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Interactions of irregular Gaiotto states in Liouville theory

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Abstract We compute the correlation functions of irregular Gaiotto states appearing in the colliding limit of the Liouville theory by using "regularizing" conformal transformations mapping the irregular (coherent) states to regular vertex operators in the Liouville theory. The N-point correlation functions of the irregular vertex operators of arbitrary ranks are expressed in terms of N-point correlators of primary fields times the factor that involves regularized higher-rank Schwarzians of the above conformal transformation. In particular, in the case of three-point functions the general answer is expressed in terms of DOZZ (Dorn-Otto-Zamolodchikov-Zamolodchikov) structure constants times exponents of regularized higher-derivative Schwarzians. The explicit examples of the regularization are given for the ranks one and two.

1 Introduction

Irregular (coherent) Gaiotto states emerge in Liouville field theory in the colliding limit, relevant to extensions of the AGT conjecture to Argyres-Douglas type of gauge theories with asymptotic freedom [1,3,6–8,14,19]. The irregular states of rank N are the simultaneous eigenstates of $N+1$ Virasoro generators:

$$L_n|U_N\rangle = \rho_N(n)|U_N\rangle \qquad N \le n \le 2N \tag{1}$$

and are annihilated by higher positive Virasoro generators ($n > 2N$). This generalizes the definition of primary fields (which technically have rank zero). Just as the regular vertex operators for primary (rank zero) fields in Liouville theory can be expressed as $V_\alpha =: e^{\alpha_0 \phi}:$ (where α_0 can be regarded as "electric" charge), the irregular vertex operators for rank N coherent states can be constructed as

$$|U_N\rangle = U_N|0\rangle \qquad U_N =: e^{\sum_{n=0}^{N} \alpha_n \partial^n \phi}: \tag{2}$$

where $\alpha_1, \alpha_2 \ldots$ correspond to coefficients of the multipole expansion and are related to the geometry of the region where the colliding operators are located (e.g. with α_1 being a characteristic size of the region). These coefficients are related to Virasoro eigenvalues $\rho_n(N)$ according to

$$\rho_n(N) = \frac{1}{2}\sum_{n=0}^{N} \alpha_n \alpha_{N-n} \tag{3}$$

Computing correlation functions describing interactions of the irregular states in Liouville theory is known to be a hard and tedious problem, especially beyond two-point functions and rank one case [5,6,9,14,15]. These correlation functions are important objects as they define the correlators in Argyres-Douglas gauge theories; namely, the N-point correlators of Argyres-Douglas theories are idenfied with the M-point colliding limit of 2d CFT so that $M = M_1 + M_2 + \cdots + M_N$. Presumably, the correlators of the irregular states are also related to instanton expansions in these asymptotically-free theories, although our understanding of this correspondence is not yet complete.

In this work we address this problem by using the conformal transformations that maps operators for coherent states into regular vertex operators, expressing the interactions of the irregular states in terms of regular correlators in Liouville theory. In particular, since the structure constants of the Liouville theory are known, this conformal transformation makes it possible to express the cubic interactions of the irregular states of arbitrary rank in terms of Dorn-Zamolodchikov correlators in Liouville theory. The final formula for the correlators, however, is complicated, since it involves the generalized higher-derivative Schwarzians of the conformal transformation that are singular at the insertion points of the correlators and have to be regularized in a rather tedious way. In our work, we limit ourselves performing this regularization for the case of 3-point functions of the rank two irregular

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vertex operators. The rest of this paper is organized as follows. In the Sect. 2, we study the behavior of correlators under the conformal transformation mapping the irregular blocks into regular and derive the general result expressing N-point correlators of irregular vertices of arbitrary ranks in terms of regular conformal blocks. The answer involves the higher-derivative generalized Schwarzians of this conformal transformation that need to be regularized. In the Appendix section we explicitly perform such a regularization for three-point correlators of the rank two states. In the concluding section we discuss the implications of our results.

2 Conformal map: irregular to regular states

Consider a rank p irregular vertex operator:

$$W_p(\alpha_0, \alpha_1, ..., \alpha_p|\xi) =: e^{\sum_{n=0}^{p}\alpha_n\partial^n\phi} : (\xi) \tag{4}$$

where ϕ is Liouville field and the N-point correlators

$$A_N = \left\langle \prod_{j=1}^{N} W_{p_j}(\alpha_0^{(j)}, ..., \alpha_{p_j}^{(j)}|\xi_j) \right\rangle \tag{5}$$

Consider conformal transformation:

$$f(z) = e^{-i\sum_{j=1}^{N}(z-\xi_j-i\epsilon)^{-1}} \tag{6}$$

$$S_{n_1|n_2}(f;z) = \frac{1}{n_1!n_2!}\sum_{k_1=1}^{n_1}\sum_{k_2=1}^{n_2}\sum_{m_1\geq 0}\sum_{m_2\geq 0}\sum_{p\geq 0}\sum_{q=1}^{p}(-1)^{k_1+m_2+q}2^{-m_1-m_2}(k_1+k_2-1)!$$
$$\times\frac{\partial^{m_1}B_{n_1|k_1}(f(z);z)\partial^{m_2}B_{n_2|k_2}(f(z);z)B_{p|q}(g_1,...,g_{p-q+1})}{m_1!m_2!p!(f'(z))^{k_1+k_2}}$$
$$\times g_s = 2^{-s-1}(1+(-1)^s)\frac{\frac{d^{s+1}f}{dz^{s+1}}}{(s+1)f'(z)};\ s=1,...,p-q+1 \tag{10}$$

with the small ϵ parameter introduced in order to control regularizations. This transformations maps the half-plane to compact Riemann surface with all the the points $\xi_1, ...\xi_N$ (originally located on the real line) are glued together at zero for $\epsilon = 0$ (and are infinitely close to each other if ϵ is nonzero). Gluing points together is the trick that will be used below to compare correlation functions before and after conformal transformations. First of all, let us check how the conformal transformation (6) acts on individual irregular vertices. It is straightforward to check that the transformation (6) maps irregular vertex operators (4) into regular. To see this, first consider the infinitezimal transformations of W_p. We have

$$\delta_\epsilon W_p(\xi) = \frac{1}{2}\left[\oint\frac{dz}{2i\pi}\epsilon(z) : \partial\phi\partial\phi : (z); W_p(\xi)\right]$$
$$= \sum_{n=0}^{p}\alpha_n\partial^n(\epsilon\partial\phi)W_p(\xi)$$
$$+\frac{1}{2}\sum_{n_1,n_2=0}^{p}\frac{n_1!n_2!}{(n_1+n_2+1)!}\alpha_{n_1}\alpha_{n_2}\partial^{n_1+n_2+1}\epsilon W_p(\xi) \tag{7}$$

This infinitezimal relation is straightforward to integrate (e.g. by imposing a composition constraint). The integrated form of (7) for finite conformal transformations $z \to f(z)$ is then given by

$$W_p(z) \to \tilde{W}_p(f(z)) = \exp\left\{\alpha_0\phi + \sum_{n=1}^{p}\sum_{q=1}^{n}B_{n|q}(f(z);z)\alpha_n\partial^q\phi(f(z)) + \sum_{n_1,n_2=0}^{p}n_1!n_2!\alpha_{n_1}\alpha_{n_2}S_{n_1|n_2}(f(z);z)\right\} \tag{8}$$

