Deforming black hole and cosmological solutions by quasiperiodic and/or pattern forming structures in modified and Einstein gravity

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Abstract We elaborate on the anholonomic frame deformation method, AFDM, for constructing exact solutions with quasiperiodic structure in modified gravity theories, MGTs, and general relativity, GR. Such solutions are described by generic off-diagonal metrics, nonlinear and linear connections and (effective) matter sources with coefficients depending on all spacetime coordinates via corresponding classes of generation and integration functions and (effective) matter sources. There are studied effective free energy functionals and nonlinear evolution equations for generating off-diagonal quasiperiodic deformations of black hole and/or homogeneous cosmological metrics. The physical data for such functionals are stated by different values of constants and prescribed symmetries for defining quasiperiodic structures at cosmological scales, or astrophysical objects in non-trivial gravitational backgrounds some similar forms as in condensed matter physics. It is shown how quasiperiodic structures determined by general nonlinear, or additive, functionals for generating functions and (effective) sources may transform black hole like configurations into cosmological metrics and inversely. We speculate on possible implications of quasiperiodic solutions in dark energy and dark matter physics. Finally, it is concluded that geometric methods for constructing exact solutions consist an important alternative tool to numerical relativity for investigating nonlinear effects in astrophysics and cosmology.

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1 Introduction

A series of recent works has been devoted to constructing black hole, BH, and cosmological solutions encoding quasiperiodic gravitational and matter fields interactions and geometric flow structures [1–7]. We elaborated a new geometric and analytic method for generating exact and parametric solutions of (modified) Einstein equations characterized by generic off-diagonal metrics and (generalized) connections with coefficients depending on all spacetime coordinates. It is called the anholonomic frame deformation method, AFDM; for reviews of results, see [8–14]. This geometric formalism provides an alternative to numerical methods and simulations in modern gravity [15–18], see a review of such results in [19]. In our works, there were considered various geometric applications in particle physics and cosmology and heterotic superstring theory, and studied various models of commutative and noncommutative and/or supersymmetric modified gravity theories, MGT, and general relativity, GR, see [20–26] and references therein.

The AFDM was formulated by developing a geometric techniques of decoupling and integrating systems of nonlinear partial differential equations, PDEs. The general goal was to study nonlinear dynamical and/or evolution properties of (modified) Einstein equations, when various classes of solutions are generated by nonholonomic (i.e. non-integrable, equivalently, anholonomic) frame transforms with deformed symmetries and distortions of metric and connection structures. In this paper, we consider that readers are familiar with main concepts and methods of differential geometry and topology, functional analysis and theory of partial differential equations, PDEs, and ordinary differential equations, ODEs, with applications in mathematical relativity, astrophysics and cosmology (summarized in [27–31]). The most important feature of the AFDM is that it allows us to construct in direct form exact solutions of nonlinear systems when there are not involving simplified diagonal metric ansatz resulting in ordinary differential equations, ODEs. We can study singular and nonsingular metrics and generalized connections, quasiperiodic (non) stationary gravitational and matter field structures, nonlinear waves and solitons, deformed horizons, nontrivial topological structures etc. Additionally, one can be analysed the conditions when and if some classes of “general solutions” (with less clear physical properties/ implications) may have certain (non) smooth limits and/or nonholonomic constraints to some well-known and more “simple” physically important solutions.

We emphasize that new classes of exact and parametric quasiperiodic and pattern forming solutions in MGTs and GR generated following the AFDM (in this and partner works [4, 6, 7]) can not be constructed by applying former methods with diagonal metric ansatz, for instance, depending only on a “radial” like coordinate, for BHs, or on a time like

coordinate, for homogeneous cosmological solutions. Prescribing from the very beginning some “simplified” diagonal ansatz, we can transform the gravitational and matter field equations into certain systems of nonlinear ODEs which can be solved in general forms with metrics parameterized by integration constants. This way, we lose various possibilities to find more general classes of nonlinear solutions, for instance, with solitonic hierarchies, pattern forming structure etc. which are determined by generating and integration functions. More realistic descriptions and understanding of physical properties of realistic nonlinear gravitational and matter field systems are possible in terms of generalized ansatz for coefficients of metrics, frames, and (non) linear connections, and off-diagonal solutions of PDEs depending on a maximal possible number of spacetime coordinates and various (non) commutative continuous/ discrete parameters.

The solutions for gravitational and matter fields with quasiperiodic structures are defined by generic off-diagonals metrics, nonlinear and (generalized) connections, and (effective) matter sources with coefficients determined by respective classes of generating and integration functions depending on all spacetime coordinates.¹ Prescribing for physical and geometric objects corresponding smooth/singularity/symmetry conditions, solving corresponding Cauchy problems, and/ or satisfying necessary boundary conditions, we can model various (non) commutative and/or semi-classical and quantum stationary configurations and/or nonlinear gravitational and matter field interactions. Such models of generalized off-diagonal spacetimes are characterized by nonlinear/discrete/continuous (semi) classical symmetries; there are computed locally anisotropically polarized (physical) constants and stated values of integration parameters; in general, such constructions can be with (non) singular and/or topologically nontrivial structures. In mentioned partner works [4,6,7] (see also citations therein) and related papers on applications of the AFDM, various examples of exact and parametric solutions, for instance, with soliton distributions and nonlinear waves, partial derivative and gravitational diffusions processes, geometric flows, and quasicrystal like structures, locally anisotropic cosmological configurations etc., have been considered.

The goal of this paper is to analyze the conditions when certain data for generating and integration functions, (effective) sources, prescribed symmetries for integration functions and various type parameters, define off-diagonal deformations of physically important solutions. We study nonholonomic deformations of BH and cosmological metrics

into new classes of exact solutions encoding quasiperiodic and pattern forming gravitational and effective matter fields. Our work reconsiders a series of ideas, methods, and results elaborated in condensed matter physics and develop them with applications in modern gravity, cosmology, and astrophysics. On relevant former results, we cite a series of papers for the physics of quasicrystals and Penrose-like tilings, see [32–45]). It is considered that various web like, filament, quasiperiodic, aperiodic, singular and nonsingular cosmological evolution, solitonic like distributions and nonlinear wave structures are distinguished /found in modern cosmological observational data [46–55]. Such generic locally anisotropic and inhomogeneous cosmological configurations can not be explained exhaustively using only higher symmetry solutions (for instance, Friedmann–Lemaître–Robertson–Walker, FLRW, and/or Bianchi type metrics) depending only on time like coordinates and constructed following methods of the theory of ODEs. Applying the AFDM, we are able to construct more general classes of exact solutions in MGTs and GR which allows us to model more rich spacetime structures and nonlinear field interactions as solutions of systems of nonlinear PDEs. In this work, it is shown how such nonlinear configurations are characterized by effective free energy functionals and nonlinear evolutions equations, which can be applied to mimic dark energy and dark matter effects, see also [2–7,44,45]. We study quasiperiodic solutions not involving small deformation parameters.

The content of the paper is organized as follows: In Sect. 2, we provide a brief review of the AFDM (relevant details with necessary N-adapted coefficient formulas and proofs are presented in A). Free energy functionals for generating functions resulting in quasiperiodic and pattern forming structures for smooth, singular and BH like structures, and cosmological solutions etc. are defined and analysed in Sect. 3. We construct a series of examples of stationary and nonstationary exact solutions describing quasiperiodic deformations of BH solutions in MGTs and GR in Sect. 4. There are analyzed different classes of off-diagonal solutions determined by nonlinear or additive functionals for generating functions and effective sources. Various classes of locally anisotropic and/or inhomogeneous cosmological quasiperiodical solutions are constructed in general form and studied in Sect. 5. Such cosmological evolution models are determined by nonstationary quasiperiodic generating functionals and effective sources. Finally, we discuss and provide concluding remarks in Sect. 6.

2 A brief review of the AFDM

We outline and reformulate the AFDM as a geometric formalism for constructing exact generic off-diagonal station-

¹ A generic off-diagonal metric can not be diagonalized by coordinate transforms on an open spacetime region. Generalized connections can be (non) linear, metric (non) compatible and with nontrivial torsion. Additional constraints can be imposed in order to extract Levi-Civita configurations for metric compatible and zero torsion connections.

ary and cosmological solutions in MGTs and GR determined by generation and integration functions and effective matter sources, and related polarization functions. The method and most important formulas are summarized in Tables 1, 2 and 3. Such constructions will be applied in order to extend the approach for solutions encoding quasiperiodic and/or pattern forming, solitonic distributions and nonlinear wave gravitational and matter field structures. Details, examples with small parametric decompositions and rigorous mathematical proofs are contained in [8–14, 20–26], see also a summary of necessary N-adapted formulas in Appendix A.

2.1 Geometric preliminaries: Lorentz manifolds with nonholonomic 2 + 2 splitting

Let us consider a 4-d Lorentzian manifold V of signature $(+ + + -)$, when the local coordinates are labeled in a form adapted to a conventional 2 + 2 splitting, when $u = (x, y) = \{u^\alpha = (x^i, y^a)\}$, for $\alpha = (i, a)$; $\beta = (j, b)$, where $i, j, \dots = 1, 2$ and $a, b, \dots = 3, 4$, with $y^4 = t$ being a time like coordinate. For a 3 + 1 splitting, we can parameterize $u^\alpha = (u^i, t)$ for spacelike coordinates $u^i = (x^i, y^3)$ when $i, j, k, \dots = 1, 2, 3$. Such a spacetime manifold can be enabled with a pseudo-Riemannian metric structure $\mathbf{g} = \hat{\mathbf{g}}$,

$$\mathbf{g} = g_{\alpha\beta}(x^i, y^a) du^\alpha \otimes du^\beta, \quad \text{for dual frame coordinate basis } du^\alpha; \quad (1)$$

$$\hat{\mathbf{g}} = \mathbf{g}_{\alpha\beta}(u) \mathbf{e}^\alpha \otimes \mathbf{e}^\beta = \mathbf{g}_i(x^k) dx^i \otimes dx^i + \mathbf{g}_a(x^k, y^b) \mathbf{e}^a \otimes \mathbf{e}^b, \quad (2)$$

$$\text{for } \mathbf{e}^\alpha = (dx^i, \mathbf{e}^a = dy^a + N_i^a(u) dx^i) \quad \text{defining a N-adapted dual frame basis.} \quad (3)$$

The 2 + 2 splitting (2) is nonholonomic (equivalently, non-integrable, or anholonomic). This follows from the conditions that the co-basis $\mathbf{e}^\alpha = (dx^i, \mathbf{e}^a)$ (3) is dual to

$$\mathbf{e}_\alpha = (\mathbf{e}_i, e_a) = (\mathbf{e}_i = \partial/\partial x^i - N_i^a(u) \partial/\partial y^a, e_a = \partial/\partial y^a), \quad (4)$$

which satisfies nonholonomy conditions

$$\mathbf{e}_{[\alpha} \mathbf{e}_{\beta]} := \mathbf{e}_\alpha \mathbf{e}_\beta - \mathbf{e}_\beta \mathbf{e}_\alpha = C_{\alpha\beta}^\gamma(u) \mathbf{e}_\gamma, \quad (5)$$

with anholonomy coefficients $C_{\alpha\beta}^\gamma = \{C_{ia}^b = \partial_a N_i^b, C_{ji}^a = \mathbf{e}_j N_i^a - \mathbf{e}_i N_j^a\}$.² In above formulas, we can consider necessary type frame transforms $\mathbf{e}^\alpha = \mathbf{e}^{\alpha'}(u) du^{\alpha'}$, when $g_{\alpha'\beta'}(u) = g_{\alpha\beta} \mathbf{e}^{\alpha'} \mathbf{e}^{\beta'}$.

² If the anholonomy coefficients are nontrivial (i.e. the N-adapted frame is not integrable and can not be transformed into a coordinate one), the respective equivalent metric parameterizations (1) and (2) are generic off-diagonal.

A set of coefficients $\mathbf{N} = \{N_i^a\}$ from (2) and/or (4) states a N-adapted decomposition of the tangent Lorentz bundle TV into conventional horizontal, h , and vertical, v , subspaces, when the Whitney sum

$$\mathbf{N} : TV = hTV \oplus vTV \quad (6)$$

define a nonlinear connection (N-connection) structure. A Lorentz manifold V is called nonholonomic if it is enabled with a nonholonomic distribution. In a particular case, we shall write $\mathbf{V} = (V, \mathbf{g}, \mathbf{N})$ for a nonholonomic spacetime manifold enabled with N-connection structure $\mathbf{N}$ (6). This concept was introduced and studied in details in generalized Finsler geometry and geometric and physical models on (co) vector/tangent bundles and their supersymmetric/noncommutative generalizations, see [12] and references therein. For (pseudo) Riemannian manifolds, a conventional h/v -splitting can be considered, for instance, in order to state a fibered structure with y^a coordinates and N-adapted frames (4). Such nonholonomic structures and deformations of geometric/ physical objects (like metrics, linear connections, (effective) sources etc.) can be introduced in such forms which allows decoupling of gravitational and matter field equations in MGTs and GR, [1–7].³

It is possible to model a nonholonomic deformation with η -polarization functions, $\hat{\mathbf{g}} \rightarrow \hat{\mathbf{g}}$, of a ‘prime’ metric, $\hat{\mathbf{g}}$, into a ‘target’ metric $\mathbf{g} = \hat{\mathbf{g}}$ (2), if

$$\hat{\mathbf{g}} = \eta_i(x^k) \hat{\mathbf{g}}_i dx^i \otimes dx^i + \eta_a(x^k, y^b) \hat{\mathbf{h}}_a \mathbf{e}^a[\eta] \otimes \mathbf{e}^a[\eta], \quad (7)$$

where the N-elongated basis (4) is represented for $N_i^a(u) = \eta_i^a(x^k, y^b) \hat{N}_i^a(x^k, y^b)$, i.e. in the form⁴ $\mathbf{e}^\alpha[\eta] = (dx^i, \mathbf{e}^a = dy^a + \eta_i^a \hat{N}_i^a dx^i)$. We shall subject a $\hat{\mathbf{g}}$ to the condition that it defines a solution of (modified) Einstein equations. A general prime metric in coordinate parametrization of type (1),

$$\hat{\mathbf{g}} = \hat{g}_{\alpha\beta}(x^i, y^a) du^\alpha \otimes du^\beta$$

³ We shall follow the Einstein rule on summation on “up-low” cross indices if there is not a contrary statement for an explicit formula. One shall not be considered summation on repeating “low-low”, or “up-up” indices. In our works, we elaborated a system of “N-adapted notations” with boldface symbols for manifolds and fiber bundles enables with a nontrivial N-connection structure $\mathbf{N}$. Details on nonholonomic differential geometry and N-connections are explained in [11–14, 20–26] and references therein. If the anholonomy coefficients $C_{\alpha\beta}^\gamma$ in (5) are non-trivial, a metric $g_{\alpha\beta}$ (1) can not be diagonalized in a local finite, or infinite, spacetime region with respect to coordinate frames. Such metrics are called generally off-diagonal and characterized by six independent nontrivial coefficients from a set $\mathbf{g} = \{g_{\alpha\beta}(u)\}$. A nonholonomic frame is holonomic if all corresponding anholonomy coefficients are zero (for instance, the coordinate frames).

⁴ In this paragraph, we do not consider summation on repeating indices if they are not written as contraction of “up-low” ones.

can be also represented in N-adapted form

$$\begin{aligned}\hat{\mathbf{g}} &= \hat{g}_\alpha(u) \hat{\mathbf{e}}^\alpha \otimes \hat{\mathbf{e}}^\beta = \hat{g}_i(x) dx^i \otimes dx^i \\ &+ \hat{g}_a(x, y) \hat{\mathbf{e}}^a \otimes \hat{\mathbf{e}}^a, \text{ for } \hat{\mathbf{e}}^\alpha = (dx^i, \mathbf{e}^a = dy^a \\ &+ \hat{N}_i^a(u) dx^i), \text{ and } \hat{\mathbf{e}}_\alpha = (\hat{\mathbf{e}}_i = \partial/\partial y^a \\ &- \hat{N}_i^b(u) \partial/\partial y^b, \mathbf{e}_a = \partial/\partial y^a).\end{aligned}\quad (8)$$

It can be, or not, a solution of some gravitational field equations in a MGT or GR.

In this work, we shall be interested in two physically important cases when $\hat{\mathbf{g}}$ (8) defines a BH solution (for instance, a vacuum Kerr, or Schwarzschild, metric), or a Friedman–Lemaître–Robertson–Walker (FLRW) type metric. For such diagonalizable metrics (the off-diagonal structure of the Kerr metric is determined by rotation frames and coordinates), we can always find a coordinate system when $\hat{N}_i^b = 0$. In order to avoid singular noholonomic deformations, it is convenient to construct exact solutions of necessary type gravitational equations with nontrivial conventional “polarization” functions $\eta_\alpha = (\eta_i, \eta_a)$, η_i^a , and nonzero coefficients $\hat{N}_i^b(u)$. This can be achieved by considering necessary type frame/ coordinate transforms. Having constructed an explicit form a d-metric (7), we can study the existence and geometric/ physical properties of solutions, for instance, when $\eta_\alpha \rightarrow 1$ and $N_i^a \rightarrow \hat{N}_i^a$, or if $\eta_\alpha = 1$ and/or $\hat{N}_i^a = 0$ are imposed as some noholonomic constraints.⁵ We denote certain nonholonomic deformations of a prime d-metrics into a target one as $\hat{\mathbf{g}} \rightarrow \hat{\mathbf{g}} = [g_\alpha = \eta_\alpha \hat{g}_\alpha, \eta_i^a \hat{N}_i^a]$. Here we emphasize that one constructs, in general, different classes of solutions for nonlinear systems of PDEs, if such approximations are considered in (modified) Einstein equations before finding solutions, or at the end (after a class of solutions has been constructed in explicit form). This is an important property of nonlinear dynamical/evolution systems which can be subjected to additional nonholonomic constraints.

⁵ It is possible to keep a physical interpretation of a target metric $\hat{\mathbf{g}}$ (7) with generic off-diagonal terms, as an “almost” BH, or FLRW cosmological, like metric by constructing parametric solutions with small nonholonomic deformations on some constant parameters $\eta_\alpha = (\eta_i, \eta_a)$, η_i^a , for $0 \leq \varepsilon_\alpha, \varepsilon_i^a \ll 1$, when $\eta_i \simeq \check{\eta}_i(x^k)[1 + \varepsilon_i \chi_i(x^k)] \simeq 1 + \varepsilon_i \chi_i(x^k)$, $\eta_a \simeq \check{\eta}_a(x^k, y^b)[1 + \varepsilon_a \chi_a(x^k, y^b)] \simeq 1 + \varepsilon_a \chi_a(x^k, y^b)$, and $\eta_i^a \simeq \check{\eta}_i^a(x^k, y^b)[1 + \varepsilon_i^a \chi_i^a(x^k, y^b)] \simeq 1 + \varepsilon_i^a \chi_i^a(x^k, y^b)$. Such parametric ε -decompositions can be performed in a self-consistent form by omitting quadratic and higher terms after a class of solutions have been found for some general data (η_α, η_i^a) . For certain subclasses of solutions, we can consider that $\varepsilon_i, \varepsilon_a, \varepsilon_i^a \sim \varepsilon$, when one small parameter is considered for all coefficients of nonholonomic deformations. It is possible to work with mixed types of solutions and model only small diagonal deformations $\varepsilon_i, \varepsilon_a, \sim \varepsilon$ of metrics, for some general η_i^a , or to work with some nontrivial η_α but $\varepsilon_i^a \sim \varepsilon$. In particular, one can be generated various classes of nonholonomic small deformations of solutions like in Refs. [1–7]. In this work, our goal is to construct and study quasiperiodic solutions with generation functions and sources without small parameters.

The standard formulation of (pseudo) Riemannian geometry is in terms of the Levi-Civita, LC, connection ∇ , which (by definition) is metric compatible and with zero torsion. Nevertheless, any Lorentzian manifold V can be enabled additionally (following certain geometric/ physical principles) with other types of linear connection structures considering any $D = \nabla + Z$, characterized by a respective distortion tensor Z . A general linear connection D can be metric noncompatible and/or with nontrivial torsion. On a nonholonomic $\mathbf{V}$, we can consider a N-adapted variant of linear connection structure, called a distinguished connection, d-connection, $\mathbf{D} = (h\mathbf{D}, v\mathbf{D})$, which is defined as a metric-affine (linear) connection preserving under parallel transports the N -connection splitting into h - and v -subspaces. We can define and compute for any d-connection $\mathbf{D}$ (in standard form) the torsion tensor, $\mathcal{T} = \{\mathbf{T}_{\beta\gamma}^\alpha\}$, and the curvature tensor, $\mathcal{R} = \{\mathbf{R}_{\beta\gamma\delta}^\alpha\}$, where the coefficients can be written in N-adapted form with respect to necessary tensor products of bases (4) and their duals.

On a nonholonomic $\mathbf{V}$, we can work in equivalent form with two different linear connections:

$$\begin{aligned}(\mathbf{g}, \mathbf{N}) \\ \rightarrow \begin{cases} \nabla: \nabla \mathbf{g} = 0; \nabla \mathcal{T} = 0, \text{ for the LC-connection} \\ \hat{\mathbf{D}}: \hat{\mathbf{D}} \mathbf{g} = 0; h\hat{\mathcal{T}} = 0, v\hat{\mathcal{T}} \\ = 0, hv\hat{\mathcal{T}} \neq 0, \text{ for the canonical d-connection,} \end{cases}\end{aligned}\quad (9)$$

where $\hat{\mathbf{D}} = (h\hat{\mathbf{D}}, v\hat{\mathbf{D}})$ is completely defined by $\mathbf{g}$ for any prescribed N-connection structure $\mathbf{N}$. In these formulas, we denote (respectively, for $\hat{\mathbf{D}}$ and ∇) the torsions, $\hat{\mathcal{T}}$ and $\nabla \mathcal{T} = 0$, and curvatures, $\hat{\mathcal{R}} = \{\hat{\mathbf{R}}_{\beta\gamma\delta}^\alpha\}$ and $\nabla \mathcal{R} = \{R_{\beta\gamma\delta}^\alpha\}$, which can be defined and computed in coordinate free and/or coefficient forms.⁶ There is a canonical distortion relation

$$\hat{\mathbf{D}} = \nabla + \hat{\mathbf{Z}}. \quad (10)$$

The distortion distinguished tensor, d-tensor, $\hat{\mathbf{Z}} = \{\hat{\mathbf{Z}}_{\beta\gamma}^\alpha[\hat{\mathbf{T}}_{\beta\gamma}^\alpha]\}$, is an algebraic combination of the coefficients of the corresponding torsion d-tensor $\hat{\mathcal{T}} = \{\hat{\mathbf{T}}_{\beta\gamma}^\alpha\}$ of $\hat{\mathbf{D}}$. Readers may consult Appendix A for a summary of relevant formulas and some details on the geometry Lorentz manifolds with N-adapted $2 + 2$ variables.

We can define two different Ricci tensors, $\hat{Ric} = \{\hat{\mathbf{R}}_{\beta\gamma} := \hat{\mathbf{R}}_{\alpha\beta\gamma}^\alpha\}$ and $Ric = \{R_{\beta\gamma} := R_{\alpha\beta\gamma}^\alpha\}$, when $\hat{Ric}$ is characterized by h - v N-adapted coefficients,

⁶ It should be noted that the well known LC-connection ∇ is a linear one but not a d-connection because it does not preserve, under general frame/coordinate transforms, a h - v -splitting. $\hat{\mathcal{T}}$ is a nonholonomically induced torsion determined by nontrivial values $(C_{\alpha\beta}^\gamma, \partial_\beta N_i^a, g_{\alpha\beta})$ which is different from the Einstein–Cartan, or string theory, when additional field equations and sources for torsion fields are considered.

$$\begin{aligned}\widehat{\mathbf{R}}_{\alpha\beta} &= \{\widehat{R}_{ij} := \widehat{R}_{ijk}^k, \widehat{R}_{ia} := -\widehat{R}_{ika}^k, \\ \widehat{R}_{ai} &:= \widehat{R}_{aib}^b, \widehat{R}_{ab} := \widehat{R}_{abc}^c\}.\end{aligned}\quad (11)$$

Respectively, there are two different scalar curvatures, $R := \mathbf{g}^{\alpha\beta} R_{\alpha\beta}$ and $\widehat{\mathbf{R}} := \mathbf{g}^{\alpha\beta} \widehat{\mathbf{R}}_{\alpha\beta} = g^{ij} \widehat{R}_{ij} + g^{ab} \widehat{R}_{ab}$. Following the two connection approach (9), we conclude that the (pseudo) Riemannian geometry can be equivalently described by two different geometric data $(\mathbf{g}, \nabla)$ and $(\mathbf{g}, \mathbf{N}, \widehat{\mathbf{D}})$. Using the canonical distortion relation (10), we can compute respective distortions of curvature and Ricci tensors,

$$\widehat{\mathcal{R}} = {}^\nabla \mathcal{R} + {}^\nabla \mathcal{Z} \quad \text{and} \quad \widehat{Ric} = Ric + \widehat{\mathcal{Z}}ic, \quad (12)$$

with corresponding distortion tensors ${}^\nabla \mathcal{Z}$ and $\widehat{\mathcal{Z}}ic$. Such formulas motivate application in GR and MGTs of nonholonomic geometric methods with multiple metric and connection structures, and adapted frames. For certain well defined nonholonomic configurations, various types of gravitational and matter field equations rewritten in nonholonomic variables $(\mathbf{g}, \mathbf{N}, \widehat{\mathbf{D}})$ can be decoupled and integrated in some general forms.⁷

2.2 MGTs in N-adapted variables and decoupling of (modified) Einstein equations

We show how the gravitational field equations can be written in nonholonomic variables and consider the key steps for generating exact solutions in Tables 1, 2, and 3.

2.2.1 Gravitational field equations for the canonical d-connection

Various models of modified gravity and acceleration cosmology (see [1–7] and references therein) have been elaborated for the so-called Starobinsky type R^2 gravity [47]. For well defined conditions on a class of conformal transforms, the gravitational field equations for the so-called quadratic gravity are equivalent to the Einstein gravity with scalar field sources. In nonholonomic variables $(\mathbf{g}, \mathbf{N}, \widehat{\mathbf{D}})$ and for interactions with scalar fields defined by Lagrange density ${}^m \mathcal{L}(\mathbf{g}, \mathbf{N}, \phi)$, the action for such theories can be written in the form

$$\mathcal{S} = M_P^2 \int d^4 u \sqrt{|\mathbf{g}|} [\widehat{\mathbf{R}}^2 + {}^m \mathcal{L}[\phi]]. \quad (13)$$

In this formula the Planck mass M_P is determined by the gravitational constant; the Lagrange density ${}^m \mathcal{L}[\phi]$ and the

⁷ It is not possible to decouple the gravitational and field equations for generic off-diagonal metrics if we work from the very beginning with the data $(\mathbf{g}, \nabla)$. Having constructed certain classes of generalized solutions (following the AFDM), necessary type LC-configurations can be extracted if the condition $\widehat{\mathcal{T}} = 0$ is imposed at the end.

action ${}^m \mathcal{S} = \int d^4 u \sqrt{|\mathbf{g}|} {}^m \mathcal{L}$ are postulated in such forms which, for simplicity, allow to find explicit solutions of (modified) Einstein equations. This is possible if there are considered ${}^m \mathcal{L}[\phi]$ depending only on the coefficients of a metric field and not on their derivatives. Applying a N-adapted variational calculus, the energy–momentum d-tensor is computed ${}^m \mathbf{T}_{\alpha\beta} := -\frac{2}{\sqrt{|\mathbf{g}_{\mu\nu}|}} \frac{\delta(\sqrt{|\mathbf{g}_{\mu\nu}|} {}^m \mathcal{L})}{\delta \mathbf{g}^{\alpha\beta}} = {}^m \mathcal{L} \mathbf{g}_{\alpha\beta} + 2 \frac{\delta({}^m \mathcal{L})}{\delta \mathbf{g}^{\alpha\beta}}$. Following N-adapted variations of $\mathcal{S}$ (13), one derives such gravitational field equations

$$\widehat{\mathbf{R}}_{\mu\nu} = \Upsilon_{\mu\nu} = {}^m \Upsilon_{\mu\nu} + \widehat{\Upsilon}_{\mu\nu}, \quad \text{for} \quad (14)$$

$${}^m \Upsilon_{\alpha\beta} = \frac{1}{2M_P^2} \left({}^m \mathbf{T}_{\alpha\beta} - \frac{1}{2} \mathbf{g}_{\alpha\beta} \mathbf{g}^{\gamma\tau} {}^m \mathbf{T}_{\gamma\tau} \right) \quad \text{and}$$

$$\widehat{\Upsilon}_{\mu\nu} = \left(\frac{1}{4} \widehat{\mathbf{R}} - \frac{\widehat{\square} \widehat{\mathbf{R}}}{\widehat{\mathbf{R}}} \right) \mathbf{g}_{\mu\nu} + \frac{\widehat{\mathbf{D}}_\mu \widehat{\mathbf{D}}_\nu \widehat{\mathbf{R}}}{\widehat{\mathbf{R}}}, \quad (15)$$

where $\widehat{\square} := \widehat{\mathbf{D}}^\mu \widehat{\mathbf{D}}_\mu$ and certain constraints/ conditions with $\widehat{\mathbf{D}}|_{\widehat{\mathcal{T}} \rightarrow 0} = \nabla$ can be imposed as in GR. The Eq. (14) for MGTs can be considered as modified Einstein equations when ∇ is changed into $\widehat{\mathbf{D}}$ and the standard energy momentum tensors are nonholonomically deformed into (effective) N-adapted sources (15).

