

Analysis of the triply-charmed pentaquark states with QCD sum rules

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Abstract In this article, we construct the scalar-diquark–scalar-diquark–antiquark type current to study the ground state triply-charmed pentaquark states with the QCD sum rules. We separate the contributions of the negative-parity and positive-parity triply-charmed pentaquark states explicitly, and take the energy scale formula $\mu = \sqrt{M_p^2 - (3\mathbb{M}_c)^2}$ to determine the optimal energy scales of the QCD spectral densities. The predicted pentaquark masses can be confronted to the experimental data in the future.

1 Introduction

The diquarks $\varepsilon^{ijk}q_j^T C\Gamma q'_k$ have five structures in Dirac spinor space, where $C\Gamma = C\gamma_5, C, C\gamma_\mu\gamma_5, C\gamma_\mu$ and $C\sigma_{\mu\nu}$ for the scalar, pseudoscalar, vector, axialvector and tensor diquarks, respectively, the i, j, k are color indexes. The attractive interaction of one-gluon exchange favors formation of the diquarks in color antitriplet $\bar{3}_c$, flavor antitriplet $\bar{3}_f$ and spin singlet 1_s or flavor sextet 6_f and spin triplet 3_s [1,2]. The calculations based on the QCD sum rules indicate that the favored configurations are the $C\gamma_5$ and $C\gamma_\mu$ diquark states [3–6], the light–light $\varepsilon^{ijk}q_j^T C\gamma_5 q'_k$ diquark states have much smaller masses than the corresponding $\varepsilon^{ijk}q_j^T C\gamma_\mu q'_k$ diquark states [5], while the heavy–light or heavy–heavy $\varepsilon^{ijk}q_j^T C\gamma_5 q'_k$ and $\varepsilon^{ijk}q_j^T C\gamma_\mu q'_k$ diquark states have almost degenerate masses [3,4,6]. All in all, the lowest states are the scalar diquark states, although the energy gaps between the scalar and axialvector diquark states are rather small in some cases. We can construct the lowest tetraquark states, pentaquark states and hexaquark states with the $C\gamma_5$ and $C\gamma_\mu$ diquark states or antiquark states. Experimentally, the Z_c^\pm (3900) and Z_c^\pm (4020/4025) observed by the BESIII collaboration [7–9], the Z_b^\pm (10610), Z_b^\pm (10650), Z_c (4200) $^\pm$ observed by the Belle collaboration [10–12], the Z_c^\pm (4430)

observed by the Belle collaboration [13] and confirmed by the LHCb collaboration [14], the P_c^+ (4380) and P_c^+ (4450) observed by the LHCb collaboration [15], provide excellent candidates for the hidden-charm or hidden-bottom tetraquark states and pentaquark states.

The QCD sum rules is a powerful nonperturbative tool in studying the ground state hadrons, and has given many successful descriptions of the hadronic parameters on the phenomenological side [16–18]. For example, the Z_c^\pm (3900) can be tentatively assigned to be the ground state $C\gamma_5 \otimes \gamma_\mu C - C\gamma_\mu \otimes \gamma_5 C$ type tetraquark state [19–25] or the $i\gamma_5 \otimes \gamma_\mu + \gamma_\mu \otimes i\gamma_5$ type molecular state [26–29]. In Ref. [19], we tentatively assign the X (3872) and Z_c^\pm (3900) to be the axialvector tetraquark states and study their masses with the QCD sum rules in a systematic way, and explore the energy scale dependence of the hidden-charm tetraquark states in details for the first time. In Ref. [30], we study the diquark–antidiquark type hidden-charm vector tetraquark states in details and suggest a formula,

$$\mu = \sqrt{M_{X/Y/Z}^2 - (2\mathbb{M}_c)^2}, \quad (1)$$

with the effective c -quark mass \mathbb{M}_c to determine the optimal energy scales of the QCD spectral densities in the QCD sum rules. The formula also works well for the diquark–diquark–antiquark type hidden-charm pentaquark states [31–35], and be extended to study the diquark–diquark–diquark type doubly-charmed hexaquark state to enhance the pole contribution [36].

In 2017, the LHCb collaboration observed the doubly-charmed baryon state Ξ_{cc}^{++} in the $\Lambda_c^+ K^- \pi^+ \pi^+$ mass spectrum [37]. The doubly heavy tetraquark state $QQ\bar{q}\bar{q}'$ is very similar to the doubly heavy baryon state QQq , where we have a light antidiquark $\bar{q}\bar{q}'$ instead of a light quark q in color triplet. The energy scale formula $\mu = \sqrt{M_{X/Y/Z}^2 - (2\mathbb{M}_Q)^2}$ also works well for the doubly heavy tetraquark states [38,39]. Recently, the triply heavy tetraquark states were studied in detailed with the QCD sum rules [40]. So it is

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interesting to study the triply heavy pentaquark states with the QCD sum rules. At the first step, we study the triply-charmed pentaquark states and explore the energy scale dependence of the QCD spectral densities. The diquark–diquark–antiquark type triply-charmed pentaquark states differ from the baryon-meson type triply-charmed molecular states remarkably, which have been studied with the potential models based on the heavy quark symmetry [41,42].