Here

$$B_{n|q}(f(z);z) = \sum_{n|n_1...n_q}\frac{\partial^{n_1}\phi...\partial^{n_q}\phi}{n_1!...n_q!r(n_1)!...r(n_q)!} \tag{9}$$

are the restricted length q Bell polynomials in derivatives of f, with the sum taken over the ordered length q partitions of $n = n_1 + ... + n_q$; with $0 < n_1 \leq n_2.... \leq n_q$ anq $r(n_i)$ is multiplicity of element n_i in the partition (e.g. for $8 = 2 + 3 + 3$ $q(1) = 0, q(2) = 1$ and $q(3) = 2$). Next, $S_{n_1|n_2}$ are the generalized rank (n_1, n_2) Schwarzians of the conformal transformation $f(z)$, defined according to [23]:

for $n_1, n_2 \neq 0$ with the sum over the non-negative numbers m_1, m_2 and p taken over all the combinations satisfying

$$m_1 + m_2 + p = k_1 + k_2$$

Also, $S_{0|0}(f;z) = ln(f'(z))$, $S_{1|0} = S_{0|1} = \frac{f''(z)}{2f'(z)}$ and $S_{1|1}$ is a usual Schwarzian derivative (up to conventional factor of $\frac{1}{6}$). As $z \to \xi_j$, the coefficients in front of derivatives $\partial^q\phi$ ($q \neq 0$) in the exponent of $\tilde{W}_p(f(z))$, determined by the length q Bell polynomials in f, are damped exponentially as $\sim e^{-\frac{q}{\epsilon}}$, so only the regular part $\sim \alpha_0\phi$ survives and the operators become regular in new coordinates. On the other hand, the price paid for the regularity is the appearance of the generalized Schwarzians $S_{n_1|n_2}(f;z)$ in the transformation law for the irregular vertices. All of these Schwarzians have inverse power behavior in ϵ and must be regularized as $\epsilon \to 0$. The final step is to compute the overlap deformation resulting from contractions of $T(z) = \frac{1}{2} : \partial\phi\partial\phi :$ with different vertex operators (i.e. each of $\partial\phi$'s contracting with different vertex) and to integrate it. This altogether is equivalent to integrating the Ward identities and, in the limit $\epsilon \to 0$, the integrated overlap deformation determines the difference between the correlators computed on the half-plane and on the Riemann surface defined by the conformal map $f(z)$, with the transformation laws (8) for the vertex operators. The relevant infinitezimal overlap transformation of the N-point correlator is given by

$$\delta_{overlap} A_N(\xi_1, ...\xi_N) = A_N(\xi_1, ...\xi_N) \sum_{j=1}^{N-1} \sum_{k=j+1}^{N} \sum_{n_j=1}^{p_j} \sum_{n_k=1}^{p_k} \alpha_{n_j}^{(j)} \alpha_{n_k}^{(k)} \times \left(n_k! \partial_{\xi_j}^{n_j} \frac{\epsilon(\xi_j)}{(\xi_j - \xi_k)^{n_k+1}} + n_j! \partial_{\xi_k}^{n_k} \frac{\epsilon(\xi_k)}{(\xi_k - \xi_j)^{n_j+1}} \right) \quad (11)$$

It is straightforward to integrate it to obtain the contribution of the overlap to the deformation of A_N under the finite conformal transformation $z \to f(z)$. The result is

$$A_N(\xi_1, ..., \xi_N) \to A_N(f(\xi_1), ..., f(\xi_N)) exp \left\{ \sum_{j=1}^{N-1} \sum_{k=j+1}^{N} \sum_{n_j=1}^{p_j} \sum_{n_k=1}^{p_k} \alpha_{n_j}^{(j)} \alpha_{n_k}^{(k)} \times \left(\sum_{p_j=0}^{n_j-1} \sum_{p_k=0}^{n_k} \left(\frac{B_{n_j|n_j-p_j}(f(\xi_j); \xi_j) B_{n_k|n_k-p_k}(f(\xi_j); \xi_j)}{(f(\xi_j) - f(\xi_k))^{n_j+n_k-p_j-p_k}} - \frac{1}{(\xi_j - \xi_k)^{n_j+n_k-p_j-p_k}} \right) \right) \right\} \quad (12)$$

The relation between the correlators of the irregular and regular states is given by the transformation law (8) divided by the overlap deformation (12). Thus for N-point correlator of irregular states of arbitrary rank one has:

$$A_N(\xi_1, ..., \xi_N) = S_N(\xi_1, ..., \xi_N) \times exp \left\{ \frac{1}{2} \sum_{j=1}^{N} \sum_{k=1^{p_j}} \sum_{l=1}^{p_j} \alpha_k^{(j)} \alpha_l^{(j)} S_{k|l}(f(z); z)|_{z=\xi_j} - \sum_{j=1}^{N-1} \sum_{k=j+1}^{N} \sum_{n_j=1}^{p_j} \sum_{n_k=1}^{p_k} \alpha_{n_j}^{(j)} \alpha_{n_k}^{(k)} \times \left(\sum_{p_j=0}^{n_j-1} \sum_{p_k=0}^{n_k} \left(\frac{B_{n_j|n_j-p_j}(f(\xi_j); \xi_j) B_{n_k|n_k-p_k}(f(\xi_j); \xi_j)}{(f(\xi_j) - f(\xi_k))^{n_j+n_k-p_j-p_k}} - \frac{1}{(\xi_j - \xi_k)^{n_j+n_k-p_j-p_k}} \right) \right) \right\} \quad (13)$$

where S_N is the N-point correlator of regular Liouville primaries:

$$S_N(\xi_1, ..., \xi_N) = \left\langle e^{\alpha_0^{(1)}\phi}(\xi_1) ... e^{\alpha_0^{(N)}\phi}(\xi_N) \right\rangle \quad (14)$$

This reduces the problem of describing the interactions of irregular states of arbitrary rank in terms of those of the regular states.

In particular, for $N = 3$, S_3 is well-known and related to the regular Liouville S-matrix [21,22]. However, the Schwarzians and overlap factors (involving sums over the restricted Bell polynomials of the conformal transformation (6)) are singular at the insertion points of the irregular vertex operators and need to be regularized. In the Appendix section, we show how to perform such a regularization explicitly for the rank two irregular states ($p_1 = p_2 = p_3 = 2$) and for $N = 3$. It already turns to be quite cumbersome. It is straightforward to show, in particular, that in a particularly simple case of two-point function of the rank 1 irregular states, this regularization procedure reproduces the inner product calculated at [9,13]. To obtain this product, one has to take the limit $\xi_1 = 0, \xi_2 \to \infty$ in (5) ($N = 2$; $p_1 = p_2 = 1$). In this limit, the overlap factor vanishes, and all the contributions stem from the regularizations of the Schwarzians at ξ_1 and ξ_2:

$$A_2(w_1, w_2)|_{w_1=0, w_2\to\infty} = e^{\frac{1}{2}\beta_1^2 S_{1|1}(f;\xi_1) + \frac{1}{2}\beta_2^2 S_{1|1}(f;\xi_2) + \alpha_1\beta_1 S_{0|1}(f;\xi_1) + \alpha_2\beta_2 S_{0|1}(f,\xi_2)} S_2$$

where

$$S_{0|1}(f; z) = \frac{f''(z)}{f'(z)}$$

$$S_{1|1}(f; z) = \frac{1}{6}(\frac{f'''}{f'}) - \frac{1}{4}(\frac{f''}{f'})^2$$

$$f(z) = e^{-\frac{i}{z-\xi_1} - \frac{i}{z-\xi_2}} \quad (15)$$

are the generalized Schwarzians that need to be regularized and S_2 is the inner product for the Liouville primaries. Using the regularization relations (18) in the Appendix along with (3), one easily recovers the inner product [9,13]. The generalizations of this regularization procedure for higher number of points and for higher ranks are in principle straightforward but require far more tedious calculations.