The key steps for constructing exact solutions using two different methods [(1) with reduction to ODEs or (2) with integration of PDEs] of constructing exact solutions of gravitational and matter field equations in MGTs and GR are briefly reviewed in Appendix and summarized in the Tables 1, 2, and 3. In references [27–31], there are provided details and main examples for constructing physically important solutions by using diagonal ansatz, for instance, with spherical/cylinder symmetries reducing the (generalized) Einstein equations to certain systems of nonlinear ODEs. Proofs of results and a number of examples how the AFDM should be applied in order to generate exact solutions of gravitational, matter field and evolution nonlinear systems of PDEs are considered in [8–14, 20–26].

2.2.2 (Off-) diagonal metric ansatz and ODEs and PDEs in (non) holonomic variables

We explain how using holonomic 3 + 1 and 2 + 2 nonholonomic variables and corresponding ansatz it is possible to transform gravitational field equations in MGTs and GR into respective systems of nonlinear ODEs and PDEs is summarized in Table 1. N-adapted formulas are provided in Appendix A.

2.2.3 Decoupling and integration of gravitational field equations and stationary solutions

The key steps of AFDM for generating stationary off-diagonal exact solutions of (modified) Einstein equations

Table 1 (Modified) Einstein eqs as systems of nonlinear PDEs and the Anholonomic Frame Deformation Method, **AFDM**, for constructing generic off-diagonal exact, parametric, and physically important solutions

Diagonal ansatz: PDEs $\rightarrow$ ODEs		AFDM: PDEs with decoupling; generating functions
Radial coordinates $u^\alpha = (r, \theta, \varphi, t)$	$u = (x, y) :$	nonholonomic 2 + 2 splitting, $u^\alpha = (x^1, x^2, y^3, y^4 = t)$
LC-connection $\hat{\nabla}$	[connections]	$\mathbf{N} : T\mathbf{V} = hT\mathbf{V} \oplus vT\mathbf{V}$, locally $\mathbf{N} = \{N_i^a(x, y)\}$ canonical connection distortion $\hat{\mathbf{D}} = \nabla + \hat{\mathbf{Z}}$
Diagonal ansatz $g_{\alpha\beta}(u)$		$g_{\alpha\beta}(x^i, y^a)$ general frames/coordinates
$= \begin{pmatrix} \hat{g}_1 & & & \\ & \hat{g}_2 & & \\ & & \hat{g}_3 & \\ & & & \hat{g}_4 \end{pmatrix}$	$\mathbf{g} \Leftrightarrow$	$g_{\alpha\beta} = \begin{bmatrix} g_{ij} + N_i^a N_j^b h_{ab} & N_i^b h_{cb} \\ N_j^a h_{ab} & h_{ac} \end{bmatrix}, \text{ 2 x 2 blocks}$
$\hat{g}_{\alpha\beta} = \begin{cases} \hat{g}_\alpha(r) & \text{for BHs} \\ \hat{g}_\alpha(t) & \text{for FLRW} \end{cases}$	[coord.frames]	$\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}], \mathbf{g} = \mathbf{g}_i(x^k)dx^i \otimes dx^i + \mathbf{g}_a(x^k, y^b)e^a \otimes e^b$ $g_{\alpha\beta} = \begin{cases} g_{\alpha\beta}(r, \theta, y^3 = \varphi) & \text{stationary conf.} \\ g_{\alpha\beta}(r, \theta, y^4 = t) & \text{cosm. conf.} \end{cases}$
coord.transfoms $e_\alpha = e_{\alpha'}^{\alpha'} \partial_{\alpha'}$,		
$e^\beta = e_{\beta'}^\beta du^{\beta'}, \hat{g}_{\alpha\beta} = \hat{g}_{\alpha'\beta'} e_{\alpha'}^{\alpha'} e_{\beta'}^{\beta'} \hat{g}_\alpha(x^k, y^a) \rightarrow \hat{g}_\alpha(r), \text{ or } \hat{g}_\alpha(t), \hat{N}_i^a(x^k, y^a) \rightarrow 0.$	[N-adapt. fr.]	$\begin{cases} \mathbf{g}_i(r, \theta), \mathbf{g}_a(r, \theta, \varphi), \\ \text{or } \mathbf{g}_i(r, \theta), \mathbf{g}_a(r, \theta, t), & \text{d-metrics} \\ N_i^3 = w_i(r, \theta, \varphi), N_i^4 = n_i(r, \theta, \varphi), \\ \text{or } N_i^3 = n_i(r, \theta, t), N_i^4 = w_i(r, \theta, t), \end{cases}$
$\hat{\nabla}, Ric = \{\hat{R}_{\beta\gamma}\}$	Ricci tensors	$\hat{\mathbf{D}}, \hat{Ric} = \{\hat{\mathbf{R}}_{\beta\gamma}\}$
${}^m\mathcal{L}[\phi] \rightarrow {}^m\mathbf{T}_{\alpha\beta}[\phi]$	Sources	$\Upsilon_\nu^\mu = \mathbf{e}_{\mu'}^\mu \mathbf{e}_{\nu'}^{\nu'} \Upsilon_{\nu'}^{\mu'} [{}^m\mathcal{L}(\varphi + \psi), \hat{\Upsilon}_{\mu\nu}]$ $= \text{diag} [{}_h\Upsilon(x^i)\delta_j^i, \Upsilon(x^i, \varphi)\delta_b^a], \text{ stationary conf.}$ $= \text{diag} [{}_h\Upsilon(x^i)\delta_j^i, \Upsilon(x^i, t)\delta_b^a], \text{ cosmological conf.}$
Trivial equations for $\hat{\nabla}$ -torsion	LC-conditions	$\hat{\mathbf{D}} _{\hat{\mathcal{T}} \rightarrow 0} = \nabla$ extracting new classes of solutions in GR

described in Appendices A.2.1, A.3 are summarized in Table 2. Following a nonholonomic deformation procedure, for instance, for a generating function $h_3(r, \theta, \varphi)$, we construct a class of off-diagonal stationary solutions with Killing symmetry on ∂_t determined by sources $({}_h\Upsilon, \Upsilon)$ and effective cosmological constant Λ ,

$$ds^2 = e^{\psi(x^k)} \left[(dx^1)^2 + (dx^2)^2 \right] + h_3 \left[dy^3 + \frac{\partial_i (|h_3^{[0]} - h_3||\Upsilon| - \int d\varphi |h_3^{[0]} - h_3||\Upsilon|^\diamond) dx^i}{\partial_\varphi (|h_3^{[0]} - h_3||\Upsilon| - \int d\varphi |h_3^{[0]} - h_3||\Upsilon|^\diamond)} - \frac{|h_3^{[0]} - h_3|^\diamond}{|\Upsilon| h_3} \right] \\ \times \left[dt + \left({}_1n_k + 4 {}_2n_k \int d\varphi \frac{\left[\left(\sqrt{|\Lambda| [h_3^{[0]} - h_3]} \right)^\diamond \right]^2}{\Upsilon \Lambda |h_3|^{5/2}} \right) dx^k \right].$$

Such solutions are, in general, with nontrivial nonholonomically induced torsion (A.2). They can be re-defined equivalently in terms of generating functions $\Psi(r, \theta, \varphi)$ or $\Phi(r, \theta, \varphi)$, see (A.19).

LC-configurations in GR can be extracted for additional zero torsion constraints with a more special class of “integrable” generating functions $(\check{h}_3, \text{ and } \check{\Psi}(r, \theta, \varphi) \text{ and/or } \check{\Phi}(r, \theta, \varphi))$ for respective sources $\check{\Upsilon}$ and Λ (A.20),

$$ds^2 = e^{\psi(x^k)} \left[(dx^1)^2 + (dx^2)^2 \right] + \check{h}_3 \left[d\varphi + (\partial_i \check{A}) dx^i \right] - \left[|h_3^{[0]} - \check{h}_3|^\diamond / |\check{\Upsilon}| h_3 \right] \left[dt + (\partial_k n) dx^k \right].$$

In terms of η -polarization functions, such d-metrics and N-connections, can be parameterized to describe nonholonomic deformations of a primary (for instance, BH) d-metric $\hat{\mathbf{g}}$ into target generic off diagonal stationary solutions $\hat{\mathbf{g}}$, see (7), as $\hat{\mathbf{g}} \rightarrow \hat{\mathbf{g}} = [g_\alpha = \eta_\alpha \hat{g}_\alpha, \eta_i^a \hat{N}_i^a]$.

2.2.4 Decoupling and integration of gravitational PDEs generating cosmological solutions

In Table 3, we state the key steps of the AFDM for generating off-diagonal locally anisotropic solutions of (modified) Einstein equations described in Appendices A.2.2 and A.3.

Applying the nonholonomic deformation procedure (for simplicity, we consider metrics determined by a generating function $\bar{h}_4(x^k, t)$, we construct a class of generic off-diagonal cosmological solutions with Killing symmetry on ∂_φ determined by sources, ${}_h\Upsilon$ and Υ , and an effective non-trivial cosmological constant $\bar{\Lambda}$,

Table 2 Off-diagonal stationary configurations. Exact solutions of $\tilde{\mathbf{R}}_{\mu\nu} = \Upsilon_{\mu\nu}$ (14) transformed into a system of nonlinear PDEs (A.13)–(A.16)

d-metric ansatz with Killing symmetry $\partial_4 = \partial_t$ Effective matter sources	$ds^2 = g_i(x^k)(dx^i)^2 + g_a(x^k, y^3)(dy^a + N_i^a(x^k, y^3)dx^i)^2, \text{ for}$ $g_i = e^{\psi(r, \theta)}, \quad g_a = h_a(r, \theta, \varphi), \quad N_i^3 = w_i(r, \theta, \varphi), \quad N_i^4 = n_i(r, \theta, \varphi),$ $\Upsilon_\nu^\mu = [-h\Upsilon(r, \theta)\delta_{ij}^i, \Upsilon(r, \theta, \varphi)\delta_b^a]; x^1 = r, x^2 = \theta, y^3 = \varphi, y^4 = t$
Nonlinear PDEs (A.13)–(A.16)	$\psi^{\bullet\bullet} + \psi'' = 2 \quad h\Upsilon;$ $\varpi^\diamond h_4^\diamond = 2h_3h_4\Upsilon; \quad \text{for}$ $\beta w_i - \alpha_i = 0;$ $n_k^\diamond + \gamma n_k^\diamond = 0;$
Generating functions: $h_3(r, \theta, \varphi)$, $\Psi(r, \theta, \varphi) = e^\varpi$, $\Phi(r, \theta, \varphi)$; integration functions: $h_3^{[0]}(x^k)$, ${}_1n_k(x^i)$, ${}_2n_k(x^i)$; & nonlinear symmetries	$PSt^2 = 4(h_3^{[0]} - h_3 \Upsilon - \int dy^3 h_3^{[0]} - h_3 \Upsilon ^\diamond),$ $\Phi^2 = 4 \Lambda h_3^{[0]} - h_3 , \text{see (A.17)};$ $h_3 = h_3^{[0]} - \Phi^2/4\Lambda, h_3^\diamond \neq 0, \Lambda \neq 0 = \text{const}$
Off-diag. solutions, d-metric N-connec	$g_i = e^{\psi(x^k)} \text{ as a solution of 2-d Poisson eqs. } \psi^{\bullet\bullet} + \psi'' = 2 \quad h\Upsilon;$ $h_3 = h_3^{[0]} - \int dy^3 (\Psi^2)^\diamond/4\Upsilon = h_3^{[0]} - \Phi^2/4\Lambda;$ $h_4 = -(\Psi^2)^\diamond/4\Upsilon^2 h_3 = -(\Phi^2)^\diamond/4\Lambda\Upsilon h_3, \text{ see (A.18)};$ $w_i = \partial_i \Psi / \partial_\varphi \Psi = \partial_i \Psi^2 / \partial_\varphi \Psi^2;$ $n_k = {}_1n_k + 2n_k \int dy^3 (\Psi^\diamond)^2 / \Upsilon^2 h_3^{[0]} - \int dy^3 (\Psi^2)^\diamond/4\Upsilon ^{5/2}.$
LC-configurations (A.11)	$\partial_\varphi w_i = (\partial_i - w_i \partial_\varphi) \ln \sqrt{ h_3 }, (\partial_i - w_i \partial_\varphi) \ln \sqrt{ h_4 } = 0,$ $\partial_k w_i = \partial_i w_k, \partial_\varphi n_i = 0, \partial_i n_k = \partial_k n_i;$ $\Psi = \check{\Psi}(x^i, \varphi), (\partial_i \check{\Psi})^\diamond = \partial_i (\check{\Psi}^\diamond) \text{ and}$ $\Upsilon(x^i, \varphi) = \Upsilon[\check{\Psi}] = \check{\Upsilon}, \text{ or } \Upsilon = \text{const.}$

Table 2 continued

N-connections, zero torsion	$w_i = \partial_i \check{A} = \begin{cases} \frac{\partial_i (h_3^{[0]} - \check{h}_3) \check{\Upsilon} - \int dy^3 h_3^{[0]} - \check{h}_3 \check{\Upsilon} ^\circ}{ h_3^{[0]} - \check{h}_3 ^\circ \check{\Upsilon} } \check{\Psi} / \check{\Psi}^\circ; \\ \partial_i (\check{\Phi}^2 \check{\Upsilon} - \int dy^3 \check{\Phi}^2 \check{\Upsilon} ^\circ) / 2 \check{\Phi} \check{\Phi}^\circ \check{\Upsilon} ; \end{cases}$ <p>and $n_k = \check{n}_k = \partial_k n(x^i)$.</p>	
polarization functions $\check{\mathbf{g}} \rightarrow \check{\mathbf{g}} = [g_a = n_a \check{g}_a, \eta_i^a \check{N}_i^a]$	$ds^2 = \eta_1(r, \theta) \check{g}_1(r, \theta) [dx^1(r, \theta)]^2 + \eta_2(r, \theta) \check{g}_2(r, \theta) [dx^2(r, \theta)]^2 + \eta_3(r, \theta) \check{g}_3(r, \theta) [d\varphi + \eta_2^3(r, \theta, \varphi) \check{N}_i^3(r, \theta) dx^i(r, \theta)]^2 + \eta_4(r, \theta, \varphi) \check{g}_4(r, \theta) [dt + \eta_1^4(r, \theta, \varphi) \check{N}_i^4(r, \theta) dx^i(r, \theta)]^2,$ <p>$[\check{g}_i(r, \theta), \check{g}_a = \check{h}_a(r, \theta); \check{N}_k^3 = \check{w}_k(r, \theta), \check{N}_k^4 = \check{n}_k(r, \theta)]$ diagonalizable by frame/coordinate transforms</p>	
Prime metric defines a BH	$\check{g}_1 = (1 - r_g/r)^{-1}, \check{g}_2 = r^2, \check{h}_3 = r^2 \sin^2 \theta, \check{h}_4 = (1 - r_g/r), r_g = \text{const}$	
Example of a prime metric	The Schwarzschild solution, or any BH solution	
Solutions for polarization funct.	$\eta_i = e^{\psi(x^k)} / \check{g}_i; \eta_3 = \eta_3(r, \theta, \varphi)$ as a generating function; $\eta_4 = -\frac{ h_3^{[0]} - \check{h}_3 ^\circ}{ \check{\Upsilon} \eta_3 \check{h}_3 \check{h}_4};$ $\eta_i^3 = \frac{\partial_i (h_3^{[0]} - \check{h}_3 \check{\Upsilon} - \int dy^3 h_3^{[0]} - \check{h}_3 \check{\Upsilon} ^\circ)}{\check{w}_i h_3^{[0]} - \eta_3 \check{h}_3 ^\circ \check{\Upsilon} };$ $\eta_k^4 = \frac{1}{\check{n}_k} + 4 \frac{2\eta_k}{\check{n}_k} \int dy^3 \frac{[(\sqrt{ \Lambda h_3^{[0]} - \eta_3 \check{h}_3 })^\circ]^2}{\check{\Upsilon} \Lambda \eta_3 \check{h}_3 ^{5/2}},$	See Appendix A.3
Polariz. funct. with zero torsion	$\eta_i = e^{\psi(x^k)} / \check{g}_i; \eta_3 = \check{\eta}_3(r, \theta, \varphi)$ as a generating function; $\eta_4 = -\frac{ h_3^{[0]} - \check{h}_3 ^\circ}{ \check{\Upsilon} \eta_3 \check{h}_3 \check{h}_4}; \eta_i^3 = \partial_i \check{A} / \check{w}_k, \eta_k^4 = \frac{\partial_k n}{\check{n}_k},$	See Appendix A.3

Table 3 Off-diagonal locally anisotropic cosmological solutions. Exact solutions of $\hat{\mathbf{R}}_{\mu\nu} = \Upsilon_{\mu\nu}$ (14) transformed into a system of nonlinear PDEs (A.28)–(A.31)

d-metric ansatz with Killing symmetry $\partial_3 = \partial_\varphi$ Effective matter sources	$ds^2 = g_i(x^k)(dx^i)^2 + g_a(x^k, y^a)(dy^a + N_i^a(x^k, t)dx^i)^2, \text{ for}$ $g_i = e^{\psi(x^k)}, \quad g_a = \bar{h}_a(x^k, t), \quad N_i^3 = \bar{n}_i(x^k, t), \quad N_i^4 = \bar{w}_i(x^k, t),$ $\bar{\Upsilon}_\nu^\mu = [{}_{\bar{h}}\bar{\Upsilon}(x^k)\delta_j^i, \bar{\Upsilon}(x^k, t)\delta_0^j], \quad x^1, x^2, y^3, y^4 = t$
Nonlinear PDEs (A.28)–(A.31)	$\psi^{\bullet\bullet} + \psi'' = 2 \quad {}_{\bar{h}}\bar{\Upsilon};$ $\frac{\bar{\omega}^\circ}{\bar{n}_k^*} \bar{h}_3 = 2\bar{h}_3\bar{h}_4\bar{\Upsilon};$ $\frac{\bar{\omega}^\circ}{\bar{n}_k^*} + \frac{\bar{\gamma}}{\bar{\gamma}\bar{n}_k^*} = 0; \quad \text{for}$ $\bar{\beta}\bar{w}_i - \bar{\alpha}_i = 0;$
Generating functions: $h_4(x^k, t), \bar{\Psi}(x^k, t) = e^{\bar{\omega}}, \bar{\Phi}(x^k, t);$	$\bar{\Psi}^2 = 4(h_4^{[0]} - \bar{h}_4 \bar{\Upsilon} - \int dt h_4^{[0]} - \bar{h}_4 \bar{\Upsilon} ^*), \bar{\Phi}^2 = 4 \Lambda[h_4^{[0]} - \bar{h}_4] , \text{ see (A.32);}$
integration functions: $h_4^{[0]}(x^k), {}_1n_k(x^i), {}_2n_k(x^i);$ & nonlinear symmetries	$\bar{h}_4 = h_4^{[0]} - \bar{\Phi}^2/4\bar{\Lambda}, \bar{h}_4^* \neq 0, \bar{\Lambda} \neq 0 = const$
Off-diag. solutions, d-metric N-conne.	$g_i = e^{\psi(x^k)} \text{ as a solution of 2-d Poisson eqs. } \psi^{\bullet\bullet} + \psi'' = 2 \quad {}_{\bar{h}}\bar{\Upsilon};$ $\bar{h}_3 = -(\bar{\Psi}^2)^*/4\bar{\Upsilon}^2\bar{h}_4 = -(\bar{\Phi}^2)^*/4\bar{\Lambda}\bar{\Upsilon}\bar{h}_4, \text{ see (A.33);}$ $\bar{h}_4 = h_4^{[0]} - \int dt (\bar{\Psi}^2)^*/4\bar{\Upsilon} = h_4^{[0]} - \bar{\Phi}^2/4\bar{\Lambda};$ $\bar{n}_k = {}_1n_k + {}_2n_k \int dt (\bar{\Psi}^*)^2/\bar{\Upsilon}^2 h_4^{[0]} - \int dt (\bar{\Psi}^2)^*/4\bar{\Upsilon} ^{5/2};$ $\bar{w}_i = \partial_i \bar{\Psi}/\partial_i \bar{\Psi} = \partial_i \bar{\Psi}^2/\partial_i \bar{\Psi}^2;$ $\partial_t \bar{w}_i = (\partial_i - \bar{w}_i \partial_t) \ln \sqrt{ \bar{h}_4 }, (\partial_i - \bar{w}_i \partial_t) \ln \sqrt{ \bar{h}_3 } = 0,$ $\partial_k \bar{w}_i = \partial_i \bar{w}_k, \partial_i \bar{n}_i = 0, \partial_i \bar{n}_k = \partial_k \bar{n}_i;$ $\Psi = \bar{\Psi}(x^i, t), (\partial_i \bar{\Psi})^* = \partial_i (\bar{\Psi}^*) \text{ and}$ $\bar{\Upsilon}(x^i, t) = \bar{\Upsilon}[\bar{\Psi}] = \bar{\Upsilon}, \text{ or } \bar{\Upsilon} = const.$

Table 3 continued

N-connections, zero torsion	$\bar{n}_k = \bar{n}_k = \partial_k \bar{n}(x^i)$		
Polarization functions $\dot{\mathbf{g}} \rightarrow \bar{\mathbf{g}} = [\bar{g}_a = \bar{\eta}_a \dot{g}_a, \bar{\eta}_i^a \dot{N}_i^a]$	and $\bar{w}_i = \partial_i \bar{A} =$	$\begin{cases} \frac{\partial_t (h_4^{[0]} - \bar{h}_4) \bar{\gamma} - \int dt h_4^{[0]} - \bar{h}_4 \bar{\gamma} ^*}{ h_4^{[0]} - \bar{h}_4 \bar{\gamma} ^*} \bar{\gamma}^* ; \\ \partial_t \bar{\Psi} / \bar{\Psi}^* ; \\ \frac{\bar{\gamma}^2}{\partial_t (\bar{\Phi} \bar{\gamma} - \int dt \bar{\Phi} \bar{\gamma} ^*) / 2 \bar{\Phi} \bar{\gamma}^*} \bar{\gamma} ^* ; \end{cases}$	
Prime metric defines a cosmological solution	$ds^2 = \bar{\eta}_i(x^k, t) \dot{N}_i^3(x^k, t) dx^i]^2 + \bar{\eta}_4(x^k, t) \dot{h}_4(x^k, t) [dt + \bar{\eta}_i^4(x^k, t) \dot{N}_i^4(x^k, t) dx^i]^2 + [\dot{g}_i(x^k, t), \dot{g}_a = \dot{h}_a(x^k, t); \dot{N}_k^3 = \dot{w}_k(x^k, t), \dot{N}_k^4 = \dot{n}_k(x^k, t)]$		
Example of a prime cosmological metric	diagonalizable by frame/coordinate transforms		
Solutions for polarization funct.	$\dot{g}_1 = a^2(t)/(1 - kr^2), \dot{g}_2 = a^2(t)r^2, \dot{h}_3 = a^2(t)r^2 \sin^2 \theta, \dot{h}_4 = c^2 = \text{const}, k = \pm 1, 0;$ any frame transform of a FLRW or a Bianchi metrics		
	$\eta_i = e^{\psi(x^k)} / \dot{g}_i; \bar{\eta}_3 = -\frac{ h_4^{[0]} - \bar{\eta}_i \dot{h}_{4i} ^*}{ \bar{\gamma} \bar{\eta}_i \dot{h}_4};$		See Appendix A.3
	$\bar{\eta}_4 = \bar{\eta}_4(x^i, t)$ as a generating function;		
	$\bar{\eta}_i^3 = \frac{1 \eta_k}{\bar{h}_k} + 4 \frac{2 \eta_k}{\bar{h}_k} \int dt \frac{[(\sqrt{ \Lambda h_4^{[0]} - \bar{\eta}_i \dot{h}_{4i}})]^{*2}}{\bar{\gamma} \Lambda \bar{\eta}_i \dot{h}_{4i} ^{5/2}};$		
	$\bar{\eta}_k^4 = \frac{\partial_t (h_4^{[0]} - \bar{\eta}_i \dot{h}_{4i}) \bar{\gamma} - \int dt h_4^{[0]} - \bar{\eta}_i \dot{h}_{4i} \bar{\gamma} ^*}{\partial_k h_4^{[0]} - \bar{\eta}_i \dot{h}_{4i} ^* \bar{\gamma} },$		
	$\bar{\eta}_i = e^{\psi} / \dot{g}_i; \bar{\eta}_3 = -\frac{ h_4^{[0]} - \bar{\eta}_i \dot{h}_{4i} ^*}{ \bar{\gamma} \bar{\eta}_i \dot{h}_4};$		
Polariz. funct. with zero torsion	$\bar{\eta}_4 = \bar{\eta}_4(x^i, t)$ as a generating function;		see Appendix A.3
	$\bar{\eta}_k^3 = (\partial_k \bar{n}) / \dot{n}_k; \bar{\eta}_k^4 = \partial_k \bar{A} / \dot{w}_k,$		

$$ds^2 = e^{\psi(x^k)} \left[(dx^1)^2 + (dx^2)^2 \right] - \frac{|h_4^{[0]} - \bar{h}_4|^*}{|\bar{\Upsilon} \bar{h}_4|} \left[dy^3 \right. \\ \left. + \left({}_{1n_k} + 4 {}_{2n_k} \int dt \frac{\left[\left(\sqrt{|\Lambda|} [h_4^{[0]} - \bar{h}_4] \right)^* \right]^2}{\bar{\Upsilon} \bar{\Lambda} y |\bar{h}_4|^{5/2}} \right) dx^k \right] \\ + \bar{h}_4 \left[dt + \frac{\partial_i \left(|h_4^{[0]} - \bar{h}_4| |\bar{\Upsilon}| - \int dt |h_4^{[0]} - \bar{h}_4| |\bar{\Upsilon}|^* \right)}{\left(|h_4^{[0]} - \bar{h}_4| |\bar{\Upsilon}| - \int dt |h_4^{[0]} - \bar{h}_4| |\bar{\Upsilon}|^* \right)^*} dx^i \right]$$

Such locally anisotropic and inhomogeneous cosmological solutions are, in general, with nontrivial nonholonomically induced torsion (A.2). This class of solutions can be re-defined equivalently in terms of generating functions $\bar{\Psi}(x^k, t)$ and/or $\bar{\Phi}(x^k, t)$, see (A.34).

Cosmological configurations in GR can be extracted by imposing additional constraints for zero torsion by restricting the class of “integrable” generating functions ($\bar{h}_4 = \bar{\bar{h}}_4$, and $\bar{\Psi}(x^k, t)$ and/or $\bar{\Phi}(x^k, t)$), for respective types of sources $\bar{\Upsilon}$ and $\bar{\Lambda}$, as in (A.35),

$$ds^2 = e^{\psi(x^k)} \left[(dx^1)^2 + (dx^2)^2 \right] \\ - \frac{|h_4^{[0]} - \bar{\bar{h}}_4|^*}{|\bar{\Upsilon} \bar{\bar{h}}_4|} \left[dy^3 + (\partial_k \bar{n}) dx^k \right] \\ + \bar{\bar{h}}_4 \left[dt + \left(\partial_i \bar{A} \right) dx^i \right].$$

In terms of η -polarization functions, the coefficients of cosmological d-metrics and N-connections can be parameterized to describe nonholonomic deformations of a primary (for instance, a FLRW) d-metric $\hat{\mathbf{g}}$ into target generic off diagonal cosmological solutions $\hat{\mathbf{g}}(x^i, t) \rightarrow \hat{\mathbf{g}}(t)$, see (7), as $\hat{\mathbf{g}} \rightarrow \hat{\mathbf{g}} = [g_\alpha = \bar{\eta}_\alpha \hat{g}_\alpha, \bar{\eta}_i^a \hat{N}_i^a]$.

3 Quasiperiodic/pattern forming generating functions and sources

The goal of this section is to analyse a series of examples of quasiperiodic and/or pattern forming distributions and related free energy functionals and systems of PDEs evolution equations on a nonholonomic Lorentzian manifold $\mathbf{V}$, which have certain analogy in condensed matter physics (see, for instance, [44,45,48,49]). For small parametric dependencies, such gravitational and matter field (non) stationary distributions and nonlinear field interactions were studied in our partner works [4,6,7]. In this paper, the approach is elaborated for generating and integration polarization functions and (effective) sources stated in certain general forms without additional assumptions on decompositions on small parameters.

3.1 Nonholonomic 3 + 1 distributions with quasiperiodic/pattern forming structures

Let us consider necessary smooth classes of functions $q = q(x^i, y^3)$, for space like distributions, and $\bar{q} = \bar{q}(x^i, y^4 = t)$, for locally anisotropic cosmological configurations, defined respectively in N-adapted coordinates on open regions of $U \subset \mathbf{V}$ and $\bar{U} \subset \mathbf{V}$. Such values will be used as generating functions and/or (effective) sources for different models of quasiperiodic and/or pattern forming spacetime structures. Additionally to a nonholonomic 2 + 2 splitting which allows us to decouple systems of nonlinear PDEs and construct exact solutions of (modified) gravitational equations, we shall consider a 3 + 1 splitting with local coordinates parameterized in the form $u^\alpha = (u^i, t)$, where the space like coordinates are $u^i = (x^i, y^3)$, with $i, j, k, \dots = 1, 2, 3$. In GR, models with 3 + 1 spacetime decomposition were elaborated with the aim to introduce values similar to the energy and momentum and thermodynamical like characteristics for gravitational and scalar fields, see details in Refs. [27,29]. In our recent works [4,6,7,50,51], double 2 + 2 and 3 + 1 spacetime decompositions (and various respective extra dimension generalizations) were considered for constructing generic off-diagonal solutions in MGTs and GR encoding quasi-periodic and aperiodic structures.