In this article, we choose the $u^T C \gamma_5 c - d^T C \gamma_5 c - \bar{c}$ type configuration to study the lowest $cc\bar{c}ud$ pentaquark state with $J^P = \frac{1}{2}^-$ by calculating the operator product expansion up to dimension 10 and extend the energy scale formula $\mu = \sqrt{M_{X/Y/Z}^2 - (2M_c)^2}$ to the new form $\mu = \sqrt{M_p^2 - (3M_c)^2}$ to determine the ideal energy scale of the QCD spectral density of the triply-charmed pentaquark state, as a byproduct, we also study the $J^P = \frac{1}{2}^+ cc\bar{c}ud$ pentaquark state with QCD sum rules. One may expect to study the lowest $cc\bar{c}ud$ pentaquark state with the configuration $c^T C \gamma_\mu c - u^T C \gamma^\mu d - \bar{c}$. Naively, we expect that the larger masses of the $C\gamma_\mu$ diquark states lead to larger tetraquark or pentaquark masses compared to the $C\gamma_5$ diquark states.

The article is arranged as follows: we derive the QCD sum rules for the masses and pole residues of the triply-charmed pentaquark states in Sect. 2; in Sect. 3, we present the numerical results and discussions; and Sect. 4 is reserved for our conclusion.

2 QCD sum rules for the $\frac{1}{2}^\pm$ pentaquark states

In the following, we write down the two-point correlation function $\Pi(p^2)$ in the QCD sum rules,

$$\Pi(p^2) = i \int d^4x e^{ip \cdot x} \langle 0 | T \{ J(x) \bar{J}(0) \} | 0 \rangle, \tag{2}$$

where

$$J(x) = \varepsilon^{ila} \varepsilon^{ijk} \varepsilon^{lmn} u_j^T(x) C \gamma_5 c_k(x) d_m^T(x) C \gamma_5 c_n(x) C \bar{c}_a^T(x), \tag{3}$$

the i, j, k, l, m, n and a are color indexes, the C is the charge conjugation matrix. In this article, we choose the scalar-diquark-scalar-diquark-antiquark type current $J(x)$ with $J^P = \frac{1}{2}^-$ to study the lowest triply-charmed pentaquark states with $J^P = \frac{1}{2}^\pm$ in a consistent way.

The current $J(0)$ has negative parity, and couples potentially to the $J^P = \frac{1}{2}^-$ triply-charmed pentaquark state P^- ,

$$\langle 0 | J(0) | P^-(p) \rangle = \lambda_- U^-(p, s), \tag{4}$$

the λ_- is the pole residue, the spinor $U^-(p, s)$ satisfies the Dirac equation $(\not{p} - M_-)U^-(p, s) = 0$, the s is the polarization or spin index of the spinor, and should be distinguished

from the s quark or the energy s . On the other hand, the current $J(0)$ also couples potentially to the $J^P = \frac{1}{2}^+$ triply-charmed pentaquark state P^+ , as multiplying $i\gamma_5$ to the current $J(0)$ changes its parity [31–35,43–50],

$$\langle 0 | J(0) | P^+(p) \rangle = \lambda_+ i\gamma_5 U^+(p, s), \tag{5}$$

the spinors $U^\pm(p, s)$ have analogous properties.

On the phenomenological side, we insert a complete set of intermediate pentaquark states with the same quantum numbers as the current operators $J(x)$ and $i\gamma_5 J(x)$ into the correlation function $\Pi(p^2)$ to obtain the hadronic representation [16–18]. After isolating the pole terms of the lowest states of the triply-charmed pentaquark states, we obtain the result:

$$\Pi(p^2) = \lambda_-^2 \frac{\not{p} + M_-}{M_-^2 - p^2} + \lambda_+^2 \frac{\not{p} - M_+}{M_+^2 - p^2} + \dots \tag{6}$$

Now we obtain the hadronic spectral density through dispersion relation,

$$\begin{aligned} \frac{\text{Im}\Pi(s)}{\pi} &= \not{p} \left[\lambda_-^2 \delta(s - M_-^2) + \lambda_+^2 \delta(s - M_+^2) \right] \\ &\quad + \left[M_- \lambda_-^2 \delta(s - M_-^2) - M_+ \lambda_+^2 \delta(s - M_+^2) \right] \\ &= \not{p} \rho_H^1(s) + \rho_H^0(s), \end{aligned} \tag{7}$$

where the subscript index H denotes the hadron side, then we introduce the weight function $\exp\left(-\frac{s}{T^2}\right)$ to obtain the QCD sum rules at the hadron side,

$$\begin{aligned} \int_{9m_c^2}^{s_0} ds \left[\sqrt{s} \rho_H^1(s) + \rho_H^0(s) \right] \exp\left(-\frac{s}{T^2}\right) \\ = 2M_- \lambda_-^2 \exp\left(-\frac{M_-^2}{T^2}\right), \end{aligned} \tag{8}$$

$$\begin{aligned} \int_{9m_c^2}^{s_0} ds \left[\sqrt{s} \rho_H^1(s) - \rho_H^0(s) \right] \exp\left(-\frac{s}{T^2}\right) \\ = 2M_+ \lambda_+^2 \exp\left(-\frac{M_+^2}{T^2}\right), \end{aligned} \tag{9}$$

where the s_0 are the continuum threshold parameters and the T^2 are the Borel parameters. We separate the contributions of the negative-parity (positive-parity) pentaquark states from the positive-parity (negative-parity) pentaquark states explicitly. There is no contamination comes from the positive or negative parity triply-charmed pentaquark state.

