

On stability of a neutron star system in Palatini gravity

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Abstract We formulate the generalized Tolman–Oppenheimer–Volkoff equations for the $f(\hat{R})$ Palatini gravity in the case of static and spherical symmetric geometry. We also show that a neutron star *can be a stable system* independently of the form of the functional $f(\hat{R})$.

1 Introduction

Einstein by formulating his General Relativity [1, 2] showed that there is a relation between geometry of spacetime and matter contribution. Until now, many astronomical observations have tested GR giving us that it is the best matching theory for explaining gravitational phenomena which we have in hand. However, there are still many issues in fundamental physics, astrophysics and cosmology (for review, see e.g. [3–7]) indicating that GR is not a final theory of gravitational interactions. There are such unsolved problems as the dark matter puzzle [8, 9], inflation [10, 11], another one is the late-time cosmic acceleration [12, 13] which is explained by the assumption that there exists an exotic fluid called dark energy [3–5]. It is introduced to the dynamics throughout adding the cosmological constant to the standard Einstein’s field equations considered in Friedmann–Robertson–Lemaître–Walker (FRLW) spacetime. Due to the above shortcomings one looks for different approaches in order to find a good theory which will be able to answer the above problems and to show us directions of future research. The additional argument in a favor of searching generalizations of gravity is non-renormalization of Einstein’s theory. The renormalization problem seems to be solved if extra high curvature terms are added [14]. One would also like to unify gravity theory with the other ones (electromagnetism, weak and strong interactions has been already unified into the Standard Model) but no satisfactory result has been obtained so

far as e.g. string theory, supersymmetry, that could combine particle physics and gravitation.

One of many ways to deal with the mentioned problems are Extended Theories of Gravity (ETG) [15, 16] which have been gained a lot of attention. The main arguments which advocate them are a possibility to explain problematic issues without introducing exotic, immeasurable entities as well as some approaches to quantum gravity [17] or the geometrization of the Standard Model (plus gravity) as derived from Chamseddine–Connes spectral action in noncommutative geometry [18, 19] which indicate that the effective action for the classical gravity should be more complicated than the one introduced by Einstein.

The geometric part of the action might be changed in many different ways. One may assume that constant of Nature are not really constant values [20–22]. A scalar field might be added into Lagrangian and moreover, it can be minimally or non-minimally coupled to gravity [23, 24]. One proposes much more complicated functionals than the simple linear one used in GR, for example $f(R)$ gravity [10, 25]. The latter approach has gained a lot of interest recently as the extra geometric terms could explain not only dark matter issue [26, 27] but also dark energy problem because it produces the accelerated late-time effect at low cosmic densities. The field equations also differ from the Einstein’s ones so they could provide different behavior of the early Universe.

The $f(R)$ gravity is usually treated in two different ways: in the metric approach [6, 7, 28–30] and Palatini one [28, 31–33]. The former arises to the fourth order differential equations which are more difficult to handle than the second-order field equations of GR¹ Another point is that one believes that physical equations of motion should be of the second order.

¹ The theory possesses the scalar-tensor representation, that is, it is mathematically equivalent to the Brans–Dicke gravity (with the parameter $\omega = 0$) with a potential of the scalar field. In this representation one deals with the differential equations of the second order on the metric components and a modified Klein–Gordon equation for the scalar field, see the details e.g. [34].

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In contrast to the metric formalism, the Palatini $f(R)$ gravity provides second order differential equations since the connection and the metric are treated as independent objects. The Riemann and Ricci tensors are constructed with the connection while for building the Ricci scalar we also use the physical metric in order to contract the indices. The Palatini approach allows to get the modified Friedmann equation [35–37] in a form that might be compared with the observational data [38–42]. It shows the potential of the Palatini formulation when it is applied to the gravitational problems [43,44].

There also exist disadvantages of such an approach: being in conflict with the Standard Model of particle physics [34,48–50], the algebraic dependence of the post-Newtonian metric on the density [51,52], and the complications with the initial values problem in the presence of matter [33,53], although the problem was already solved in [54]. Another one happens at microscopic scales, that is, the theory produces instabilities in atoms which disintegrate them. One should also mention limitations which arrive when one treats the extra terms as fluid-like [45–47]. However, it was shown [55] that high curvature corrections do not cause such problem. What is also very promising, some of the Palatini Lagrangians avoid the Big Bang singularity. Moreover, the effective dynamics of Loop Quantum Gravity can be reproduced by the Palatini theory which gives the link to one of approaches to Quantum Gravity [56]. High curvature correction of the form changes the notion of the independent connection: in the simple Palatini $f(R)$ gravity the connection is auxiliary field while in the more general Palatini theory it is dynamical without making the equations of motion second order in the fields.