3 Conclusions

In this work, we analyzed interactions of the irregular states in Liouville theory by using conformal transformations that map the irregular vertex operators into regular. In particular, this allows to express three-point functions of irregulars of arbitrary rank in terms of Liouville structure constants given by DOZZ (Dorn-Otto-Zamolodchikov-Zamolodchikov) formula. The price one pays is the appearance of the objects, such as higher-derivative Schwarzians and overlap factors (sums over restricted Bell polynomials), in the transformation law for the vertices. They are singular at the insertion points and need to be regularized by a rather cumbersome procedure. Interestingly, all these objects appear in the solution of the well-known number theory problem of finding the closed analytic expressions for numbers of restricted partitions [23]. In the current paper, we restrict ourselves to the maximum rank two and the three-point function. In our future work (currently in progress), we hope to be able to develop the algorithm which simplifies the regularization scheme and can be applied to analyze the higher-rank interactions. Irregular states, apart from being relevant to AGT conjecture (extended to Argyres-Douglas class of gauge theories) also appear in the interplay between open string field theory and higher-spin gauge theories, as the irregular vertex operators can be understood as generating wavefunctions for higher-spin vertex operators in bosonic string theory. Given certain constraints on $\alpha_n^{(m)}$, the irregular vertices of the lower ranks (1 and 2) form non-trivial solutions of open string field theory (OSFT) equations of motions, describing certain special limits of collective higher-spin configurations. Using the formalism developed in this paper we hope to extend these particular solutions to arbitrary ranks in order to describe the OSFT solutions describing general collective higher-spin vacua in string theory. These solutions, in general, are parametrized by nontrivial number theory identities, particularly involving higher-order Schwarzians and Bell polynomials. We hope that classifying these solutions may be useful to elucidate deep underlying relations, existing between number theory, bosonic strings and higher spin gauge theories.

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4. Appendix

In this section, we perform explicit regularizations of the higher-derivative Schwarzians and the overlap factors, for the case of rank 2 irregular states. We limit ourselves to 3-point functions.

4a. Transformation

For each irregular vertex operator of rank 2, the conformal transformation law is

$$e^{\alpha_0\phi+\alpha_1\partial\phi+\alpha_2\partial^2\phi} \xrightarrow{z\to f(z)} \exp\Big[\alpha_0\phi+\alpha_1 B_{1|1}\partial\phi+\alpha_2\left(B_{2|1}\partial\phi+B_{2|2}\partial^2\phi\right)+\frac{1}{2}\alpha_0^2 S_{0|0} +\alpha_0\alpha_1 S_{1|0}+2\alpha_0\alpha_2 S_{2|0}+\frac{1}{2}\alpha_1^2 S_{1|1}+2\alpha_1\alpha_2 S_{2|1}+2\alpha_2^2 S_{2|2}\Big] \tag{16}$$

where

$$S_{0|0}=\log f',\quad S_{1|0}=\frac{1}{2}\frac{f''}{f'},\quad S_{1|1}=\frac{1}{6}\left(\frac{f''}{f'}\right)'-\frac{1}{12}\left(\frac{f''}{f'}\right)^2,$$

$$S_{2|0}=\frac{1}{6}\frac{f'''}{f'}-\frac{1}{8}\left(\frac{f''}{f'}\right)^2,\quad S_{2|1}=\frac{1}{8}\left(\frac{f''}{f'}\right)^3-\frac{1}{6}\frac{f''f'''}{(f')^2}+\frac{1}{24}\frac{f^{(4)}}{f'},$$

$$S_{2|2}=\frac{1}{120}\frac{f^{(5)}}{f'}-\frac{1}{24}\frac{f^{(4)}f''}{(f')^2}-\frac{1}{24}\left(\frac{f'''}{f'}\right)^2+\frac{1}{6}\frac{(f'')^2f'''}{(f')^3}-\frac{3}{32}\left(\frac{f''}{f'}\right)^4$$

Next, consider the overlap transformation to the 3-point correlator. The overlap contribution between $(V(\xi_1), V(\xi_2))$ is given below, the overlaps between $V(\xi_1), V(\xi_2)$ and $V(\xi_2), V(\xi_3)$ are obtained similarly, by replacing variables α, β, ξ accordingly. Denote $f_1 = f(\xi_1)$ and $f_2 = f(\xi_2)$. We have:

$$\text{overlap}\left(e^{\alpha_0\phi+\alpha_1\partial\phi+\alpha_2\partial^2\phi}(\xi_1)e^{\beta_0\phi+\beta_1\partial\phi+\beta_2\partial^2\phi}(\xi_2)\right) \xrightarrow{z\to f(z)} \exp\Big[-\alpha_0\beta_0\log(f_1-f_2)-\alpha_1\beta_0\frac{f_1'}{f_1-f_2} +\alpha_0\beta_1\frac{f_2'}{f_1-f_2}-\alpha_1\beta_1\frac{f_1'f_2'}{(f_1-f_2)^2} -\alpha_2\beta_0\left(\frac{f_1''}{f_1-f_2}-\left(\frac{f_1'}{f_1-f_2}\right)^2\right) +\alpha_0\beta_2\left(\frac{f_2''}{f_1-f_2}+\left(\frac{f_2'}{f_1-f_2}\right)^2\right)$$

$$-\alpha_2\beta_1\left(\frac{f_1'' f_2'}{(f_1-f_2)^2}-\frac{2(f_1')^2 f_2'}{(f_1-f_2)^3}\right)$$
$$-\alpha_1\beta_2\left(\frac{f_1' f_2''}{(f_1-f_2)^2}+\frac{2f_1'(f_2')^2}{(f_1-f_2)^3}\right)$$
$$-\alpha_2\beta_2\left(\frac{f_1'' f_2''}{(f_1-f_2)^2}-\frac{2(f_1')^2 f_2''}{(f_1-f_2)^3}\right.$$
$$\left.+\frac{2f_1''(f_2')^2}{(f_1-f_2)^3}-\frac{6(f_1')^2(f_2')^2}{(f_1-f_2)^4}\right)$$
$$+\alpha_0\beta_0\log(\xi_1-\xi_2)+\frac{\alpha_1\beta_0}{\xi_1-\xi_2}$$
$$-\frac{\alpha_0\beta_1}{\xi_1-\xi_2}+\frac{\alpha_1\beta_1}{(\xi_1-\xi_2)^2}-\frac{\alpha_2\beta_0}{(\xi_1-\xi_2)^2}$$
$$\left.-\frac{\alpha_0\beta_2}{(\xi_1-\xi_2)^2}-\frac{2\alpha_2\beta_1}{(\xi_1-\xi_2)^3}+\frac{2\alpha_1\beta_2}{(\xi_1-\xi_2)^3}-\frac{6\alpha_2\beta_2}{(\xi_1-\xi_2)^4}\right] \tag{17}$$

4b. Regularization: 2-point function

We start by regularilizing the 2-point function first. We have:

$$A_2=\left\langle e^{\alpha_0\phi+\alpha_1\partial\phi+\alpha_2\partial^2\phi}(\xi_1)e^{\beta_0\phi+\beta_1\partial\phi+\beta_2\partial^2\phi}(\xi_2)\right\rangle,$$
$$f(z)=e^{-i\left(\frac{1}{z-\xi_1}+\frac{1}{z-\xi_2}\right)} \tag{18}$$

Regularizing generalized Schwarzians:

For $e^{\alpha_0\phi+\alpha_1\partial\phi+\alpha_2\partial^2\phi}(\xi_1)$,

$$S_{0|0}\sim\log\left(ie^{-i/\xi}\right),\quad S_{1|0}\sim\frac{i}{2\xi^2},\quad S_{1|1}\sim\frac{7}{12\xi^4}+\frac{1}{\xi^2},$$
$$S_{2|0}\sim-\frac{7}{24\xi^4}-\frac{i}{2\xi^3},\quad S_{2|1}\sim-\frac{1}{4\xi^5}-\frac{1}{\xi_{12}^3},$$
$$S_{2|2}\sim-\frac{323}{480\xi^8}-\frac{19}{12\xi^6}+\frac{1}{2\xi^4} \tag{19}$$

For $e^{\beta_0\phi+\beta_1\partial\phi+\beta_2\partial^2\phi}(\xi_2)$, $\xi_{12}\to-\xi_{12}$.