In the following, we briefly outline the operator product expansion for the correlation function $\Pi(p^2)$ in perturbative QCD. Firstly, we contract the u, d and c quark fields in the correlation function $\Pi(p^2)$ with Wick theorem, and obtain the result:

$$\begin{aligned} \Pi(p^2) = & -i \varepsilon^{ila} \varepsilon^{ijk} \varepsilon^{lmn} \varepsilon^{i'l'd'} \varepsilon^{j'k'} \varepsilon^{l'm'n'} \int d^4x e^{ip \cdot x} \text{times} \left\{ \text{Tr} \left[\gamma_5 C U_{jj'}^T(x) C \gamma_5 C_{kk'}(x) \right] \right. \\ & \times \text{Tr} \left[\gamma_5 C D_{mm'}^T(x) C \gamma_5 C_{nn'}(x) \right] C C_{a'a}^T(-x) C \\ & \left. - \text{Tr} \left[\gamma_5 C U_{jj'}^T(x) C \gamma_5 C_{kn'}(x) \gamma_5 C D_{mm'}^T(x) C \gamma_5 C_{nk'}(x) \right] C C_{a'a}^T(-x) C \right\}, \end{aligned} \tag{10}$$

where the $U_{ij}(x)$, $D_{ij}(x)$ and $C_{ij}(x)$ are the full u , d and c quark propagators, respectively (we can set $S_{ij}(x) = U_{ij}(x)$, $D_{ij}(x)$ in the chiral limit $m_u = m_d = 0$),

$$\begin{aligned} S_{ij}(x) = & \frac{i \delta_{ij} \not{x}}{2\pi^2 x^4} - \frac{\delta_{ij} \langle \bar{q} q \rangle}{12} - \frac{\delta_{ij} x^2 \langle \bar{q} g_s \sigma G q \rangle}{192} \\ & - \frac{i g_s G_{\alpha\beta}^a t_{ij}^a (\not{x} \sigma^{\alpha\beta} + \sigma^{\alpha\beta} \not{x})}{32\pi^2 x^2} \\ & - \frac{1}{8} \langle \bar{q}_j \sigma^{\mu\nu} q_i \rangle \sigma_{\mu\nu} + \dots, \end{aligned} \tag{11}$$

$$\begin{aligned} C_{ij}(x) = & \frac{i}{(2\pi)^4} \int d^4k e^{-ik \cdot x} \left\{ \frac{\delta_{ij}}{\not{k} - m_c} \right. \\ & - \frac{g_s G_{\alpha\beta}^a t_{ij}^a \sigma^{\alpha\beta} (\not{k} + m_c) + (\not{k} + m_c) \sigma^{\alpha\beta}}{4(k^2 - m_c^2)^2} \\ & \left. - \frac{g_s^2 (t^a t^b)_{ij} G_{\alpha\beta}^a G_{\mu\nu}^b (f^{\alpha\beta\mu\nu} + f^{\alpha\mu\beta\nu} + f^{\alpha\nu\mu\beta})}{4(k^2 - m_c^2)^5} + \dots \right\}, \\ f^{\alpha\beta\mu\nu} = & (\not{k} + m_c) \gamma^\alpha (\not{k} + m_c) \gamma^\beta (\not{k} + m_c) \gamma^\mu (\not{k} + m_c) \gamma^\nu (\not{k} + m_c), \end{aligned} \tag{12}$$

and $t^n = \frac{\lambda^n}{2}$, the λ^n is the Gell-Mann matrix [18, 19, 51]. We retain the term $\langle \bar{q}_j \sigma_{\mu\nu} q_i \rangle$ originates from Fierz re-ordering of the $\langle q_i \bar{q}_j \rangle$ to absorb the gluons emitted from other quark lines to form $\langle \bar{q}_j g_s G_{\alpha\beta}^a t_{mn}^a \sigma_{\mu\nu} q_i \rangle$ to extract the mixed condensate $\langle \bar{q} g_s \sigma G q \rangle$ [19]. Then we compute the integrals both in the coordinate and momentum spaces to obtain the correlation function $\Pi(p^2)$, therefore the QCD spectral densities $\rho_{QCD}^1(s)$ and $\rho_{QCD}^0(s)$ at the quark level through dispersion relation,

$$\frac{\text{Im}\Pi(s)}{\pi} = \not{p} \rho_{QCD}^1(s) + m_c \rho_{QCD}^0(s). \tag{13}$$

In this article, we carry out the operator product expansion to the vacuum condensates up to dimension-10, and assume vacuum saturation for the higher dimensional vacuum condensates. The condensates $\langle g_s^3 G G G \rangle$, $\langle \frac{\alpha_s G G}{\pi} \rangle^2$, $\langle \frac{\alpha_s G G}{\pi} \rangle \langle \bar{q} g_s \sigma G q \rangle$ have the dimensions 6, 8, 9 respectively, but they are the vacuum expectations of the operators of the order $\mathcal{O}(\alpha_s^{3/2})$, $\mathcal{O}(\alpha_s^2)$, $\mathcal{O}(\alpha_s^{3/2})$ respectively, and discarded [19].