Any metric and/or equivalent d-metric structures parameterized in a form (2) or (7) can be re-written in certain forms with nonholonomic 3 + 1 splitting,

$$\mathbf{g} = b_i(x^k) dx^i \otimes dx^i + b_3(x^k, y^3, t) \mathbf{e}^3 \otimes \mathbf{e}^3 \\ - \check{N}^2(x^k, y^3, t) \mathbf{e}^4 \otimes \mathbf{e}^4, \\ \mathbf{e}^3 = dy^3 + N_i^3(x^k, y^3, t) dx^i, \\ \mathbf{e}^4 = dt + N_i^4(x^k, y^3, t) dx^i. \quad (16)$$

For such configurations, a 4-d metric $\mathbf{g}$ can be considered as an extension of a 3-d metric $b_{ij} = \text{diag}(b_i) = (b_i, b_3)$ on a family of 3-d hypersurfaces $\hat{\Sigma}_t$ parameterized by coordinate t considered as a parameter, and when $b_3 = h_3$ and $\check{N}^2(u) = -h_4$ is defined by a lapse function $\check{N}(u)$. We can impose additional conditions in order to transform stationary d-metrics (A.19), or locally anisotropic cosmological d-metrics (A.34), into respective 3 + 1 versions (16). Correspondingly, we shall work with a Killing symmetry on ∂_4 and lapse functions of type $\check{N}(x^k, y^3)$, or with a Killing symmetry on ∂_3 and lapse functions of type $\check{N}(x^k, t)$. Such a decomposition results in a representation $\hat{\mathbf{D}} = ({}^3\hat{\mathbf{D}}, {}^t\hat{\mathbf{D}})$, where ${}^3\hat{\mathbf{D}}$ defines the action of the canonical d-connection covariant derivative on space like coefficients and ${}^t\hat{\mathbf{D}}$ of time like coefficients. For LC-configurations, the covariant derivative operator splits as $\nabla = ({}^3\nabla, {}^t\nabla)$, when the action on a scalar field $q(u)$ can be parameterized via frame/ coordinate transforms as $\nabla q = ({}^3\nabla q, {}^t\nabla q = \partial_t q = q^*)$.

3.1.1 Many pattern-forming nonlinear gravitational and matter fields systems

In condensed matter physics, models with tree-waves interactions, 3WIs, for many pattern-forming systems were elaborated for explaining experimental observations of certain microscopic and quasi-classical quantum systems, see [44] and references therein. Modern cosmological data show a very complex web like quasiperiodic and/or aperiodic like structure formation, geometric anisotropic evolution and nonlinear gravitational and matter field interactions, including dark energy and dark matter configurations [46, 52]. We can apply very similar mathematical methods for geometric modeling of quasi-crystal matter or (super) galactic clusters and 3-d distributions of dark energy and dark matter and generating solutions of systems on nonlinear PDEs describing such physical systems.

A prime pattern-forming field can be taken in the form

$$\bar{q}(x^i, t) = \sum_{l=1, |\mathbf{k}_l|=1}^{\infty} z_l(t) e^{i\mathbf{k}_l \cdot \mathbf{u}} + \sum_{l=1, |\mathbf{c}_l|=c} v_l(t) e^{i\mathbf{c}_l \cdot \mathbf{u}} + \text{higher order terms} \quad (17)$$

defining in flat spaces 3WIs involving two comparable wavelengths. We consider systems with two wave numbers $k = 1$ and $k = c$,⁸ for $0 < c < 1$, when 3WIs are modelled in two forms:

1. two waves (with wave number 1, on the outer circle) interact nonlinearly with a wave on the inner circle, with wave number c (for instance, wave vectors configurations $\mathbf{k}_1$ and $\mathbf{k}_2$ interact with $\mathbf{c}_1 = \mathbf{k}_1 + \mathbf{k}_2$);
2. for $1/2 < c < 1$, two waves on the inner circle interact with another wave on the outer circle (for instance, wave vectors configurations $\mathbf{c}_2$ and $\mathbf{c}_3$ interact with $\mathbf{k}_1 = \mathbf{c}_2 + \mathbf{c}_3$).

For (17), we can parameterize the coefficients and respective values in such forms that there are modelled two types of 3WIs. There are involved (in the case 1 above) a triad of wave vectors $\mathbf{k}_1$, $\mathbf{k}_2$ and $\mathbf{c}_1 = \mathbf{k}_1 + \mathbf{k}_2$, when amplitudes are subjected to equations

$$z_1^* = \mu z_1 + Q_{zv} \bar{z}_2 v_1 + \text{cubic terms}, z_2^* = \mu z_2 + Q_{zv} \bar{z}_1 v_1 + \text{cubic terms}, \text{ and } v_1^* = \xi z_1 + Q_{zz} z_1 z_2 + \text{cubic terms}. \quad (18)$$

Other configuration can be defined (in the case 2) for wave vectors $\mathbf{c}_2$, $\mathbf{c}_3$ and $\mathbf{k}_1 = \mathbf{c}_2 + \mathbf{c}_3$ with respective equations for

amplitudes

$$v_2^* = \xi v_2 + Q_{vz} \bar{v}_2 z_1 + \text{cubic terms}, \\ v_3^* = \xi v_3 + Q_{vz} \bar{v}_1 z_1 + \text{cubic terms}, \text{ and} \\ z_1^* = \mu z_1 + Q_{zz} v_2 v_3 + \text{cubic terms}. \quad (19)$$

In these formulas, the coefficients μ and ξ determine, respectively, the growth rates of amplitudes corresponding to wave numbers 1 and c ; and, for instance, Q_{zv} and Q_{zz} are quadratic elements. For simplicity, we omit the cubic terms even they play also an important role in the nonlinear dynamics of such waves and result in observable effects both in condensed matter physics and at cosmological distances. Similar equations can be replicated for all possible combinations of modes describing 3WIs. Different stationary and dynamical models with nonlinear wave interactions are generated for certain ranges of values of coefficients μ , ξ , Q_{zv} , Q_{zz} , ... and the signs of quadratic coefficients for products of type $Q_{zv} Q_{zz}$ etc. were analyzed in [44] (on experimental data in condensed matter physics, see references therein).

In a more general context, we can consider nonlinear deformations of prime waves (17), $\bar{q}(x^i, t) \rightarrow {}^P \bar{q}(x^i, t)$, when the target field ${}^P \bar{q} = \bar{q}$ describes pattern forming configurations as solutions of nonlinear PDE,

$$\bar{q}^* = \mathcal{L} \bar{q} + {}^1 Q \bar{q}^2 + {}^2 Q ({}^3 \hat{\mathbf{D}}_i {}^3 \hat{\mathbf{D}}^i \bar{q}) + {}^3 Q |({}^3 \hat{\mathbf{D}}_i \bar{q})|^2 - \bar{q}^3, \quad (20)$$

with summation on up-low index i . This equation can be parameterized with a linear part $\mathcal{L}$ acting on a mode e^{ikx} with an eigenvalue $\sigma(k)$ specified by $\sigma(1) = \mu$ and $\sigma(c) = \xi$. Such a specification controls growth rates of the modes of interest; with $d\sigma/dx = 0$, for $k = 1$ and $k = c$, and $\sigma(0) = \sigma_0 < 0$, one controls the depths of minimum between $k = 1$ and $k = c$. For even functions σ of k , we can consider a 4th order polynomial on k^2 ,

$$\sigma(k) = \frac{[\mu A(k) + \xi B(k)] k^2}{(1 - c^2)^3 c^4} + \frac{\sigma_0}{c^4} (1 - k^2) (c^2 - k^2)^2,$$

where

$$A(k) = \left[(c^2 - 3) k^2 - 2c^2 + 4 \right] (k^2 - c^2)^2 c^4 \text{ and} \\ B(k) = \left[(3c^2 - 1) k^2 + 2c^2 - 4c^4 \right] (k^2 - 1)^2.$$

The linear operator $\mathcal{L}$ in the linear part of (20) is defined by replacing k^2 by $-{}^3 \hat{\mathbf{D}}_i {}^3 \hat{\mathbf{D}}^i$. The nonlinear terms in that PDE can be boosted as quadratic and cubic combinations of $\bar{q}$ and its derivatives, which can re-parameterized as certain nonlinear deformations of $\bar{q}$ (17). A standard weakly nonlinear theory expresses the values ${}^1 Q$, ${}^2 Q$, and ${}^3 Q$ as certain products Q_{zv} , Q_{zz} etc. when the solutions of (18) and (19) are nonlinearly mixed which results in different signs and constant coefficients.

⁸ We emphasize that in this work, the constant c should be not confused with the speed of light.

Using ${}^P\bar{q} = \bar{q}$ as a solution of (20), we can generate various type pattern forming configurations in a nonholonomic spacetime. For instance, there are possible 3 important results (for simplicity, we take values of constants reproducing the results explained in details in [44] and references therein; there are cited also some papers analyzing similar structures in modern gravity, cosmology and astrophysics):

1. A bifurcation pattern can be seen for $c = 0.66$, $\sigma_0 = -2$, ${}^1Q = 0.3$, ${}^2Q = 1.3$, and ${}^3Q = 1.7$. There are: z hexagons, for $k = 1$; v hexagons for $k = c$; certain spatiotemporal chaos, STC, with mixed patterns defined by v stripes with patches of z rectangles; two super lattice patterns (the first one is with 6 modes on the outer circle and with 12 modes on the inner; the second one is with 6 modes on the inner circle and with 12 ones on the outer). Here it should be noted that the scales stated by μ and ξ are not uniform and that we can model additionally various diffusion and additional nonlinear wave interactions [9, 10].
2. One can be found a STC pattern with $\mu = \xi = 0.000707$, when the correlation length is about 1-2 wave-lengths. We can add new terms with fractional chaos and diffusion [21, 22].
3. Another interesting configuration with two critical circles can be reproduced in the power section [53].

Solutions of (20) constructed as nonholonomic deformations of prime generating functions (17) with nonlinear superpositions of (18) and (19) reproduce, for explicit parameterizations of constants, various type of nonlinear interactions and pattern forming configurations. In condensed matter physics, such interactions between two waves of one wavelength with a third wave of the other wavelength are known both experimentally and theoretically. Both in condensed matter physics and at cosmological scales, that they play a key role in producing a rich variety of interesting phenomena such as web structures [52]. In this and partner works [4, 6, 7], we show that one can be reproduced also quasipatterns, superlattice patterns, and STC. The geometric methods and mechanism we elaborate in our works can be applied to any systems in which such 3WIs can occur at microscopic or cosmological scales. In condensed matter physics, such effects are confirmed via Faraday wave experiments, for coupled Turing systems, and some optical systems, see details in [44] and references therein. At cosmological scales, similar configurations are observed and modeled theoretically [46, 52, 53].

3.1.2 Quasicrystal like configurations in MGTs and GR

We can consider other types of quasi-periodic, or aperiodic, generating functions (not) related to 3WIs.

Formation of quasicrystalline structures and analogous dynamic phase field crystal models:

Quasicrystal, QC, structures can be modeled by generating functions ${}^Q\bar{q} = \bar{q}(x^i, t)$ defined as solutions of an evolution equation with conserved dynamics,

$$\frac{\partial \bar{q}}{\partial t} = {}^b\hat{\Delta} \left[\frac{\delta F}{\delta \bar{q}} \right] = - {}^b\hat{\Delta} (\Theta \bar{q} + Q \bar{q}^2 - \bar{q}^3). \quad (21)$$

In this formula, the canonically nonholonomically deformed hypersurface Laplace operator ${}^b\hat{\Delta} := ({}^b\hat{D})^2 = {}^{b^i j} \hat{D}_i \hat{D}_j$ is defined by (16) as a distortion of ${}^b\Delta := ({}^b\nabla)^2$. Such operators can be computed for any family of hypersurfaces $\hat{\Xi}_t$ using formulas (10).

The functional F in (21) defines an effective free energy

$$F[\bar{q}] = \int \left[-\frac{1}{2} \bar{q} \Theta \bar{q} - \frac{Q}{3} \bar{q}^3 + \frac{1}{4} \bar{q}^4 \right] \sqrt{b} dx^1 dx^2 dy^3, \quad (22)$$

where $b = \det |b_{ij}|$, $\delta y^3 = \mathbf{e}^3$ and the operator Θ will be defined below. Such nonlinear interactions can be stabilized by the cubic term when the second order resonant interactions are varied by setting the value of Q . The average value $\langle \bar{q} \rangle$ of the generating function $\bar{q}$ is conserved for any fixed t . In result, we can consider $\bar{q}$ as an effective parameter of the system and that we can choose $\langle \bar{q} \rangle|_{t=t_0} = 0$ since other values can be redefined and accommodated by altering Θ and Q . Using the functional (22), we can elaborate on models of dark energy, DE, for certain locally anisotropic and inhomogeneous cosmological configurations [4, 6, 7]. Similarly to QCs in condensed matter physics [45], we concluded that the effective free energy $F[\bar{q}]$ characterizes a 3-d phase gravitational field crystal model, when modulations are generated with two length scales for off-diagonal cosmological structures. This model consists of a nonlinear PDE with conserved nonholonomic dynamics resulting in evolution equation. It describes a time evolution of $\bar{q}$ over diffusive time scales.

3-d phase field like quasicrystal structures and evolution:

In cosmological theories, there are studied scalar fields potentials $V(\varphi)$ modified by effective quasicrystal structures, $\varphi \rightarrow \varphi = \varphi_0 + \psi$, where $\psi(x^i, y^3, t)$ with (quasi) crystal like phases described by periodic or quasi-periodic modulations. Such modifications can be modelled in dynamical phase field crystal, PFC, like form [45, 54]. Applying such mathematical methods in modern cosmology [4, 6, 7], we can elaborate models of 3-d nonrelativistic dynamics which determined by Laplace like operators ${}^3\Delta := ({}^3\nabla)^2$, or ${}^b\hat{\Delta}$ (the left label 3 emphasizes that such an operator is for a 3-d hypersurface). We write ψ instead of $\bar{q}$ in order to distinguish such QC structures (which can be generated both by gravitational and

matter field with two length scales) from the class of models considered above.

In N-adapted frames with 3 + 1 splitting the equations for a local minimum conserving dynamics,

$$\begin{aligned}\partial_t \psi &= {}^3\Delta \left[\frac{\delta F[\psi]}{\delta \psi} \right], \text{ or in a nonholonomic variant} \\ \partial_t \psi &= {}^b\widehat{\Delta} \left[\frac{\delta F[\psi]}{\delta \psi} \right],\end{aligned}\quad (23)$$

with two length scales $l_i = 2\pi/k_i$, for $i = 1, 2$. We can elaborate on local diffusion processes determined by a free energy functional

$$\begin{aligned}F[\psi] &= \int \sqrt{|{}^3g|} dx^1 dx^2 dy^3 \\ &\times \left[\frac{1}{2} \psi \left\{ -\epsilon + \prod_{i=1,2} (k_i^2 + {}^3\Delta)^2 \right\} \psi + \frac{1}{4} \psi^4 \right], \\ \text{or } {}^bF[\psi] &= \int \sqrt{|{}^3g|} dx^1 dx^2 dy^3 \\ &\times \left[\frac{1}{2} \psi \left\{ -\epsilon + \prod_{i=1,2} (k_i^2 + {}^b\widehat{\Delta})^2 \right\} \psi + \frac{1}{4} \psi^4 \right],\end{aligned}\quad (24)$$

where $|{}^3g|$ is the determinant of the 3-d space metric and ϵ is a constant. For simplicity, we can restrict our constructions to only non-relativistic diffusion processes, see [22] for relativistic and N-adapted generalizations. The functional ${}^bF[\psi]$ is defined by a nonholonomic deformation of the Laplace operator, ${}^3\Delta \rightarrow {}^b\widehat{\Delta}$, resulting in a nonholonomic distortion of $F[\psi]$.

3.1.3 Solitonic space like distributions and nonlinear waves

Off-diagonal interactions determined by generating and integration functions and nontrivial effective sources in MGTs and GR heterotic string gravity may result in various effects with solitonic like distributions, cosmological and geometric evolution models [2–4, 6, 7, 21].

Stationary solitonic distributions:

We shall use distributions ${}^{sd}q = q(r, \vartheta, \varphi)$ as solutions of a respective class of solitonic 3-d equations

$$\begin{aligned}\partial_{rr}^2 q + \epsilon \partial_\varphi (\partial_\vartheta q + 6q \partial_\varphi q + \partial_{\varphi\varphi}^3 q) &= 0, \\ \partial_{rr}^2 q + \epsilon \partial_\vartheta (\partial_\varphi q + 6q \partial_\vartheta q + \partial_{\vartheta\vartheta}^3 q) &= 0, \\ \partial_{\vartheta\vartheta}^2 q + \epsilon \partial_\varphi (\partial_r q + 6q \partial_\varphi q + \partial_{\varphi\varphi}^3 q) &= 0, \\ \partial_{\vartheta\vartheta}^2 q + \epsilon \partial_r (\partial_\varphi q + 6q \partial_r q + \partial_{rr}^3 q) &= 0, \\ \partial_{\varphi\varphi}^2 q + \epsilon \partial_r (\partial_\vartheta q + 6q \partial_r q + \partial_{rr}^3 q) &= 0, \\ \partial_{\varphi\varphi}^2 q + \epsilon \partial_\vartheta (\partial_r q + 6q \partial_\vartheta q + \partial_{\vartheta\vartheta}^3 q) &= 0,\end{aligned}\quad (25)$$

for $\epsilon = \pm 1$. The left label *sd* states that such a function is defined as a “solitonic distribution” when in N-adapted frames a function ${}^{sd}q$ does not depend on time coordinate. The equations (25) and their solutions can be redefined via frame/coordinate transforms for stationary generating functions parameterized in non-spherical coordinates, ${}^{sd}q = q(x^i, y^3)$.

Generating nonlinear solitonic waves:

3-d solitonic waves with explicit dependence on time coordinate t are solutions of such nonlinear PDEs:

$$\begin{aligned}{}^{sw}\bar{q} &= \begin{cases} \bar{q}(t, \vartheta, \varphi) \text{ as a solution of } \partial_{tt}^2 \bar{q} + \epsilon \frac{\partial}{\partial \varphi} [\partial_\vartheta \bar{q} + 6 \bar{q} \frac{\partial}{\partial \varphi} \bar{q} + \frac{\partial^3}{(\partial \varphi)^3} \bar{q}] = 0; \\ \bar{q}(\vartheta, t, \varphi) \text{ as a solution of } \partial_{\vartheta\vartheta}^2 \bar{q} + \epsilon \frac{\partial}{\partial \varphi} [\partial_t \bar{q} + 6 \bar{q} \frac{\partial}{\partial \varphi} \bar{q} + \frac{\partial^3}{(\partial \varphi)^3} \bar{q}] = 0; \\ \bar{q}(t, r, \varphi) \text{ as a solution of } \partial_{tt}^2 \bar{q} + \epsilon \frac{\partial}{\partial \varphi} [\partial_r \bar{q} + 6 \bar{q} \frac{\partial}{\partial \varphi} \bar{q} + \frac{\partial^3}{(\partial \varphi)^3} \bar{q}] = 0; \\ \bar{q}(r, t, \varphi) \text{ as a solution of } \partial_{rr}^2 \bar{q} + \epsilon \frac{\partial}{\partial \varphi} [\partial_t \bar{q} + 6 \bar{q} \frac{\partial}{\partial \varphi} \bar{q} + \frac{\partial^3}{(\partial \varphi)^3} \bar{q}] = 0; \\ \bar{q}(t, \varphi, \vartheta) \text{ as a solution of } \partial_{tt}^2 \bar{q} + \epsilon \frac{\partial}{\partial \vartheta} [\partial_\varphi \bar{q} + 6 \bar{q} \frac{\partial}{\partial \vartheta} \bar{q} + \frac{\partial^3}{(\partial \vartheta)^3} \bar{q}] = 0; \\ \bar{q}(\varphi, t, \vartheta) \text{ as a solution of } \partial_{\varphi\varphi}^2 \bar{q} + \epsilon \frac{\partial}{\partial \vartheta} [\partial_t \bar{q} + 6 \bar{q} \frac{\partial}{\partial \vartheta} \bar{q} + \frac{\partial^3}{(\partial \vartheta)^3} \bar{q}] = 0. \end{cases}\end{aligned}\quad (26)$$

Applying general frame/coordinate transforms, solitonic waves of type ${}^{sw}\bar{q} = \bar{q}(x^i, t) = \bar{q}(x^1, y^3, t)$, or $= \bar{q}(x^2, y^3, t)$, can be used as generating functions for certain classes of nonholonomic deformations of stationary, or cosmological metrics, and as generating sources.

3.2 Effective sources with effective quasiperiodic free energy

We can prescribe respective generating functions $\Phi(r, \theta, \varphi)$, or $\Psi(r, \theta, \varphi)$, (for stationary configurations with nonlinear symmetry (A.17)), and $\overline{\Phi}(x^i, t)$, or $\overline{\Psi}(x^i, t)$, (for cosmological solutions with nonlinear symmetry (A.32)) for quasiperiodic and/or aperiodic effective sources (15) in (14). Such configurations can be determined by additive source functionals and effective cosmological constants, or by nonlinear functionals.

3.2.1 Additive effective sources and cosmological constants

We shall be able to integrate in explicit form gravitational and matter fields systems of nonlinear PDEs for parameterizations of N-adapted sources (A.6) as functionals of type

$$\Upsilon(r, \theta, \varphi) = {}^P\Upsilon[{}^P\bar{q}_0] + {}^{QC}\Upsilon[{}^{QC}\bar{q}_0] + {}^\psi\Upsilon[\psi_0] + {}^{sd}\Upsilon[{}^{sd}q] + {}^{sw}\Upsilon[{}^{sw}\bar{q}_0],\quad (27)$$

$$\begin{aligned}\overline{\Upsilon}(x^i, t) &= {}^P\overline{\Upsilon}[{}^P\bar{q}] + {}^{QC}\overline{\Upsilon}[{}^{QC}\bar{q}_0] \\ &+ {}^\psi\overline{\Upsilon}[\psi] + {}^{sd}\overline{\Upsilon}[{}^{sd}q_0] + {}^{sw}\overline{\Upsilon}[{}^{sw}\bar{q}].\end{aligned}\quad (28)$$

In these formulas, the left labels emphasize what types of effective v-sources are considered. For simplicity, the h-sources can be taken any general ones ${}_h\Upsilon(r, \theta)$, or ${}_h\overline{\Upsilon}(x^i)$. The functional dependence [...] is parameterized for such classes of functions: ${}^P\bar{q}(x^i, t)$ (20)

with stationary configurations ${}^P\bar{q}_0 := {}^P\bar{q}(x^i, t = t_0)$ for a fixed value t_0 ; ${}^{QC}\bar{q} = \bar{q}(x^i, t)$ (21) with stationary configurations ${}^{QC}\bar{q}_0 := {}^{QC}\bar{q}(x^i, t_0)$; $\psi(x^i, y^3, t)$ (23) with stationary $\psi_0 := \psi(x^i, y^3, t_0)$; ${}^{sd}q(r, \vartheta, \varphi)$ (25) when cosmological configurations are generated by any source ${}^{sd}q_0 := {}^{sd}q(r, \vartheta, \varphi_0)$ for any fixed value $\varphi = \varphi_0$; and ${}^{sw}\bar{q} = \bar{q}(x^i, t)$ (25) with stationary ${}^{sw}\bar{q}_0 := {}^{sw}\bar{q}(x^i, t_0)$.

For additive sources (27) and (28), respective stationary and cosmological configurations possess nonlinear symmetries:

$$\begin{aligned} \Lambda \Psi^2 &= \Phi^2 \left(|{}^P\Upsilon| + |{}^{QC}\Upsilon| + |\psi\Upsilon| + |{}^{sd}\Upsilon| + |{}^{sw}\Upsilon| \right) \\ &\quad - \int d\varphi \Phi^2 \left(|{}^P\Upsilon|^\diamond + |{}^{QC}\Upsilon|^\diamond + |\psi\Upsilon|^\diamond + |{}^{sd}\Upsilon|^\diamond + |{}^{sw}\Upsilon|^\diamond \right), \\ \Lambda &= {}^P\Lambda + {}^{QC}\Lambda + \psi\Lambda + {}^{sd}\Lambda \\ &\quad + {}^{sw}\Lambda, \text{ with effective cosmological constants;} \end{aligned} \quad (29)$$

$$\begin{aligned} \bar{\Lambda} \bar{\Psi}^2 &= \bar{\Phi}^2 \left(|{}^P\bar{\Upsilon}| + |{}^{QC}\bar{\Upsilon}| + |\psi\bar{\Upsilon}| + |\bar{\Upsilon}| + |\bar{\Upsilon}| \right) \\ &\quad - \int dt \bar{\Phi}^2 \left(|{}^P\bar{\Upsilon}|^* + |{}^{QC}\bar{\Upsilon}|^* + |\psi\bar{\Upsilon}|^* + |{}^{sd}\bar{\Upsilon}|^* + |{}^{sw}\bar{\Upsilon}|^* \right), \\ \bar{\Lambda} &= {}^P\bar{\Lambda} + {}^{QC}\bar{\Lambda} + \psi\bar{\Lambda} + {}^{sd}\bar{\Lambda} \\ &\quad + {}^{sw}\bar{\Lambda}, \text{ with effective cosmological constants.} \end{aligned} \quad (30)$$

Such formulas are of type (A.17) and (A.32) and can be stated separately for all sources. The generating functions are chosen in a general form Ψ , or Φ , and, correspondingly, $\bar{\Psi}$, or $\bar{\Phi}$. The QC like components of such quasiperiodic/aperiodic structures are characterized by free energy functionals (22), for ${}^{QC}\bar{q}$, and (24), for ψ .

3.2.2 Nonlinear functionals for effective sources and cosmological constants

The modified Einstein equations can be integrated in explicit form for general nonlinear functionals

$${}^{qp}\Upsilon(r, \theta, \varphi) = \Upsilon \left[{}^P\bar{q}_0, {}^{QC}\bar{q}_0, \psi_0, {}^{sd}q, {}^{sw}\bar{q}_0 \right] \text{ and} \quad (31)$$

$${}^{qp}\bar{\Upsilon}(x^i, t) = \bar{\Upsilon} \left[{}^P\bar{q}, {}^{QC}\bar{q}_0, \psi, {}^{sd}q_0, {}^{sw}\bar{q} \right]. \quad (32)$$

The respective nonlinear symmetries (A.17) and/or (A.32) are parameterized

$$\begin{aligned} \Lambda \Psi^2 &= \Phi^2 |{}^{qp}\Upsilon| - \int d\varphi \Phi^2 |{}^{qp}\Upsilon|^\diamond, \text{ for} \\ \Lambda &= \Lambda \left({}^P\Lambda, {}^{QC}\Lambda, \psi\Lambda, {}^{sd}\Lambda, {}^{sw}\Lambda \right); \end{aligned} \quad (33)$$

$$\begin{aligned} \bar{\Lambda} \bar{\Psi}^2 &= \bar{\Phi}^2 |{}^{qp}\bar{\Upsilon}| - \int dt \bar{\Phi}^2 |{}^{qp}\bar{\Upsilon}|^*, \text{ for} \\ \bar{\Lambda} &= \bar{\Lambda} \left({}^P\bar{\Lambda}, {}^{QC}\bar{\Lambda}, \psi\bar{\Lambda}, {}^{sd}\bar{\Lambda}, {}^{sw}\bar{\Lambda} \right), \end{aligned} \quad (34)$$

resulting in functional dependencies of effective cosmological constants.

The formulas (31) and (32) transform respectively in (27) and (28) for additional effective sources and cosmological constants. We emphasize that nonlinear effects are very important in structure formation and for multi-wave nonlinear (solitonic or other types) matter fields and gravitational interactions. Nonlinear dependencies and running of physical constants can be considered also in quantum models. Such stationary and/or cosmological solutions (without contributions of quasiperiodic fields) were studied in a series of our works [1, 13, 24, 26], see also recent results for quasiperiodic structures [4–7].

For nonlinear effective sources (31) and respective nonlinear symmetries (33), we can define additionally free energy functionals (22), for ${}^{QC}\bar{q}$, and (24), for ψ . Such values and constants have to be determined in explicit form for astrophysical and/or cosmological configurations in order to describe observational data for dark matter and dark energy distributions with respective scales and/or quasiperiodic/aperiodic structures.

3.3 Quasiperiodic generating functions

We can generate various types of gravitational field stationary and/or cosmological configurations using respective classes of generating functions. Such quasiperiodic/aperiodic configurations can be defined by additive quadratic functionals, or in some general nonlinear forms.

3.3.1 Additive quasiperiodic quadratic generating functions

We can prescribe a nontrivial cosmological constant Λ , or $\bar{\Lambda}$, and consider generating functions of type

$$\begin{aligned} \Phi^2(r, \theta, \varphi) &= {}^a\Phi^2 = {}^P\Phi^2[{}^P\bar{q}_0] + {}^{QC}\Phi^2[{}^{QC}\bar{q}_0] \\ &\quad + \psi\Phi^2[\psi_0] + {}^{sd}\Phi^2[{}^{sd}q] + {}^{sw}\Phi^2[{}^{sw}\bar{q}_0], \text{ or} \end{aligned} \quad (35)$$

$$\begin{aligned} \bar{\Phi}^2(x^i, t) &= {}^a\bar{\Phi}^2 = {}^P\bar{\Phi}^2[{}^P\bar{q}] + {}^{QC}\bar{\Phi}^2[{}^{QC}\bar{q}_0] \\ &\quad + \psi\bar{\Phi}^2[\psi] + {}^{sd}\bar{\Phi}^2[{}^{sd}q_0] + {}^{sw}\bar{\Phi}^2[{}^{sw}\bar{q}], \end{aligned} \quad (36)$$

where the left label “a” emphasizes that we consider additions of functionals. Nonlinear symmetries of type (A.17) and/or (A.32) allow to compute respectively corresponding data (${}^a\Psi, \Upsilon$), or (${}^a\bar{\Psi}, \bar{\Upsilon}$), for certain fixed effective sources $\Upsilon = \Upsilon(r, \theta, \varphi)$, or $\bar{\Upsilon} = \bar{\Upsilon}(x^i, t)$. The respective formulas are

$$\begin{aligned} \Lambda \left({}^P\Psi^2 + {}^{QC}\Psi^2 + \psi\Psi^2 + {}^{sd}\Psi^2 + {}^{sw}\Psi^2 \right) \\ = \left({}^P\Phi^2 + {}^{QC}\Phi^2 + \psi\Phi^2 + {}^{sd}\Phi^2 + {}^{sw}\Phi^2 \right) |\Upsilon| \end{aligned}$$

$$-\int d\varphi \left({}^P\Phi^2 + {}^{QC}\Phi^2 + {}^\psi\Phi^2 + {}^{sd}\Phi^2 + {}^{sw}\Phi^2 \right) |\Upsilon|^\diamond, \quad (37)$$

$$\begin{aligned} \bar{\Lambda} \left({}^P\bar{\Psi}^2 + {}^{QC}\bar{\Psi}^2 + {}^\psi\bar{\Psi}^2 + {}^{sd}\bar{\Psi}^2 + {}^{sw}\bar{\Psi}^2 \right) \\ = \left({}^P\bar{\Phi}^2 + {}^{QC}\bar{\Phi}^2 + {}^\psi\bar{\Phi}^2 + {}^{sd}\bar{\Phi}^2 + {}^{sw}\bar{\Phi}^2 \right) |\bar{\Upsilon}| \\ - \int dt \left({}^P\bar{\Phi}^2 + {}^{QC}\bar{\Phi}^2 + {}^\psi\bar{\Phi}^2 + {}^{sd}\bar{\Phi}^2 + {}^{sw}\bar{\Phi}^2 \right) |\bar{\Upsilon}|^*. \end{aligned} \quad (38)$$

The QC like components of such quasiperiodic/ aperiodic gravitational structures are also characterized by respective effective free energy functionals (22), for ${}^{QC}\bar{q}$, and (24), for ψ encoding nontrivial vacuum structures with effective cosmological constant.

