligned} \tag{81}$$

Then after some lengthy calculations we find that the last expression is equal to

$$4\tau_p^2(N\partial_i M - M\partial_i N)\mathbf{a}^{ij} p_\mu \tau^{\mu\nu}\tau_{\nu\sigma}\partial_j x^\sigma \det \mathbf{a}_{ij}. \tag{82}$$

Inserting this result into (81) we obtain final result

$$\{\mathbf{T}_T(N), \mathbf{T}_T(M)\} = \mathbf{T}_S((N\partial_i M - M\partial_i N)4\tau_p^2\mathbf{a}^{ij}\det \mathbf{a}_{ij}). \tag{83}$$

These results show that $\mathcal{H}_0$ and $\mathcal{H}_i$ are the first class constraints that reflect diffeomorphism invariance of non-relativistic p-brane.

4 Conclusion

In this paper we formulated non-relativistic actions for string and p-brane in Newton–Cartan background. Then we found their Hamiltonian formulations and we determined structure of constraints in the special case where the gauge field $m_\mu^{\ a}$ is zero. We argued that we restricted to this case since we were not able to express time derivatives of x^μ or their combinations as functions of canonical variables in the case when $m_\mu^{\ a} \neq 0$. Certainly this more general case deserves further study. One possibility is to address this problem from different point of view when we start with the Hamiltonian formulation string in general background, then we perform limiting procedure on the background metric and NSNS two form field and derive corresponding Hamiltonian. This problem is currently under study and we hope to report on this analysis in near future.

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References

1. S.A. Hartnoll, A. Lucas, S. Sachdev, Holographic quantum matter. arXiv:1612.07324 [hep-th]
2. D.T. Son, Toward an AdS/cold atoms correspondence: a geometric realization of the Schrodinger symmetry. Phys. Rev. D **78**, 046003 (2008). https://doi.org/10.1103/PhysRevD.78.046003. [arXiv:0804.3972 [hep-th]]
3. K. Balasubramanian, J. McGreevy, Gravity duals for non-relativistic CFTs. Phys. Rev. Lett. **101**, 061601 (2008). https://doi.org/10.1103/PhysRevLett.101.061601. [arXiv:0804.4053 [hep-th]]
4. C.P. Herzog, M. Rangamani, S.F. Ross, Heating up Galilean holography. JHEP **0811**, 080 (2008). https://doi.org/10.1088/1126-6708/2008/11/080. [arXiv:0807.1099 [hep-th]]
5. D.T. Son, Newton–Cartan Geometry and the Quantum Hall Effect. arXiv:1306.0638 [cond-mat.mes-hall]
6. S. Janiszewski, A. Karch, Non-relativistic holography from Horava gravity. JHEP **1302**, 123 (2013). https://doi.org/10.1007/JHEP02(2013)123. [arXiv:1211.0005 [hep-th]]
7. E. Cartan, Sur les varits connexion affine et la thorie de la relativit gnralise. (premire partie). Annales Sci. Ecole Norm. Sup **40**, 325 (1923)
8. P. Horava, Quantum Gravity at a Lifshitz Point. Phys. Rev. D **79**, 084008 (2009). https://doi.org/10.1103/PhysRevD.79.084008. [arXiv:0901.3775 [hep-th]]
9. E. Bergshoeff, A. Chatzistavrakidis, L. Romano, J. Rosseel, Newton–Cartan gravity and torsion. JHEP **1710**, 194 (2017). [arXiv:1708.05414 [hep-th]]
10. E.A. Bergshoeff, J. Rosseel, Three-dimensional extended Bargmann supergravity. Phys. Rev. Lett **116**(25), 251601 (2016). https://doi.org/10.1103/PhysRevLett.116.251601. [arXiv:1604.08042 [hep-th]]
11. J. Hartong, Y. Lei, N.A. Obers, Nonrelativistic Chern-Simons theories and three-dimensional Hoava–Lifshitz gravity. Phys. Rev. D **94**(6), 065027 (2016). https://doi.org/10.1103/PhysRevD.94.065027. [arXiv:1604.08054 [hep-th]]
12. J. Hartong, N.A. Obers, Hoava-Lifshitz gravity from dynamical Newton–Cartan geometry. JHEP **1507**, 155 (2015). https://doi.org/10.1007/JHEP07(2015)155. [arXiv:1504.07461 [hep-th]]
13. H.R. Afshar, E.A. Bergshoeff, A. Mehra, P. Parekh, B. Rollier, A Schrdinger approach to Newton–Cartan and Hoava–Lifshitz gravities. JHEP **1604**, 145 (2016). https://doi.org/10.1007/JHEP04(2016)145. [arXiv:1512.06277 [hep-th]]
14. E. Bergshoeff, J. Rosseel, T. Zojer, Newton-Cartan supergravity with torsion and Schrdinger supergravity. JHEP **1511**, 180 (2015). https://doi.org/10.1007/JHEP11(2015)180. [arXiv:1509.04527 [hep-th]]
15. E. Bergshoeff, J. Rosseel, T. Zojer, Newton–Cartan (super)gravity as a non-relativistic limit. Class. Quant. Grav. **32**(20), 205003 (2015). https://doi.org/10.1088/0264-9381/32/20/205003. [arXiv:1505.02095 [hep-th]]