Once the analytical QCD spectral densities $\rho_{QCD}^1(s)$ and $\rho_{QCD}^0(s)$ are obtained, we can take the quark-hadron duality below the continuum thresholds s_0 and introduce the weight function $\exp\left(-\frac{s}{T^2}\right)$ to obtain the QCD sum rules:

$$\begin{aligned} 2M_- \lambda_-^2 \exp\left(-\frac{M_-^2}{T^2}\right) = & \int_{9m_c^2}^{s_0} ds \left[\sqrt{s} \rho_{QCD}^1(s) \right. \\ & \left. + m_c \rho_{QCD}^0(s) \right] \exp\left(-\frac{s}{T^2}\right), \end{aligned} \tag{14}$$

$$\begin{aligned} 2M_+ \lambda_+^2 \exp\left(-\frac{M_+^2}{T^2}\right) = & \int_{9m_c^2}^{s_0} ds \left[\sqrt{s} \rho_{QCD}^1(s) - m_c \rho_{QCD}^0(s) \right] \\ & \times \exp\left(-\frac{s}{T^2}\right), \end{aligned} \tag{15}$$

where

$$\begin{aligned} \rho_{QCD}^1(s) = & \rho_0^1(s) + \rho_3^1(s) + \rho_4^1(s) + \rho_5^1(s) \\ & + \rho_6^1(s) + \rho_8^1(s) + \rho_{10}^1(s), \\ \rho_{QCD}^0(s) = & \rho_0^0(s) + \rho_3^0(s) + \rho_4^0(s) + \rho_5^0(s) \\ & + \rho_6^0(s) + \rho_8^0(s) + \rho_{10}^0(s), \end{aligned} \tag{16}$$

$$\begin{aligned} \rho_0^1(s) = & \frac{1}{40960\pi^8} \int dztr ztr (1-r-t-z)^2 (s - \widehat{m}_c^2)^4 (8s - 3\widehat{m}_c^2), \\ \rho_0^0(s) = & \frac{1}{40960\pi^8} \int dztr zt (1-r-t-z)^2 (s - \widehat{m}_c^2)^4 (7s - 2\widehat{m}_c^2), \end{aligned} \tag{17}$$

$$\begin{aligned} \rho_3^1(s) = & -\frac{m_c \langle \bar{q} q \rangle}{128\pi^6} \int dztr rz (1-r-t-z) (s - \widehat{m}_c^2)^2 (2s - \widehat{m}_c^2), \\ \rho_3^0(s) = & -\frac{m_c \langle \bar{q} q \rangle}{384\pi^6} \int dztr z (1-r-t-z) (s - \widehat{m}_c^2)^2 (5s - 2\widehat{m}_c^2), \end{aligned} \tag{18}$$

$$\begin{aligned} \rho_4^1(s) = & -\frac{m_c^2}{6144\pi^6} \left\langle \frac{\alpha_s G G}{\pi} \right\rangle \int dztr t \left(\frac{r}{z^2} + \frac{z}{2r^2} \right) \\ & \times (1-r-t-z)^2 (s - \widehat{m}_c^2)^2 (5s - 3\widehat{m}_c^2) \\ & + \frac{1}{2048\pi^6} \left\langle \frac{\alpha_s G G}{\pi} \right\rangle \int dztr rt (1-r-t-z) \\ & \times (s - \widehat{m}_c^2)^2 (2s - \widehat{m}_c^2) \\ & + \frac{3}{65536\pi^6} \left\langle \frac{\alpha_s G G}{\pi} \right\rangle \int dztr rtz \\ & \times (s - \widehat{m}_c^2)^2 (2s - \widehat{m}_c^2) \\ & + \frac{3m_c^2}{65536\pi^6} \left\langle \frac{\alpha_s G G}{\pi} \right\rangle \int dztr r \\ & \times (s - \widehat{m}_c^2)^2 \end{aligned}$$

$$\begin{aligned} & -\frac{3}{131072\pi^6} \left\langle \frac{\alpha_s GG}{\pi} \right\rangle \int dztr r(1-r-t-z)^2 \\ & \times (s - \widehat{m}_c^2)^2 (2s - \widehat{m}_c^2) \\ \rho_4^0(s) = & -\frac{m_c^2}{3072\pi^6} \left\langle \frac{\alpha_s GG}{\pi} \right\rangle \int dztr t \left(\frac{1}{z^2} + \frac{z}{2r^3} \right) (1-r-t-z)^2 \\ & \times (s - \widehat{m}_c^2)^2 (2s - \widehat{m}_c^2) \\ & + \frac{1}{12288\pi^6} \left\langle \frac{\alpha_s GG}{\pi} \right\rangle \int dztr \\ & \times \left(\frac{tz}{r^2} - \frac{3}{32} \right) (1-r-t-z)^2 \\ & \times (s - \widehat{m}_c^2)^2 (5s - 2\widehat{m}_c^2) \\ & + \frac{1}{6144\pi^6} \left\langle \frac{\alpha_s GG}{\pi} \right\rangle \int dztr t(1-r-t-z) \\ & \times (s - \widehat{m}_c^2)^2 (5s - 2\widehat{m}_c^2) \\ & + \frac{1}{65536\pi^6} \left\langle \frac{\alpha_s GG}{\pi} \right\rangle \int dztr tz (s - \widehat{m}_c^2)^2 (5s - 2\widehat{m}_c^2) \\ & + \frac{3m_c^2}{65536\pi^6} \left\langle \frac{\alpha_s GG}{\pi} \right\rangle \int dztr (s - \widehat{m}_c^2)^2, \end{aligned} \tag{19}$$