3.3.2 Nonlinear functionals for quasiperiodic quadratic generating functions

We shall be able to generate in explicit form solutions of modified Einstein equations for nonlinear functionals

$$\begin{aligned} \Phi^2(r, \theta, \varphi) &= {}^{qp}\Phi^2 \\ &= \Phi^2 \left[{}^P\bar{q}_0, {}^{QC}\bar{q}_0, \psi_0, {}^{sd}q, {}^{sw}\bar{q}_0 \right] \text{ and } \end{aligned} \quad (39)$$

$$\begin{aligned} {}^{qp}\bar{\Phi}^2(x^i, t) &= {}^{qp}\bar{\Phi}^2 \\ &= \bar{\Phi}^2 \left[{}^P\bar{q}, {}^{QC}\bar{q}, \psi, {}^{sd}q_0, {}^{sw}\bar{q} \right]. \end{aligned} \quad (40)$$

Respective nonlinear symmetries (A.17) and/or (A.32) involve correspondingly an effective cosmological constant, Λ , or $\bar{\Lambda}$, and nonholonomic constraints for v-sources, $\Upsilon(r, \theta, \varphi)$, or $\bar{\Upsilon}(x^i, t)$,

$$\Lambda \quad {}^{qp}\Psi^2 = {}^{qp}\Phi^2 |\Upsilon| - \int d\varphi \quad {}^{qp}\Phi^2 |\Upsilon|^\diamond \text{ and/or} \quad (41)$$

$$\bar{\Lambda} \quad {}^{qp}\bar{\Psi}^2 = {}^{qp}\bar{\Phi}^2 |\bar{\Upsilon}| - \int dt \quad {}^{qp}\bar{\Phi}^2 |\bar{\Upsilon}|^*, \quad (42)$$

resulting in functional dependencies of effective cosmological constants.

The formulas (39)–(42) can be re-parameterized respectively as (35)–(38) when the values of effective cosmological constant and matter sources are prescribed to be compatible with experimental data. For such configurations, we can consider structures described additionally by a free energy functional (22), for ${}^{QC}\bar{q}$, modeling QC like gravitational nonholonomic deformations.

4 (Non)stationary black hole deformations and quasiperiodic structures

Generic off-diagonal nonholonomic deformations of BH like solutions were constructed in [2, 3, 8, 9, 11–13, 23, 25] by

applying the AFDM in various theories of (non) commutative, generalized Finsler, supergravity and superstring MGTs and GR. For small parametric decompositions, such solutions define black ellipsoid stationary configurations, deformations of BH horizons and locally anisotropic polarizations of physical constants, deformations of vacuum solutions into nonvacuum ones. Different classes of solutions were generated with nontrivial (non)commutative backgrounds, containing solitonic distributions and/or describing propagation of black holes in extra dimensions, geometric flows of black holes etc. The approach was generalized for various quasiperiodic, quasicrystal and other type aperiodic solutions in (super) string gravity and for nonholonomic Ricci soliton configurations [4, 5].

The goal of this section is to construct and study physical implications of (non) stationary generic off-diagonal solutions describing deformations of some prime BH solutions by quasiperiodic / aperiodic structures. The necessary geometric formalism is summarized in Table 2 and Appendices A.2.1 and A.3.1. In this work, the solutions are generated in general forms (not depending on small parameters, see footnote 5).

4.1 Nonlinear PDEs for quasiperiodic/aperiodic stationary configurations

There are two possibilities to transform the (modified) Einstein equations (14) into systems of nonlinear PDEs (A.7)–(A.10) with quasiperiodic solutions. In the first case, one considers quasiperiodic sources determined by some additive or general nonlinear functionals. In the second case, additive/general nonlinear functionals on quasiperiodic solutions are prescribed for generating functions. It is possible also to construct certain classes of solutions involving nonlinear functionals both for generating functions and (effective) sources.

4.1.1 Gravitational eqs and nonsingular solutions for stationary quasiperiodic sources

Stationary solutions with additive sources:

Considering a source of type (27) (the left label as is used for “additive stationary”), when

$$\begin{aligned} {}^P_0\Upsilon &:= {}^P\Upsilon[{}^P\bar{q}_0], \quad {}^{QC}_0\Upsilon := {}^{QC}\Upsilon[{}^{QC}\bar{q}_0], \quad {}^\psi_0\Upsilon \\ &:= {}^\psi\Upsilon[\psi_0], \quad {}^{sd}_0\Upsilon := {}^{sd}\Upsilon[{}^{sd}q], \quad {}^{sw}_0\Upsilon := {}^{sw}\Upsilon[{}^{sw}\bar{q}_0], \\ {}^{as}\Upsilon &= {}^P_0\Upsilon + {}^{QC}_0\Upsilon + {}^\psi_0\Upsilon + {}^{sd}_0\Upsilon + {}^{sw}_0\Upsilon, \end{aligned} \quad (43)$$

the equation (A.14) transforms into $\varpi^\diamond h_4^\diamond = 2h_3h_4 \quad {}^{as}\Upsilon$. This equation can be integrated on $y^3 = \varphi$.

Exact solutions for stationary configurations of the systems of nonlinear PDEs (A.13)–(A.16) can be constructed following the procedure summarized in Table 2. We can generate such off-diagonal metrics and generalized connections

in general form for a generating function $h_3(r, \theta, \varphi)$ with Killing symmetry on ∂_t determined by sources $(h_3\Upsilon, {}^{as}\Upsilon)$ and effective cosmological constant ${}^{as}\Lambda := {}^P_0\Lambda + {}^{QC}_0\Lambda + {}^\psi_0\Lambda + {}^{sd}_0\Lambda + {}^{sw}_0\Lambda$ related to ${}^{as}\Upsilon$ via nonlinear symmetry transforms (29). The corresponding class of quadratic elements defining stationary solutions can be written in the form

$$ds^2 = e^{\psi(x^k)} \left[(dx^1)^2 + (dx^2)^2 \right] + h_3 \left[dy^3 + \frac{\partial_i (|h_3^{[0]} - h_3| |{}^{as}\Upsilon| - \int d\varphi |h_3^{[0]} - h_3| |{}^{as}\Upsilon|^\diamond)}{\partial_\varphi (|h_3^{[0]} - h_3| |{}^{as}\Upsilon| - \int d\varphi |h_3^{[0]} - h_3| |{}^{as}\Upsilon|^\diamond)} dx^i \right] - \frac{|h_3^{[0]} - h_3|^\diamond}{|{}^{as}\Upsilon| h_3} \times \left[dt + \left({}_1n_k + 4 {}_2n_k \int d\varphi \frac{[\sqrt{|{}^{as}\Lambda| h_3^{[0]} - h_3|}]^\diamond}{|{}^{as}\Upsilon| |{}^{as}\Lambda| h_3^{5/2}} \right) dx^k \right]. \quad (44)$$

LC-configurations in GR determined by quasisperiodic sources can be extracted for additional zero torsion constraints resulting in a more special class of “integrable” generating functions $(\check{h}_3, \text{ and } \check{\Psi}(r, \theta, \varphi) \text{ and/or } \check{\Phi}(r, \theta, \varphi))$ for respective sources ${}^{as}\check{\Upsilon}$ and ${}^{as}\check{\Lambda}$ (A.20),

$$ds^2 = e^{\psi(x^k)} \left[(dx^1)^2 + (dx^2)^2 \right] + \check{h}_3 \left[d\varphi + (\partial_i \check{A}) dx^i \right] - \frac{|h_3^{[0]} - \check{h}_3|^\diamond}{|{}^{as}\check{\Upsilon}| \check{h}_3} \left[dt + (\partial_k n) dx^k \right]. \quad (45)$$

Above classes of solutions define stationary off-diagonal gravitational solutions generated by quasiperiodic/apperiodic additive sources ${}^{as}\Upsilon$. The term ${}^\psi_0\Upsilon$ can be used for standard and/or dark matter fields but other ones $({}^P_0\Upsilon, {}^{QC}_0\Upsilon, {}^{sd}_0\Upsilon, {}^{sw}_0\Upsilon)$ may model dark matter stationary distributions with respective quasiperiodic/apperiodic/solitonic configurations. Considering smooth classes of generating/integration functions and sources, we can construct various classes of non-singular exact solutions. Applying similar methods, we can generate, for instance, generalizations of stationary models with nonlinear diffusion, fractional, self-organizing and other type processes Refs. [21, 23].

Stationary solutions with nonlinear functional sources:

We can work with general nonlinear quasiperiodic/apperiodic/soliton functionals for effective source of type ${}^{qp}\Upsilon(r, \theta, \varphi) = {}^{qp}\Upsilon[{}^P\bar{q}_0, {}^{QC}\bar{q}_0, \psi_0, {}^{sd}q, {}^{sw}\bar{q}_0]$ (31) with nonlinear symmetries (33). Stationary solutions of the nonlinear system PDEs (A.13)–(A.16) can be written as in Table 2,

$$ds^2 = e^{\psi(x^k)} \left[(dx^1)^2 + (dx^2)^2 \right] + h_3 \left[dy^3 + \frac{\partial_i (|h_3^{[0]} - h_3| |{}^{qp}\Upsilon| - \int d\varphi |h_3^{[0]} - h_3| |{}^{qp}\Upsilon|^\diamond)}{\partial_\varphi (|h_3^{[0]} - h_3| |{}^{qp}\Upsilon| - \int d\varphi |h_3^{[0]} - h_3| |{}^{qp}\Upsilon|^\diamond)} dx^i \right] - \frac{|h_3^{[0]} - h_3|^\diamond}{|{}^{qp}\Upsilon| h_3} \times \left[dt + \left({}_1n_k + 4 {}_2n_k \int d\varphi \frac{[\sqrt{|{}^{as}\Lambda| h_3^{[0]} - h_3|}]^\diamond}{|{}^{as}\Upsilon| |{}^{as}\Lambda| h_3^{5/2}} \right) dx^k \right]. \quad (46)$$

$$\times \left[dt + \left({}_1n_k + 4 {}_2n_k \int d\varphi \frac{[\sqrt{|{}^{qp}\Lambda| h_3^{[0]} - h_3|}]^\diamond}{|{}^{qp}\Upsilon| |{}^{qp}\Lambda| h_3^{5/2}} \right) dx^k \right]. \quad (46)$$

This formula is similar to (44) but with another type of nonlinear generation functions for (effective) sources for dark and/or usual matter sources, when ${}^{as}\Upsilon \rightarrow {}^{qp}\Upsilon$ and ${}^{as}\Lambda \rightarrow {}^{qp}\Lambda$. Similar re-definitions of additive sources and cosmological constants in (45) into nonlinear functionals generate nonlinear LC-configurations

$$ds^2 = e^{\psi(x^k)} \left[(dx^1)^2 + (dx^2)^2 \right] + \check{h}_3 \left[d\varphi + (\partial_i \check{A}) dx^i \right] - \frac{|h_3^{[0]} - \check{h}_3|^\diamond}{|{}^{qp}\check{\Upsilon}| \check{h}_3} \left[dt + (\partial_k n) dx^k \right]. \quad (47)$$

We note that formulas (46) and (47) provide respective generalizations of some classes of solutions (44) and (45) considering general “functionals of functionals” with quasiperiodic (effective) real and dark matter structures which can be organized by corresponding parameters in certain forms with cosmic webs, filaments, quasiperiodic/apperiodic and/or solitonic distributions etc.

4.1.2 Nonsingular solutions for stationary quasiperiodic generating functions

Stationary solutions with additive generating functions:

For this class of solutions, the quasiperiodic/apperiodic/solitonic structure is stated via generating functions (35) for nonlinear gravitational field interactions without similar prescriptions for (effective) sources as in the previous section. We write in brief

$$\begin{aligned} {}^P_0\Phi^2 &:= {}^P\Phi^2[{}^P\bar{q}_0], \quad {}^{QC}_0\Phi^2 := {}^{QC}\Phi^2[{}^{QC}\bar{q}_0], \\ {}^\psi_0\Phi^2 &:= {}^\psi\Phi^2[\psi_0], \quad {}^{sd}_0\Phi^2 := {}^{sd}\Phi^2[{}^{sd}q], \quad {}^{sw}_0\Phi^2 := {}^{sw}\Phi^2[{}^{sw}\bar{q}_0], \\ {}^a\Phi^2 &:= {}^P_0\Phi^2 + {}^{QC}_0\Phi^2 + {}^\psi_0\Phi^2 + {}^{sd}_0\Phi^2 + {}^{sw}_0\Phi^2 \end{aligned} \quad (48)$$

for additive generating functions subjected to nonlinear symmetries (37).

The equation (A.14) transforms into a functional equation $\varpi^\diamond[{}^a\Phi, \Lambda] h_4^\diamond[{}^a\Phi, \Lambda] = 2h_3[{}^a\Phi, \Lambda] h_4[{}^a\Phi, \Lambda] \Upsilon$, which can be written in terms of functionals of type $h_{3,4}[{}^a\Upsilon, \Lambda]$, and respective nonlinear functionals for coefficients in (A.15) and (A.16). The solutions of such equations can be parameterized in the form (see the third parametrization in (A.19))

$$ds^2 = e^{\psi(x^k)} \left[(dx^1)^2 + (dx^2)^2 \right] + \left(h_3^{[0]}(x^k) - \frac{{}^a\Phi^2}{4\Lambda} \right) \times \left[dy^3 + \frac{\partial_i ({}^a\Phi^2|\Upsilon| - \int dy^3 {}^a\Phi^2|\Upsilon|^\diamond)}{({}^a\Phi^2|\Upsilon| - \int dy^3 {}^a\Phi^2|\Upsilon|^\diamond)^\diamond} dx^i \right]$$

$$\begin{aligned}
& - \frac{\partial_\varphi({}^a\Phi^2)}{\Upsilon(4\Lambda h_3^{[0]} - {}^a\Phi^2)} \left[dt + \left(n_k \right. \right. \\
& \left. \left. + 2n_k \int dy^3 \frac{({}^a\Phi^\diamond)^2}{\Upsilon\Lambda} \left| \frac{\Upsilon(4\Lambda h_3^{[0]} - {}^a\Phi^2)}{({}^a\Phi^2)^\diamond} \right|^{5/2} \right) dx^k \right], \quad (49)
\end{aligned}$$

For LC-configurations, we obtain (see the third parametrization in (A.20))

$$\begin{aligned}
ds^2 = e^{\psi(x^k)} & \left[(dx^1)^2 + (dx^2)^2 \right] \\
& + \left(h_3^{[0]} - \frac{{}^a\check{\Phi}^2}{4\Lambda} \right) \times \left[dy^3 + (\partial_i {}^a\check{A}) dx^i \right] \\
& - \frac{({}^a\check{\Phi}^2)^\diamond}{\check{\Upsilon}(4\Lambda h_3^{[0]} - {}^a\check{\Phi}^2)} \left[dt + (\partial_k n) dx^k \right]. \quad (50)
\end{aligned}$$

We emphasize that the (effective) source Υ in formulas (49) and (50) is not obligatory quasiperiodic. Such a source is considered for general matter fields (including both types of standard and dark matter) and stationary distributions described by an effective cosmological constant Λ . In another turn, the gravitational field distributions are with quasiperiodic/apperiodic/solitonic structure. For such classes of solutions, the gravitational fields encode certain dark energy nonlinear distributions with rich nonholonomic geometric structure and generalized nonlinear symmetries. The coefficients of this class of d-metrics can be chosen to be of necessary smooth class (for instance, nonsingular ones).

Stationary solutions with nonlinear quasiperiodic functionals for generating functions:

Above classes of generic off-diagonal solutions can be generalized for nonlinear quasiperiodic generating functionals ${}^{qp}\Phi^2[{}^P\bar{q}_0, {}^{Qc}\bar{q}_0, \psi_0, {}^{sd}q, {}^{sw}\bar{q}_0]$ (39) characterized by nonlinear symmetries of type (41). The equation (A.14) transforms into a functional equation $\varpi^\diamond[{}^{qp}\Phi, \Lambda] h_4^\diamond[{}^{qp}\Phi, \Lambda] = 2h_3[{}^{qp}\Phi, \Lambda] h_4[{}^{qp}\Phi, \Lambda] \Upsilon$, which can be solved together with other equations form the system (A.13)–(A.16) following geometric methods summarized in Table 2.

The solutions for such stationary configurations with general nonlinear functionals for generating functions can be written the form (A.19) (for simplicity, we consider only the third type parametrization)

$$\begin{aligned}
ds^2 = e^{\psi(x^k)} & \left[(dx^1)^2 + (dx^2)^2 \right] + \left(h_3^{[0]}(x^k) - \frac{{}^{qp}\Phi^2}{4\Lambda} \right) \\
& \times \left[dy^3 + \frac{\partial_i ({}^{qp}\Phi^2|\Upsilon| - \int d\varphi {}^{qp}\Phi^2|\Upsilon|^\diamond)}{(\Phi^2|\Upsilon| - \int d\varphi \Phi^2|\Upsilon|^\diamond)} dx^i \right] \\
& - \frac{\partial_\varphi({}^{qp}\Phi^2)}{\Upsilon(4\Lambda h_3^{[0]} - {}^{qp}\Phi^2)}
\end{aligned}$$

$$\begin{aligned}
& \times \left[dt + \left(n_k \right. \right. \\
& \left. \left. + 2n_k \int d\varphi \frac{({}^{qp}\Phi^\diamond)^2}{\Upsilon\Lambda} \left| \frac{\Upsilon(4\Lambda h_3^{[0]} - {}^{qp}\Phi^2)}{({}^{qp}\Phi^2)^\diamond} \right|^{5/2} \right) dx^k \right]. \quad (51)
\end{aligned}$$

We can impose additional zero torsion constraints and extract LC-configurations as in (A.20),

$$\begin{aligned}
ds^2 = e^{\psi(x^k)} & \left[(dx^1)^2 + (dx^2)^2 \right] \\
& + \left(h_3^{[0]} - \frac{{}^{qp}\check{\Phi}^2}{4\Lambda} \right) \left[d\varphi + (\partial_i {}^{qp}\check{A}) dx^i \right] \\
& - \frac{({}^{qp}\check{\Phi}^2)^\diamond}{\check{\Upsilon}(4\Lambda h_3^{[0]} - {}^{qp}\check{\Phi}^2)} \left[dt + (\partial_k n) dx^k \right]. \quad (52)
\end{aligned}$$

Considering additional assumptions and approximations for additive functionals, the formulas (51) and (52) transform respectively into (49) and (50). For such classes of solutions, the gravitational fields encode certain dark energy nonlinear distributions with a more rich nonholonomic geometric structure and generalized nonlinear symmetries when quasiperiodicity is induced from “quasiperiodicity” of matter fields. The coefficients of this class of d-metrics can be chosen to be of necessary smooth class (for instance, nonsingular ones) but can involve certain stochastic sources and fractional derivative processes.

4.1.3 Stationary solutions from nonlinear functionals for quasiperiodic coefficients and sources

In a more general context, we can generate nonsingular stationary off-diagonal generalized quasiperiodic solutions of the (modified) Einstein equations determined both by nonlinear functionals for generating functions, ${}^{qp}\Phi$ (39) and nonlinear functionals for (effective) sources, ${}^{qp}\Upsilon$ (31). The quasiperiodic data (for instance, scales, interaction constants and associated free energies) for the generating functions are different from the quasiperiodic data for sources. Nevertheless, such data can not be arbitrary independent ones by subjected to nonlinear symmetries generalizing (41) and (33), ${}^{qp}\Lambda \ {}^{qp}\Psi^2 = {}^{qp}\Phi^2 |{}^{qp}\Upsilon| - \int d\varphi {}^{qp}\Phi^2 |{}^{qp}\Upsilon|^\diamond$. For additive functionals both in the gravitational and (effective sources), such a nonlinear symmetry transforms into ${}^a\Lambda \ {}^a\Psi^2 = {}^a\Phi^2 |{}^a\Upsilon| - \int d\varphi {}^a\Phi^2 |{}^a\Upsilon|^\diamond$, which is a generalization of (37) and (29).

Following again the procedure summarized in Table 2 but for the data $({}^{qp}\Psi, {}^{qp}\Upsilon)$, and/or, equivalently, $({}^{qp}\Phi, {}^{qp}\Lambda)$, the general multi-functional nonlinear generalization of stationary solutions (46) and (51) are constructed in the form

$$\begin{aligned}
ds^2 = & e^{\psi(x^k)} \left[(dx^1)^2 + (dx^2)^2 \right] \\
& + \left(h_3^{[0]}(x^k) - \frac{q^p \Phi^2}{4 q^p \Lambda} \right) \\
& \times \left[dy^3 + \frac{\partial_i (q^p \Phi^2 |q^p \Upsilon| - \int d\varphi q^p \Phi^2 |q^p \Upsilon|^\diamond)}{(q^p \Phi^2 |q^p \Upsilon| - \int d\varphi q^p \Phi^2 |q^p \Upsilon|^\diamond)} dx^i \right] \\
& - \frac{\partial_\varphi (q^p \Phi^2)}{q^p \Upsilon (4 q^p \Lambda h_3^{[0]} - q^p \Phi^2)} \\
& \times \left[dt + \left(n_k \right. \right. \\
& \left. \left. + 2n_k \int d\varphi \frac{(q^p \Phi^\diamond)^2}{q^p \Upsilon q^p \Lambda} \left| \frac{q^p \Upsilon (4 q^p \Lambda h_3^{[0]} - q^p \Phi^2)}{(q^p \Phi^2)^\diamond} \right|^{5/2} \right) dx^k \right]. \quad (53)
\end{aligned}$$

For LC-configurations, we obtain multi-functional nonlinear generalizations of (47) and (52), for stationary solutions in GR,

$$\begin{aligned}
ds^2 = & e^{\psi(x^k)} \left[(dx^1)^2 + (dx^2)^2 \right] \\
& + \left(h_3^{[0]} - \frac{q^p \check{\Phi}^2}{4 q^p \Lambda} \right) \left[d\varphi + (\partial_i q^p \check{A}) dx^i \right] \\
& - \frac{(q^p \check{\Phi}^2)^\diamond}{q^p \check{\Upsilon} (4 q^p \Lambda h_3^{[0]} - q^p \check{\Phi}^2)} \left[dt + (\partial_k n) dx^k \right]. \quad (54)
\end{aligned}$$

The class of solutions (53) describes off-diagonal stationary configurations determined by multi-functional nonlinear quasiperiodic structures both for the dark energy (nonlinear gravitational distributions) and for the dark (and standard) matter fields. In explicit form, such data can be stated to be compatible with observations in modern astrophysics and cosmology.

4.2 BHs in (off-) diagonal quasiperiodic media

Various classes of generic off-diagonal stationary quasiperiodic solutions can be described in terms of η -polarization functions as in Appendix A.3.1 and following the geometric method summarized in Tables 1 and 2. In this section, we consider a primary BH d-metric $\check{\mathbf{g}}$ (8) defined by data $[\check{g}_i(r, \theta, \varphi), \check{g}_a = \check{h}_a(r, \theta, \varphi); \check{N}_k^3 = \check{w}_k(r, \theta, \varphi), \check{N}_k^4 = \check{n}_k(r, \theta, \varphi)]$ which can be diagonalized (for simplicity, we consider the Schwarzschild metric) by frame/ coordinate transforms. The stationary quasiperiodic solutions will be determined by target metrics $\widehat{\mathbf{g}}$ generated by nonholonomic deformations $\check{\mathbf{g}} \rightarrow \widehat{\mathbf{g}} = [g_i(x^k) = \eta_i \check{g}_i, g_b(x^k, y^3) = \eta_b \check{g}_b, N_i^a(x^k, y^3) = \eta_i^a \check{N}_i^a]$. The quadratic elements corresponding to by $\widehat{\mathbf{g}}$ are parameterized in some forms similar to (7),

$$\begin{aligned}
ds^2 = & \eta_i(r, \theta, \varphi) \check{g}_i(r, \theta, \varphi) \left[dx^i(r, \theta, \varphi) \right]^2 \\
& + \eta_a(r, \theta, \varphi) \check{g}_a(r, \theta, \varphi) [d\varphi \\
& + \eta_k^a(r, \theta, \varphi) \check{N}_k^a(r, \theta, \varphi) dx^k(r, \theta, \varphi)]^2, \quad (55)
\end{aligned}$$

with summation on repeating contracted low-up indices.

4.2.1 Singular solutions generated by stationary quasiperiodic sources

We consider quasiperiodic sources of type $q^p \Upsilon [{}^P \bar{q}_0, {}^Q \bar{q}_0, \psi_0, {}^{sd} q, {}^{sw} \bar{q}_0]$ as in (46) and compute the coefficients of (55) following formulas.

$$\begin{aligned}
\eta_i &= e^{\psi(x^k)} / \check{g}_i; \\
\eta_3 &= \eta_3(r, \theta, \varphi) \text{ as a generating function;} \\
\eta_4 &= - \frac{|h_3^{[0]} - \eta_3 \check{h}_3|^\diamond}{|q^p \Upsilon | \eta_3 \check{h}_3 \check{h}_4}; \\
\eta_i^3 &= \frac{\partial_i (|h_3^{[0]} - \eta_3 \check{h}_3| |q^p \Upsilon| - \int d\varphi |h_3^{[0]} - \eta_3 \check{h}_3| |q^p \Upsilon|^\diamond)}{\check{w}_i |h_3^{[0]} - \eta_3 \check{h}_3| |q^p \Upsilon|}; \\
\eta_k^4 &= \frac{1n_k}{\check{n}_k} + 4 \frac{2n_k}{\check{n}_k} \int d\varphi \frac{\left[\left(\sqrt{|q^p \Lambda [h_3^{[0]} - \eta_3 \check{h}_3]|} \right)^\diamond \right]^2}{q^p \Upsilon q^p \Lambda |\eta_3 \check{h}_3|^{5/2}}. \quad (56)
\end{aligned}$$

In these formulas, $\eta_3(r, \theta, \varphi)$ is taken as a (non) singular generating function. Other types of generating functions are determined with nonlinear symmetries (A.17) and functionals of $\eta_3(r, \theta, \varphi)$ and data for the prime d-metric,

$$\begin{aligned}
\Phi^2 &= 4 |q^p \Lambda [h_3^{[0]}(r, \theta, \varphi) - \eta_3(r, \theta, \varphi) \check{h}_3(r, \theta, \varphi)]|, \\
\Psi^2 &= 4 \left(|h_3^{[0]}(r, \theta, \varphi) - \eta_3(r, \theta, \varphi) \check{h}_3(r, \theta, \varphi)| |q^p \Upsilon(r, \theta, \varphi)| \right. \\
&\quad \left. - \int d\varphi |h_3^{[0]}(r, \theta, \varphi) - \eta_3(r, \theta, \varphi) \check{h}_3(r, \theta, \varphi)| |q^p \Upsilon(r, \theta, \varphi)|^\diamond \right).
\end{aligned}$$

We can constrain the coefficients (56) to a subclass of data generating target stationary off-diagonal metrics of type (A.20) with zero torsion,

$$\begin{aligned}
\eta_i &= e^{\psi(x^k)} / \check{g}_i; \eta_3 = \check{\eta}_3(r, \theta, \varphi) \text{ as a generating function;} \\
\eta_4 &= - \frac{|h_3^{[0]} - \check{\eta}_3 \check{h}_3|^\diamond}{|q^p \check{\Upsilon} | \check{\eta}_3 \check{h}_3 \check{h}_4}; \eta_i^3 = \partial_i \check{A} / \check{w}_k, \eta_k^4 = \frac{\partial_k n}{\check{n}_k}.
\end{aligned}$$

In above formulas, the nonlinear functionals for the quasiperiodic v-source and (effective) cosmological constant can be changed into additive functionals $q^p \Upsilon \rightarrow {}^{as} \Upsilon$ and $q^p \Lambda \rightarrow {}^{as} \Lambda$. The singular behaviour of such solutions is generated by some prime BH data. For certain classes of generating functions and sources, the same type of singularity is preserved. Such examples have been studied in some general forms for small parametric deformations. Similar stationary configurations can be computed for general quasiperiodic structures.

The constructions depend on the type of explicit model we construct (for instance, with certain web/filament/solitonic stationary distributions). We can treat such generic off-diagonal stationary solutions as certain conventional nonholonomically deformed BH configurations imbedded into certain (non) singular media determined by stationary dark and usual matter quasiperiodic distributions.