The problem which is our interest in this work concerns astrophysical aspects of Palatini gravity, for example black holes [57–60], wormholes [61] and neutron stars [62]. We would like to examine the last objects in the context of Palatini gravity. In general, the maximal mass of neutron stars is still an open problem: Einstein’s gravity gives us, together with the recent observations, the limit as $2 M_{\odot}$. The pulsar PSR J1614-2230 is found to have the limit $1.97 M_{\odot}$ [63] while another one is Vela X-1 with the mass $\sim 1.8 M_{\odot}$ [64]. One also supposes the existence more massive neutron stars as for instance the ones with masses around $2.1 M_{\odot}$ [65] and $2.4 M_{\odot}$ (B1957+20) [66]. For the recent neutron stars mass determination see [67]. There also exists “hyperon puzzle”: it turns out that equations of state including hyperons make the maximal mass limit in the case of neutron stars without magnetic field lower than $2 M_{\odot}$ [68–70]. Due to that fact it seems that these equations of state could not be feasible for massive neutron stars in the framework of General Relativity. Some solutions were already proposed, like for example hyperon-vector coupling, chiral quark-meson coupling and the existence of strong magnetic fields inside the star. An

emissary for the latter idea are works indicating an influence of magnetic fluid which increases masses of the stars [71–73]. This arises to a question if neutron stars without strong magnetic field with mass larger than $2 M_{\odot}$ can exist [74–76] whose structure can be still described by the equations derived from General Relativity.

Moreover, because of the very recent neutron stars’ merger observation [77] the confrontation of gravitational theories with the observational data is an appealing task. Neutron stars are perfect objects for tests at high density regimes thus one may perceive possible deviations from GR. It also seems that using General Relativity as a description of strong gravitational fields [76,79] can be an extrapolation for fields sourced by neutron stars since they are many orders of magnitude larger than the one we may probe by solar system tests. Therefore, likely GR should be modified in a strong gravity regime and in the case of large spacetime curvature [80].

Since we are interested in Palatini gravity, one should mention another problem which arose throughout the studies on neutron stars in this formalism. In [81,82] it was shown that there exist surface singularities of static spherically symmetric objects in the case of polytropic equation of state. It would seem that all Palatini $f(\hat{R})$ functionals besides Einstein-Hilbert one could be ruled out since they provide unphysical behavior. However in [83] it was demonstrated that the problem appears because of the particular equation of state rather than the dynamics of the theory itself. Further approach to the problem [84] indicates that polytropic equation of state is not a fundamental issue but only an approximation of the matter content in the star. Nevertheless, they examined the case of problematic matter description according to the Ehlers–Pirani–Schild (EPS) interpretation [85–87], which we also follow in our work. Considering the conformal metric as the one responsible for the free fall they revealed that in this case the singularities are not generated in comparison to metric which was used in [81].

Due to the above brief considerations on neutron stars, there are two main problems concerning these objects: how to model dense matter inside the star, which one would like to describe by an equation of state, and viable equations which depict the macroscopic properties, that is, mass and radius. The alternative equations, or rather modified Tolman–Oppenheimer–Volkoff equations, describing the macroscopic values of relativistic stars are actively proposed [88–94] and tested with the observational data. In this work we would also like to challenge this problem and hence in the next section we will present briefly $f(\hat{R})$ gravity in Palatini formalism and then we are going to focus on stars’ stability problem. We will also provide appropriate TOV equations for this generalization of General Relativity. We are using the Weinberg’s [95] signature convention, that is, $(-, +, +, +)$.