$$\begin{aligned} \rho_5^1(s) = & \frac{m_c \langle \bar{q} g_s \sigma G q \rangle}{512\pi^6} \int dztr rz \\ & \times \left(1 - \frac{1-r-t-z}{t} - \frac{3}{16} \right) \\ & \times (s - \widehat{m}_c^2) (5s - 3\widehat{m}_c^2), \\ \rho_5^0(s) = & \frac{m_c \langle \bar{q} g_s \sigma G q \rangle}{256\pi^6} \int dztr z \\ & \times \left(1 - \frac{1-r-t-z}{t} - \frac{3}{16} \right) \\ & \times (s - \widehat{m}_c^2) (2s - \widehat{m}_c^2), \end{aligned} \tag{20}$$

$$\begin{aligned} \rho_6^1(s) = & \frac{m_c^2 \langle \bar{q} q \rangle^2}{96\pi^4} \int dz t (1-t-z) (s - \overline{m}_c^2), \\ \rho_6^0(s) = & \frac{m_c^2 \langle \bar{q} q \rangle^2}{96\pi^4} \int dz t (s - \overline{m}_c^2), \end{aligned} \tag{21}$$

$$\begin{aligned} \rho_8^1(s) = & -\frac{m_c^2 \langle \bar{q} q \rangle \langle \bar{q} g_s \sigma G q \rangle}{64\pi^4} \int dz t (1-t-z) \\ & \times \left[1 + \frac{s}{3} \delta(s - \overline{m}_c^2) \right] \\ & + \frac{m_c^2 \langle \bar{q} q \rangle \langle \bar{q} g_s \sigma G q \rangle}{192\pi^4} \int dz t \frac{1-t-z}{t}, \\ \rho_8^0(s) = & -\frac{m_c^2 \langle \bar{q} q \rangle \langle \bar{q} g_s \sigma G q \rangle}{96\pi^4} \int dz t \\ & \times \left[1 + \frac{s}{2} \delta(s - \overline{m}_c^2) \right] \\ & + \frac{m_c^2 \langle \bar{q} q \rangle \langle \bar{q} g_s \sigma G q \rangle}{192\pi^4} \int dz t \frac{1}{t}, \end{aligned} \tag{22}$$

$$\begin{aligned} \rho_{10}^1(s) = & \frac{m_c^2 \langle \bar{q} g_s \sigma G q \rangle^2}{256\pi^4} \int dz t (1-t-z) \\ & \times \left(1 + \frac{2s}{3T^2} + \frac{s^2}{6T^4} \right) \delta(s - \overline{m}_c^2) \\ & - \frac{m_c^2 \langle \bar{q} g_s \sigma G q \rangle^2}{384\pi^4} \int dz t \frac{1-t-z}{t} \\ & \times \left(1 + \frac{s}{2T^2} \right) \delta(s - \overline{m}_c^2) \\ & + \frac{11m_c^2 \langle \bar{q} g_s \sigma G q \rangle^2}{18432\pi^4} \int dz t \frac{1-t-z}{tz} \delta(s - \overline{m}_c^2), \\ \rho_{10}^0(s) = & \frac{m_c^2 \langle \bar{q} g_s \sigma G q \rangle^2}{768\pi^4} \int dz t \\ & \times \left(1 + \frac{s}{T^2} + \frac{s^2}{2T^4} \right) \delta(s - \overline{m}_c^2) \\ & - \frac{m_c^2 \langle \bar{q} g_s \sigma G q \rangle^2}{768\pi^4} \int dz t \frac{1}{t} \left(1 + \frac{s}{T^2} \right) \delta(s - \overline{m}_c^2) \\ & + \frac{11m_c^2 \langle \bar{q} g_s \sigma G q \rangle^2}{18432\pi^4} \int dz t \frac{1}{tz} \delta(s - \overline{m}_c^2), \end{aligned} \tag{23}$$

$$\begin{aligned} \int dztr = & \int_{z_i}^{z_f} dz \int_{t_i}^{t_f} dt \int_{r_i}^{r_f} dr, \\ \int dz t = & \int_{z_i}^{z_f} dz \int_{t_i}^{t_f} dt, \end{aligned} \tag{24}$$

$$\begin{aligned} z_{f/i} = & \frac{\hat{s} - 3 \pm \sqrt{(\hat{s} - 3)^2 - 4\hat{s}}}{2\hat{s}}, \\ t_{f/i} = & \frac{1 - z \pm \sqrt{(1-z)^2 - 4\frac{z-z^2}{z\hat{s}-1}}}{2}, \\ r_i = & \frac{tz}{tz\hat{s} - t - z}, \\ r_f = & 1 - z - t, \end{aligned} \tag{25}$$

$$\begin{aligned} \widehat{m}_c^2 = & \frac{m_c^2}{t} + \frac{m_c^2}{z} + \frac{m_c^2}{r}, \\ \overline{m}_c^2 = & \frac{m_c^2}{t} + \frac{m_c^2}{z} + \frac{m_c^2}{1-t-z}, \\ \hat{s} = & \frac{s}{m_c^2}, \end{aligned} \tag{26}$$

$$\int dz t \rightarrow \int_0^1 dz \int_0^{1-z} dt, \tag{27}$$

when the δ function $\delta(s - \overline{m}_c^2)$ appears. We derive Eqs. (14, 15) with respect to $\tau = \frac{1}{T^2}$, then eliminate the pole residues λ_{\pm} and obtain the QCD sum rules for the masses of the triply-charmed pentaquark states,