4.2.2 BH solutions deformed by stationary quasiperiodic generating functions

Let us consider another class of solutions (55) when the coefficients of the d-metrics are determined by nonlinear generating functionals ${}^{qp}\Phi^2[{}^P\bar{q}_0, {}^Q\bar{q}_0, \psi_0, {}^{sd}q, {}^{sw}\bar{q}_0]$ (39). Similarly, we can consider additive functionals ${}^a\Phi^2$ (48) for some prescribed data $\Upsilon(r, \theta, \varphi)$ and Λ . The formulas for nonlinear symmetries (41) allow us to find (recurrently) corresponding nonlinear functionals, ${}^{qp}\eta_3(r, \theta, \varphi)$, or additive functionals, ${}^a\eta_3(r, \theta, \varphi)$, and related polarization functions,

$$\begin{aligned} {}^{qp}\eta_3 &= h_3^{[0]}(r, \theta)/\dot{h}_3(r, \theta, \varphi) \mp {}^{qp}\Phi^2(r, \theta, \varphi)/4\Lambda\dot{h}_3(r, \theta, \varphi), \\ {}^{qp}\Psi^2 &= 4(|h_3^{[0]}(r, \theta) - {}^{qp}\eta_3(r, \theta, \varphi)\dot{h}_3(r, \theta, \varphi)| |\Upsilon(r, \theta, \varphi)| \\ &\quad - \int d\varphi |h_3^{[0]}(r, \theta, \varphi) \\ &\quad - {}^{qp}\eta_3(r, \theta, \varphi)\dot{h}_3(r, \theta, \varphi)| |\Upsilon(r, \theta, \varphi)|^\diamond). \end{aligned}$$

The coefficients of (55) are computed

$$\begin{aligned} \eta_i &= e^{\psi(x^k)}/\dot{g}_i; \eta_3 = {}^{qp}\eta_3(r, \theta, \varphi) \text{ as a generating function;} \\ \eta_4 &= -\frac{|h_3^{[0]} - {}^{qp}\eta_3\dot{h}_3|^\diamond}{|\Upsilon| {}^{qp}\eta_3\dot{h}_3\dot{h}_4}; \\ \eta_i^3 &= \frac{\partial_i \left(|h_3^{[0]} - {}^{qp}\eta_3\dot{h}_3| |\Upsilon| - \int dy^3 |h_3^{[0]} - {}^{qp}\eta_3\dot{h}_3| |\Upsilon|^\diamond \right)}{\dot{w}_i |h_3^{[0]} - {}^{qp}\eta_3\dot{h}_3|^\diamond |\Upsilon|}; \\ \eta_k^4 &= \frac{1}{\dot{n}_k} + 4 \frac{2n_k}{\dot{n}_k} \int dy^3 \frac{\left[\left(\sqrt{|\Lambda[h_3^{[0]} - {}^{qp}\eta_3\dot{h}_3]|} \right)^\diamond \right]^2}{\Upsilon\Lambda |{}^{qp}\eta_3\dot{h}_3|^{5/2}}. \end{aligned} \quad (57)$$

Target stationary off-diagonal metrics (A.20) with zero torsion can be generated by polarization functions subjected to additional integrability conditions,

$$\begin{aligned} \eta_i &= e^{\psi(x^k)}/\dot{g}_i; \eta_3 = \check{\eta}_3(r, \theta, \varphi) \text{ as a generating function;} \\ \eta_4 &= -\frac{|h_3^{[0]} - {}^{qp}\check{\eta}_3\dot{h}_3|^\diamond}{|\check{\Upsilon}| {}^{qp}\check{\eta}_3\dot{h}_3\dot{h}_4}; \eta_i^3 = \partial_i {}^{qp}\check{A}/\dot{w}_k, \eta_k^4 = \frac{\partial_k n}{\dot{n}_k}. \end{aligned}$$

The solutions determined in this section describe certain nonholonomically deformed BH configurations self-consistently imbedded into a quasiperiodic gravitational (dark energy) media.

4.2.3 Stationary BH deformations by quasiperiodic sources and generating functions

The most general class of nonholonomic stationary quasiperiodic deformations of BHs is determined by corresponding nonlinear quasiperiodic functionals both for the generating functions and (effective) sources. Nonlinear superpositions of solutions of type (55) and (57) are determined by coefficients of (55) computed

$$\begin{aligned} \eta_i &= e^{\psi(x^k)}/\dot{g}_i; \eta_3 = {}^{qp}\eta_3(r, \theta, \varphi) \text{ as a generating function;} \\ \eta_4 &= -\frac{|h_3^{[0]} - {}^{qp}\eta_3\dot{h}_3|^\diamond}{|\Upsilon| {}^{qp}\eta_3\dot{h}_3\dot{h}_4}; \\ \eta_i^3 &= \frac{\partial_i \left(|h_3^{[0]} - {}^{qp}\eta_3\dot{h}_3| |\Upsilon| - \int dy^3 |h_3^{[0]} - {}^{qp}\eta_3\dot{h}_3| |\Upsilon|^\diamond \right)}{\dot{w}_i |h_3^{[0]} - {}^{qp}\eta_3\dot{h}_3|^\diamond |\Upsilon|}; \\ \eta_k^4 &= \frac{1}{\dot{n}_k} + 4 \frac{2n_k}{\dot{n}_k} \int dy^3 \frac{\left[\left(\sqrt{|{}^{qp}\Lambda[h_3^{[0]} - {}^{qp}\eta_3\dot{h}_3]|} \right)^\diamond \right]^2}{{}^{qp}\Upsilon {}^{qp}\Lambda |{}^{qp}\eta_3\dot{h}_3|^{5/2}}. \end{aligned} \quad (58)$$

In such formulas, there are considered nonlinear generating functionals ${}^{qp}\Phi^2[{}^P\bar{q}_0, {}^Q\bar{q}_0, \psi_0, {}^{sd}q, {}^{sw}\bar{q}_0]$ (39). Similarly, we can consider additive functionals ${}^a\Phi^2$ (48) for some prescribed nonlinear functionals ${}^{qp}\Upsilon(r, \theta, \varphi)$ and ${}^{qp}\Lambda$ (in particular, additive nonlinear functionals, ${}^{as}\Upsilon$ (43) and ${}^{as}\Lambda$ can be taken). All such data are related via nonlinear symmetries generalizing (41) which allows to find (recurrently) corresponding nonlinear functionals, ${}^{qp}\eta_3(r, \theta, \varphi)$, or additive functionals, ${}^a\eta_3(r, \theta, \varphi)$, for the polarization function,

$$\begin{aligned} {}^{qp}\eta_3 &= h_3^{[0]}(r, \theta)/\dot{h}_3(r, \theta, \varphi) \mp {}^{qp}\Phi^2(r, \theta, \varphi)/4 {}^{qp}\Lambda\dot{h}_3(r, \theta, \varphi), \\ {}^{qp}\Psi^2 &= 4(|h_3^{[0]}(r, \theta) \\ &\quad - {}^{qp}\eta_3(r, \theta, \varphi)\dot{h}_3(r, \theta, \varphi)| |\Upsilon(r, \theta, \varphi)| \\ &\quad - \int d\varphi |h_3^{[0]}(r, \theta) - {}^{qp}\eta_3(r, \theta, \varphi)\dot{h}_3(r, \theta, \varphi)| |{}^{qp}\Upsilon(r, \theta, \varphi)|^\diamond). \end{aligned}$$

Imposing additional conditions for zero torsion, target stationary off-diagonal metrics (A.20) are generated

$$\begin{aligned} \eta_i &= e^{\psi(x^k)}/\dot{g}_i; \eta_3 = \check{\eta}_3(r, \theta, \varphi) \text{ as a generating function;} \\ \eta_4 &= -\frac{|h_3^{[0]} - {}^{qp}\check{\eta}_3\dot{h}_3|^\diamond}{|\check{\Upsilon}| {}^{qp}\check{\eta}_3\dot{h}_3\dot{h}_4}; \eta_i^3 = \partial_i {}^{qp}\check{A}/\dot{w}_k, \eta_k^4 = \frac{\partial_k n}{\dot{n}_k}. \end{aligned}$$

Finally, we conclude that the class of stationary solutions of type (58) describes nonholonomic deformations of a BH self-consistently imbedded into quasiperiodic gravitational (dark energy) backgrounds and quasiperiodic dark/ standard matter. Such distributions can be with a very rich geometric structure (in general, with singular, or nonsingular configurations) and described by respective nonlinear symmetries.

4.3 Nonstationary deformations of BH metrics into quasiperiodic cosmological solutions

Prime BH metrics can be deformed nonholonomically into certain classes of exact quasiperiodic solutions in MGTs and GR depending in explicit form on a time like coordinate t . For such configurations, nonlinear quasiperiodic interactions can transform black hole spacetimes into locally anisotropic cosmological ones. In order to construct such exact solutions, we consider a primary BH d-metric $\hat{\mathbf{g}}$ (8) which via coordinate transforms is parameterized by data $[\hat{g}_i(x^k), \hat{g}_a = \hat{h}_a(x^k); \hat{N}_k^3 = \hat{w}_k(x^i), \hat{N}_k^4 = \hat{n}_k(x^i)]$. The non-stationary quasiperiodic solutions will be determined by target metrics $\hat{\mathbf{g}}$ generated by nonholonomic deformations

$$\hat{\mathbf{g}} \rightarrow \hat{\mathbf{g}} = \left[g_i(x^k) = \eta_i \hat{g}_i, g_b(x^k, y^4 = t) = \bar{\eta}_b \hat{g}_b, N_i^a(x^k, y^4 = t) = \bar{\eta}_i^a \hat{N}_i^a \right].$$

The quadratic elements generated by such $\hat{\mathbf{g}}$ are parameterized,

$$ds^2 = \eta_i(x^k) \hat{g}_i(x^k) [dx^i]^2 + \bar{\eta}_a(x^k, t) \hat{g}_a(x^i) \times \left[dt + \bar{\eta}_k^a(x^i, t) \hat{N}_k^a(x^i) dx^k \right]^2, \quad (59)$$

with summation on low-up indices.

The quasiperiodic sources are given by nonlinear functionals for effective sources,

$${}^{qp}\bar{\Upsilon}(x^i, t) = {}^{qp}\bar{\Upsilon}[{}^P\bar{\Upsilon}[{}^P\bar{q}(x^i, t)], {}^{QC}\bar{\Upsilon}[{}^{QC}\bar{q}(x^i, t)], \\ \psi \bar{\Upsilon}[\psi(x^i, t)], {}^{sd}\bar{\Upsilon}[{}^{sd}q(x^i)], {}^{sw}\bar{\Upsilon}[{}^{sw}\bar{q}(x^i, t)]],$$

and nonlinear functionals on cosmological constants ${}^{qp}\bar{\Lambda} = \bar{\Lambda}[{}^P\Lambda, {}^{QC}\Lambda, \psi\Lambda, {}^{sd}\Lambda, {}^{sw}\Lambda]$. In general, we can consider some nonlinear functionals for generating functions ${}^{qp}\bar{\Phi}^2[{}^P\bar{q}(x^i, t), {}^{QC}\bar{q}(x^i, t), \psi(x^i, t), {}^{sd}q(x^i), {}^{sw}\bar{q}(x^i, t)]$, when the nonlinear symmetries are stated by formulas

$${}^{qp}\bar{\eta}_4 = \bar{h}_4^{[0]}(x^i)/\bar{h}_4(x^i) \mp {}^{qp}\bar{\Phi}^2(x^i, t)/4 {}^{qp}\bar{\Lambda}\bar{h}_4(x^i, t), \\ {}^{qp}\bar{\Psi}^2 = 4 \left(|\bar{h}_4^{[0]}(x^i) - {}^{qp}\bar{\eta}_4(x^i, t)\bar{h}_4(x^i)| |{}^{qp}\bar{\Upsilon}(x^i, t)| \right. \\ \left. - \int dt |\bar{h}_4^{[0]}(x^i) - {}^{qp}\bar{\eta}_4(x^i, t)\bar{h}_4(x^i, t)| |{}^{qp}\bar{\Upsilon}(x^i, t)|^* \right).$$

The equation (A.14) transforms into $\bar{\omega}^* \bar{h}_3^* = 2h_3h_4 {}^{qp}\bar{\Upsilon}$. This equation can be integrated on $y^4 = t$. Exact solutions for non-stationary configurations of the systems of nonlinear PDEs (A.13)–(A.16) can be constructed following the procedure summarized as an interference in Tables 2 and 3 and Appendices A.2.2 and A.3.2. The respective coefficients for (59) are computed

$$\eta_i = e^{\psi(x^k)}/\hat{g}_i; \bar{\eta}_3 = - \frac{|\bar{h}_4^{[0]} - {}^{qp}\bar{\eta}_4\bar{h}_4|^*}{|{}^{qp}\bar{\Upsilon}|} {}^{qp}\bar{\Upsilon}$$

$\bar{\eta}_4\bar{h}_4\bar{h}_3; \bar{\eta}_4 = {}^{qp}\bar{\eta}_4(x^i, t)$ as a generating function;

$$\bar{\eta}_k^3 = \frac{1}{\bar{n}_k} + 4 \frac{2\bar{n}_k}{\bar{n}_k} \int dt \frac{[(\sqrt{|{}^{qp}\bar{\Lambda}[\bar{h}_4^{[0]} - {}^{qp}\bar{\eta}_4\bar{h}_4]|})^*]^2}{{}^{qp}\bar{\Upsilon} {}^{qp}\bar{\Lambda} |{}^{qp}\bar{\eta}_4\bar{h}_4|^{5/2}}; \\ \bar{\eta}_i^4 = \frac{\partial_i(|\bar{h}_4^{[0]} - {}^{qp}\bar{\eta}_4\bar{h}_4|) {}^{qp}\bar{\Upsilon} - \int dt |\bar{h}_4^{[0]} - {}^{qp}\bar{\eta}_4\bar{h}_4| |{}^{qp}\bar{\Upsilon}|^*}{\bar{w}_i|\bar{h}_4^{[0]} - {}^{qp}\bar{\eta}_4\bar{h}_4|^* |{}^{qp}\bar{\Upsilon}|}. \quad (60)$$

For additional zero torsion conditions, we generate target non-stationary off-diagonal metrics (A.35) defined by coefficients

$$\eta_i = e^{\psi(x^k)}/\hat{g}_i; \bar{\eta}_3 = - \frac{|\bar{h}_4^{[0]} - {}^{qp}\bar{\eta}_4\bar{h}_4|^*}{|{}^{qp}\bar{\Upsilon}|} {}^{qp}\bar{\Upsilon};$$

$\bar{\eta}_4 = \bar{\eta}_4(x^i, t)$ as a generating function;

$$\bar{\eta}_k^3 = \frac{\partial_k n}{\bar{n}_k}, \bar{\eta}_i^4 = \partial_i {}^{qp}\bar{A}/\bar{w}_k.$$

Non-stationary solutions generated from BH prime metrics (with zero or non-zero nonholonomically induced torsions) can be constructed in a similar manner for additive functionals for (effective) sources and generating functions which are similar to those for locally anisotropic spacetime. The integration functions and primary data are taken for BH stationary configurations.

Off-diagonal solutions of type (60) describe nonholonomic deformations of a BH solution by imbedding it self-consistently in a dynamical quasiperiodic gravitational (dark energy) background and quasiperiodic dark/ standard matter interacting fields. Such locally anisotropic cosmological like spacetimes contain BH like structures are described by relativistic evolution of geometric structure, singular, or non-singular, and described by respective nonlinear symmetries. For certain special parameterizations, we can model “dissipation” of singular BH structure into certain types (locally anisotropic) cosmological spacetimes.

5 Off-diagonal quasiperiodic cosmological spacetimes

Locally anisotropic and inhomogeneous cosmological solutions and accelerating universe scenarios (in MGTs, GR, and geometric flow theories) were studied in a series of works [1, 12, 24, 26, 50, 51], see also references therein. Various classes of generic off-diagonal cosmological metrics were constructed by applying the AFDM as a geometric alternative to numeric methods [19]. Recently, the approach was developed by constructing quasiperiodic cosmological solutions with small parametric deformations [6, 7].

The goal of this section is to study physical implications of (non) stationary generic off-diagonal solutions describing deformations of some prime cosmological spacetimes

by quasiperiodic/apperiodic/solitonic and/or pattern forming structures. The necessary geometric formalism is summarized in Table 3 and Appendices A.2.2 and A.3.2. We emphasize that in this work the cosmological solutions are constructed for general nonlinear or additive nonlinear functionals for generating functions and (effective) sources without additional assumptions on modelling small parameter configurations (see footnote 5).

5.1 Nonlinear PDEs for quasiperiodic/apperiodic cosmological configurations

There are two possibilities to transform the (modified) Einstein equations (14) into systems of nonlinear PDEs (A.21)–(A.24) with quasiperiodic solutions depending in explicit form on a time like variable. In the first case, one considers quasiperiodic sources determined by some additive or general nonlinear functionals. In the second case, respective nonlinear functionals determining quasiperiodic solutions are prescribed for generating functions. It is also possible to construct certain classes of locally anisotropic and inhomogeneous cosmological solutions by considering nonlinear/additive functionals both for generating functions and (effective) sources.

5.1.1 Cosmological solutions for quasiperiodic sources

Cosmological solutions generated by additive functionals for sources:

Let us consider an additive functional for a quasiperiodic source of type $\bar{\Upsilon}(x^i, t)$ (28),

$${}^{as}\bar{\Upsilon} = {}^P\bar{\Upsilon} + {}^Q\bar{\Upsilon} + {}^\psi\bar{\Upsilon} + {}^{sd}\bar{\Upsilon}[{}^{sd}q_0] + {}^{sw}\bar{\Upsilon}[{}^{sw}q], \quad (61)$$

for ${}^Q\bar{\Upsilon} = {}^Q\bar{\Upsilon}[{}^Q\bar{q}_0]$ and ${}^{sd}\bar{\Upsilon} = {}^{sd}\bar{\Upsilon}[{}^{sd}q_0]$, and an associated additive cosmological constant ${}^{as}\bar{\Lambda} = {}^P\bar{\Lambda} + {}^Q\bar{\Lambda} + {}^\psi\bar{\Lambda} + {}^{sd}\bar{\Lambda} + {}^{sw}\bar{\Lambda}$. Such values are related to different types of generating functions via nonlinear symmetries of type (30). The equation (A.29) transforms into $\bar{\omega}^* \bar{h}_3 = 2\bar{h}_3 \bar{h}_4 {}^{as}\bar{\Upsilon}$ which can be integrated on time like variable $y^4 = t$. The systems of nonlinear PDEs (A.28)–(A.31) can be integrated following the procedure summarized in Table 3. Such generic off-diagonal solutions are parameterized in the form

$$ds^2 = e^{\psi(x^k)} \left[(dx^1)^2 + (dx^2)^2 \right] - \frac{|h_4^{[0]} - \bar{h}_4|^*}{|{}^{as}\bar{\Upsilon} \bar{h}_4|} \times \left[dy^3 + \left({}_{1n_k} + 4 {}_{2n_k} \int dt \frac{\left[\left(\sqrt{|{}^{as}\bar{\Lambda} [h_4^{[0]} - \bar{h}_4]|} \right)^* \right)^2}{{}^{as}\bar{\Upsilon} {}^{as}\bar{\Lambda} |\bar{h}_4|^{5/2}} \right) dx^k \right]$$

$$+ \bar{h}_4 \left[dt + \frac{\partial_i \left(|h_4^{[0]} - \bar{h}_4| |{}^{as}\bar{\Upsilon}| - \int dt |h_4^{[0]} - \bar{h}_4| |{}^{as}\bar{\Upsilon}|^* \right)}{\left(|h_4^{[0]} - \bar{h}_4| |{}^{as}\bar{\Upsilon}| - \int dt |h_4^{[0]} - \bar{h}_4| |{}^{as}\bar{\Upsilon}|^* \right)^*} dx^i \right]. \quad (62)$$

For local pseudo-Riemannian configurations, we have to fix respective sign of the coefficient $\bar{h}_4(x^k, t)$ which can be considered as a generating function with Killing symmetry on ∂_3 determined by sources $({}_h\bar{\Upsilon}, {}^{as}\bar{\Upsilon})$. Such solutions are of type (A.34) and can be re-written equivalently with coefficients stated as functionals of ${}^{as}\bar{\Phi}$ and ${}^{as}\bar{\Psi}$ define by nonlinear symmetry formulas determined by additive (effective) sources and cosmological constants.

We can extract from (62) locally anisotropic generic off-diagonal cosmological LC-configurations in GR determined by quasiperiodic sources by imposing additional zero torsion constraints. This restricts the respective classes of “integrable” generating functions $(\bar{h}_4, \bar{\Psi}(x^i, t))$ and/or $(\bar{\Phi}(x^i, t))$ for sources ${}^{as}\bar{\Upsilon}$ and ${}^{as}\bar{\Lambda}$ (A.35),

$$ds^2 = e^{\psi(x^k)} [(dx^1)^2 + (dx^2)^2] - \frac{|h_4^{[0]} - \bar{h}_4|^*}{|{}^{as}\bar{\Upsilon} \bar{h}_4|} \times \left[dy^3 + (\partial_k \bar{n}) dx^k \right] + \bar{h}_4 [dt + (\partial_i \bar{A}) dx^i]. \quad (63)$$

Above linear quadratic elements define off-diagonal cosmological solutions generated by quasiperiodic additive sources ${}^{as}\bar{\Upsilon}$. The term ${}^\psi\bar{\Upsilon}$ encodes contributions of standard and/or dark matter fields. The sources $({}^P\bar{\Upsilon}, {}^Q\bar{\Upsilon}, {}^{sd}\bar{\Upsilon}, {}^{sw}\bar{\Upsilon})$ can be chosen to model dark matter stationary distributions with respective quasiperiodic/apperiodic/solitonic nonlinear wave interactions. Considering smooth classes of generating/integration functions and sources, we can generate nonsingular cosmological exact solutions. Applying similar methods, we can study, for instance, generalizations of cosmological models to effects determined by nonlinear diffusion, fractional, self-organizing and other type processes as we proved in Refs. [21, 23].

Cosmological solutions generated by nonlinear functionals for quasiperiodic sources:

Exact solutions can be generated by nonlinear quasiperiodic functionals for effective sources, ${}^{qp}\bar{\Upsilon}(x^i, t) = {}^{qp}\bar{\Upsilon}[{}^P\bar{\Upsilon}, {}^Q\bar{\Upsilon}, {}^\psi\bar{\Upsilon}, {}^{sd}\bar{\Upsilon}, {}^{sw}\bar{\Upsilon}]$ (32) with nonlinear symmetries (34). Using the AFDM summarized in Table 3, we construct cosmological solutions of the nonlinear system PDEs (A.28)–(A.31),

$$ds^2 = e^{\psi(x^k)} \left[(dx^1)^2 + (dx^2)^2 \right] - \frac{|h_4^{[0]} - \bar{h}_4|^*}{|{}^{qp}\bar{\Upsilon} \bar{h}_4|} \times \left[dy^3 + \left({}_{1n_k} + 4 {}_{2n_k} \int dt \frac{\left[\left(\sqrt{|{}^{qp}\bar{\Lambda} [h_4^{[0]} - \bar{h}_4]|} \right)^* \right)^2}{{}^{qp}\bar{\Upsilon} {}^{qp}\bar{\Lambda} |\bar{h}_4|^{5/2}} \right) dx^k \right]$$

$$+ \bar{h}_4 \left[dt + \frac{\partial_i (|h_4^{[0]} - \bar{h}_4|^{qp} \bar{\Upsilon}| - \int dt |h_4^{[0]} - \bar{h}_4|^{qp} \bar{\Upsilon}|^*)}{(|h_4^{[0]} - \bar{h}_4|^{qp} \bar{\Upsilon}| - \int dt |h_4^{[0]} - \bar{h}_4|^{qp} \bar{\Upsilon}|^*)^*} dx^i \right]. \quad (64)$$

This formula is similar to (44) but with another type of nonlinear generation functions for (effective) sources for dark and/or usual matter sources, when ${}^{as}\Upsilon \rightarrow {}^{qp}\Upsilon$ and ${}^{as}\Lambda \rightarrow {}^{qp}\Lambda$. Similar re-definitions of additive sources and cosmological constants in (45) into nonlinear functionals generate nonlinear LC-configurations

$$ds^2 = e^{\psi(x^k)} [(dx^1)^2 + (dx^2)^2] - \frac{|h_4^{[0]} - \bar{h}_4|^*}{|{}^{qp}\bar{\Upsilon}| \bar{h}_4} \left[dy^3 + (\partial_k \bar{n}) dx^k \right] + \bar{h}_4 [dt + (\partial_i \bar{A}) dx^i]. \quad (65)$$

For additive functionals for cosmological sources, the formulas (64) and (65) transforms respectively into quadratic linear elements (62) and (63). Considering small parametric deformations for 4-d cosmological solutions, we can reproduce the results from [6,7].

5.1.2 Cosmological solutions for nonstationary quasiperiodic generating functions

In this section, the sources are with arbitrary data $\bar{\Upsilon}^\mu_v = [{}_h\bar{\Upsilon}(x^k), \bar{\Upsilon}(x^k, t)]$ but the generating functions are considered for some additive or general nonlinear functionals with quasiperiodic structure.

Locally anisotropic and inhomogeneous cosmological metrics with additive generating functions:

For this class of solutions, the quasiperiodic/apperiodic/solitonic structure is stated via generating functions (36) for nonlinear quasiperiodic gravitational field interactions both without explicit prescriptions on any quasiperiodic structure for (effective) sources. In brief, such additive functionals are written

$${}^a\bar{\Phi}^2(x^i, t) = {}^P\bar{\Phi}^2[{}^P\bar{q}] + {}^{QC}\bar{\Phi}^2[{}^{QC}\bar{q}] + {}^\psi\bar{\Phi}^2[\psi] + {}^{sd}\bar{\Phi}^2[{}^{sd}q] + {}^{sw}\bar{\Phi}^2[{}^{sw}q]$$

with nonlinear symmetries (38), where ${}^{sd}\bar{\Phi}^2[{}^{sd}q]$ is taken for any functions ${}^{sd}q(x^k)$ but other components are considered as functionals on respective functions ${}^P\bar{q}$, ${}^{QC}\bar{q}$, ψ , and ${}^{sw}q$ on (x^i, t) . In result, the equation (A.29) transforms into a functional equation $\bar{\omega}^*[{}^a\bar{\Phi}, \bar{\Lambda}] \bar{h}_4^*[{}^a\bar{\Phi}, \bar{\Lambda}] = 2\bar{h}_3[{}^a\bar{\Phi}, \bar{\Lambda}] \bar{h}_4[{}^a\bar{\Phi}, \bar{\Lambda}] \bar{\Upsilon}$, which can be written equivalently in different forms with functionals of type $\bar{h}_{3,4}[{}^a\bar{\Psi}, \bar{\Lambda}]$ and respective nonlinear functionals for coefficients in (A.30) and

(A.31). The solutions of such equations can be parameterized in a form (A.34),

$$ds^2 = - \frac{({}^a\bar{\Phi}^2)^*}{\bar{\Upsilon}(4\bar{\Lambda}h_4^{[0]} - {}^{qp}\bar{\Phi}^2)} \times \left[dy^3 + \left(-1n_k + {}_2n_k \int dt \frac{({}^{qp}\bar{\Phi}^*)^2}{\bar{\Upsilon}\bar{\Lambda}} \left| \frac{\bar{\Upsilon}(4\bar{\Lambda}h_4^{[0]} - {}^{qp}\bar{\Phi}^2)}{({}^{qp}\bar{\Phi}^2)^*} \right|^{5/2} \right) dx^k \right] + \left(h_4^{[0]} - \frac{{}^a\bar{\Phi}^2}{4\bar{\Lambda}} \right) \times \left[dt + \frac{\partial_i ({}^a\bar{\Phi}^2|\bar{\Upsilon}| - \int dt {}^a\bar{\Phi}^2|\bar{\Upsilon}|^*)}{({}^a\bar{\Phi}^2|\bar{\Upsilon}| - \int dt {}^a\bar{\Phi}^2|\bar{\Upsilon}|^*)^*} dx^i \right] e^{\psi(x^k)} [(dx^1)^2 + (dx^2)^2]. \quad (66)$$

For LC-configurations, we obtain (see the third parametrization in (A.35))

$$ds^2 = e^{\psi(x^k)} [(dx^1)^2 + (dx^2)^2] - \frac{({}^a\bar{\Phi}^2)^*}{\bar{\Upsilon} \left(4\bar{\Lambda}h_4^{[0]} - {}^a\bar{\Phi}^2 \right)} \times \left[dy^3 + (\partial_k \bar{n}) dx^k \right] + \left(h_4^{[0]} - \frac{\bar{\Phi}^2}{4\bar{\Lambda}} \right) \left[dt + (\partial_i \bar{A}) dx^i \right]. \quad (67)$$

We can consider small parametric decompositions and frame/coordinate transforms in order to related such solutions to some well known (off) diagonal cosmological metrics.