2 A brief description of Palatini gravity: frames

As we have already mentioned, we are interested in Palatini gravity which is a particular case of EPS formalism [85]. That means that geometry of spacetime is described by two structures which in Palatini gravity turns out to be the metric g and the connection $\hat{\Gamma}$ which are independent objects. In addition, the connection is a Levi-Civita connection of a metric conformally related to g . Due to this interpretation, one considers motion of a mass particle appointed by the connection while clocks and distances are ruled by the metric g . Let us briefly, but in a quite detailed way, introduce the basic of the Palatini $f(\hat{R})$ gravity. The action is given by

$$S = S_g + S_m = \frac{1}{2} \int \sqrt{-g} f(\hat{R}) d^4x + S_m, \tag{1}$$

where $\hat{R} = \hat{R}^{\mu\nu} g_{\mu\nu}$ is the generalized Ricci scalar. The variation of Eq. (1) with respect to the metric $g_{\mu\nu}$ provides

$$f'(\hat{R}) \hat{R}_{\mu\nu} - \frac{1}{2} f(\hat{R}) g_{\mu\nu} = T_{\mu\nu}, \tag{2}$$

where $T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta}{\delta g_{\mu\nu}} S_m = (\rho + p)u^\mu u^\nu + p g^{\mu\nu}$ is energy momentum tensor, p pressure while ρ energy density of the fluid. The vector u^α is a 4-velocity of the observer co-moving with the fluid ($g_{\mu\nu} u^\mu u^\nu = -1$). Taking the trace of the Eq. (2) with respect to $g_{\mu\nu}$ yields a structural equation, which is given by

$$f'(\hat{R}) \hat{R} - 2f(\hat{R}) = T. \tag{3}$$

Assuming that we are able to solve Eq. (3) as $\hat{R}(T)$ we see that $f(\hat{R})$ is also a function of the trace of the energy momentum tensor, where $T = g^{\mu\nu} T_{\mu\nu} \equiv 3p - \rho$.

The variation with respect to the independent connection gives

$$\hat{\nabla}_\alpha (\sqrt{-g} f'(\hat{R}) g^{\mu\nu}) = 0, \tag{4}$$

from which we immediately notice that $\hat{\nabla}_\alpha$ is the covariant derivative calculated with respect to $\hat{\Gamma}$, that is, it is the Levi-Civita connection of the conformal metric $h_{\mu\nu} = f'(\hat{R}) g_{\mu\nu}$.

It is well known [29] that the action (1) is dynamically equivalent to the constraint system with first order Palatini gravitational Lagrangian:

$$S(g_{\mu\nu}, \Gamma_{\rho\sigma}^\lambda, \chi) = \frac{1}{2\kappa} \int d^4x \sqrt{-g} (f'(\chi)(\hat{R} - \chi) + f(\chi)) + S_m(g_{\mu\nu}, \psi), \tag{5}$$

providing that $f''(\hat{R}) \neq 0$.² Introducing further a scalar field $\Phi = f'(\chi)$ and taking into account the constraint equation $\chi = \hat{R}$, one can rewrite the action in dynamically equivalent way as a Palatini action

$$S(g_{\mu\nu}, \Gamma_{\rho\sigma}^\lambda, \Phi) = \frac{1}{2\kappa} \int d^4x \sqrt{-g} (\Phi \hat{R} - U(\Phi)) + S_m(g_{\mu\nu}, \psi), \tag{6}$$

where the potential $U(\Phi)$ “remembers” the form of function $f(\hat{R})$. It is defined as

$$U(\Phi) = \chi(\Phi)\Phi - f(\chi(\Phi)) \tag{7}$$

where $\chi \equiv \hat{R} = \frac{dU(\Phi)}{d\Phi}$ and $\Phi = \frac{df(\chi)}{d\chi}$. Palatini variation of this action provides

$$\Phi \left(\hat{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \hat{R} \right) + \frac{1}{2} g_{\mu\nu} U(\Phi) - \kappa T_{\mu\nu} = 0 \tag{8}$$

$$\hat{\nabla}_\lambda (\sqrt{-g} \Phi g^{\mu\nu}) = 0 \tag{9}$$

$$\hat{R} - U'(\Phi) = 0 \tag{10}$$

The last equation due to the constraint $\hat{R} = \chi = U'(\phi)$ is automatically satisfied. The middle Eq. (9) implies that the connection $\hat{\Gamma}$ is a metric connection for the new metric $h_{\mu\nu} = \Phi g_{\mu\nu}$. The g -trace of the first equation $-\Phi \hat{R} + 2U(\Phi) = \kappa T$, can be recast as

$$2U(\Phi) - U'(\Phi)\Phi = \kappa T. \tag{11}$$

which provides an analog of the structure Eq. (3).