$$M_-^2 = \frac{-\frac{d}{d\tau} \int_{9m_c^2}^{s_0} ds \left[\sqrt{s} \rho_{QCD}^1(s) + m_c \rho_{QCD}^0(s) \right] \exp(-s\tau)}{\int_{9m_c^2}^{s_0} ds \left[\sqrt{s} \rho_{QCD}^1(s) + m_c \rho_{QCD}^0(s) \right] \exp(-s\tau)}, \tag{28}$$

$$M_+^2 = \frac{-\frac{d}{d\tau} \int_{9m_c^2}^{s_0} ds \left[\sqrt{s} \rho_{QCD}^1(s) - m_c \rho_{QCD}^0(s) \right] \exp(-s\tau)}{\int_{9m_c^2}^{s_0} ds \left[\sqrt{s} \rho_{QCD}^1(s) - m_c \rho_{QCD}^0(s) \right] \exp(-s\tau)}. \tag{29}$$

3 Numerical results and discussions

We take the standard values of the vacuum condensates $\langle \bar{q}q \rangle = -(0.24 \pm 0.01 \text{ GeV})^3$, $\langle \bar{q}g_s\sigma Gq \rangle = m_0^2 \langle \bar{q}q \rangle$, $m_0^2 = (0.8 \pm 0.1) \text{ GeV}^2$, $\langle \frac{\alpha_s G G}{\pi} \rangle = (0.33 \text{ GeV})^4$ at the energy scale $\mu = 1 \text{ GeV}$ [16–18, 52], and take the \overline{MS} mass $m_c(m_c) = (1.28 \pm 0.03) \text{ GeV}$ from the Particle Data Group [53]. Furthermore, we set $m_u = m_d = 0$ due to the small current quark masses. We take into account the energy-scale dependence of the input parameters from the renormalization group equation,

$$\begin{aligned} \langle \bar{q}q \rangle(\mu) &= \langle \bar{q}q \rangle(Q) \left[\frac{\alpha_s(Q)}{\alpha_s(\mu)} \right]^{\frac{12}{25}}, \\ \langle \bar{q}g_s\sigma Gq \rangle(\mu) &= \langle \bar{q}g_s\sigma Gq \rangle(Q) \left[\frac{\alpha_s(Q)}{\alpha_s(\mu)} \right]^{\frac{2}{25}}, \\ m_c(\mu) &= m_c(m_c) \left[\frac{\alpha_s(\mu)}{\alpha_s(m_c)} \right]^{\frac{12}{25}}, \\ \alpha_s(\mu) &= \frac{1}{b_0 t} \\ &\times \left[1 - \frac{b_1 \log t}{b_0^2 t} + \frac{b_1^2 (\log^2 t - \log t - 1) + b_0 b_2}{b_0^4 t^2} \right], \end{aligned} \tag{30}$$

where $t = \log \frac{\mu^2}{\Lambda^2}$, $b_0 = \frac{33-2n_f}{12\pi}$, $b_1 = \frac{153-19n_f}{24\pi^2}$, $b_2 = \frac{2857 - \frac{5033}{9}n_f + \frac{325}{27}n_f^2}{128\pi^3}$, $\Lambda = 210 \text{ MeV}$, 292 MeV and 332 MeV for the flavors $n_f = 5, 4$ and 3 , respectively [53–55], and evolve all the input parameters to the optimal energy scales to extract the masses of the triply-charmed pentaquark states with the flavor $n_f = 4$.

In the article, we study the scalar-diquark–scalar-diquark–antiquark type pentaquark state, which consists of two charmed diquark states and a charmed antiquark. In the heavy quark limit, the c -quark serves as a static well potential and combines with a light quark q to form a charmed diquark in color antitriplet, or combines with a light antiquark \bar{q} to form a charmed meson in color singlet (meson-like state in color octet),

$$\begin{aligned} q^j + c^k &\rightarrow \varepsilon^{ijk} q^j c^k, \\ \bar{q}^j + c^k &\rightarrow \bar{q}^j \delta_{jk} c^k \quad (\bar{q}^j \lambda_{jk}^a c^k), \end{aligned} \tag{31}$$

where the i, j, k are color indexes, the λ^a is Gell-Mann matrix. Then

$$\varepsilon^{ijk} q^j c^k + \varepsilon^{lmn} q'^m c^n + \bar{c}^a \rightarrow \text{compact pentaquark states.} \tag{32}$$

The five-quark systems $qq'cc\bar{c}$ are characterized by the effective charmed quark mass \mathbb{M}_c (or constituent quark mass) and the virtuality $V = \sqrt{M_P^2 - (3\mathbb{M}_c)^2}$ (or bound energy not as robust), where the P denotes the triply-charmed pentaquark states. It is natural to set the energy scales of the QCD spectral densities to be $\mu = V$. In Refs. [19, 20, 26, 27, 30–36, 38, 39, 56, 57], we study the acceptable energy scales of the QCD spectral densities for the hidden-charm (hidden-bottom) tetraquark states and molecular states, hidden-charm pentaquark states, hidden-charm hexaquark states, and doubly-heavy tetraquark states in the QCD sum rules in details, and suggest an energy scale formula $\mu = \sqrt{M_{X/Y/Z/P}^2 - (2\mathbb{M}_Q)^2}$ to determine the optimal energy scales, which works well. The updated values of the effective heavy quark masses are $\mathbb{M}_c = 1.82 \text{ GeV}$ and $\mathbb{M}_b = 5.17 \text{ GeV}$ for the multi-quark states having heavy–light diquark states [58, 59]. Now we use the energy scale formula,

$$\mu = \sqrt{M_P^2 - (3\mathbb{M}_c)^2}, \tag{33}$$

to determine the ideal energy scales of the QCD spectral densities.