Regularizing the overlap part:

Denote $X_k\equiv 1/\xi_{12}^k$ and $F_1\equiv e^{-iX_1}$. Here we omit the free correlator part. Below, one by one, we present the regularized coefficients in front of $\alpha_i\beta_j$; $i,j=0,1,2$ in the exponential.

$\alpha_0\beta_0$

$$-\log\left(F_1-\frac{1}{F_1}\right) \tag{20}$$

$\alpha_1\beta_0$

$$-\frac{iF_1^2\left(\left(F_1^2-1\right)X_2+2iX_3\right)}{\left(F_1^2-1\right)^2} \tag{21}$$

$\alpha_0\beta_1$

$$\frac{i\left(\left(F_1^2-1\right)X_2+2iF_1^2X_3\right)}{\left(F_1^2-1\right)^2} \tag{22}$$

$\alpha_1\beta_1$

$$\frac{F_1^2}{\left(F_1^2-1\right)^4}\left(4i\left(F_1^4-1\right)X_2X_3+\left(F_1^2-1\right)^2X_2^2\right.$$
$$-2\left(-i\left(F_1^4-1\right)X_5-3\left(F_1^2-1\right)^2\right.$$
$$\left.\left.\times X_4+\left(F_1^4+4F_1^2+1\right)X_3^2\right)\right) \tag{23}$$

$\alpha_2\beta_0$

$$-\frac{F_1^2}{\left(F_1^2-1\right)^4}\left(4i\left(F_1^4-1\right)\right.$$
$$\times X_2X_3+\left(F_1^2-1\right)^2X_2^2-2\Big(-i\left(F_1^4-1\right)X_5$$
$$+i\left(F_1^2-1\right)^3X_3-3\left(F_1^2-1\right)^2$$
$$\left.\times X_4+\left(F_1^4+4F_1^2+1\right)X_3^2\Big)\right) \tag{24}$$

$\alpha_0\beta_2$

$$-\frac{1}{\left(F_1^2-1\right)^4}\left(4i\left(F_1^4-1\right)F_1^2X_2X_3+\left(F_1^2-1\right)^2\right.$$
$$\times F_1^2X_2^2-2\Big(-i\left(F_1^4-1\right)F_1^2X_5$$
$$+i\left(F_1^2-1\right)^3X_3-3\left(F_1^2-1\right)^2F_1^2$$
$$\left.\times X_4+\left(F_1^4+4F_1^2+F_1\right)F_1^2X_3^2\Big)\right) \tag{25}$$

$\alpha_2\beta_1$

$$-\frac{F_1^2}{3\left(F_1^2-1\right)^6}\Big[3i\left(F_1^2+1\right)\left(F_1^2-1\right)^3$$
$$\times X_2^3+24i\left(F_1^2+1\right)\left(F_1^2-1\right)^3X_3^2$$
$$-18\left(F_1^4+4F_1^2+1\right)\left(F_1^2-1\right)^2X_2^2X_3$$
$$+3\left(F_1^2-1\right)^2\Big(15i\left(F_1^4-1\right)X_6+20\left(F_1^2-1\right)^2X_5$$
$$-2\left(F_1^4+4F_1^2+1\right)X_7\Big)+6\left(F_1^2-1\right)X_2\Big(\left(F_1^2-1\right)^3$$
$$\times X_3+3i\left(F_1^2-1\right)(3\left(F_1^4-1\right)X_4$$
$$+i\left(F_1^4+4F_1^2+1\right)X_5)$$

$$
\begin{aligned}
&-3i\left(F_1^6+11F_1^4+11F_1^2+1\right)X_3^2\Big)\\
&-6\left(F_1^2-1\right)X_3\Big(9\left(F_1^6+3F_1^4-3F_1^2-1\right)\\
&\times X_4+2i\left(F_1^6+11F_1^4+11F_1^2+1\right)X_5\Big)\\
&+4\left(F_1^8+26F_1^6+66F_1^4+26F_1^2+1\right)X_3^3\Big]
\end{aligned}
\tag{26}
$$

$\alpha_1\beta_2$

$$
\begin{aligned}
&-\frac{F_1^2}{3\left(F_1^2-1\right)^6}\Big[-3i\left(F_1^2+1\right)\left(F_1^2-1\right)^3\\
&\times X_2^3-24i\left(F_1^2+1\right)\left(F_1^2-1\right)^3X_3^2\\
&+18\left(F_1^4+4F_1^2+1\right)\left(F_1^2-1\right)^2\\
&\times X_2^2X_3-3\left(F_1^2-1\right)^2\Big(15i\left(F_1^4-1\right)\\
&\times X_6+20\left(F_1^2-1\right)^2X_5\\
&-2\left(F_1^4+4F_1^2+1\right)X_7\Big)+6\left(F_1^2-1\right)\\
&\times X_2\Big(-\left(F_1^2-1\right)^3X_3+3\left(F_1^2-1\right)\left(-3i\left(F_1^4-1\right)X_4\right.\\
&\left.+\left(F_1^4+4F_1^2+1\right)X_5\right)\\
&+3i\left(F_1^6+11F_1^4+11F_1^2+1\right)X_3^2\Big)\\
&+6\left(F_1^2-1\right)X_3\left(9\left(F_1^6+3F_1^4-3F_1^2-1\right)\right.\\
&\left.\times X_4+2i\left(F_1^6+11F_1^4+11F_1^2+1\right)X_5\right)\\
&-4\left(F_1^8+26F_1^6+66F_1^4+26F_1^2+1\right)X_3^3\Big]
\end{aligned}
\tag{27}
$$