Now the Eq. (8) can be rewritten as a dynamical equation for the metric $h_{\mu\nu}$ [41,42]

$$\bar{R}_{\mu\nu} - \frac{1}{2} h_{\mu\nu} \bar{R} = \kappa \bar{T}_{\mu\nu} - \frac{1}{2} h_{\mu\nu} \bar{U}(\Phi) \tag{12a}$$

$$\Phi \bar{R} - \left(\Phi^2 \bar{U}(\Phi) \right)' = 0 \tag{12b}$$

where we have introduced $\bar{U}(\phi) = U(\phi)/\Phi^2$ and appropriate energy momentum tensor $\bar{T}_{\mu\nu} = \Phi^{-1} T_{\mu\nu}$. Moreover, one notices that $\hat{R}_{\mu\nu} = \bar{R}_{\mu\nu}$, $\bar{R} = h^{\mu\nu} \bar{R}_{\mu\nu} = \Phi^{-1} \hat{R}$ and $h_{\mu\nu} \bar{R} = g_{\mu\nu} \hat{R}$. The last equation, together with the trace of Eq. (12a), can be replaced by

$$\Phi \bar{U}'(\Phi) + \kappa \bar{T} = 0. \tag{13}$$

Thus the system Eqs. (12a)–(13) corresponds to a scalar-tensor action for the metric $h_{\mu\nu}$ and (non-dynamical) scalar field Φ

² In that case the linear Einstein–Hilbert Lagrangian $f(\hat{R}) = \hat{R} - 2\Lambda$ is excluded on that level.

$$S(h_{\mu\nu}, \Phi) = \frac{1}{2\kappa} \int d^4x \sqrt{-h} (\bar{R} - \bar{U}(\Phi)) + S_m(\Phi^{-1}h_{\mu\nu}, \psi), \tag{14}$$

where

$$\bar{T}^{\mu\nu} = -\frac{2}{\sqrt{-h}} \frac{\delta}{\delta h_{\mu\nu}} S_m = (\bar{\rho} + \bar{p}) \bar{u}^\mu \bar{u}^\nu + \bar{p} h^{\mu\nu} = \Phi^{-3} T^{\mu\nu}, \tag{15}$$

and $\bar{u}^\mu = \Phi^{-\frac{1}{2}} u^\mu$, $\bar{\rho} = \Phi^{-2} \rho$, $\bar{p} = \Phi^{-2} p$, $\bar{T}_{\mu\nu} = \Phi^{-1} T_{\mu\nu}$, $\bar{T} = \Phi^{-2} T$ (see e.g. [96]). Further, the trace of Eq. (12a), provides

$$\bar{R} = 2\bar{U}(\Phi) - \kappa \bar{T}. \tag{16}$$

3 Stability condition

Following [95] and [98] we wish to show that Palatini neutron stars whose matter is modeled by perfect fluid are in dynamical equilibrium and are described by modified TOV equations. Moreover, we are going to demonstrate that this equilibrium is stable which means that a considered configuration is in thermodynamic equilibrium.

3.1 TOV equations in ETGs

It was shown in [97] how some modifications of the TOV equations influence a mass and radius of a neutron star. The introduced parameters to the equations not only allow to see how the mass-radius diagram changes with respect to them but also can be useful to give some clue on a kind of ETG which we should look for in order to get the desired results which are in agreement with the observations. The details and discussion on gravitational theories which are related to the parameters can be also found in [97,99].

The TOV equations can be generalized for a certain class of theories after the generalization of the energy density and pressure [100]:

$$Q(r) := \rho(r) + \frac{\sigma(r)W_{tt}(r)}{\kappa B(r)}, \tag{17}$$

$$\Pi(r) := p(r) + \frac{\sigma(r)W_{rr}(r)}{\kappa A(r)} \tag{18}$$

where the extra terms above are provided by the modified Einstein field equations of the particular form [101–103]

$$\sigma(\Psi^i)(G_{\mu\nu} - W_{\mu\nu}) = \kappa T_{\mu\nu}. \tag{19}$$

As usually, the tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ is the Einstein tensor, $\kappa = -8\pi G$, the factor $\sigma(\Psi^i)$ is a coupling to the gravity while Ψ^i represents for instance curvature invariants

or other fields, like scalar ones. The symmetric tensor $W_{\mu\nu}$ denotes additional geometrical terms which might appear in considered ETG. It should be noted that Eq. (19) provides a parameterization of gravitational theories at the level of field equations. The energy–momentum tensor $T_{\mu\nu}$ will be considered as the one of a perfect fluid discussed in the previous section. Because the tensor $W_{\mu\nu}$ usually includes extra fields like scalar or electromagnetic ones, apart the modified Einstein’s field Eq. (19) we also deal with equations for the additional fields.