Cosmological solutions with nonlinear quasiperiodic functionals for generating functions:

The formulas (66) and (67) can be generalized in order to generate solutions from nonlinear functionals for generating functions when nonlinear quasiperiodic generating functionals ${}^{qp}\bar{\Phi}^2[{}^P\bar{\Phi}, {}^{QC}\bar{\Phi}, \psi\bar{\Phi}, {}^{sd}\bar{\Phi}^2, {}^{sw}\bar{\Phi}^2]$ (40) characterized by nonlinear symmetries of type (42). The equation (A.29) transforms into a more general functional equation, $\bar{\omega}^*[{}^{qp}\bar{\Phi}, \bar{\Lambda}] \bar{h}_4^*[{}^{qp}\bar{\Phi}, \bar{\Lambda}] = 2\bar{h}_3[{}^{qp}\bar{\Phi}, \bar{\Lambda}] \bar{h}_4[{}^{qp}\bar{\Phi}, \bar{\Lambda}] \bar{\Upsilon}$. Such a nonlinear system of PDEs can be solved together with other equations form the system (A.28)–(A.31) following the steps summarized in Table 3. We obtain such solutions:

$$ds^2 = e^{\psi(x^k)} [(dx^1)^2 + (dx^2)^2] - \frac{({}^{qp}\bar{\Phi}^2)^*}{\bar{\Upsilon} (4\bar{\Lambda}h_4^{[0]} - {}^{qp}\bar{\Phi}^2)} \times \left[dy^3 + ({}_1n_k + {}_2n_k \int dt \frac{({}^{qp}\bar{\Phi}^*)^2}{\bar{\Upsilon}\bar{\Lambda}} \left| \frac{\bar{\Upsilon}(4\bar{\Lambda}h_4^{[0]} - {}^{qp}\bar{\Phi}^2)}{({}^{qp}\bar{\Phi}^2)^*} \right|^{5/2} \right) dx^k \right]$$

$$+ \left(h_4^{[0]} - \frac{qp\bar{\Phi}^2}{4\bar{\Lambda}} \right) \times \left[dt + \frac{\partial_i \left(qp\bar{\Phi}^2 |\bar{\Upsilon}| - \int dt qp\bar{\Phi}^2 |\bar{\Upsilon}|^* \right)}{\left(qp\bar{\Phi}^2 |\bar{\Upsilon}| - \int dt qp\bar{\Phi}^2 |\bar{\Upsilon}|^* \right)^*} dx^i \right]. \quad (68)$$

For zero torsion constraints in order to extract LC-configurations,

$$ds^2 = e^{\psi(x^k)} \left[(dx^1)^2 + (dx^2)^2 \right] - \frac{\left(qp\bar{\Phi}^2 \right)^*}{\bar{\Upsilon} \left(4\bar{\Lambda} h_4^{[0]} - qp\bar{\Phi}^2 \right)} \left[dy^3 + (\partial_k \bar{n}) dx^k \right] + \left(h_4^{[0]} - \frac{qp\bar{\Phi}^2}{4\bar{\Lambda}} \right) \left[dt + (\partial_i \bar{A}) dx^i \right]. \quad (69)$$

Here we emphasize that there is certain duality between formulas when $y^4 = t \leftrightarrow y^3$ and, respectively, “overlined” values are changed into “not overlined” ones, and inversely. For such nonholonomic dual transforms, the formulas (51) and (52) transform into corresponding (68) and (69) [inverse maps can be also considered]. The coefficients of this class of d-metrics can be chosen to be of necessary smooth class and involve certain stochastic sources and fractional derivative processes. Such nonholonomic deformation and generalized transform may change the topological spacetime structure and encode dark energy and dark matter effects.

5.1.3 Cosmology from nonstationary functionals for quasiperiodic coefficients and sources

Conventionally, all classes of considered above cosmological solutions can be formulated in terms of generalized quasiperiodic nonlinear functionals both for generating functions, $qp\bar{\Phi}$ (40) and nonlinear functionals for (effective) sources, $qp\bar{\Upsilon}$ (32). Such data are subjected to conditions of nonlinear symmetries generalizing (42) and (34), when $qp\bar{\Lambda} qp\bar{\Psi}^2 = qp\bar{\Phi}^2 qp\bar{\Upsilon}| - \int dt qp\bar{\Phi}^2 qp\bar{\Upsilon}|^*$. Similar nonlinear symmetries can be considered for additive functionals both for the gravitational fields and (effective) sources, when (38) and (30) transform into $a\bar{\Lambda} a\bar{\Psi}^2 = a\bar{\Phi}^2 a\bar{\Upsilon}| - \int dt a\bar{\Phi}^2 a\bar{\Upsilon}|^*$. The procedure summarized in Table 3 and generalized for the data $(qp\bar{\Psi}, qp\bar{\Upsilon})$, and/or, equivalently, $(qp\bar{\Phi}, qp\bar{\Lambda})$, allows us to construct multi-functional nonlinear quasiperiodic cosmological configurations,

$$ds^2 = - \frac{\left(qp\bar{\Phi}^2 \right)^*}{\bar{\Upsilon} \left(4\bar{\Lambda} h_4^{[0]} - qp\bar{\Phi}^2 \right)}$$

$$\times \left[dy^3 + (\partial_k \bar{n}) dx^k \right] + 2n_k \int dt \frac{\left(qp\bar{\Phi}^2 \right)^*}{qp\bar{\Upsilon} qp\bar{\Lambda}} \left| \frac{qp\bar{\Upsilon} \left(4qp\bar{\Lambda} h_4^{[0]} - qp\bar{\Phi}^2 \right)}{\left(qp\bar{\Phi}^2 \right)^*} \right|^{5/2} dx^k \left[\left(h_4^{[0]} - \frac{qp\bar{\Phi}^2}{4qp\bar{\Lambda}} \right) \times \left[dt + \frac{\partial_i \left(qp\bar{\Phi}^2 qp\bar{\Upsilon}| - \int dt qp\bar{\Phi}^2 qp\bar{\Upsilon}|^* \right)}{\left(qp\bar{\Phi}^2 qp\bar{\Upsilon}| - \int dt qp\bar{\Phi}^2 qp\bar{\Upsilon}|^* \right)^*} dx^i \right] \right] \times e^{\psi(x^k)} \left[(dx^1)^2 + (dx^2)^2 \right]. \quad (70)$$

For LC-configurations, we obtain multi-functional nonlinear generalizations of (68) and (69) which result in more general classes of locally anisotropic and inhomogeneous solutions in GR,

$$ds^2 = e^{\psi(x^k)} \left[(dx^1)^2 + (dx^2)^2 \right] - \frac{\left(qp\bar{\Phi}^2 \right)^*}{\bar{\Upsilon} \left(4\bar{\Lambda} h_4^{[0]} - qp\bar{\Phi}^2 \right)} \times \left[dy^3 + (\partial_k \bar{n}) dx^k \right] + \left(h_4^{[0]} - \frac{qp\bar{\Phi}^2}{4\bar{\Lambda}} \right) \left[dt + (\partial_i \bar{A}) dx^i \right]. \quad (71)$$

The classes (70), of cosmological solutions in MGTs, and (71), for cosmological solutions in GR, describe off-diagonal non-stationary configurations determined by multi-functional nonlinear quasiperiodic structures. Such rich geometric nonholonomically dynamical structures can be described both for the dark energy (nonlinear gravitational distributions) and for the dark (and standard) matter fields. In general, such cosmological solutions may not have smooth limits to well-known cosmological metric (for instance, FLRW, or any Bianchi type; we shall study such configurations in next sections). In explicit form, the geometric data for such generic off-diagonal cosmological solutions can be stated to explain various observations in modern cosmology.

5.2 Cosmological metrics evolving in (off-) diagonal quasiperiodic media

In this section, generic off-diagonal quasiperiodic cosmological solutions are constructed in terms of η -polarization functions as in Appendix A.3.2 and following the geometric method summarized in Tables 1 and 3. We consider a primary cosmological d-metric $\hat{\mathbf{g}}$ (8) defined by data $[\hat{g}_i(x^k, t), \hat{g}_a = \hat{h}_a(x^k, t); \hat{N}_k^3 = \hat{n}_k(x^i, t), \hat{N}_k^4 = \hat{w}_k(x^i, t)]$ which can be diagonalized for a FLRW cosmological metric (in general, we can consider off-diagonal Bianchi anisotropic metrics) by frame/ coordinate transforms. The cosmological quasiperiodic solutions will be determined by target metrics $\hat{\mathbf{g}}$ gener-

ated by nonholonomic deformations $\hat{\mathbf{g}} \rightarrow \bar{\mathbf{g}} = [\bar{g}_i(x^k) = \bar{\eta}_i \hat{g}_i, \bar{g}_b(x^k, t) = \bar{\eta}_b \hat{g}_b, \bar{N}_i^a(x^k, t) = \bar{\eta}_i^a \hat{N}_i^a]$. The quadratic elements corresponding to cosmological metrics $\hat{\mathbf{g}}$ are similar to (2) but with a corresponding parametrization in terms of polarization functions,

$$ds^2 = \bar{\eta}_1(x^i, t) \hat{g}_1(x^i, t) [dx^1]^2 + \bar{\eta}_2(x^i, t) \hat{g}_2(x^i, t) [dx^1]^2 \\ + \bar{\eta}_3(x^i, t) \hat{h}_3(x^i, t) \left[dy^3 \right. \\ \left. + \bar{\eta}_i^3(x^i, t) \hat{N}_i^3(x^k, t) dx^i \right]^2 + \bar{\eta}_4(x^i, t) \hat{h}_4(x^i, t) \\ \left[dt + \bar{\eta}_i^4(x^k, t) \hat{N}_i^4(x^k, t) dx^i \right]^2.$$

The target d-metrics $\hat{\mathbf{g}} = \bar{\mathbf{g}}(x^k, t)$ are characterized by N-adapted coefficients

$$\bar{g}_i(x^k) = \bar{\eta}_i \hat{g}_i, h_a(x^i, t) = \bar{\eta}_a \hat{h}_a, \\ \bar{N}_i^3 = \bar{\eta}_i^3(x^i, t) \hat{N}_i^3(x^k, t) = \bar{n}_i(x^k, t), \\ \bar{N}_i^4 = \bar{\eta}_i^4(x^i, t) \hat{N}_i^4(x^k, t) = \bar{w}_i(x^k, t).$$

5.2.1 Cosmological evolutions generated by nonstationary quasiperiodic sources

We consider quasiperiodic cosmological sources of type ${}^{qp}\bar{\Upsilon}(x^i, t) = {}^{qp}\bar{\Upsilon}[{}^P\bar{\Upsilon}, {}^Q\bar{\Upsilon}, {}^\psi\bar{\Upsilon}, {}^{sd}\bar{\Upsilon}, {}^{sw}\bar{\Upsilon}]$ (32) as in (64) and (65) and compute the η -polarization functions following formulas

$$\bar{\eta}_i = e^{\psi/\hat{g}_i}; \bar{\eta}_3 = -\frac{|h_4^{[0]} - \bar{\eta}_4 \hat{h}_4|^*}{|{}^{qp}\bar{\Upsilon}| \bar{\eta}_4 \hat{h}_4}; \\ \bar{\eta}_4 = \bar{\eta}_4(x^i, t) \text{ as a generating function}; \\ \bar{\eta}_i^3 = \frac{1 n_k}{\hat{n}_k} \\ + 4 \frac{2 n_k}{\hat{n}_k} \int dt \frac{\left[\left(\sqrt{|{}^{qp}\bar{\Lambda}| [h_4^{[0]} - \bar{\eta}_4 \hat{h}_4]} \right)^* \right]^2}{{}^{qp}\bar{\Upsilon} |{}^{qp}\bar{\Lambda}| \bar{\eta}_4 \hat{h}_4 |^{5/2}}; \\ \bar{\eta}_k^4 = \frac{\partial_i \left(|h_4^{[0]} - \bar{\eta}_4 \hat{h}_4| |{}^{qp}\bar{\Upsilon}| - \int dt |h_4^{[0]} - \bar{\eta}_4 \hat{h}_4| |{}^{qp}\bar{\Upsilon}|^* \right)}{\hat{w}_k |h_4^{[0]} - \bar{\eta}_4 \hat{h}_4|^* |{}^{qp}\bar{\Upsilon}|}. \quad (72)$$

There are other types of f generating functions determined by $\bar{\eta}_4(x^i, t)$, data for the prime d-metric, subjected to nonlinear symmetries (A.32),

$$\bar{\Phi}^2 = 4 |{}^{qp}\bar{\Lambda}| \left[h_4^{[0]}(x^k) - \bar{\eta}_4(x^i, t) \hat{h}_4(x^k, t) \right], \\ \bar{\Psi}^2 = 4 \left(|h_4^{[0]}(x^k) - \bar{\eta}_4(x^i, t) \hat{h}_4(x^i, t)| |{}^{qp}\bar{\Upsilon}(x^i, t)| \right. \\ \left. - \int dt |h_4^{[0]}(x^k) - \bar{\eta}_4(x^i, t) \hat{h}_4(x^i, t)| |{}^{qp}\bar{\Upsilon}(x^i, t)|^* \right).$$

Restricting the classes of generating functionals and sources to subclasses of solutions of (A.25) (see Table 3 and

Appendix A.3.2) for target off-diagonal cosmological metrics (A.35) with zero torsion, we obtain

$$\bar{\eta}_i = e^{\psi/\hat{g}_i}; \\ \bar{\eta}_3 = -\frac{|h_4^{[0]} - \bar{\eta}_4 \hat{h}_4|^*}{|{}^{qp}\bar{\Upsilon}| \bar{\eta}_4 \hat{h}_4}; \bar{\eta}_4 = \bar{\eta}_4(x^i, t) \text{ as a generating function}; \\ \bar{\eta}_k^3 = (\partial_k \bar{n})/\hat{n}_k; \bar{\eta}_k^4 = \partial_k \bar{A}/\hat{w}_k.$$

In (72) and next formula for LC-configurations, the nonlinear functionals for the quasiperiodic v-source and (effective) cosmological constant can be changed into additive functionals ${}^{qp}\bar{\Upsilon} \rightarrow {}^{as}\bar{\Upsilon}$ and ${}^{qp}\bar{\Lambda} \rightarrow {}^{as}\bar{\Lambda}$. We generate solutions of type (62) and, respectively, (63) but with a prime cosmological structure modified by certain classes of quasiperiodic generating functions and sources. For cosmological scenarios, we can generate solutions without singularities. Such examples have been studied in details for small parametric deformations in [1, 12, 24, 26, 50, 51] and, for quasiperiodic configurations, in [6, 7]. In this work, cosmological configurations are studied for general quasiperiodic structures. The constructions depend on the type of model we construct (for instance, different classes of generating functionals and sources have to be prescribed for certain web/filament/solitonic stationary distributions). We can treat such generic off-diagonal locally anisotropic solutions as nonholonomically deformed prime cosmological configurations imbedded into certain (non) singular media determined by evolving dark and usual matter with quasiperiodic interactions.

5.2.2 Cosmology from nonstationary quasiperiodic generating functions

We can construct other classes of locally anisotropic and inhomogeneous cosmological solutions as nonholonomic deformations of some prime cosmological metrics when the coefficients of the d-metrics are determined nonlinear generating functionals ${}^{qp}\bar{\Phi}^2[{}^P\bar{q}, {}^Q\bar{q}, \psi, {}^{sd}\bar{q}, {}^{sw}\bar{q}]$ (40). In a similar manner, we can generate similar cosmological metrics by additive functionals ${}^a\bar{\Phi}^2$ (35) for a prescribed effective source $\bar{\Upsilon}(x^i, t)$ and cosmological constant $\bar{\Lambda}$. The formulas for nonlinear symmetries (42) allow us to find (recurrently) corresponding nonlinear functionals, ${}^{qp}\bar{\eta}_4(x^i, t)$, or additive functionals, ${}^a\bar{\eta}_4(x^i, t)$, and related polarization functions,

$${}^{qp}\bar{\eta}_4 = h_4^{[0]}(x^i)/\hat{h}_4(x^i, t) \mp {}^{qp}\bar{\Phi}^2(x^i, t)/4\bar{\Lambda}\hat{h}_4(x^i, t), \\ {}^{qp}\bar{\Psi}^2 = 4 \left(|h_4^{[0]}(x^k) - {}^{qp}\bar{\eta}_4(x^i, t) \hat{h}_4(x^i, t)| |\bar{\Upsilon}(x^i, t)| \right. \\ \left. - \int dt |h_4^{[0]}(x^k) - {}^{qp}\bar{\eta}_4(x^i, t) \hat{h}_4(x^i, t)| |\bar{\Upsilon}(x^i, t)|^* \right).$$

The coefficients of quadratic elements of type (A.34) are recurrently computed,

$$\begin{aligned}\bar{\eta}_i &= e^{\psi/\hat{g}_i}; \bar{\eta}_3 = -\frac{|h_4^{[0]} - q^p \bar{\eta}_4 \hat{h}_4|^*}{|\bar{\Upsilon}| q^p \bar{\eta}_4 \hat{h}_4}; \\ \bar{\eta}_4 &= q^p \bar{\eta}_4(x^i, t) \text{ as a generating function;} \\ \bar{\eta}_i^3 &= \frac{1n_k}{\hat{n}_k} + 4 \frac{2n_k}{\hat{n}_k} \int dt \frac{\left[\left(\sqrt{|\Lambda[h_4^{[0]} - q^p \bar{\eta}_4 \hat{h}_4]|} \right)^* \right]^2}{\bar{\Upsilon} \bar{\Lambda} |q^p \bar{\eta}_4 \hat{h}_4|^{5/2}}; \\ \bar{\eta}_k^4 &= \frac{\partial_i \left(|h_4^{[0]} - q^p \bar{\eta}_4 \hat{h}_4| |\bar{\Upsilon}| - \int dt |h_4^{[0]} - q^p \bar{\eta}_4 \hat{h}_4| |\bar{\Upsilon}|^* \right)}{\hat{w}_k |h_4^{[0]} - q^p \bar{\eta}_4 \hat{h}_4|^* |\bar{\Upsilon}|}.\end{aligned}\quad (73)$$

Target off-diagonal cosmological metrics (A.35) with zero torsion can be generated by polarization functions

$$\begin{aligned}\bar{\eta}_i &= e^{\psi/\hat{g}_i}; \bar{\eta}_3 = -\frac{|h_4^{[0]} - q^p \bar{\eta}_4 \hat{h}_4|^*}{|\bar{\Upsilon}| q^p \bar{\eta}_4 \hat{h}_4}; \\ \bar{\eta}_4 &= q^p \bar{\eta}_4(x^i, t) \text{ as a generating function;} \\ \bar{\eta}_k^3 &= (\partial_k \bar{n})/\hat{n}_k; \bar{\eta}_k^4 = \partial_k \bar{A}/\hat{w}_k,\end{aligned}$$

see Appendix A.3.2 for explanations on conventions and nonholonomic constraints for functions with “inverse hats” and transforming general quasiperiodic functionals into additive ones.

The solutions generated in this section describe certain nonholonomically deformed prime cosmological configurations (for instance, a FLRW, or Bianchi, type metric) self-consistently imbedded into a quasiperiodic gravitational (dark energy) media.

5.2.3 Cosmological solutions for nonstationary quasiperiodic sources and generating functions

We can construct very general classes of nonholonomic deformations of prime cosmological metrics generated by nonlinear quasiperiodic functionals both for the generating functions and (effective) sources. Nonlinear superpositions of cosmological solutions (72) and (73) are determined by coefficients of (64) computed,

$$\begin{aligned}\bar{\eta}_i &= e^{\psi/\hat{g}_i}; \bar{\eta}_3 = -\frac{|h_4^{[0]} - q^p \bar{\eta}_4 \hat{h}_4|^*}{|q^p \bar{\Upsilon}| q^p \bar{\eta}_4 \hat{h}_4}; \\ \bar{\eta}_4 &= q^p \bar{\eta}_4(x^i, t) \text{ as a generating function;} \\ \bar{\eta}_i^3 &= \frac{1n_k}{\hat{n}_k} + 4 \frac{2n_k}{\hat{n}_k} \int dt \frac{\left[\left(\sqrt{|q^p \bar{\Upsilon} \Lambda[h_4^{[0]} - q^p \bar{\eta}_4 \hat{h}_4]|} \right)^* \right]^2}{q^p \bar{\Upsilon} q^p \bar{\Lambda} |q^p \bar{\eta}_4 \hat{h}_4|^{5/2}}; \\ \bar{\eta}_k^4 &= \frac{\partial_i \left(|h_4^{[0]} - q^p \bar{\eta}_4 \hat{h}_4| |q^p \bar{\Upsilon}| - \int dt |h_4^{[0]} - q^p \bar{\eta}_4 \hat{h}_4| |q^p \bar{\Upsilon}|^* \right)}{\hat{w}_k |h_4^{[0]} - q^p \bar{\eta}_4 \hat{h}_4|^* |q^p \bar{\Upsilon}|}.\end{aligned}\quad (74)$$

In such formulas, there are considered nonlinear generating functionals $q^p \bar{\Phi}^2 [{}^P \bar{\Phi}, {}^{QC} \bar{\Phi}, \psi \bar{\Phi}, {}^{sd} \bar{\Phi}^2, {}^{sw} \bar{\Phi}^2]$ (40) characterized by nonlinear symmetries of type (42) for some prescribed nonlinear functionals $q^p \bar{\Upsilon}(x^i, t)$ and $q^p \bar{\Lambda}$. Instead of $({}^q \bar{\Phi}, {}^q \bar{\Upsilon}, {}^q \bar{\Lambda})$, we can consider additive data $({}^a \bar{\Phi}, {}^a \bar{\Upsilon}, {}^a \bar{\Lambda})$. Using formulas for nonlinear symmetries, we can define general nonlinear, $q^p \bar{\eta}_4(x^i, t)$, or additive functionals, ${}^a \bar{\eta}_4(x^i, t)$, for the polarization function,

$$\begin{aligned}q^p \bar{\eta}_4 &= h_4^{[0]}(x^i)/\hat{h}_4(x^i, t) \mp q^p \bar{\Phi}^2(x^i, t)/4 q^p \bar{\Lambda} \hat{h}_4(x^i, t), \\ q^p \bar{\Psi}^2 &= 4(|h_4^{[0]}(x^k) - q^p \bar{\eta}_4(x^i, t) \hat{h}_4(x^i, t)| |q^p \bar{\Upsilon}(x^i, t)| \\ &\quad - \int dt |h_4^{[0]}(x^k) - q^p \bar{\eta}_4(x^i, t) \hat{h}_4(x^i, t)| |q^p \bar{\Upsilon}(x^i, t)|^*).\end{aligned}$$

LC-configurations with zero torsion for target off-diagonal cosmological metrics (A.35) are generated

$$\begin{aligned}\bar{\eta}_i &= e^{\psi/\hat{g}_i}; \bar{\eta}_3 = -\frac{|h_4^{[0]} - q^p \bar{\eta}_4 \hat{h}_4|^*}{|q^p \bar{\Upsilon}| q^p \bar{\eta}_4 \hat{h}_4}; \\ \bar{\eta}_4 &= q^p \bar{\eta}_4(x^i, t) \text{ as a generating function;} \\ \bar{\eta}_k^3 &= (\partial_k \bar{n})/\hat{n}_k; \bar{\eta}_k^4 = \partial_k \bar{A}/\hat{w}_k.\end{aligned}$$

We note that the class of cosmological solutions (74) is “dual” on y^3 and y^4 coordinates to stationary solutions (58). For certain classes of parameterizations, such classes describe nonholonomic deformations of cosmological spacetimes, or BHs, self-consistently imbedded into quasiperiodic cosmological (dark energy) backgrounds and quasiperiodic dark/standard matter.

6 Discussion and concluding remarks

Cosmology theories rely on the cosmological principle stating that our universe is sufficiently homogeneous and isotropic on large scales. Geometric models and computation algorithms in numerical relativity encode observational data and theoretical assumptions that expansion is that for the FLWR model governed by (modified) Friedmann equations and employing a Newtonian approximation of gravity. The transition to cosmic homogeneity begins on scales $\sim 80 \text{ h}^{-1} \text{ Mpc}$ when the Universe is inhomogeneous and anisotropic on smaller scales, see Planck2015 data [56–59]. Modern telescopes reached a precision which shows that nonlinear general relativistic effects from inhomogeneities could be important.

There are more extreme hypotheses that inhomogeneities may provide an alternative explanation for the accelerating expansion of the Universe (for instance, it is considered the back reaction or replaced the role assigned to dark energy in the standard Λ CDM model). Such alternative approaches are based on a number of cosmological observations during the last 20 years emphasizing important phenomena of the accel-

erating Universe and dark energy and dark matter. The dark energy physics yields a late time acceleration of the space-time. In its turn, the dark matter physics is “hidden” as an invisible matter (there are various models of dust, cold and hot matter, gravitational polarizations etc.) which favour the process of gravitational clustering. It is not clear how such intriguing physical effects (substances and (non) linear interactions) could emerge in the general relativity theory, GR, or should complement such a theory. The search for alternative modified gravity theories, MGTs, has become an active area for theoretical and experimental investigations. A very important task is to elaborate on new methods of constructing exact and parametric solutions of motion and evolution field equations in MGT and GR describing nonlinear gravitational and matter field interactions (as we motivated in the Introduction section). A large number of MGTs is available and studied in various details in modern literature [1–3, 24–26, 47, 60–70].

The predominantly accepted Λ CDM paradigm predicts for the cosmological structure that dark matter is organized as a cosmic web structure of walls, filaments and halos [46–55]. Numerous images of the visually striking cosmic web have been created and studied theoretically in a framework of research of the large-scale structure of the universe. Recently, an advanced observational and numerical simulation technique was elaborated for pseudo-3d visualisations aimed to minimal loss of information and accurate representation of cosmic web fundamental shapes and components. In the cosmic web, one observes large scale filaments with lengths that can reach tens of Mps, see [55] and references therein. Such filaments are observed indirectly through the galaxies positions (using large galaxy surveys), or through absorption features in the spectra of high red shift sources (with direct detecting of intergalactic medium filaments through their emission on the HI 21 cm line). One estimates that gas in filaments of length $l \geq 15h^{-1}$ Mpc with relatively small inclinations to the line of sight ($\leq 10^\circ$) can be observed during 40–100 h with modern (radio) telescopes. It is revealed from observations of the local Universe that galaxies reside in a complex network of filamentary structures (with cosmic web). Such structures can be explained in the Λ CDM framework as resulting from nonlinear gravitational evolution. The dark matter halos (within which galaxies reside) are connected to each other through a patchwork of filaments and sheets and quasiperiodic/aperiodic structures that constitute the structure of the intergalactic medium, IGM. In modern astronomy it is explored the possibility of using the HI 21 cm line to directly observe ICM filaments.

Various alternative approaches to structure formation in modern cosmology were elaborated using numerical relativity [15–18]. This involves modelling by numerical methods which began with evolutions of planar and spherically symmetric spacetime following the Arnolitt–Deser–Misner,

ADM, formalism [27]. Latter, there were considered generalizations with Kasner and matter-fields, and for the propagation and collision and gravitational wave perturbations and linearised perturbations to homogeneous spacetimes. In order to simplify the numerical calculations there are included linear and nonlinear symmetries. Simulations with (non) linear symmetries have been performed in order to study the evolution of small perturbations to an FLRW spacetime, to explore observational implications showed differential expansion in an inhomogeneous universe. Such works indicate that the effects of nonlinear inhomogeneities may be significant.

Modern cosmological observational data and related phenomenological models emphasize a crucial importance of nonlinear physics and related mathematical methods. Distinguishing general relativistic effects determined by nonlinear structures requires more general classes of solutions of Einstein’s equations in GR and application of advanced geometric and numeric methods elaborated recently for theoretical studies in MGT, non standard particle physics, astrophysics and cosmology. Post-Newtonian, small parametric (perturbative and non-perturbative) numeric approximations consist a worthwhile approach extended with methods including density perturbations in a highly nonlinear form. However the validity of nonlinear effects must be checked against more precise exact and parametric solutions constructed in analytic form.

In this work, we performed a brief review and feasibility study of geometric methods of constructing exact off-diagonal solutions to the (modified) Einstein equations for stationary, locally anisotropic BH and cosmological (in)homogeneous cosmology by geometric modeling and comparing with numeric simulating the growths of quasiperiodic/aperiodic structures and comparing to known analytic solutions. We also presented a study on the evolution of nonlinear stationary and/ or locally anisotropic configurations and analysed the resulting new classes of exact solutions in MGTs and GR. Such a research was performed by developing the anholonomic frame method, AFDM, for constructing new classes of stationary and nonstationary (cosmological) solutions of (modified) Einstein equations. Such solutions are described by generic off-diagonal metrics and generalized connections and depend, in principle, on all possible 4-d and extra dimensional space coordinates; see details, examples and various applications in [8–14, 20–26].

A crucial difference from former approaches to constructing exact solutions [27–31] is that the AFDM allows us to work with generating and integration functions for coefficients of generic off-diagonal metrics, generalized connections and (effective) sources transforming motion and geometric evolution equations into nonlinear systems of partial differential equations, PDEs, with decoupling properties. Following this geometric method, we perform such nonholo-

nomic deformations of some prime stationary and nonstationary solutions (for instance, BH and/or (an) isotropic cosmological solutions) when the (generalized) Einstein equations can be decoupled in general forms and integrated for various classes of metrics $g_{\alpha\beta}(x^i, y^3, t)$. In particular, we can reproduce former results for diagonalizable ansatz with dependence on radial and warping coordinates as solutions of ordinary differential equations, ODEs. Nevertheless, the AFDM, is more than a constructive interference of geometric and analytic methods for constructing exact solutions for certain classes of important nonlinear systems of PDEs, in mathematical relativity and cosmology. It reflects new and formerly unknown properties and nonlinear symmetries of the (modified) Einstein equations when generic off-diagonal interactions and mixed continuous and discrete structures (quasiperiodic/apperiodic/pattern forming/solitonic ones) are considered for vacuum and non-vacuum gravitational interactions. The AFDM is appealing in some sense it is “economical and very efficient”; allowing us to proceed in the same manner but with fractional/random/noncommutative sources and their respective interaction parameters.

Finally, we emphasize that there are a number of directions in (modified) gravity, cosmology and astrophysics which can be pursued using as starting points the methods and solutions elaborated in this works. This includes noncommutative and nonassociative generalizations defined by possible modified dispersion relations and/ or extra dimension contributions to dark energy/matter physics and/or quantum models with quasiperiodic and pattern forming structures.

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A MGTs in nonholonomic 2 + 2 N-adapted variables

We summarize most important definitions and formulas necessary for generating solutions of gravitational field equations following the anholonomic frame deformation method, AFDM, see details and proofs in [8–14, 20–26].