$\alpha_2\beta_2$

$$
\begin{aligned}
&-\frac{F_1^2}{3\left(F_1^2-1\right)^8}\Big[-3\left(F_1^2-1\right)^4\left(F_1^4+4F_1^2+1\right)\\
&\times X_2^4-24i\left(F_1^2-1\right)^3\left(F_1^6+11F_1^4+11F_1^2+1\right)X_2^3X_3\\
&+12\left(F_1^2-1\right)^2X_2^2\Big(i\left(F_1^2+1\right)\left(F_1^2-1\right)^3\\
&\times X_3-3\left(F_1^2-1\right)\left(3\left(F_1^6+3F_1^4-3F_1^2-1\right)X_4\right.\\
&\left.+i\left(F_1^6+11F_1^4+11F_1^2+1\right)X_5\right)\\
&+3\left(F_1^8+26F_1^6+66F_1^4+26F_1^2+1\right)X_3^2\Big)\\
&+4iX_2\Big(24i\left(F_1^4+4F_1^2+1\right)\left(F_1^2-1\right)^4\\
&\times X_3^2+3\left(F_1^2-1\right)^3\Big(20\left(F_1^2+1\right)\left(F_1^2-1\right)^2X_5\\
&+i\Big(15\left(F_1^6+3F_1^4-3F_1^2-1\right)\\
&\times X_6+2i\left(F_1^6+11F_1^4+11F_1^2+1\right)X_7\Big)\Big)\\
&-6\left(F_1^2-1\right)^2X_3\Big(9\left(F_1^8+10F_1^6-10F_1^2-1\right)\\
&\times X_4+2i\left(F_1^8+26F_1^6+66F_1^4+26F_1^2+1\right)X_5\Big)\\
&+4\left(F_1^{12}+56F_1^{10}+245F_1^8-245F_1^4-56F_1^2-1\right)X_3^3\Big)\\
&-2\Big(28i\left(F_1^2-1\right)^3\Big(F_1^6+11F_1^4\\
&+11F_1^2+1\Big)X_3^3+6\left(F_1^2-1\right)^2\\
&\times X_3\Big(10iF_1^8X_6-F_1^8X_7+100iF_1^6X_6-26F_1^6X_7-66F_1^4X_7\\
&-18i\left(F_1^2-1\right)^3\left(F_1^2+1\right)X_4-100iF_1^2X_6-26F_1^2X_7\\
&+26\left(F_1^2-1\right)^2\left(F_1^4+4F_1^2+1\right)X_5\\
&-10iX_6-X_7\Big)+3\left(F_1^2-1\right)^2\\
&\times\Big(27\left(F_1^2-1\right)^2\left(F_1^4+4F_1^2+1\right)X_4^2\\
&+12i\left(F_1^8+10F_1^6-10F_1^2-1\right)X_4\\
&\times X_5-\left(F_1^2-1\right)\Big(40\left(F_1^2-1\right)^3\\
&\times X_6+i\Big(50\left(F_1^2+1\right)\left(F_1^2-1\right)^2X_7\\
&+i\Big(14\left(F_1^6+3F_1^4-3F_1^2-1\right)\\
&\times X_8+i\left(F_1^6+11F_1^4+11F_1^2+1\right)X_9\Big)\Big)\Big)\\
&-\left(F_1^8+26F_1^6+66F_1^4+26F_1^2+1\right)X_5^2\Big)\\
&-6\left(F_1^2-1\right)X_3^2\Big(F_1^{10}(6X_4+iX_5+1)\\
&+F_1^8(150X_4+57iX_5-5)+2F_1^6(120X_4\\
&+151iX_5+5)-2F_1^4(120X_4-151iX_5+5)\\
&+F_1^2(-150X_4+57iX_5+5)-6X_4+iX_5-1\Big)\\
&+\Big(F_1^{12}+120F_1^{10}+1191F_1^8\\
&+2416F_1^6+1191F_1^4+120F_1^2+1\Big)X_3^4\Big)\Big]
\end{aligned}
\tag{28}
$$

4c. Regularization: 3-point function

Now we generalize the regularization, performed above, to the case of the 3-point forrelator of rank 2 operators. The correlator and the conformal transformation, in the limit $\epsilon \to 0$, are given by:

$$
\begin{aligned}
&\Big\langle e^{\alpha_0\phi+\alpha_1\partial\phi+\alpha_2\partial^2\phi}(\xi_1)e^{\beta_0\phi+\beta_1\partial\phi+\beta_2\partial^2\phi}\\
&\times(\xi_2)e^{\gamma_0\phi+\gamma_1\partial\phi+\gamma_2\partial^2\phi}(\xi_3)\Big\rangle,
\end{aligned}
\tag{29}
$$

$$
f(z)=e^{-i\left(\frac{1}{z-\xi_1}+\frac{1}{z-\xi_2}+\frac{1}{z-\xi_3}\right)}
\tag{30}
$$

Regularization of the generalized Schwarzians

Denoting $\xi_{ab} \equiv \xi_a - \xi_b$ and

$$X_k \equiv \frac{1}{\xi_{12}^k} + \frac{1}{\xi_{13}^k}, \quad Y_k \equiv \frac{1}{\xi_{21}^k} + \frac{1}{\xi_{23}^k}, \quad Z_k \equiv \frac{1}{\xi_{31}^k} + \frac{1}{\xi_{32}^k} \tag{31}$$

For $e^{\alpha_0\phi+\alpha_1\partial\phi+\alpha_2\partial^2\phi}(\xi_1)$,

$$\begin{aligned}
S_{0|0} &\sim \log\left(ie^{-iX_1}\right), \quad S_{1|0} \sim \frac{i}{2}X_2,\\
S_{1|1} &\sim \frac{X_4}{2} + \frac{X_2^2}{12} + X_2,\\
S_{2|0} &\sim -\frac{1}{2}\left(\frac{X_4}{2} + \frac{X_2^2}{12} + iX_3\right),\\
S_{2|1} &\sim -\frac{X_5}{6} - \frac{X_2X_3}{12} - X_3,\\
S_{2|2} &\sim \frac{146X_8}{480} - \frac{23X_2X_6}{60} - \frac{2X_3X_5}{5}\\
&\quad -\frac{31X_4^2}{160} - \frac{20X_6}{12} + \frac{X_3^2}{12} + X_4 - \frac{X_2^2}{2}
\end{aligned} \tag{32}$$

For $e^{\beta_0\phi+\beta_1\partial\phi+\beta_2\partial^2\phi}(\xi_2)$ and $e^{\gamma_0\phi+\gamma_1\partial\phi+\gamma_2\partial^2\phi}(\xi_3)$, replace X_k with Y_k and Z_k, repectively.

Regularization of the overlap part

We show explicitly in the following the overlap contributions between the irregular vertex operators at ξ_1 and ξ_2. The other contributions are given by simply replacing X_k, Y_k with X_k, Z_k for ξ_1, ξ_3 and with Y_k, Z_k for ξ_2, ξ_3.

Denote $D_k \equiv X_k - Y_k$ and $R \equiv e^{iD_1}$. Again, free correlators are omitted.

$\alpha_0\beta_0$ regularization

$$-\log\left(e^{-iX_1} - e^{-iY_1}\right) \tag{33}$$

$\alpha_0\beta_1$ regularization

$$\frac{R}{(1-R)^3}\left[i(R-1)^2Y_2 - \frac{1}{2}iD_2^2(R+1) + D_3(R-1)\right] \tag{34}$$

For $\alpha_1\beta_0$ regularization, interchange $X_k \leftrightarrow Y_k$.

$\alpha_1\beta_1$ regularization

$$\begin{aligned}
&\frac{R^3}{(1-R)^6}\left[\frac{11D_2^4}{4} - 3D_2^3 + 6D_2^2X_2\right.\\
&+\frac{1}{R^2}\left(\frac{D_2^4}{24} + \frac{D_2^3}{2} + D_2^2\left(-X_2 - \frac{1}{2}(iD_3)\right)\right.\\
&+D_2(-3iD_3 - D_4 - X_2 + 4iX_3) - \frac{D_3^2}{2}\\
&\left.+2iD_3X_2 - 3D_4 + iD_5 + X_2^2 + 6X_4\right)\\
&+R^2\left(\frac{D_2^4}{24} + \frac{D_2^3}{2} + \frac{1}{2}iD_2^2(D_3 + 2iX_2)\right.\\
&+D_2(3iD_3 - D_4 - X_2 - 4iX_3) - \frac{D_3^2}{2}\\
&\left.-2iD_3X_2 - 3D_4 - iD_5 + X_2^2 + 6X_4\right)\\
&+R\left(\frac{13D_2^4}{12} + D_2^3 + D_2^2(-2X_2 + 5iD_3)\right.\\
&+D_2(-6iD_3 - 2D_4 + 4X_2 + 8iX_3)\\
&\left.-D_3^2 + 4iD_3X_2 + 12D_4 + 2iD_5 - 4X_2^2 - 24X_4\right)\\
&+\frac{1}{R}\left(\frac{13D_2^4}{12} + D_2^3 + D_2^2(-2X_2 - 5iD_3)\right.\\
&+D_2(6iD_3 - 2D_4 + 4X_2 - 8iX_3) - D_3^2\\
&\left.-4iD_3X_2 + 12D_4 - 2iD_5 - 4X_2^2 - 24X_4\right)\\
&+6D_2(D_4 - X_2)\\
&\left.+3\left(D_3^2 + 2\left(-3D_4 + X_2^2 + 6X_4\right)\right)\right]
\end{aligned} \tag{35}$$