As shown [97], in the case of the spherical-symmetric metric

$$ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2d\theta^2 + r^2 \sin^2 \theta d\phi^2, \tag{20}$$

the modified Einstein field equations of the form Eq. (19) reproduces the generalized TOV equations

$$\left(\frac{\Pi}{\sigma}\right)' = -\frac{G\mathcal{M}}{r^2} \left(\frac{Q}{\sigma} + \frac{\Pi}{\sigma}\right) \left(1 + \frac{4\pi r^3 \frac{\Pi}{\sigma}}{\mathcal{M}}\right) \left(1 - \frac{2G\mathcal{M}}{r}\right)^{-1} + \frac{2\sigma}{\kappa r} \left(\frac{W_{\theta\theta}}{r^2} - \frac{W_{rr}}{A}\right) \tag{21}$$

$$\mathcal{M}(r) = \int_0^r 4\pi \tilde{r}^2 \frac{Q(\tilde{r})}{\sigma(\tilde{r})} d\tilde{r}. \tag{22}$$

A specification of a gravitational theory allows us to study stellar configurations. We would like to emphasize the importance of extra field equations which may appear in the considered theory of gravity, as we have already shown it in the previous work as well as we will always take them into account further. Moreover, let us notice that the modified mass Eq. (22) comes directly from the solution of the field Eqs. (19) with the metric ansatz (20) so the extra term proportional to W_{tt} should not be skipped in the examination of the star’s configuration. One may treat them as a contribution from additional fields to the theory, for example electromagnetic or scalar ones, as caused by the influence of modified equation of state [104–106] or as just pure geometric modifications.

3.2 Stability of Palatini stars

Let us focus our attention on the discussed stability problem in the case of Palatini gravity. From the structural Eq. (3) we expect that the crucial quantity in the Palatini formalism, that is, the conformal factor $\phi = f'(\hat{R})$, will depend on the trace of the energy momentum tensor which in the case of the perfect fluid has a form $T = 3p - \rho$. As we assumed the spherical-symmetric geometry, all physical quantities depend on the radial coordinate r only and hence $\phi = \phi(r)$ which makes the both metric spherical-symmetric. Another important observation is that the “Palatini–Einstein” tensor $\hat{G}_{\mu\nu} = \hat{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\hat{R}$ built of both, Palatini-Ricci tensor and

the metric $g_{\mu\nu}$ turns out to be equal to the Einstein tensor considered in so-called Einstein frame $\bar{G}_{\mu\nu} = \bar{R}_{\mu\nu} - \frac{1}{2}h_{\mu\nu}\bar{R}$. Due to that fact the equations in both frames give a solution on the metric $h_{\mu\nu}$ while the metric $g_{\mu\nu}$ can be obtained from the conformal relation Eq. (4). Therefore we will refer to the metric Eq. (20) as the metric $h_{\mu\nu}$ which enters the Einstein equations in the Einstein frame (12a). This is a metric responsible for a geodesic motion, that is, it is a Levi-Civita metric of the connection $\hat{\nabla}_\mu$. According to this interpretation [85] motion of particles should be considered with respect to the Palatini connection while the metric g is related to light cones and distances which are measured by a normalized observer u^μ , that is, $u^\mu u^\nu g_{\mu\nu} = -1$.