A.1 N-adapted coefficients for curvatures and torsions

Using standard formulas, we can define and compute both in abstract and coordinate forms the torsion, $\mathcal{T}$, the nonmetric-

ity, $\mathcal{Q}$, and the curvature, $\mathcal{R}$, tensors for any d-connection $\mathbf{D} = (hD, vD)$,

$$\begin{aligned}\mathcal{T}(\mathbf{X}, \mathbf{Y}) &:= \mathbf{D}_\mathbf{X} \mathbf{Y} - \mathbf{D}_\mathbf{Y} \mathbf{X} - [\mathbf{X}, \mathbf{Y}], \quad \mathcal{Q}(\mathbf{X}) := \mathbf{D}_\mathbf{X} \mathbf{g}, \mathcal{R}(\mathbf{X}, \mathbf{Y}) \\ &:= \mathbf{D}_\mathbf{X} \mathbf{D}_\mathbf{Y} - \mathbf{D}_\mathbf{Y} \mathbf{D}_\mathbf{X} - \mathbf{D}_{[\mathbf{X}, \mathbf{Y}]}\end{aligned}$$

In literature [8–14], there are used terms like distinguished tensor, d-tensor, and distinguished (geometric) object, d-object, (also d-metric, d-connection) etc. for geometric and physical values defined in N-adapted form, i.e. when all values are defined in some coefficient forms preserving under parallelism the N-connection splitting (6)). Using N-adapted coefficients of a d-connection, $\mathbf{D} = \{\Gamma^\gamma_{\alpha\beta} = (L^i_{jk}, L^a_{bk}, C^i_{jc}, C^a_{bc})\}$, we can compute with respect to N-adapted frames (4) and (3) corresponding N-adapted coefficients

$$\begin{aligned}\mathcal{T} &= \{\mathbf{T}^\gamma_{\alpha\beta} = (T^i_{jk}, T^i_{ja}, T^a_{ji}, T^a_{bi}, T^a_{bc})\}, \quad \mathcal{Q} = \{\mathbf{Q}^\gamma_{\alpha\beta}\}, \\ \mathcal{R} &= \{\mathbf{R}^\alpha_{\beta\gamma\delta} = (R^i_{hjk}, R^a_{bjk}, R^i_{hja}, R^c_{bj a}, R^i_{hba}, R^c_{bea})\}.\end{aligned}$$

We omit such cumbersome formulas which can be found in above mentioned references.

The coefficients of the canonical d-connection $\hat{\mathbf{D}} = \{\hat{\Gamma}^\gamma_{\alpha\beta} = (\hat{L}^i_{jk}, \hat{L}^a_{bk}, \hat{C}^i_{jc}, \hat{C}^a_{bc})\}$ in (9) are

$$\begin{aligned}\hat{L}^i_{jk} &= \frac{1}{2} g^{ir} (\mathbf{e}_k g_{jr} + \mathbf{e}_j g_{kr} - \mathbf{e}_r g_{jk}), \\ \hat{C}^a_{bc} &= \frac{1}{2} g^{ad} (e_c g_{bd} + e_b g_{cd} - e_d g_{bc}), \\ \hat{C}^i_{jc} &= \frac{1}{2} g^{ik} e_c g_{jk}, \quad \hat{L}^a_{bk} = e_b (N^a_k) \\ &\quad + \frac{1}{2} g^{ac} (\mathbf{e}_k g_{bc} - g_{dc} e_b N^d_k - g_{db} e_c N^d_k).\end{aligned}\quad (\text{A.1})$$

The coefficients of the distortion d-tensor, $\hat{\mathbf{Z}}^\gamma_{\alpha\beta} = \hat{\Gamma}^\gamma_{\alpha\beta} - \Gamma^\gamma_{\alpha\beta}$ (10) can be written in N-adapted form using (A.1) and LC-connection $\nabla = \{\Gamma^\gamma_{\alpha\beta}\}$, all computed with respect to (4) and (3). Using such values, we find the nontrivial d-torsion coefficients $\hat{\mathbf{T}}^\gamma_{\alpha\beta}$ of $\hat{\mathbf{D}}$,

$$\begin{aligned}\hat{T}^i_{jk} &= \hat{L}^i_{jk} - \hat{L}^i_{kj}, \quad \hat{T}^i_{ja} = \hat{C}^i_{jb}, \quad \hat{T}^a_{ji} = -\Omega^a_{ji}, \\ \hat{T}^c_{aj} &= \hat{L}^c_{aj} - e_a (N^c_j), \quad \hat{T}^a_{bc} = \hat{C}^a_{bc} - \hat{C}^a_{cb}.\end{aligned}\quad (\text{A.2})$$

We note that the d-torsion coefficients (A.2) vanish if in N-adapted form there are satisfied the conditions

$$\hat{L}^c_{aj} = e_a (N^c_j), \quad \hat{C}^i_{jb} = 0, \quad \Omega^a_{ji} = 0. \quad (\text{A.3})$$

Following similar formulas, we can compute (see details in above references) the nontrivial coefficients of the Riemann d-tensor, $\hat{\mathbf{R}}^\alpha_{\beta\gamma\delta}$, the Ricci d-tensor, $\hat{\mathbf{R}}_{\alpha\beta}$ (11), and the Einstein d-tensor $\hat{\mathbf{E}}_{\alpha\beta} := \hat{\mathbf{R}}_{\alpha\beta} - \frac{1}{2} \mathbf{g}_{\alpha\beta} \hat{R}$.

A.2 Decoupling property of (modified) Einstein equations

For general assumptions and using frame/ coordinate transforms, any d-metric $\hat{\mathbf{g}}$ (2) can be parameterized

$$\begin{aligned} g_i &= e^{\psi(r,\theta)}, \quad g_a = \omega(r, \theta, y^b) h_a(r, \theta, \varphi), \\ N_i^3 &= w_i(r, \theta, \varphi), \\ N_i^4 &= n_i(r, \theta, \varphi), \text{ for } \omega = 1, \text{ stationary conf.}; \end{aligned} \quad (\text{A.4})$$

$$\begin{aligned} g_i &= e^{\psi(x^k)}, \quad g_a = \omega(x^k, y^b) \bar{h}_a(x^k, t), \\ N_i^3 &= \bar{n}_i(x^k, t), \\ N_i^4 &= \bar{w}_i(x^k, t), \text{ for } \omega = 1, \text{ cosmological conf.} \end{aligned} \quad (\text{A.5})$$

In order to write certain formulas in compact forms, we shall use also brief notations of partial derivatives $\partial_\alpha q = \partial q / \partial u^\alpha$ (for instance, for a function $q(x^k, y^a)$)

$$\begin{aligned} \partial_1 q &= q^\bullet = \partial q / \partial x^1, \quad \partial_2 q = q' = \partial q / \partial x^2, \quad \partial_3 q = \partial q / \partial y^3 \\ &= \partial q / \partial \varphi = q^\diamond, \quad \partial_4 q = \partial q / \partial t = \partial_t q = q^*, \\ \partial_{33}^2 &= \partial^2 q / \partial \varphi^2 = \partial_{\varphi\varphi}^2 q = q^{\diamond\diamond}, \quad \partial_{44}^2 = \partial^2 q / \partial t^2 \\ &= \partial_{tt}^2 q = q^{**}. \end{aligned}$$

The sources (15) for (effective) matter field configurations can be parameterized via frame transforms in respective N-adapted forms

$$\begin{aligned} \Upsilon_\nu^\mu &= \mathbf{e}_{\mu'}^\mu \mathbf{e}_{\nu'}^{\nu'} [{}^m \Upsilon_{\nu'}^{\mu'} + \hat{\Upsilon}_{\nu'}^{\mu'}] \\ &= \begin{cases} [{}_h \Upsilon(r, \theta) \delta_j^i, \Upsilon(r, \theta, \varphi) \delta_b^a], & \text{stationary configurations;} \\ [{}_h \bar{\Upsilon}(x^i) \delta_j^i, \bar{\Upsilon}(x^i, t) \delta_b^a], & \text{cosmological configurations.} \end{cases} \end{aligned} \quad (\text{A.6})$$

In these formulas, there are considered necessary type vielbein transforms $\mathbf{e}_{\mu'}^\mu(u^\nu)$ and their duals $\mathbf{e}_{\nu'}^{\nu'}(u^\nu)$, when $\mathbf{e}^\mu = \mathbf{e}_{\mu'}^\mu du^{\mu'}$, and $\Upsilon_{\nu'}^{\mu'} = {}^m \Upsilon_{\nu'}^{\mu'} + \hat{\Upsilon}_{\nu'}^{\mu'}$. The values $[{}_h \Upsilon(r, \theta), \Upsilon(r, \theta, \varphi)]$, or $[{}_h \bar{\Upsilon}(x^i), \bar{\Upsilon}(x^i, t)]$, are considered as generating functions for (effective) matter sources imposing nonholonomic frame constraints on stationary distributions or cosmological dynamics of (effective) matter fields. For simplicity, we shall generate in explicit form certain classes of generic off-diagonal solutions with $\omega = 1$ (i.e. with at least one Killing symmetry on ∂_t or ∂_φ) when the frame/coordinate systems and transforms are compatible with the conditions $\partial_b h_a \neq 0$ and $[{}_h \Upsilon(r, \theta), \Upsilon(r, \theta, \varphi)] \neq 0$, or $[{}_h \bar{\Upsilon}(x^i), \bar{\Upsilon}(x^i, t)] \neq 0$. In next sections, we shall prove that above introduced parameterizations of d-metrics and (effective) sources allows us to integrate explicitly the gravitational field equations (14).⁹

⁹ It is possible to construct various classes of physically important vacuum and nonvacuum solutions if such conditions are violated with respect to certain systems of references, or in some points, open regions of nonholonomic spacetime models. This requests more cumbersome

A.2.1 Off-diagonal stationary configurations

In this section, we outline key steps for proofs of general decoupling and integrability of (modified) Einstein equations for general assumptions on coefficients d-metrics and N-connections which do not depend on y^4 with respect to a class of N-adapted frames.

Nontrivial components of the Ricci d-tensor and (modified) Einstein equations:

Introducing d-metric data (A.4) with $\omega = 1$ into formulas (A.1) and (11) (with respective sources (A.6), and considering nontrivial N-adapted coefficients of the Ricci d-tensor), we transform the modified Einstein equations (14) into such a system of nonlinear PDEs

$$\begin{aligned} \hat{R}_1^1 &= \hat{R}_2^2 = \frac{1}{2g_1 g_2} \\ &\times \left[\frac{g_1^\bullet g_2^\bullet}{2g_1} + \frac{(g_2^\bullet)^2}{2g_2} - g_2^{\bullet\bullet} + \frac{g_1' g_2'}{2g_2} + \frac{(g_1')^2}{2g_1} - g_1'' \right] = -{}_h \Upsilon, \end{aligned} \quad (\text{A.7})$$

$$\hat{R}_3^3 = \hat{R}_4^4 = \frac{1}{2h_3 h_4} \left[\frac{(h_4^\diamond)^2}{2h_4} + \frac{h_3^\diamond h_4^\diamond}{2h_3} - h_4^{\diamond\diamond} \right] = -\Upsilon, \quad (\text{A.8})$$

$$\begin{aligned} \hat{R}_{3k} &= -\frac{w_k}{2h_4} \left[\frac{(h_4^\diamond)^2}{2h_4} + \frac{h_3^\diamond h_4^\diamond}{2h_3} - h_4^{\diamond\diamond} \right] + \frac{h_4^\diamond}{4h_4} \left(\frac{\partial_k h_3}{h_3} + \frac{\partial_k h_4}{h_4} \right) \\ &- \frac{\partial_k h_4^\diamond}{2h_4} = 0, \end{aligned} \quad (\text{A.9})$$

$$\hat{R}_{4k} = \frac{h_4}{2h_4} n_k^{\diamond\diamond} + \left(\frac{h_4}{h_3} h_3^\diamond - \frac{3}{2} h_4^\diamond \right) \frac{n_k^\diamond}{2h_3} = 0. \quad (\text{A.10})$$

In N-adapted frames, this system possesses a **decoupling property**: The equations (A.7) allow us to find g_1 (or, inversely, g_2) for any prescribed h-source ${}_h \Upsilon(r, \theta)$ and given coefficient g_2 (or, inversely, g_1). Integrating on variable y^3 in (A.8), we can define $h_3(r, \theta, \varphi)$ as a solution of first order PDE for any prescribed v-source $\Upsilon(r, \theta, \varphi)$ and given coefficient $h_4(r, \theta, \varphi)$ [we can define $h_4(r, \theta, \varphi)$ if, inversely, $h_3(r, \theta, \varphi)$ is given but in such cases we have to solve a second order PDE]. For well-defined values of h_3 and h_4 , the equations (A.9) transform into an algebraic linear equation for $w_k(r, \theta, \varphi)$. We have to integrate two times on y^3 in order to compute $n_k(r, \theta, \varphi)$ for any well-defined h_3 and h_4 . So, the decoupling property of the system (A.7)–(A.10) reflects the possibility to integrate such PDEs step by step by defining the h-coefficients, g_i , and v-coefficients, h_a , of a d-metric $[\mathbf{g}_i = g_i(x^k), \mathbf{g}_a = g_a(x^k, y^3)]$ (2) (with data (A.4)), and, finally, the N-connection coefficients, $N_i^a = [w_i(x^k, y^3), n_i(x^k, y^3)]$.

Extracting torsionless configurations:

technical considerations. We shall not study such solutions in this work, see examples in Refs. [10–14, 20–26].

We note that the LC-conditions (A.3) for stationary configurations transform into a system of 1st order PDEs,

$$\begin{aligned}\partial_\varphi w_i &= (\partial_i - w_i \partial_\varphi) \ln \sqrt{|h_3|}, (\partial_i - w_i \partial_\varphi) \ln \sqrt{|h_4|} = 0, \\ \partial_k w_i &= \partial_i w_k, \partial_\varphi n_i = 0, \partial_i n_k = \partial_k n_i,\end{aligned}\quad (\text{A.11})$$

imposing additional constraints on off-diagonal coefficients of metrics of type (1). Such conditions can be imposed on d-metric and N-connection coefficients after a class of off-diagonal solutions for (14) has been constructed in an explicit form. For certain well-defined N-adapted parameterizations, the equations (A.11) can be solved in explicit forms.

Nonlinear gravitational PDEs with explicit decoupling:

Let us show how the system of nonlinear PDEs (A.7)–(A.10) can be integrated in explicit form. We introduce the coefficients $\alpha_i = (\partial_\varphi h_4) (\partial_i \varpi)$, $\beta = (\partial_\varphi h_4) (\partial_\varphi \varpi)$, $\gamma = \partial_\varphi (\ln |h_4|^{3/2}/|h_3|)$, where

$$\varpi = \ln |\partial_3 h_4 / \sqrt{|h_3 h_4|}|. \quad (\text{A.12})$$

Using such nontrivial and nonsingular values for $\partial_3 h_a \neq 0$ and $\partial_i \varpi \neq 0$,¹⁰ we obtain

$$\psi^{\bullet\bullet} + \psi'' = 2 {}_h \Upsilon, \quad (\text{A.13})$$

$$\varpi^\diamond h_4^\diamond = 2 h_3 h_4 \Upsilon, \quad (\text{A.14})$$

$$\beta w_i - \alpha_i = 0, \quad (\text{A.15})$$

$$n_k^\diamond + \gamma n_k^\diamond = 0. \quad (\text{A.16})$$

This system can be integrated in explicit form (see below) for any generating function $\Psi(r, \theta, \varphi) := e^\varpi$ and generating sources ${}_h \Upsilon(r, \theta)$ and $\Upsilon(r, \theta, \varphi)$.

Nonlinear symmetries for stationary generating functions and sources with effective cosmological constant:

We emphasize that the system of two equations (A.12) and (A.8) relates four functions (h_3 , h_4 , Υ , and Ψ) and possesses an important nonlinear symmetry for re-defining generating functions, $(\Psi, \Upsilon) \iff (\Phi, \Lambda)$, when

$$\Lambda (\Psi^2)^\diamond = |\Upsilon| (\Phi^2)^\diamond, \text{ or } \Lambda \Psi^2 = \Phi^2 |\Upsilon| - \int dy^3 \Phi^2 |\Upsilon|^\diamond, \quad (\text{A.17})$$

which allows to introduce a new generating function $\Phi(r, \theta, \varphi)$ and an (effective) cosmological constant $\Lambda \neq 0$. The value of Λ can be chosen from certain physical considerations. It can be positive or negative. Solutions with $\Lambda = 0$ have to be studied by special methods, see details and examples in [10–14, 20–26]. We can describe nonlinear systems of PDEs by two equivalent generating data (Ψ, Υ) or (Φ, Λ) [for different classes of solutions, one can be convenient to work

with different types of such data]. Modules in such formulas are taken in certain forms which allows to work with real functions.

Stationary solutions for off-diagonal metrics and N-coefficients:

By explicit verifications (see similar details and rigorous proofs in [8–14, 20–26]), we can prove integrating “step by step” the system (A.13)–(A.16) that there are generated generic off-diagonal solutions the solutions of the nonlinear PDEs (A.7)–(A.10), i.e. the gravitational field equations (14), if the d-metric coefficients are computed

$$\begin{aligned}g_i &= e^{\psi(x^k)} \text{ as a solution of 2-d Poisson eqs. } \psi^{\bullet\bullet} \\ &+ \psi'' = 2 {}_h \Upsilon; \\ g_3 &= h_3(r, \theta, \varphi) = h_3^{[0]}(x^k) \\ &- \int dy^3 (\Psi^2)^\diamond / 4 \Upsilon = h_3^{[0]}(x^k) - \Phi^2 / 4 \Lambda; \\ g_4 &= h_4(r, \theta, \varphi) \\ &= -(\Psi^2)^\diamond / 4 \Upsilon^2 h_3 = -\partial_\varphi (\Psi^2) / 4 \Upsilon^2 \left(h_3^{[0]}(x^k) \right. \\ &\quad \left. - \int dy^3 (\Psi^2)^\diamond / 4 \Upsilon \right) \\ &= -(\Phi^2)^\diamond / 4 \Lambda \Upsilon h_3 \\ &= -\partial_\varphi (\Phi^2) / \Upsilon \left(4 \Lambda h_3^{[0]}(x^k) - \Phi^2 \right); \quad (\text{A.18})\end{aligned}$$

and the N-connection coefficients are

$$\begin{aligned}N_i^3 &= w_i(r, \theta, \varphi) \\ &= \partial_i \Psi / \partial_\varphi \Psi = \partial_i \Psi^2 / \partial_\varphi \Psi^2 = \partial_i (\Phi^2 |\Upsilon| \\ &\quad - \int dy^3 \Phi^2 |\Upsilon|^\diamond) / 2 \Phi \Phi^\diamond |\Upsilon|; \\ N_k^4 &= n_k(r, \theta, \varphi) = {}_1 n_k(x^i) \\ &+ {}_2 n_k(x^i) \int dy^3 (\Psi^\diamond)^2 / \Upsilon^2 |h_3^{[0]}(x^i) \\ &\quad - \int dy^3 (\Psi^2)^\diamond / 4 \Upsilon|^{5/2} \\ &= {}_1 n_k(x^i) + {}_2 n_k(x^i) \int dy^3 (\Phi^\diamond)^2 / \Upsilon \Lambda |h_3|^{5/2}.\end{aligned}$$

In these formulas, there are considered also integration functions $h_3^{[0]}(x^k)$, ${}_1 n_k(x^i)$, and ${}_2 n_k(x^i)$ encoding various possible sets of (non) commutative parameters and integration constants. Such values, together with generating data (Ψ, Υ) , or (Φ, Λ) , related by nonlinear differential/integral transforms (A.17) can be stated in explicit form following certain topology/symmetry/asymptotic conditions for some classes of exact/parametric solutions of gravitational field equations. The coefficients (A.18) define generic off-diagonal solutions if the corresponding anholonomy coefficients $C_{\alpha\beta}^\gamma(u)$ (5) are not trivial. Such solutions are with nontrivial nonholonomically induced d-torsion (A.2) with N-adapted coefficients which can be computed in explicit form.

¹⁰ We can construct nontrivial solutions if such conditions are not satisfied; for simplicity, we omit a study of such more special cases.

Quadratic line elements for off-diagonal stationary configurations:

As a matter of principle, we can consider any coefficient $h_3 = h_3^{[0]}(x^k) - \Phi^2/4\Lambda$, $h_3^\circ \neq 0$, as a generating function. Using this formula, we find $\Phi = \pm 2\sqrt{|\Lambda[h_3^{[0]}(x^k) - h_3(r, \theta, \varphi)]|}$ which transform (A.17) in

$$\Psi^2 = 4(|h_3^{[0]}(x^k) - h_3(r, \theta, \varphi)| |\Upsilon| - \int dy^3 |h_3^{[0]}(x^k) - h_3(r, \theta, \varphi)| |\Upsilon|^\circ).$$

Introducing such a functional $\Psi[h_3^{[0]}, h_3, \Upsilon]$ into respective formulas for h_a and Υ in (A.18), we express possible generating functions and the respective d-metric (2), with data (A.4), in terms of h_3 , integration functions and effective sources. Respective quadratic elements can be expressed in three equivalent forms

$$ds^2 = e^{\psi(x^k)} [(dx^1)^2 + (dx^2)^2] + \left\{ \begin{array}{ll} h_3 \left[dy^3 + \frac{\partial_i (|h_3^{[0]} - h_3| |\Upsilon| - \int dy^3 |h_3^{[0]} - h_3| |\Upsilon|^\circ)}{\partial_\varphi (|h_3^{[0]} - h_3| |\Upsilon| - \int dy^3 |h_3^{[0]} - h_3| |\Upsilon|^\circ)} dx^i \right] - & \text{gener. funct. } h_3, \\ \frac{|h_3^{[0]} - h_3|^\circ}{|\Upsilon| h_3} \left[dt + \left({}_1n_k + 4 {}_2n_k \int dy^3 \frac{[\sqrt{|\Lambda[h_3^{[0]} - h_3| |\Upsilon|^\circ]}]^2}{\Upsilon \Lambda |h_3|^{5/2}} \right) dx^k \right], & \text{source } \Upsilon, \text{ or } \Lambda; \\ \text{or} & \\ \frac{\partial_\varphi(\Psi^2)}{4\Upsilon^2 (h_3^{[0]} - \int dy^3 \frac{(\Psi^2)^\circ}{4\Upsilon})} \left[dt + ({}_1n_k + {}_2n_k \int dy^3 \frac{(\Psi^2)^\circ}{\Upsilon^2 |h_3^{[0]} - \int dy^3 \frac{(\Psi^2)^\circ}{4\Upsilon}|^{5/2}}) dx^k \right], & \begin{array}{l} \text{gener. funct. } \Psi, \\ \text{source } \Upsilon; \end{array} \\ \text{or} & \\ \frac{\partial_\varphi(\Phi^2)}{\Upsilon (4\Lambda h_3^{[0]} - \Phi^2)} \left[dt + \left({}_1n_k + {}_2n_k \int dy^3 \frac{(\Phi^2)^\circ}{\Upsilon \Lambda} \left| \frac{\Upsilon (4\Lambda h_3^{[0]} - \Phi^2)}{(\Phi^2)^\circ} \right|^{5/2} \right) dx^k \right], & \begin{array}{l} \text{gener. funct. } \Phi \\ \text{effective } \Lambda \text{ for } \Upsilon. \end{array} \end{array} \right. \quad (\text{A.19})$$

If we consider nonholonomic deformations of a primary d-metric $\hat{\mathbf{g}}$ into a target one $\hat{\mathbf{g}} = [g_\alpha = \eta_\alpha \hat{g}_\alpha, \eta_i^a \hat{N}_i^a]$ with Killing symmetry on ∂_t , we can re-write all formulas (A.18) and (A.19) in terms of η -polarization functions (η_α and η_i^a , determined by generation and integration functions and respective sources) and encoding primary data $[\hat{g}_\alpha, \hat{N}_i^a]$. For instance, $\hat{\mathbf{g}}$ can be chosen for a BH in GR. Off-diagonal nonholonomic deformations may preserve the singular structure of a primary metric (with certain possible deformations of the

horizons, for certain classes of solutions), or (for more general classes of solutions) to eliminate the singular structure, or to change the topology in the resulting target solutions. In section 4, we analyze explicit examples for quasiperiodic off-diagonal deformations.

Off-diagonal Levi-Civita stationary configurations:

We can impose additional constraints on generating functions and sources in order to extract solutions with zero torsion. The Eq. (A.11) can be solved for a special class of generating functions and sources when, for instance, $\Psi = \check{\Psi}(x^i, \varphi)$, $(\partial_i \check{\Psi})^\circ = \partial_i (\check{\Psi}^\circ)$ and $\Upsilon(x^i, \varphi) = \Upsilon[\check{\Psi}] = \check{\Upsilon}$, or $\Upsilon = \text{const}$. If such conditions are imposed, the nonlinear symmetries (A.17) result in formulas

$$\Lambda \check{\Psi}^2 = \check{\Phi}^2 |\check{\Upsilon}| - \int dy^3 \check{\Phi}^2 |\check{\Upsilon}|^\circ, \check{\Phi}^2 = 4|\Lambda[h_3^{[0]}(r, \theta) - \check{h}_3(r, \theta, \varphi)]|$$

$$\check{\Psi}^2 = 4(|h_3^{[0]}(r, \theta) - \check{h}_3(r, \theta, \varphi)| |\check{\Upsilon}| - \int dy^3 |h_3^{[0]}(r, \theta) - \check{h}_3(r, \theta, \varphi)| |\check{\Upsilon}|^\circ),$$

where $h_3 = \check{h}_3(r, \theta, \varphi)$ can be considered also as generating function. For such LC-configurations, we find some functions $\check{A}(r, \theta, \varphi)$ and $n(r, \theta)$ when the N-connection coefficients are

$$w_i = \partial_i \check{A} = \begin{cases} \partial_i (|h_3^{[0]} - \check{h}_3| |\check{\Upsilon}| - \int dy^3 |h_3^{[0]} - \check{h}_3| |\check{\Upsilon}|^\diamond) / [|h_3^{[0]} - \check{h}_3|^\diamond |\check{\Upsilon}|]; \\ \partial_i \check{\Psi} / \check{\Psi}^\diamond; \\ \partial_i (\check{\Phi}^2 |\check{\Upsilon}| - \int dy^3 \check{\Phi}^2 |\check{\Upsilon}|^\diamond) / 2\check{\Phi} \check{\Phi}^\diamond |\check{\Upsilon}|; \end{cases} \quad \text{and } n_k = \check{n}_k = \partial_k n(x^i).$$

In result, we can construct new classes of off-diagonal stationary solutions in GR defined as subclasses of solutions (A.19) with zero torsion,

[${}_h \overline{\Upsilon}(x^i) \delta_j^i, \overline{\Upsilon}(x^i, t) \delta_b^a$] (A.6), we obtain from the modified Einstein equations (14) a system of nonlinear PDEs ¹¹

$$ds^2 = e^{\psi(x^k)} [(dx^1)^2 + (dx^2)^2] + \begin{cases} \check{h}_3 [dy^3 + (\partial_i \check{A}) dx^i] - \frac{|h_3^{[0]} - \check{h}_3|^\diamond}{|\check{\Upsilon}| h_3} [dt + (\partial_k n) dx^k], & \text{gener. funct. } \check{h}_3, \\ \text{or} & \text{source } \check{\Upsilon}, \text{ or } \Lambda; \\ (h_3^{[0]} - \int dy^3 \frac{(\check{\Psi}^2)^\diamond}{4\check{\Upsilon}}) [dy^3 + (\partial_i \check{A}) dx^i] - \frac{(\check{\Psi}^2)^\diamond}{4\check{\Upsilon}^2 (h_3^{[0]} - \int dy^3 \frac{(\check{\Psi}^2)^\diamond}{4\check{\Upsilon}})} [dt + (\partial_k n) dx^k], & \text{gener. funct. } \check{\Psi}, \\ \text{or} & \text{source } \check{\Upsilon}; \\ + (h_3^{[0]} - \frac{\check{\Phi}^2}{4\Lambda}) [dy^3 + (\partial_i \check{A}) dx^i] - \frac{(\check{\Phi}^2)^\diamond}{\check{\Upsilon} (4\Lambda h_3^{[0]} - \check{\Phi}^2)} [dt + (\partial_k n) dx^k], & \text{gener. funct. } \check{\Phi} \\ & \text{effective } \Lambda \text{ for } \check{\Upsilon}. \end{cases} \quad (\text{A.20})$$

Such stationary metrics are generic off-diagonal and define new classes of solutions different, for instance, from the Kerr metric (defined by rotation coordinates, or other equivalent ones) if the anholonomy coefficients $C_{\alpha\beta}^\gamma = \{C_{ia}^b = \partial_a N_i^b, C_{ji}^a = \mathbf{e}_j N_i^a - \mathbf{e}_i N_j^a\}$, see formulas (5), are not zero for $N_i^3 = \partial_i \check{A}$ and $N_k^4 = \partial_k n$. We can analyze certain non-holonomic configurations determined, for instance, by data $(\check{\Upsilon}, \check{\Psi}, h_3^{[0]}, \check{n}_k)$, when $w_i = \partial_i \check{A} \rightarrow 0$ and $\partial_k n \rightarrow 0$ (or 0 values are considered as certain additional constraints).

A.2.2 Off-diagonal cosmological solutions

In this paper, for simplicity, there are considered solutions $g_{\alpha\beta}(x^i, t)$ with Killing symmetry on ∂_3 , i.e. with $\omega = 1$ in (A.5), which allows us to generate exact solutions in explicit form. Solutions depending on all spacetime coordinates, $g_{\alpha\beta}(x^i, y^3, t)$, can be constructed for nontrivial vertical conformal factors $\omega(x^i, y^a)$, see details and examples in [10–14, 20–26]. It should be noted that if certain classes of off-diagonal solutions for such nonholonomic cosmological configurations have been constructed in explicit form, we can impose additional nonholonomic constraints, or limits with necessary smooth classes of functions, when $g_{\alpha\beta}(x^i, t) \approx g_{\alpha\beta}(t)$ are related to Bianchi type, or FLRW, like cosmological metrics.