$\alpha_2\beta_1$ regularization

$$\begin{aligned}
&-\frac{R}{(1-R)^9}\left[i(R+1)X_2^3(R-1)^6 + 12i(R+1)\right.\\
&\times X_3^2(R-1)^6 + 15i(R+1)X_6(R-1)^6\\
&+4\left(3D_2\left(R^2+4R+1\right) - (R-1)^2\right)\\
&\times X_5(R-1)^5 - \frac{1}{2}i\left(3\left(R^3+11R^2+11R+1\right)D_2^2\right.\\
&+2(R-1)^2(R+1)D_2\\
&\left.+6D_3i\left(R^3+3R^2-3R-1\right)\right)X_2^2(R-1)^4\\
&-\frac{3}{2}i\left(3\left(R^3+11R^2+11R+1\right)\right.\\
&\times D_2^2 + 4(R-1)^2(R+1)D_2\\
&\left.+6D_3i\left(R^3+3R^2-3R-1\right)\right)X_4(R-1)^4\\
&-\left(\left(R^4+26R^3+66R^2+26R+1\right)D_2^3\right.\\
&+5(R-1)^2\left(R^2+4R+1\right)D_2^2 + 2(R-1)
\end{aligned}$$

$$
\begin{aligned}
&\times \Big(3 i D_3 \left(R^3+11R^2+11R+1\right)-(R-1)^3\Big)\, D_2 \\
&+2i(R-1)^2 \Big(5 D_3 \left(R^2-1\right) \\
&+3 D_4 i \left(R^2+4R+1\right)\Big)\Big) X_3 (R-1)^3 \\
&+\frac{1}{720}\Big[-i \Big(R^7+247R^6+4293R^5+15619R^4 \\
&+15619R^3+4293R^2+247R+1\Big) D_2^6 \\
&-42i(R-1)^2 \Big(R^5+57R^4 \\
&+302R^3+302R^2+57R+1\Big) D_2^5 \\
&+30(R-1)\Big(D_3 \Big(R^6+120R^5+1191R^4 \\
&+2416R^3+1191R^2+120R+1\Big) \\
&-8i(R-1)^3 \left(R^3+11R^2+11R+1\right)\Big) \\
&\times D_2^4+120(R-1)^2 \Big(7 D_3 \big(R^5+25R^4+40R^3 \\
&-40R^2-25R-1\big)+D_4 i \Big(R^5+57R^4+302R^3 \\
&+302R^2+57R+1\Big)\Big) D_2^3 \\
&+180(R-1)^2 \Big(16 D_3 \left(R^2+4R+1\right)(R-1)^3 \\
&+2i\big(7 D_4 \left(R^4+10R^3-10R-1\right) \\
&+D_5 i \left(R^4+26R^3+66R^2+26R+1\right)\big)(R-1) \\
&+D_3^2 i\big(R^5+57R^4+302R^3+302R^2 \\
&+57R+1\big)\Big) D_2^2+360 i(R-1)^3 \\
&\times\Big(7 \left(R^4+10R^3-10R-1\right) D_3^2 \\
&+2 D_4 i \left(R^4+26R^3+66R^2+26R+1\right) \\
&\times D_3+2(R-1)\big(8 D_4(R+1)(R-1)^2 \\
&+i \Big(7 D_5 \left(R^3+3R^2-3R-1\right) \\
&+D_6 i \left(R^3+11R^2+11R+1\right)\Big)\big)\Big) D_2 \\
&-120(R-1)^3 \Big(\left(R^4+26R^3+66R^2+26R+1\right) \\
&\times D_3^3-24i(R-1)^3(R+1) D_3^2 \\
&+6(R-1)\Big(7 D_4 \left(R^3+3R^2-3R-1\right) \\
&+D_5 i \left(R^3+11R^2+11R+1\right)\Big) D_3 \\
&+3(R-1)\Big(i \left(R^3+11R^2+11R+1\right) \\
&\times D_4^2+2(R-1)\big(8 D_5(R-1)^2+7 D_6 i \left(R^2-1\right) \\
&-D_7 \left(R^2+4R+1\right)\big)\Big)\Big)\Big] \\
&+X_2\Big[18i(R+1) X_4(R-1)^6+2\Big(6 D_2 \left(R^2+4R+1\right) \\
&-(R-1)^2\Big) X_3(R-1)^5 \\
&+\frac{1}{24} i \Big(3 \left(R^5+57R^4+302R^3+302R^2+57R+1\right) D_2^4 \\
&+32(R-1)^2 \left(R^3+11R^2+11R+1\right) \\
&\times D_2^3+36 D_3 i \big(R^5+25R^4+40R^3-40R^2 \\
&-25R-1\big) D_2^2+24i(R-1)^2\Big(8 D_3 \left(R^3+3R^2-3R-1\right) \\
&+3 D_4 i \left(R^3+11R^2+11R+1\right)\Big) D_2 \\
&-12(R-1)^2\Big(3 \left(R^3+11R^2+11R+1\right) D_3^2 \\
&+2(R-1)\Big(8 D_4 \left(R^2-1\right) \\
&+3 D_5 i \left(R^2+4R+1\right)\Big)\Big)\Big)(R-1)^2\Big]\Big]
\end{aligned}
\tag{36}
$$

For $\alpha_1\beta_2$ regularization, interchange $X_k \leftrightarrow Y_k$.

$\alpha_2\beta_2$ regularization

$$
\begin{aligned}
&-\frac{R}{40320(1-R)^{12}}\Big[-\big(R^{10}+2036R^9+152637R^8 \\
&+2203488R^7+9738114R^6+15724248R^5 \\
&+9738114R^4+2203488R^3+152637R^2+2036R+1\big) \\
&\times D_2^8-112(R-1)^2\big(R^8+502R^7 \\
&+14608R^6+88234R^5+156190R^4+88234R^3 \\
&+14608R^2+502R+1\big) D_2^7 \\
&+56(R-1)\Big(2(R-1)\big((2X_2-25)R^8 \\
&+2(502X_2-1475)R^7+8(3652X_2-2975)R^6 \\
&+2(88234X_2-1925)R^5+10(31238X_2+6125)R^4 \\
&+2(88234X_2-1925)R^3 \\
&+8(3652X_2-2975)R^2+2(502X_2-1475)R \\
&+2X_2-25\big)-i D_3\big(R^9+1013R^8+47840R^7 \\
&+455192R^6+1310354R^5+1310354R^4 \\
&+455192R^3+47840R^2+1013R+1\big)\Big) D_2^6 \\
&+336(R-1)^2\Big(-14 D_3 i \Big(R^8+246R^7+4046R^6 \\
&+11326R^5-11326R^3-4046R^2-246R-1\Big) \\
&+D_4 \Big(R^8+502R^7+14608R^6+88234R^5 \\
&+156190R^4+88234R^3+14608R^2+502R+1\Big) \\
&+2(R-1)\big((15X_2+4i X_3-20)R^7 \\
&+(1785X_2+988i X_3-460)R^6 \\
&+9(1785X_2+1908i X_3+20)R^5 \\
&+(18375X_2+62476i X_3+1900)R^4 \\
&+(-18375X_2+62476i X_3-1900)R^3 \\
&-9(1785X_2-1908i X_3+20)R^2
\end{aligned}
$$