Writing the Eq. (12a) as

$$\bar{R}_{\mu\nu} = \kappa \bar{T}_{\mu\nu} - \frac{1}{2}h_{\mu\nu}(\bar{T} + \bar{U}(\phi)) \tag{23}$$

and using the spherical-symmetric ansatz on the metric $h_{\mu\nu}$ one may write

$$\frac{\bar{R}_{rr}}{2A} + \frac{\bar{R}_{00}}{2B} + \frac{\bar{R}_{\theta\theta}}{r^2} = -\frac{A'}{rA^2} - \frac{1}{r^2} + \frac{1}{Ar^2} = \kappa \bar{\rho} + \frac{1}{2}r^2 \bar{U} \tag{24}$$

from which immediately we get that

$$A(r) = \left(1 - \frac{2G\mathcal{M}(r)}{r}\right)^{-1}, \tag{25}$$

where

$$\mathcal{M}(r) = \int_0^r \left(4\pi \tilde{r}^2 \bar{\rho}(\tilde{r}) + \frac{\tilde{r}^2 \bar{U}(\tilde{r})}{4G}\right) d\tilde{r} \tag{26}$$

or, if one prefers to write the mass distribution for the physical energy density

$$\mathcal{M}(r) = \int_0^r \left(4\pi \tilde{r}^2 \frac{\rho}{\phi(\tilde{r})^2}(\tilde{r}) + \frac{\tilde{r}^2 U(\tilde{r})}{4G\phi(\tilde{r})^2}\right) d\tilde{r}. \tag{27}$$

Let us underline that we took into account the transformations of Eq. (15) together with the conformal transformation of the metric, that is, the diagonal element of the metric g will be denoted by $\bar{A}(r) = \phi^{-1}A(r)$ and so on, which will be also transparent in the further considerations.

Due to the relations (17) and the Eq. (19) one has

$$\bar{Q} = \bar{\rho} + \frac{1}{2}\bar{U} = \frac{\rho}{\phi^2} + \frac{U}{2\phi^2} = \phi^{-2}Q, \tag{28a}$$

$$\bar{\Pi} = \bar{p} - \frac{1}{2}\bar{U} = \frac{p}{\phi^2} - \frac{U}{2\phi^2} = \phi^{-2}\Pi. \tag{28b}$$

We are dealing with two conserved quantities in Einstein frame

$$\hat{\nabla}_\mu \bar{J}^\mu = \hat{\nabla}_\mu n \bar{u}^\mu = 0, \tag{29}$$

$$\bar{u}_\nu \hat{\nabla}_\mu \bar{T}_{\text{eff}}^{\mu\nu} = \bar{u}_\nu \hat{\nabla}_\mu \left(\bar{T}^{\mu\nu} - \frac{1}{2}h^{\mu\nu}\bar{U}\right) = 0, \tag{30}$$

where the first equation is the nucleon number current conservation law while the second one is the Bianchi identity contracted with the vector field \bar{u}^α , which naively could be thought as an observer with respect to the metric h but she is not the one responsible for the measurements. Since the relation between the physical observer and the vector field \bar{u}^α is as already mentioned

$$\bar{u}^\alpha = \phi^{-\frac{1}{2}}u^\alpha \tag{31}$$

we may apply the vector field \bar{u}^α to the Eq. (30) in order to simplify the calculations. The another observation is that the proper nucleon number density n appearing in Eq. (29) is frame independent, that is, $n = -u_\alpha J_N^\alpha = -\bar{u}_\alpha \bar{J}_N^\alpha$. From Eq. (30)

$$\bar{u}^\mu \left(-n\bar{p}\hat{\nabla}_\mu\left(\frac{1}{n}\right) - n\hat{\nabla}_\mu\left(\frac{\bar{\rho}}{n}\right) - \frac{1}{2\kappa}\bar{U}'\hat{\nabla}_\mu\phi\right) = 0 \tag{32}$$

which we write as

$$\hat{\nabla}_\mu n = \frac{n}{\bar{\rho} + \bar{p}}\hat{\nabla}_\mu \bar{\rho} + \frac{1}{2\kappa}\frac{n}{\bar{\rho} + \bar{p}}\bar{U}'\hat{\nabla}_\mu\phi, \tag{33}$$

where we may also use the relation (13) in the last term. Thus the infinitesimal changes with respect to the infinitesimal changes of the energy density are

$$\delta n = \frac{n}{\bar{\rho} + \bar{p}}\delta\bar{\rho} + \frac{1}{2\kappa}\frac{n}{\bar{\rho} + \bar{p}}\bar{U}'\delta\phi, \tag{34}$$

which turns out to be

$$\delta n = \frac{n}{\bar{Q} + \bar{\Pi}}\delta\bar{Q} \tag{35}$$

after applying the definitions (28) and $\delta\bar{\rho} = \delta\bar{Q} - \frac{\bar{U}'}{2\kappa}\delta\phi$.