In this article, we take the continuum threshold parameters as $\sqrt{s_0} = M_P + (0.4 \sim 0.7) \text{ GeV}$, and vary the parameters $\sqrt{s_0}$ to obtain the optimal Borel parameters T^2 to satisfy the following four criteria:

1. Pole dominance on the phenomenological side;
2. Convergence of the operator product expansion;
3. Appearance of the Borel platforms;
4. Satisfying the energy scale formula.

In calculations, we observe that

$$\begin{aligned} \mu \uparrow & \quad M_P \downarrow, \\ \mu \downarrow & \quad M_P \uparrow, \end{aligned} \tag{34}$$

from the QCD sum rules in Eqs. (28, 29). On the other hand, the energy scale formula indicates that

$$\begin{aligned} \mu \uparrow & \quad M_P \uparrow, \\ \mu \downarrow & \quad M_P \downarrow, \end{aligned} \tag{35}$$

as it can be rewritten as

$$M_P = \sqrt{\mu^2 + (3\mathbb{M}_c)^2}. \tag{36}$$

Table 1 The Borel parameters (Borel windows), continuum threshold parameters, optimal energy scales, pole contributions, masses and pole residues for the triply-charmed pentaquark states

	T^2 (GeV ²)	$\sqrt{s_0}$ (GeV)	μ (GeV)	pole	M (GeV)	λ (GeV ⁶)
$\frac{1}{2}^-$	2.9–3.3	6.3 ± 0.1	1.3	(71–87)%	5.61 ± 0.10	$(2.38 \pm 0.31) \times 10^{-3}$
$\frac{1}{2}^+$	4.1–4.5	6.4 ± 0.1	1.7	(42–61)%	5.72 ± 0.10	$(1.45 \pm 0.28) \times 10^{-3}$

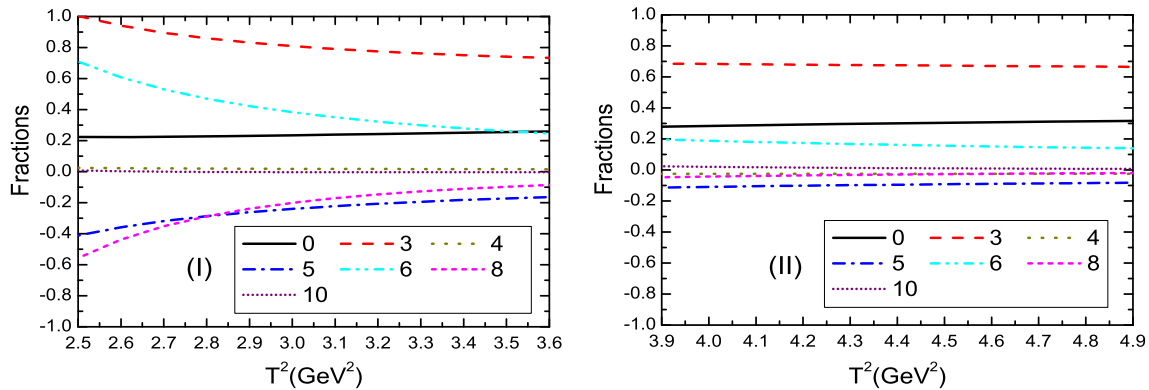


Fig. 1 The contributions of the vacuum condensates of dimension n with $n = 0, 3, 4, 5, 6, 8$ and 10 , where the (I) and (II) denote the negative parity and positive parity pentaquark states, respectively.

It is difficult to obtain the optimal energy scales μ and masses M_P , however, the optimal energy scales μ and masses M_P do exist. The resulting Borel parameters or Borel windows T^2 , continuum threshold parameters s_0 , optimal energy scales of the QCD spectral densities, pole contributions of the ground states are shown explicitly in Table 1.

In Fig. 1, we plot the contributions of the vacuum condensates D_n of dimension n in the operator product expansion for the central values of the input parameters,

$$D_n = \frac{\int_{9m_c^2}^{s_0} ds [\sqrt{s}\rho_n^1(s) \pm m_c\rho_n^0(s)] \exp\left(-\frac{s}{T^2}\right)}{\int_{9m_c^2}^{s_0} ds [\sqrt{s}\rho_{QCD}^1(s) \pm m_c\rho_{QCD}^0(s)] \exp\left(-\frac{s}{T^2}\right)}. \tag{37}$$

From the figure, we can see that the dominant contributions come from the quark condensate D_3 , the contributions of the perturbative terms (or D_0) are about (20–30)%, so in this article we approximate the continuum contributions as $[\sqrt{s}\rho_{QCD}^1(s) \pm m_c\rho_{QCD}^0(s)] \Theta(s - s_0)$, and define the pole contributions PC as

$$PC = \frac{\int_{9m_c^2}^{s_0} ds [\sqrt{s}\rho_{QCD}^1(s) \pm m_c\rho_{QCD}^0(s)] \exp\left(-\frac{s}{T^2}\right)}{\int_{9m_c^2}^{\infty} ds [\sqrt{s}\rho_{QCD}^1(s) \pm m_c\rho_{QCD}^0(s)] \exp\left(-\frac{s}{T^2}\right)}. \tag{38}$$

From Table 1, we can see that the pole dominance condition can be well satisfied. Although the contributions of the vacuum condensate of dimension $n = 3$ are very large, the contributions of the vacuum condensates of dimensions 6, 8, 10

have the hierarchy $D_6 \gg |D_8| \gg |D_{10}|$, the operator product expansion is convergent. Now the criterion 1 and criterion 2 are satisfied.