$$+(-1785X_2+988iX_3+460)$$
$$\times R-15X_2+4iX_3+20)\Big)D_2^5$$
$$+840(R-1)^2\Big(\big(R^8+502R^7+14608R^6$$
$$+88234R^5+156190R^4+88234R^3+14608R^2$$
$$+502R+1\big)D_3^2+4i\left(R^2-1\right)\big((2X_2-25)R^6$$
$$+6(82X_2-225)R^5+(8094X_2-3375)R^4$$
$$+4(5786X_2+2375)R^3+(8094X_2-3375)R^2$$
$$+6(82X_2-225)R+2X_2-25\big)D_3$$
$$+2(R-1)\Big(14D_4\left(R^7+119R^6+1071R^5\right.$$
$$\left.+1225R^4-1225R^3-1071R^2-119R-1\right)$$
$$+D_5i\left(R^7+247R^6+4293R^5+15619R^4\right.$$
$$\left.+15619R^3+4293R^2+247R+1\right)$$
$$-2(R-1)\big(-20\left(R^4+26R^3+66R^2+26R+1\right)$$
$$\times X_2(R-1)^2+3\big(R^6+120R^5+1191R^4$$
$$+2416R^3+1191R^2+120R+1\big)X_2^2+6$$
$$\times\big((R^6+120R^5+1191R^4+2416R^3+1191R^2$$
$$+120R+1)X_4-4i\left(R^6+56R^5+245R^4\right.$$
$$\left.-245R^2-56R-1\right)X_3)\big)\Big)\Big)D_2^4$$
$$+6720(R-1)^3\Big(7\left(R^7+119R^6+1071R^5\right.$$
$$\left.+1225R^4-1225R^3-1071R^2-119R-1\right)D_3^2$$
$$+i\Big(D_4\left(R^7+247R^6+4293R^5+15619R^4\right.$$
$$\left.+15619R^3+4293R^2+247R+1\right)$$
$$+2(R-1)\big((15X_2+4iX_3-20)R^6$$
$$+40(21X_2+12iX_3-4)R^5$$
$$+(3675X_2+4764iX_3+380)R^4+9664iX_3R^3$$
$$+(-3675X_2+4764iX_3-380)R^2$$
$$-40(21X_2-12iX_3-4)R-15X_2+4iX_3+20\big)\Big)$$
$$\times D_3-(R-1)\Big(18X_2^2R^6+D_6R^6$$
$$-44iX_3R^6+24iX_2X_3R^6+54X_4R^6+16iX_5R^6$$
$$+432X_2^2R^5+120D_6R^5-352iX_3R^5$$
$$+1344iX_2X_3R^5+1296X_4R^5+896iX_5R^5$$
$$+270X_2^2R^4+1191D_6R^4+836iX_3R^4$$
$$+5880iX_2X_3R^4+810X_4R^4+3920iX_5R^4$$
$$-1440X_2^2R^3+2416D_6R^3-4320X_4R^3$$
$$+270X_2^2R^2+1191D_6R^2-836iX_3R^2$$
$$-5880iX_2X_3R^2+810X_4R^2-3920iX_5R^2$$
$$+432X_2^2R+120D_6R+352iX_3R-1344iX_2X_3R$$
$$+1296X_4R-896iX_5R+18X_2^2+D_6$$
$$-14iD_5\left(R^6+56R^5+245R^4-245R^2-56R-1\right)$$
$$+2D_4\big((2X_2-25)R^6+120(2X_2-5)R^5$$
$$+3(794X_2-125)R^4+16(302X_2+125)R^3$$
$$+3(794X_2-125)R^2+120(2X_2-5)R$$
$$+2X_2-25\big)+44iX_3-24iX_2X_3+54X_4-16iX_5\Big)\Big)$$
$$\times D_2^3+3360(R-1)^3\Big(i(R^7+247R^6$$
$$+4293R^5+15619R^4+15619R^3+4293R^2$$
$$+247R+1)D_3^3-6(R-1)\big((2X_2-25)R^6$$
$$+120(2X_2-5)R^5+3(794X_2-125)R^4$$
$$+16(302X_2+125)R^3+3(794X_2-125)R^2$$
$$+120(2X_2-5)R+2X_2-25\big)D_3^2+6i(R-1)$$
$$\times\Big(14D_4(R^6+56R^5+245R^4-245R^2$$
$$-56R-1)+D_5i\left(R^6+120R^5+1191R^4\right.$$
$$\left.+2416R^3+1191R^2+120R+1\right)$$
$$-2(R-1)\big(-20\left(R^3+11R^2+11R+1\right)$$
$$X_2(R-1)^2+3(R^5+57R^4+302R^3+302R^2$$
$$+57R+1)X_2^2-6i(4\left(R^5+25R^4+40R^3\right.$$
$$\left.-40R^2-25R-1\right)X_3+i(R^5+57R^4+302R^3$$
$$+302R^2+57R+1)X_4)\big)\Big)D_3-3(R-1)$$
$$\times\Big((R^6+120R^5+1191R^4+2416R^3+1191R^2$$
$$+120R+1)D_4^2+4(R-1)\big((15X_2+4iX_3-20)R^5$$
$$+(375X_2+228iX_3-20)R^4$$
$$+8(75X_2+151iX_3+20)R^3$$
$$-8(75X_2-151iX_3+20)R^2$$
$$+(-375X_2+228iX_3+20)R$$
$$-15X_2+4iX_3+20\big)D_4+2(R-1)\big(-4X_2^3R^5$$
$$+2X_2^2R^5-24X_3^2R^5+D_7iR^5+4iX_3R^5$$
$$+48iX_2X_3R^5-36X_2X_4R^5+36X_4R^5$$
$$+48iX_5R^5-20X_6R^5-100X_2^3R^4+2X_2^2R^4$$
$$-600X_3^2R^4+57D_7iR^4-12iX_3R^4$$
$$+432iX_2X_3R^4-900X_2X_4R^4+36X_4R^4+432iX_5R^4$$
$$-500X_6R^4-160X_2^3R^3-16X_2^2R^3$$
$$-960X_3^2R^3+302D_7iR^3+8iX_3R^3-480iX_2X_3R^3$$
$$-1440X_2X_4R^3-288X_4R^3-480iX_5R^3$$

$$
\begin{aligned}
&-800X_6R^3+160X_2^3R^2+16X_2^2R^2+960X_3^2R^2\\
&+302D_7iR^2+8iX_3R^2-480iX_2X_3R^2\\
&+1440X_2X_4R^2+288X_4R^2-480iX_5R^2+800X_6R^2\\
&+100X_2^3R-2X_2^2R+600X_3^2R+57D_7i\\
&\times R-12iX_3R+432iX_2X_3R+900X_2X_4R-36X_4R\\
&+432iX_5R+500X_6R+4X_2^3-2X_2^2+24X_3^2\\
&+D_7i+14D_6(R^5+25R^4+40R^3-40R^2\\
&-25R-1)+2D_5i(R+1)\big((2X_2-25)R^4\\
&+8(14X_2-25)R^3+6(82X_2+75)R^2\\
&+8(14X_2-25)R+2X_2-25\big)+4iX_3\\
&+48iX_2X_3+36X_2X_4-36X_4+48iX_5+20X_6)\Big)\Big)D_2^2\\
&+6720(R-1)^4\Bigg(14i\left(R^6+56R^5+245R^4\right.\\
&\left.-245R^2-56R-1\right)D_3^3-3\Big(D_4(R^6+120R^5\\
&+1191R^4+2416R^3+1191R^2+120R+1)\\
&+2(R-1)\big((15X_2+4iX_3-20)R^5\\
&+(375X_2+228iX_3-20)R^4+8(75X_2\\
&+151iX_3+20)R^3-8(75X_2-151iX_3+20)R^2\\
&+(-375X_2+228iX_3+20)R-15X_2\\
&+4iX_3+20)\Big)D_3^2-6i(R-1)\Big(2D_4(R+1)\big((2X_2\\
&-25)R^4+8(14X_2-25)R^3+6(82X_2+75)R^2\\
&+8(14X_2-25)R+2X_2-25\big)\\
&-i\big(14D_5\left(R^5+25R^4+40R^3-40R^2-25R-1\right)\\
&+D_6i(R^5+57R^4+302R^3+302R^2\\
&+57R+1)+2(R-1)(27iX_4R^4-8X_5R^4\\
&+270iX_4R^3-208X_5R^3-528X_5R^2\\
&-270iX_4R-208X_5R+9i\left(R^4+10R^3-10R-1\right)\\
&\times X_2^2+22(R-1)^2\left(R^2+4R+1\right)X_3\\
&-12\left(R^4+26R^3+66R^2+26R+1\right)X_2X_3\\
&-27iX_4-8X_5))\Big)D_3\\
&-6(R-1)\Big(7\left(R^5+25R^4+40R^3-40R^2-25R-1\right)\\
&\times D_4^2+\big(iD_5(R^5+57R^4+302R^3\\
&+302R^2+57R+1)-2(R-1)(-20\left(R^2+4R+1\right)\\
&\times X_2(R-1)^2+3(R^4+26R^3\\
&+66R^2+26R+1)X_2^2+6((R^4+26R^3+66R^2\\
&+26R+1)X_4\\
&-4i(R^4+10R^3-10R-1)X_3))\big)D_4+i(R-1)\\
&\times\big(D_5((30X_2+8iX_3-40)R^4
\end{aligned}
$$