In order to examine the stability condition in the case of Palatini gravity, we will follow the approach presented, for example, in [95], that is, the Lagrange multipliers method. Following Weinberg, let us formulate the proposition:

Proposition 1 *A particular stellar configuration in the Palatini gravity with an arbitrary function $f(\hat{R})$, with uniform*

entropy per nucleon and chemical composition, will satisfy the equations ($\phi = f'(\tilde{R})$)

$$\mathcal{M}(r) = \int_0^r 4\pi \tilde{r}^2 \frac{Q(\tilde{r})}{\phi(\tilde{r})^2} d\tilde{r},$$

$$\frac{d}{dr} \left(\frac{\Pi(r)}{\phi^2(r)} \right) = -\frac{\tilde{A}G\mathcal{M}}{r^2} \left(\frac{\Pi + Q}{\phi(r)^2} \right) \left(1 + 4\pi r^3 \frac{\Pi}{\phi(r)^2 \mathcal{M}} \right)$$

for equilibrium, if the quantity \mathcal{M} , defined by

$$\mathcal{M} \equiv \int 4\pi r^2 \frac{Q(r)}{\phi(r)^2} dr$$

is stationary with respect to all variations of $\bar{Q}(r)$ that leave unchanged the quantity (nuclear number)

$$N = \int_0^R 4\pi r^2 \left(1 - \frac{2G\mathcal{M}(r)}{r} \right)^{-\frac{1}{2}} n(r) dr$$

and that leave the entropy per nucleon and the chemical composition uniform and unchanged. The equilibrium is stable with respect to radial oscillations if \mathcal{M} is a minimum with respect to all such variations.

In order to demonstrate the validity of the proposition, let us firstly notice that the nuclear number N also depends on modified geometry via the relation Eqs. (26) or (27). Moreover, the extra term appearing in Eq. (21) vanishes in our particular theory. Then we see that \mathcal{M} is stationary with respect to all variations that leave the nuclear number fixed if there exists a constant λ such that $\mathcal{M} - \lambda N$ is stationary with respect to all variations. It means that we are interested in the case when \mathcal{M} can be an extremum for the unchanged nuclear number, together with uniform entropy per nucleon mass and the chemical decomposition. Hence we deal with

$$0 = \delta\mathcal{M} - \lambda\delta N = \int_0^\infty 4\pi r^2 \delta\bar{Q} dr - \lambda \int_0^\infty 4\pi r^2 \left(1 - \frac{2G\mathcal{M}(r)}{r} \right)^{-\frac{1}{2}} \delta n(r) dr - \lambda G \int_0^\infty 4\pi r \left(1 - \frac{2G\mathcal{M}(r)}{r} \right)^{-\frac{3}{2}} n(r) \delta\mathcal{M}(r) dr. \tag{36}$$

Using the relation Eq. (35) and gravitational mass definition (26)

$$\delta\mathcal{M}(r) = \int_0^r 4\pi r'^2 \delta\bar{Q}(r') dr', \tag{37}$$

we interchange the r and r' integrals in the last term after replacing it by the above to get

$$\delta\mathcal{M} - \lambda\delta N = \int_0^\infty 4\pi r^2 \left(1 - \frac{\lambda n(r)}{\bar{\Pi}(r) + \bar{Q}(r)} \left[1 - \frac{2G\mathcal{M}(r)}{r} \right]^{-\frac{1}{2}} - \lambda G \int_r^\infty 4\pi r' n(r') \left[1 - \frac{2G\mathcal{M}(r')}{r'} \right]^{-\frac{3}{2}} dr' \right) \delta\bar{Q}(r) dr. \tag{38}$$