Nontrivial components of the Ricci d-tensor for nonholonomic cosmological configurations:

Let us consider d-metric data (A.5) with $\omega = 1$ in order to compute the N-adapted and nontrivial coefficients $\hat{\mathbf{D}} = \{\hat{\Gamma}_{\alpha\beta}^\gamma\}$ (A.1) and $\hat{\mathbf{R}}_{\alpha\beta}$ (11). For nontrivial sources

$$\hat{R}_1^1 = \hat{R}_2^2 = \frac{1}{2g_1 g_2} \left[\frac{g_1^\bullet g_2^\bullet}{2g_1} + \frac{(g_2^\bullet)^2}{2g_2} - g_2^{\bullet\bullet} + \frac{g_1' g_2'}{2g_2} + \frac{(g_1')^2}{2g_1} - g_1'' \right] = -{}_h \overline{\Upsilon}, \quad (\text{A.21})$$

$$\hat{R}_3^3 = \hat{R}_4^4 = \frac{1}{2\bar{h}_3 \bar{h}_4} \left[\frac{(\bar{h}_3^*)^2}{2\bar{h}_3} + \frac{\bar{h}_3^* \bar{h}_4^*}{2\bar{h}_4} - \bar{h}_3^{**} \right] = -\overline{\Upsilon} \quad (\text{A.22})$$

$$\hat{R}_{3k} = \frac{\bar{h}_3}{2\bar{h}_4} \bar{n}_k^{**} + \left(\frac{\bar{h}_3}{\bar{h}_4} \bar{h}_4^* - \frac{3}{2} \bar{h}_3^* \right) \frac{\bar{n}_k^*}{2\bar{h}_4} = 0, \quad (\text{A.23})$$

$$\hat{R}_{4k} = -\frac{\bar{w}_k}{2\bar{h}_3} \left[\frac{(\bar{h}_3^*)^2}{2\bar{h}_3} + \frac{\bar{h}_3^* \bar{h}_4^*}{2\bar{h}_4} - \bar{h}_3^{**} \right] + \frac{\bar{h}_3^*}{4\bar{h}_3} \left(\frac{\partial_k \bar{h}_3}{\bar{h}_3} + \frac{\partial_k \bar{h}_4}{\bar{h}_4} \right) - \frac{\partial_k \bar{h}_3^*}{2\bar{h}_3} = 0. \quad (\text{A.24})$$

These equations can be transformed, respectively, into the system (A.7)–(A.10) if ${}_h \overline{\Upsilon}(x^i) \rightarrow {}_h \Upsilon(x^i)$, $\overline{\Upsilon}(x^i, t) \rightarrow \Upsilon(x^i, y^3 = \varphi)$, $\bar{h}_3(x^i, t) \rightarrow h_4(x^i, \varphi)$, $\bar{h}_4(x^i, t) \rightarrow h_3(x^i, \varphi)$, $\bar{h}_3^*(x^i, t) \rightarrow h_4^\diamond(x^i, \varphi)$, $\bar{h}_4^*(x^i, t) \rightarrow h_3^\diamond(x^i, \varphi)$, $\bar{w}_k(x^i, t) \rightarrow n_k(x^i, \varphi)$, $\bar{n}_k(x^i, t) \rightarrow w_k(x^i, \varphi)$ etc. The AFDM allows to redefine the procedure considered in the previous section for stationary nonholonomic configurations in order to generate solutions with explicit dependence on time like coordinate.

¹¹ With partial derivatives $\partial_i q = \partial_4 q = q^*$ and $\partial_i q = (\partial_1 q = q^\bullet, \partial_2 q = q')$.

Extracting torsionless locally anisotropic cosmological configurations:

The LC-conditions (A.3) for data (A.5) transform into

$$\begin{aligned}\partial_t \bar{w}_i &= (\partial_i - \bar{w}_i \partial_t) \ln \sqrt{|\bar{h}_4|}, (\partial_i - \bar{w}_i \partial_t) \ln \sqrt{|\bar{h}_3|} = 0, \\ \partial_k \bar{w}_i &= \partial_i \bar{w}_k, \partial_t \bar{n}_i = 0, \partial_i \bar{n}_k = \partial_k \bar{n}_i.\end{aligned}\quad (\text{A.25})$$

Such nonlinear first order PDEs can be solved in explicit form by imposing additional nonholonomic constraints on cosmological d-metrics and N-coefficients of (modified) Einstein equations.

Decoupling of nonlinear PDEs for off-diagonal cosmological solutions:

The system of nonlinear PDE (A.21)–(A.24) can be decoupled and integrated following the AFDM. Let us introduce the coefficients

$$\begin{aligned}\bar{\alpha}_i &= (\partial_t \bar{h}_3) (\partial_i \bar{\omega}), \bar{\beta} = (\partial_t \bar{h}_3) (\partial_t \bar{\omega}), \bar{\gamma} \\ &= \partial_t \left(\ln |h_3|^{3/2} / |h_4| \right),\end{aligned}\quad (\text{A.26})$$

$$\text{where } \bar{\omega} = \ln |\partial_t \bar{h}_3 / \sqrt{|\bar{h}_3 \bar{h}_4|}|. \quad (\text{A.27})$$

For $\partial_t h_a \neq 0$ and $\partial_t \omega \neq 0$,¹² we rewrite the equations in the form

$$\psi^{\bullet\bullet} + \psi'' = 2 {}_h \bar{\gamma} \quad (\text{A.28})$$

$$\bar{\omega}^* \bar{h}_3^* = 2 \bar{h}_3 \bar{h}_4 \bar{\gamma} \quad (\text{A.29})$$

$$\bar{n}_i^{**} + \bar{\gamma} \bar{n}_i^* = 0, \quad (\text{A.30})$$

$$\bar{\beta} \bar{w}_i - \bar{\alpha}_i = 0. \quad (\text{A.31})$$

We can integrate this system for any generating function $\bar{\Psi}(x^i, t) := e^{\bar{\omega}}$ and sources ${}_h \bar{\gamma}(x^i)$ and $\bar{\gamma}(x^k, t)$.

Nonlinear symmetries for generating functions and sources with effective cosmological constant:

The system of two Eqs. (A.27) and (A.29) relates four functions ($\bar{h}_3, \bar{h}_4, \bar{\gamma}$, and $\bar{\Psi}$) which emphasizes an important nonlinear symmetry for locally anisotropic cosmological solutions and respective generating functions, $(\bar{\Psi}, \bar{\gamma}) \iff (\bar{\Phi}, \bar{\Lambda})$, when

$$\bar{\Lambda} (\bar{\Psi}^2)^* = |\bar{\gamma}| (\bar{\Phi}^2)^*, \text{ or } \bar{\Lambda} \bar{\Psi}^2 = \bar{\Phi}^2 |\bar{\gamma}| - \int dt \bar{\Phi}^2 |\bar{\gamma}|^*. \quad (\text{A.32})$$

This allows to introduce a new generating function $\bar{\Phi}(x^i, t)$ and an (effective) cosmological constant $\bar{\Lambda} \neq 0$, which can be applied for generating exact off-diagonal solutions in explicit forms.

¹² We have to consider other special methods for generating solutions if such conditions are not satisfied.

Off-diagonal metrics and N-coefficients for locally anisotropic cosmological solutions:

Integrating “step by step” the system (A.28)–(A.31), we generate exact solutions of the nonlinear PDEs (A.21)–(A.24), i.e. for the (modified) Einstein equations (14), determined by d-metric coefficients,

$$\begin{aligned}g_i &= e^{\psi(x^k)} \text{ as a solution of 2-d Poisson eqs. } \psi^{\bullet\bullet} + \psi'' = 2 {}_h \bar{\gamma}; \\ g_3 &= \bar{h}_3(x^i, t) = -(\bar{\Psi}^2)^* / 4 \bar{\gamma}^2 \bar{h}_4 = -(\bar{\Psi}^2)^* / 4 \bar{\gamma}^2 (h_4^{[0]}(x^k) \\ &\quad - \int dt (\bar{\Psi}^2)^* / 4 \bar{\gamma}) \\ &= -(\bar{\Phi}^2)^* / 4 \bar{\Lambda} \bar{\gamma} \bar{h}_4 = -(\bar{\Phi}^2)^* / \bar{\gamma} (4 \bar{\Lambda} h_4^{[0]}(x^k) - \bar{\Phi}^2); \\ g_4 &= \bar{h}_4(x^i, t) = h_4^{[0]}(x^k) \\ &\quad - \int dt (\bar{\Psi}^2)^* / 4 \bar{\gamma} = h_4^{[0]}(x^k) - \bar{\Phi}^2 / 4 \bar{\Lambda};\end{aligned}\quad (\text{A.33})$$

and N-connection coefficients,

$$\begin{aligned}N_k^3 &= \bar{n}_k(x^i, t) = {}_1 n_k(x^i) + {}_2 n_k(x^i) \int dt (\bar{\Psi}^*)^2 / \bar{\gamma}^2 |h_4^{[0]}(x^i) \\ &\quad - \int dt (\bar{\Psi}^2)^* / 4 \bar{\gamma} |^{5/2} \\ &= {}_1 n_k(x^i) + {}_2 n_k(x^i) \int dt (\bar{\Phi}^*)^2 / \bar{\gamma} \bar{\Lambda} |\bar{h}_4|^{5/2}; \\ N_i^4 &= \bar{w}_i(x^i, t) = \partial_i \bar{\Psi} / \bar{\Psi}^* = \partial_i \bar{\Psi}^2 / (\bar{\Psi}^2)^* = \partial_i (\bar{\Phi}^2 |\bar{\gamma}|) \\ &\quad - \int dt \bar{\Phi}^2 |\bar{\gamma}|^* / 2 \bar{\Phi} \bar{\Phi}^* |\bar{\gamma}|,\end{aligned}$$

where $h_4^{[0]}(x^k)$, ${}_1 n_k(x^i)$, and ${}_2 n_k(x^i)$ are integration functions encoding various possible sets of (non) commutative parameters and integration constants. We can choose some generating data $(\bar{\Psi}, \bar{\gamma})$, or $(\bar{\Phi}, \bar{\Lambda})$, related by nonlinear differential/integral transforms (A.32), and respective integration functions in explicit form following certain topology/ symmetry/asymptotic conditions for some classes of exact/parametric cosmological solutions. The coefficients (A.33) define generic off-diagonal cosmological solutions if the corresponding anholonomy coefficients $C_{\alpha\beta}^\gamma(x^i, t)$ (5) are not trivial. Such locally cosmological solutions are with nontrivial nonholonomically induced d-torsion (A.2) with N-adapted coefficients which can be computed in explicit form. In order to generate as particular cases some well-known cosmological FLRW, or Bianchi, type metrics, we have to consider data of type $(\bar{\Psi}(t), \bar{\gamma}(t))$, or $(\bar{\Phi}(t), \bar{\Lambda})$, with integration functions which allow frame/ coordinate transforms to respective (off-) diagonal configurations $g_{\alpha\beta}(t)$.

Quadratic line elements for off-diagonal locally anisotropic cosmological configurations:

Any coefficient $\bar{h}_4 = h_4^{[0]}(x^k) - \bar{\Phi}^2 / 4 \bar{\Lambda}$, $\bar{h}_4^* \neq 0$, can be considered also as a generating function. Using (A.33), we find $\bar{\Phi} = \pm 2 \sqrt{|\bar{\Lambda} [h_4^{[0]}(x^k) - \bar{h}_4(x^k, t)]|}$ transforming (A.32) in

$$\bar{\Psi}^2 = 4 (|h_4^{[0]}(x^k) - \bar{h}_4(x^i, t)| |\bar{\gamma}| - \int dt [|h_4^{[0]}(x^k) - \bar{h}_4(x^i, t)| |\bar{\gamma}|^*).$$

Introducing such values into respective formulas for $\bar{h}_a$ and $\bar{\Upsilon}$ in (A.18), we construct locally anisotropic cosmological solutions of type d-metric (2), with data (A.5),

$$\begin{aligned} \bar{\Lambda} \bar{\Psi}^2 &= \bar{\Phi}^2 |\bar{\Upsilon}| \\ &- \int dt \bar{\Phi}^2 |\bar{\Upsilon}|^\circ, \quad \bar{\Phi}^2 = 4|\bar{\Lambda}[h_4^{[0]}(x^i) - \check{h}_4(x^i, t)]|, \end{aligned}$$

$$\begin{aligned} ds^2 &= e^{\psi(x^k)} [(dx^1)^2 + (dx^2)^2] \\ &+ \left\{ \begin{aligned} & - \frac{|h_4^{[0]} - \bar{h}_4|^*}{|\bar{\Upsilon}| \bar{h}_4} \left[dy^3 + \left({}_1n_k + 4 {}_2n_k \int dt \frac{[(\sqrt{|\Lambda[h_4^{[0]} - \bar{h}_4]|})^*]^2}{\bar{\Upsilon} \bar{\Lambda} |\bar{h}_4|^{5/2}} \right) dx^k \right] && \text{gener. funct. } \bar{h}_4, \\ & + \bar{h}_4 \left[dt + \frac{\partial_i (|h_4^{[0]} - \bar{h}_4| |\bar{\Upsilon}| - \int dt |h_4^{[0]} - \bar{h}_4| |\bar{\Upsilon}|^*)}{(|h_4^{[0]} - \bar{h}_4| |\bar{\Upsilon}| - \int dt |h_4^{[0]} - \bar{h}_4| |\bar{\Upsilon}|^*)^*} dx^i \right], && \text{source } \bar{\Upsilon}, \text{ or } \bar{\Lambda}; \\ \text{or} \\ & - \frac{(\bar{\Psi}^2)^*}{4\bar{\Upsilon}^2 (h_4^{[0]} - \int dt \frac{(\bar{\Psi}^2)^*}{4\bar{\Upsilon}})} \left[dy^3 + ({}_1n_k + {}_2n_k \int dt \frac{(\bar{\Psi}^2)^*}{\bar{\Upsilon}^2 |h_4^{[0]} - \int dt \frac{(\bar{\Psi}^2)^*}{4\bar{\Upsilon}}|^{5/2}}) dx^k \right] && \text{gener. funct. } \bar{\Psi}, \\ & + (h_4^{[0]} - \int dt \frac{(\bar{\Psi}^2)^*}{4\bar{\Upsilon}}) \left[dt + \frac{\partial_i \bar{\Psi}}{\bar{\Psi}^*} dx^i \right], && \text{source } \bar{\Upsilon}; \\ \text{or} \\ & - \frac{(\bar{\Phi}^2)^*}{\bar{\Upsilon} (4\bar{\Lambda} h_4^{[0]} - \bar{\Phi}^2)} \left[dy^3 + \left({}_1n_k + {}_2n_k \int dt \frac{(\bar{\Phi}^2)^*}{\bar{\Upsilon} \bar{\Lambda}} \left| \frac{\bar{\Upsilon} (4\bar{\Lambda} h_4^{[0]} - \bar{\Phi}^2)}{(\bar{\Phi}^2)^*} \right|^{5/2} \right) dx^k \right] && \text{gener. funct. } \bar{\Phi} \\ & + \left(h_4^{[0]} - \frac{\bar{\Phi}^2}{4\bar{\Lambda}} \right) \left[dt + \frac{\partial_i (\bar{\Phi}^2 |\bar{\Upsilon}| - \int dt \bar{\Phi}^2 |\bar{\Upsilon}|^*)}{(\bar{\Phi}^2 |\bar{\Upsilon}| - \int dt \bar{\Phi}^2 |\bar{\Upsilon}|^*)^*} dx^i \right], && \text{effective } \bar{\Lambda} \text{ for } \bar{\Upsilon}. \end{aligned} \right. \quad (\text{A.34}) \end{aligned}$$

Such solutions possess a Killing symmetry on ∂_3 and can be re-written in terms of η -polarization function functions for target locally anisotropic cosmological metrics $\hat{\mathbf{g}} = [g_\alpha = \eta_\alpha \hat{g}_\alpha, \eta_i^a \hat{N}_i^a]$ encoding primary cosmological data $[g_\alpha, \hat{N}_i^a]$. In Sect. 5, we analyze explicit examples for quasiperiodic off-diagonal nonholonomic deformations of cosmological solutions.

Off-diagonal Levi-Civita locally anisotropic cosmological configurations:

We can extract cosmological spacetimes in GR, with zero torsion, if the equations (A.25) are solved for a special class of generating functions and sources when, for instance, $\bar{\Psi} = \bar{\Psi}(x^i, t)$, when $(\partial_i \bar{\Psi})^* = \partial_i (\bar{\Psi}^*)$ and $\bar{\Upsilon}(x^i, t) = \bar{\Upsilon}[\bar{\Psi}] = \bar{\Upsilon}$, or $\bar{\Upsilon} = \text{const}$. For such classes of generating functions and sources, the nonlinear symmetries (A.32) result in formulas

$$\begin{aligned} \bar{\Psi}^2 &= 4(|h_4^{[0]}(x^i) - \check{h}_4(x^i, t)| |\bar{\Upsilon}| \\ &- \int dt |h_4^{[0]}(x^i) - \check{h}_4(x^i, t)| |\bar{\Upsilon}|^*), \end{aligned}$$

where $\bar{h}_4 = \check{h}_4(x^i, t)$ can be considered also as generating function for cosmological solutions. For such LC-configurations, there are some functions $\bar{A}(x^i, t)$ and $n(x^i)$ when the N-connection coefficients are

$$\begin{aligned} \bar{n}_k &= \bar{n}_k = \partial_k \bar{n}(x^i) \text{ and } \bar{w}_i = \partial_i \bar{A} \\ &= \begin{cases} \partial_i (|h_4^{[0]} - \bar{h}_4| |\bar{\Upsilon}| - \int dt |h_4^{[0]} - \bar{h}_4| |\bar{\Upsilon}|^*) / |h_4^{[0]} - \bar{h}_4|^* |\bar{\Upsilon}|; \\ \partial_i \bar{\Psi} / \bar{\Psi}^*; \\ \partial_i (\bar{\Phi}^2 |\bar{\Upsilon}| - \int dt \bar{\Phi}^2 |\bar{\Upsilon}|^*) / 2\bar{\Phi} \bar{\Phi}^* |\bar{\Upsilon}|. \end{cases} \end{aligned}$$

In result, we can construct new classes of off-diagonal locally anisotropic cosmological solutions in GR defined as subclasses of solutions (A.34) with zero torsion,

$$\begin{aligned} ds^2 &= e^{\psi(x^k)} [(dx^1)^2 + (dx^2)^2] \\ &+ \left\{ \begin{aligned} & - \frac{|h_4^{[0]} - \bar{h}_4|^*}{|\bar{\Upsilon}| \bar{h}_4} [dy^3 + (\partial_k \bar{n}) dx^k] + \bar{h}_4 [dt + (\partial_i \bar{A}) dx^i], && \text{gener. funct. } \bar{h}_4, \\ & \text{or} && \text{source } \bar{\Upsilon}, \text{ or } \bar{\Lambda}; \\ & - \frac{(\bar{\Psi}^2)^*}{4\bar{\Upsilon}^2 (h_4^{[0]} - \int dt \frac{(\bar{\Psi}^2)^*}{4\bar{\Upsilon}})} [dy^3 + (\partial_k \bar{n}) dx^k] + (h_4^{[0]} - \int dt \frac{(\bar{\Psi}^2)^*}{4\bar{\Upsilon}}) [dt + (\partial_i \bar{A}) dx^i], && \text{gener. funct. } \bar{\Psi}, \\ & \text{or} && \text{source } \bar{\Upsilon}; \\ & - \frac{(\bar{\Phi}^2)^*}{\bar{\Upsilon} (4\bar{\Lambda} h_4^{[0]} - \bar{\Phi}^2)} [dy^3 + (\partial_k \bar{n}) dx^k] + (h_4^{[0]} - \frac{\bar{\Phi}^2}{4\bar{\Lambda}}) [dt + (\partial_i \bar{A}) dx^i], && \text{gener. funct. } \bar{\Phi} \\ & && \text{effective } \bar{\Lambda} \text{ for } \bar{\Upsilon}. \end{aligned} \right. \quad (\text{A.35}) \end{aligned}$$

Such cosmological metrics are generic off-diagonal and define new classes of solutions if the anholonomy coefficients $C_{\alpha\beta}^\gamma$, see formulas (5), are not zero for $N_k^3 = \partial_k \bar{n}$ and $N_i^4 = \partial_i \bar{A}$. We can analyze certain nonholonomic cosmological configurations determined, for instance, by data $(\bar{\Upsilon}, \bar{\Psi}, h_4^{[0]}, \bar{n}_k)$, when $\partial_k \bar{n} \rightarrow 0$ and $\bar{w}_i = \partial_i \bar{A} \rightarrow 0$ (we notat that 0 values can be fixed by certain additional constraints). Choosing data $(\bar{\Upsilon}(t), \bar{\Psi}(t), h_4^{[0]} = \text{const}, \bar{n}_k = \text{const})$, we can generate (off-) diagonal metrics of Bianchi, or FLRW, types and generalizations to other type configurations $g_{\alpha\beta}(t)$ in GR.

A.3 General polarization functions for stationary and cosmological solutions

Quadratic linear elements for exact off-diagonal solutions constructed in previous section can be parameterized in the form (7) [in terms of polarization functions $\eta_\alpha = (\eta_i, \eta_a)$ and η_i^a] defining nonholonomic deformations of a prime d-metric, $\hat{g}$, into a target d-metric, $\bar{g} = [g_\alpha = \eta_\alpha \hat{g}_\alpha, \eta_i^a \hat{N}_i^a] \rightarrow \bar{g}$. Such parameterizations are useful for analyzing possible physical implications of general off-diagonal deformations of some physically important solutions when, for instance, $\hat{g}$ is taken for a standard black hole, BH, or cosmological, solution in GR, or a MGT.

A.3.1 Stationary polarization functions

We write the stationary d-metrics in the forms

$$ds^2 = \eta_1(r, \theta) \hat{g}_1(r, \theta) [dx^1(r, \theta)]^2 + \eta_2(r, \theta) \hat{g}_2(r, \theta) [dx^2(r, \theta)]^2 \\ + \eta_3(r, \theta, \varphi) \hat{g}_3(r, \theta) \left[d\varphi + \eta_i^3(r, \theta, \varphi) \hat{N}_i^3(r, \theta) dx^i(r, \theta) \right]^2 \\ + \eta_4(r, \theta, \varphi) \hat{g}_4(r, \theta) \left[dt + \eta_i^4(r, \theta, \varphi) \hat{N}_i^4(r, \theta) dx^i(r, \theta) \right]^2,$$

where data $[\hat{g}_i(r, \theta), \hat{g}_a = \hat{h}_a(r, \theta); \hat{N}_k^3 = \hat{w}_k(r, \theta), \hat{N}_k^4 = \hat{n}_k(r, \theta)]$ define a BH metric diagonalizable by frame/ coordinate transforms.

The polarization functions for a general target stationary d-metric (A.19) can be parameterized

$$\eta_i = e^{\psi(x^k)} / \hat{g}_i; \eta_3 = \eta_3(r, \theta, \varphi) \text{ as a generating function;} \\ \eta_4 = -\frac{|h_3^{[0]} - \eta_3 \hat{h}_3|^\diamond}{|\Upsilon| \eta_3 \hat{h}_3 \hat{h}_4}; \\ \eta_i^3 = \frac{\partial_i (|h_3^{[0]} - \eta_3 \hat{h}_3| |\Upsilon| - \int dy^3 |h_3^{[0]} - \eta_3 \hat{h}_3| |\Upsilon|^\diamond)}{\hat{w}_i |h_3^{[0]} - \eta_3 \hat{h}_3|^\diamond |\Upsilon|}; \\ \eta_i^4 = \frac{1}{\hat{n}_k} + 4 \frac{2n_k}{\hat{n}_k} \int dy^3 \frac{\left[\left(\sqrt{|\Lambda[h_3^{[0]} - \eta_3 \hat{h}_3|]} \right)^\diamond \right]^2}{\Upsilon \Lambda |\eta_3 \hat{h}_3|^{5/2}}.$$

Other type generating functions with nonlinear symmetries (A.17) are functionals of $\eta_3(r, \theta, \varphi)$ and data for the prime d-metric,

$$\Phi^2 = 4|\Lambda[h_3^{[0]}(r, \theta) - \eta_3(r, \theta, \varphi) \hat{h}_3(r, \theta)]|, \\ \Psi^2 = 4(|h_3^{[0]}(r, \theta) - \eta_3(r, \theta, \varphi) \hat{h}_3(r, \theta)| |\Upsilon(r, \theta, \varphi)| \\ - \int dy^3 |h_3^{[0]}(r, \theta) - \eta_3(r, \theta, \varphi) \hat{h}_3(r, \theta)| |\Upsilon(r, \theta, \varphi)|^\diamond).$$

Target stationary off-diagonal metrics (A.20) with zero torsion can be generated by polarization functions

$$\eta_i = e^{\psi(x^k)} / \hat{g}_i; \eta_3 = \eta_3(r, \theta, \varphi) \text{ as a generating function;} \\ \eta_4 = -\frac{|h_3^{[0]} - \eta_3 \hat{h}_3|^\diamond}{|\Upsilon| \eta_3 \hat{h}_3 \hat{h}_4}; \eta_i^3 = \partial_i \bar{A} / \hat{w}_k, \eta_k^4 = \frac{\partial_k n}{\hat{n}_k}.$$

Above formulas can be simplified by choosing the integration functions in the form $h_3^{[0]} = \hat{h}_3$ and $1n_k = \hat{n}_k = \partial_k n$.

A.3.2 Locally anisotropic cosmological polarization functions

The locally anisotropic cosmological metrics are parameterized in the forms

$$ds^2 = \bar{\eta}_1(x^i, t) \bar{g}_1(x^i, t) [dx^1]^2 + \bar{\eta}_2(x^i, t) \bar{g}_2(x^i, t) [dx^1]^2 \\ + \bar{\eta}_3(x^i, t) \bar{h}_3(x^i, t) [dy^3 + \bar{\eta}_i^3(x^i, t) \bar{N}_i^3(x^k, t) dx^i]^2 \\ + \bar{\eta}_4(x^i, t) \bar{h}_4(x^i, t) [dt + \bar{\eta}_i^4(x^k, t) \bar{N}_i^4(x^k, t) dx^i]^2,$$

where data $[\bar{g}_i(x^i, t), \bar{g}_a = \bar{h}_a(x^i, t); \bar{N}_k^3 = \bar{n}_k(x^i), \bar{N}_k^4 = \bar{w}_k(x^i, t)]$ define, in general, a locally cosmological solution (in particular, it can be a frame/coordinate transform of the FLRW, or a Bianchi type metric, of a metric $\hat{g}_{\alpha\beta}(t)$ in GR). The target d-metrics $\bar{g}(x^k, t)$ are characterized by N-adapted coefficients $\bar{g}_i(x^k) = \bar{\eta}_i \bar{g}_i, h_a(x^i, t) = \bar{\eta}_a \bar{h}_a, \bar{N}_i^3 = \bar{\eta}_i^3(x^i, t) \bar{N}_i^3(x^k, t) = \bar{n}_i(x^k), \bar{N}_i^4 = \bar{\eta}_i^4(x^i, t) \bar{N}_i^4(x^k, t) = \bar{w}_i(x^k, t)$.

The polarization functions for a general target locally anisotropic cosmological d-metric (A.34) can be parameterized

$$\bar{\eta}_i = e^{\psi} / \bar{g}_i; \bar{\eta}_3 = -\frac{|h_4^{[0]} - \bar{\eta}_4 \hat{h}_4|^*}{|\Upsilon| \bar{\eta}_4 \hat{h}_4}; \\ \bar{\eta}_4 = \bar{\eta}_4(x^i, t) \text{ as a generating function;} \\ \bar{\eta}_i^3 = \frac{1}{\bar{n}_k} + 4 \frac{2n_k}{\bar{n}_k} \int dt \frac{[(\sqrt{|\Lambda[h_4^{[0]} - \bar{\eta}_4 \hat{h}_4]|})^*]^2}{\Upsilon \Lambda |\bar{\eta}_4 \hat{h}_4|^{5/2}}; \\ \bar{\eta}_k^4 = \frac{\partial_i (|h_4^{[0]} - \bar{\eta}_4 \hat{h}_4| |\Upsilon| - \int dt |h_4^{[0]} - \bar{\eta}_4 \hat{h}_4| |\Upsilon|^*)}{\hat{w}_k |h_4^{[0]} - \bar{\eta}_4 \hat{h}_4|^* |\Upsilon|}.$$

Corresponding generating functions subjected to transforms (A.32) are expressed in terms of polarization coefficient $\bar{\eta}_4$,

$$\bar{\Phi}^2 = 4|\Lambda[h_4^{[0]}(x^k) - \bar{\eta}_4(x^i, t) \hat{h}_4(x^k, t)]|,$$

$$\bar{\Psi}^2 = 4(|h_4^{[0]}(x^k) - \bar{\eta}_4(x^i, t)\dot{h}_4(x^i, t)|\bar{\Upsilon}(x^i, t)| \\ - \int dt |h_4^{[0]}(x^k) - \bar{\eta}_4(x^i, t)\dot{h}_4(x^i, t)|\bar{\Upsilon}(x^i, t)|^*).$$

Target off-diagonal cosmological metrics (A.35) with zero torsion can be generated by polarization functions

$$\bar{\eta}_i = e^{\psi/\dot{g}_i}; \bar{\eta}_3 = -\frac{|h_4^{[0]} - \bar{\eta}_4\dot{h}_4|^*}{|\bar{\Upsilon}|\bar{\eta}_4\dot{h}_4}; \\ \bar{\eta}_4 = \bar{\eta}_4(x^i, t) \text{ as a generating function; } \bar{\eta}_k^3 \\ = (\partial_k \bar{\eta})/\dot{n}_k; \\ \bar{\eta}_k^4 = \partial_k \bar{A}/\dot{w}_k.$$

Above formulas can be simplified by choosing the integration functions in the form $h_4^{[0]} = \dot{h}_4$ and ${}_1n_k = \dot{n}_k = \partial_k \bar{\eta}$. We use symbols with inverse “hat” following the conventions on integrability of generating functions which are similar to proofs of formula (A.20).

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