$$
\begin{aligned}
&+4(75X_2+52iX_3+20)R^3+528iX_3R^2\\
&-4(75X_2-52iX_3+20)R-30X_2+8iX_3+40)\\
&+i\big(2X_2^3R^4+24X_3^2R^4+D_8R^4+24iX_2^2X_3R^4\\
&+4iX_2X_3R^4+36X_2X_4R^4+72iX_3X_4R^4\\
&-40iX_5R^4+48iX_2X_5R^4+30X_6R^4+24iX_7R^4\\
&+4X_2^3R^3+48X_3^2R^3+26D_8R^3\\
&+240iX_2^2X_3R^3-8iX_2X_3R^3+72X_2X_4R^3\\
&+720iX_3X_4R^3+80iX_5R^3+480iX_2X_5R^3\\
&+60X_6R^3+240iX_7R^3-12X_2^3R^2-144X_3^2R^2\\
&+66D_8R^2-216X_2X_4R^2-180X_6R^2\\
&+4X_2^3R+48X_3^2R+26D_8R-240iX_2^2X_3R\\
&+8iX_2X_3R+72X_2X_4R-720iX_3X_4R\\
&-80iX_5R-480iX_2X_5R+60X_6R-240iX_7\\
&\times R+2X_2^3+24X_3^2+D_8\\
&-14iD_7\left(R^4+10R^3-10R-1\right)+2D_6((2X_2-25)\\
&R^4+(52X_2-50)R^3+6(22X_2+25)R^2\\
&+(52X_2-50)R+2X_2-25)-24iX_2^2X_3-4iX_2X_3\\
&+36X_2X_4-72iX_3X_4+40iX_5\\
&-48iX_2X_5+30X_6-24iX_7))\Big)\Big)D_2-1680(R-1)^4\\
&\times\Big((R^6+120R^5+1191R^4+2416R^3\\
&+1191R^2+120R+1)D_3^4+8i\left(R^2-1\right)\\
&\times\big((2X_2-25)R^4+8(14X_2-25)R^3\\
&+6(82X_2+75)R^2+8(14X_2-25)R+2X_2-25\big)\\
&\times D_3^3+12(R-1)\Big(14D_4(R^5+25R^4\\
&+40R^3-40R^2-25R-1)+\\
&\times D_5i\left(R^5+57R^4+302R^3+302R^2+57R+1\right)\\
&-2(R-1)\big(-20\left(R^2+4R+1\right)X_2(R-1)^2\\
&+3\left(R^4+26R^3+66R^2+26R+1\right)X_2^2\\
&+6\left(\left(R^4+26R^3+66R^2+26R+1\right)X_4\right.\\
&\left.-4i\left(R^4+10R^3-10R-1\right)X_3\right)\big)\Big)D_3^2\\
&+12i(R-1)\Big(\left(R^5+57R^4+302R^3+302R^2\right.\\
&\left.+57R+1\right)D_4^2+4(R-1)((15X_2+4iX_3\\
&-20)R^4+2(75X_2+52iX_3+20)R^3+264iX_3\\
&\times R^2-2(75X_2-52iX_3+20)R-15X_2\\
&+4iX_3+20)D_4+2i(R-1)\big(2D_5((2X_2-25)\\
&\times R^4+(52X_2-50)R^3+6(22X_2+25)R^2\\
&+(52X_2-50)R+2X_2-25)
\end{aligned}
$$

$$
-i\left(14D_6\left(R^4+10R^3-10R-1\right)\right.
+D_7 i(R^4+26R^3+66R^2
+26R+1)-2(R-1)(2\left(R^3+11R^2+11R+1\right)
\times X_2^3-(R-1)^2(R+1)X_2^2
+6\left(3\left(R^3+11R^2+11R+1\right)\right.
\times X_4-4i\left(R^3+3R^2-3R-1\right)X_3\big)
\times X_2+2(-iX_3(R-1)^3
-9(R+1)X_4(R-1)^2
+6\left(R^3+11R^2+11R+1\right)X_3^2-12
\times iR^3X_5-36iR^2X_5+12iX_5
+36iRX_5+5R^3X_6+55R^2X_6+55RX_6+5X_6))))\big)
\times D_3-12(R-1)^2\big(2((2X_2-25)R^4
+(52X_2-50)R^3+6(22X_2+25)R^2
+(52X_2-50)R+2X_2-25)D_4^2
+2\big(-14D_5 i\left(R^4+10R^3-10R-1\right)
+D_6\left(R^4+26R^3+66R^2+26R+1\right)
+2(R-1)(27X_4R^3+8iX_5R^3+81X_4R^2
+88iX_5R^2-81X_4R+88iX_5R
+9\left(R^3+3R^2-3R-1\right)X_2^2-22i(R-1)^2
\times(R+1)X_3+12i\left(R^3+11R^2+11R+1\right)X_2X_3
-27X_4+8iX_5))D_4
+D_5^2\left(R^4+26R^3+66R^2+26R+1\right)
+4D_5 i(R-1)\big(-20(R+1)X_2(R-1)^2
+3\left(R^3+11R^2+11R+1\right)X_2^2
+6\big(\left(R^3+11R^2+11R+1\right)
\times X_4-4i\left(R^3+3R^2-3R-1\right)X_3\big)\big)
+2(R-1)\big(-R^3X_2^4-3R^2X_2^4+3RX_2^4
+X_2^4-36R^3X_4X_2^2-108R^2X_4X_2^2+108RX_4X_2^2
+36X_4X_2^2-48R^3X_3^2X_2-144R^2X_3^2X_2
+144RX_3^2X_2+48X_3^2X_2-60R^3X_6X_2
-180R^2X_6X_2+180RX_6X_2+60X_6X_2
+14D_8R^3+D_9iR^3+42D_8R^2+11D_9iR^2
-4R^3X_3^2+12R^2X_3^2-12RX_3^2+4X_3^2-54R^3X_4^2
-162R^2X_4^2+162RX_4^2
+54X_4^2-14D_8
+D_9 i-42D_8R+11D_9iR+2D_7i(R+1)
\big((2X_2-25)R^2+10(2X_2+5)R+2X_2-25\big)
+2D_6((15X_2+4iX_3-20)R^3+(45X_2+44iX_3+60)R^2
+(-45X_2+44iX_3-60)R
-15X_2+4iX_3+20)-96R^3X_3X_5-288R^2X_3X_5
+288RX_3X_5+96X_3X_5+80R^3X_6
-240R^2X_6+240RX_6-80X_6-28R^3X_8
-84R^2X_8+84RX_8+28X_8))\big)\big)\Big] \quad (37)
$$

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