From the above expression one sees that $\delta\mathcal{M} - \lambda\delta N$ vanishes for all $\delta\bar{Q}(r)$ if and only if

$$\frac{1}{\lambda} = \frac{n(r)}{\bar{\Pi} + \bar{Q}} \left[1 - \frac{2G\mathcal{M}(r)}{r} \right]^{-\frac{1}{2}} + G \int_r^\infty 4\pi r' n(r') \left[1 - \frac{2G\mathcal{M}(r')}{r'} \right]^{-\frac{3}{2}} dr' \tag{39}$$

for some the Lagrange multiplier λ . That means that the right-hand side the above result should be independent of r and hence

$$0 = \left(\frac{n'}{\bar{\Pi} + \bar{Q}} - \frac{n(\bar{\Pi}' + \bar{Q}')}{(\bar{\Pi} + \bar{Q})^2} \right) \left[1 - \frac{2G\mathcal{M}(r)}{r} \right]^{-\frac{1}{2}} + \frac{Gn}{\bar{\Pi} + \bar{Q}} \left(4\pi r \bar{Q} - \frac{\mathcal{M}}{r^2} \right) \left[1 - \frac{2G\mathcal{M}(r)}{r} \right]^{-\frac{3}{2}} - 4\pi Grn \left[1 - \frac{2G\mathcal{M}(r)}{r} \right]^{-\frac{3}{2}}. \tag{40}$$

Since the entropy per nucleon is uniform, thus using

$$n'(r) = \frac{n(r)\bar{Q}'(r)}{\bar{\Pi} + \bar{Q}} \tag{41}$$

we write Eq. (40) in the form of the TOV equations in Palatini gravity with an arbitrary Lagrangian as

$$\left(\frac{\Pi}{\phi(r)^2} \right)' = -\frac{G\tilde{A}\mathcal{M}}{r^2} \left(\frac{Q + \Pi}{\phi(r)^2} \right) \left(1 + \frac{4\pi r^3 \frac{\Pi}{\phi(r)^2}}{\mathcal{M}} \right)$$

$$\mathcal{M}(r) = \int_0^r 4\pi \tilde{r}^2 \frac{Q(\tilde{r})}{\phi(\tilde{r})^2} d\tilde{r}, \tag{42}$$

where $\tilde{A} = \phi^{-1}A(r)$. Thus, $\delta\mathcal{M}$ vanishes for all δQ with the constraint $\delta N = 0$ if and only if one deals with Eq. (42). We see from Eq. (38) that one deals with a stationary point of \mathcal{M} if the expression on the right-hand side vanishes. It will be so if we are equipped with the generalized TOV equations in the given form since this expression can be transformed into it.

We have just shown that the mass \mathcal{M} appearing in the TOV equations is stationary, so the necessary condition for the star to pass from stability to instability with respect to some radial perturbation is satisfied. In order to have the TOV equations which describe a stable star's configuration, the constrained extremum must be a minimum: therefore one deals with $\delta^2 \mathcal{M} > 0$. We see from the form of (27) that the second order in $\delta \bar{Q}$ has to be positive-definite for all perturbations, analogously to the $\delta \rho$ in General Relativity. It has to be so since if any of the squared frequencies ω_n^2 from the all perturbation modes takes negative values then the frequency is purely imaginary and the time variation of this mode would grow exponentially leading to an unstable system.

4 Conclusions

We have examined the stability condition of neutron stars in the case of the Palatini gravity with an arbitrary Lagrangian functional $f(\hat{R})$. We immediately notice that the obtained result Eq. (42) reduces to the General Relativity equations describing equilibrium of the relativistic stars, specifically, when $f(\hat{R}) = \hat{R}$ since $\phi \equiv f'(\hat{R}) = 1$, where prime denotes here the derivative with respect to the Palatini scalar \hat{R} . Then also the term proportional to $U(\phi)$, which appears in the definitions of generalized energy density and pressure, vanishes.

The performed analysis on the stability criterion for Palatini gravity, which is the simplest representation of EPS formulation, shows that a stability condition of a neutron star is very similar to the one in General Relativity. Let us notice that in the case of General Relativity one gives the final statement if the star's system is stable or not after the examination of the stability conditions with respect to an equation of state. In Palatini gravity one has to do even more: we see that the generalized energy density Q depends on ρ and the form of the $f(\hat{R})$ via the potential U and coupling ϕ . Thus, besides the equation of state we need to choose a model which we wish to study in Palatini formulation. We would like to add here that in the case when we are able to solve the master equation to get $\hat{R} = \hat{R}(T)$, the generalized energy density appears finally to depend on the equation of state only.

With the all discussed points above, and together with the previous result presented in [100], where we considered scalar-tensor gravity with a minimally coupled scalar field and an arbitrary potential, we conclude that there are other gravitational theories besides General Relativity which allow us to consider possible stable stars' configurations with respect to radial oscillations. Future and more detailed work along these lines is currently underway.

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