16. E.A. Bergshoeff, J. Hartong, J. Rosseel, Torsional Newton–Cartan geometry and the Schrdinger algebra. Class. Quant. Grav. **32**(13), 135017 (2015). https://doi.org/10.1088/0264-9381/32/13/135017. [arXiv:1409.5555 [hep-th]]
17. R. Andringa, E. Bergshoeff, S. Panda, M. de Roo, Newtonian gravity and the Bargmann algebra. Class. Quant. Grav. **28**, 105011 (2011). https://doi.org/10.1088/0264-9381/28/10/105011. [arXiv:1011.1145 [hep-th]]
18. R. Andringa, E. Bergshoeff, J. Gomis, M. de Roo, Stringy Newton–Cartan gravity. Class. Quant. Grav. **29**, 235020 (2012). https://doi.org/10.1088/0264-9381/29/23/235020. [arXiv:1206.5176 [hep-th]]
19. E. Bergshoeff, J. Rosseel, T. Zojer, Non-relativistic fields from arbitrary contracting backgrounds. Class. Quant. Grav. **33**(17), 175010 (2016). https://doi.org/10.1088/0264-9381/33/17/175010. [arXiv:1512.06064 [hep-th]]
20. A. Bagchi, R. Basu, A. Kakkar, A. Mehra, Galilean Yang–Mills Theory. JHEP **1604**, 051 (2016). https://doi.org/10.1007/JHEP04(2016)051. [arXiv:1512.08375 [hep-th]]
21. G. Festuccia, D. Hansen, J. Hartong, N.A. Obers, Symmetries and couplings of non-relativistic electrodynamics. JHEP **1611**, 037 (2016). https://doi.org/10.1007/JHEP11(2016)037. [arXiv:1607.01753 [hep-th]]
22. M. Geracie, K. Prabhu, M.M. Roberts, Curved non-relativistic spacetimes, Newtonian gravitation and massive matter. J. Math. Phys **56**(10), 103505 (2015). https://doi.org/10.1063/1.4932967. [arXiv:1503.02682 [hep-th]]
23. K. Jensen, On the coupling of Galilean-invariant field theories to curved spacetime. arXiv:1408.6855 [hep-th]
24. J. Hartong, E. Kiritsis, N.A. Obers, Schrdinger invariance from Lifshitz isometries in holography and field theory. Phys. Rev. D **92**, 066003 (2015). https://doi.org/10.1103/PhysRevD.92.066003. [arXiv:1409.1522 [hep-th]]
25. K. Kuchar, Gravitation, geometry, and nonrelativistic quantum theory. Phys. Rev. D **22**, 1285 (1980). https://doi.org/10.1103/PhysRevD.22.1285
26. A. Barducci, R. Casalbuoni, J. Gomis, Non-relativistic spinning particle in a Newton–Cartan background. arXiv:1710.10970 [hep-th]
27. E. Bergshoeff, J. Gomis, M. Kovacevic, L. Parra, J. Rosseel, T. Zojer, Nonrelativistic superparticle in a curved background. Phys. Rev. D **90**(6), 065006 (2014). https://doi.org/10.1103/PhysRevD.90.065006. [arXiv:1406.7286 [hep-th]]
28. J. Kluson, Canonical analysis of non-relativistic particle and superparticle. arXiv:1709.09405 [hep-th]
29. J. Gomis, H. Ooguri, Nonrelativistic closed string theory. J. Math. Phys. **42**, 3127 (2001). https://doi.org/10.1063/1.1372697. [arXiv:hep-th/0009181]
30. J. Gomis, F. Passerini, T. Ramirez, A. Van Proeyen, Non relativistic Dp branes. JHEP **0510**, 007 (2005). https://doi.org/10.1088/1126-6708/2005/10/007. [arXiv:hep-th/0507135]
31. J. Gomis, J. Gomis, K. Kamimura, Non-relativistic superstrings: a new soluble sector of AdS(5) x S**5. JHEP **0512**, 024 (2005). https://doi.org/10.1088/1126-6708/2005/12/024. [arXiv:hep-th/0507036]
32. C. Batlle, J. Gomis, D. Not, Extended Galilean symmetries of non-relativistic strings. JHEP **1702**, 049 (2017). https://doi.org/10.1007/JHEP02(2017)049. [arXiv:1611.00026 [hep-th]]
33. J. Kluson, Remark about non-relativistic p-Brane. arXiv:1707.04034 [hep-th]
34. J. Kluson, Note about integrability of non-relativistic string. arXiv:1705.10951 [hep-th]
35. J. Kluson, Hamiltonian analysis of non-relativistic non-BPS Dp-brane. JHEP **1707**, 007 (2017). https://doi.org/10.1007/JHEP07(2017)007. [arXiv:1704.08003 [hep-th]]
36. T. Harmark, J. Hartong, N.A. Obers, Nonrelativistic strings and limits of the AdS/CFT correspondence. Phys. Rev. D **96**(8), 086019 (2017). https://doi.org/10.1103/PhysRevD.96.086019. [arXiv:1705.03535 [hep-th]]