We take into account all uncertainties of the input parameters, and obtain the values of the masses and pole residues of the triply-charmed pentaquark states, which are shown explicitly in Table 1 and Figs. 2, 3. In Figs. 2, 3, we plot the masses and pole residues of the triply-charmed pentaquark states in much larger ranges than the Borel windows. From the figures, we can see that the platforms for the mass and pole residue of the $J^P = \frac{1}{2}^-$ pentaquark state appear as the minimum values, the Borel platforms are very flat, while the predicted mass and pole residue of the $J^P = \frac{1}{2}^+$ pentaquark state increase slowly with the increase of the Borel parameter, we determine the platform by requiring the uncertainty $\frac{\delta M_P}{M_P}$ induced by the Borel parameter is less than 1%. The criterion 3 is also satisfied, furthermore, the energy scale formula $\mu = \sqrt{M_P^2 - (3M_c)^2}$ is well satisfied. Now the four criteria are all satisfied, we expect to make reliable predictions.

If the energy scale formula survives, the masses of the lowest triply-charmed pentaquark states should be larger than $\sqrt{(1\text{GeV})^2 + (3M_c)^2} = 5.55\text{ GeV}$. In Fig. 4, we plot the predicted masses of the negative parity and positive parity triply-charmed pentaquark states with variations of the Borel parameters T^2 for the threshold parameters $\sqrt{s_0} = 6.3\text{ GeV}$ and 6.4 GeV , respectively. From the figure, we can see that the most flat platform appears at the energy scale $\mu = 1.3\text{ GeV}$ for the $J^P = \frac{1}{2}^-$ triply-charmed pentaquark state, which happens to be the optimal energy scale deter-

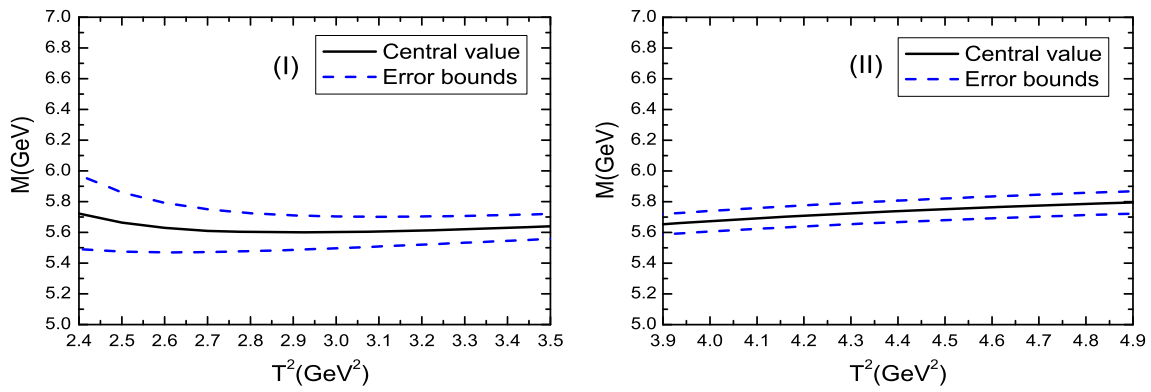


Fig. 2 The masses of the triply-charmed pentaquark states with variations of the Borel parameters, where the (I) and (II) denote the negative parity and positive parity pentaquark states, respectively

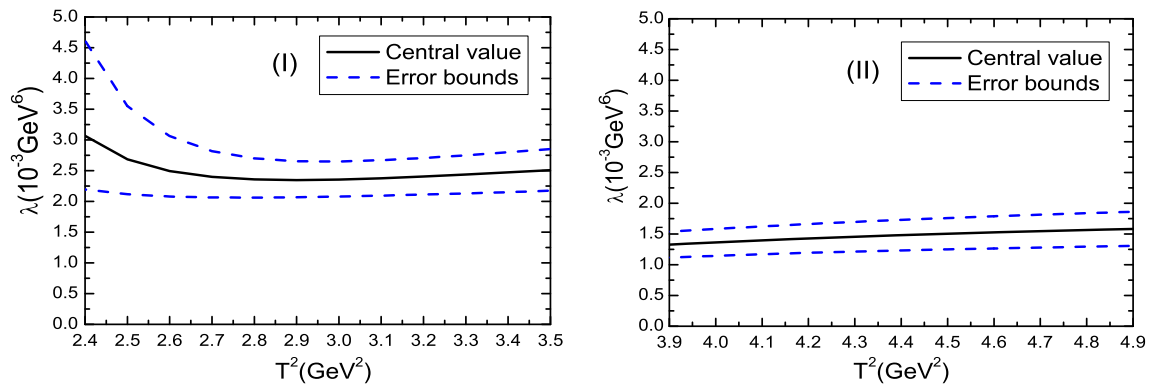


Fig. 3 The pole residues of the triply-charmed pentaquark states with variations of the Borel parameters, where the (I) and (II) denote the negative parity and positive parity pentaquark states, respectively

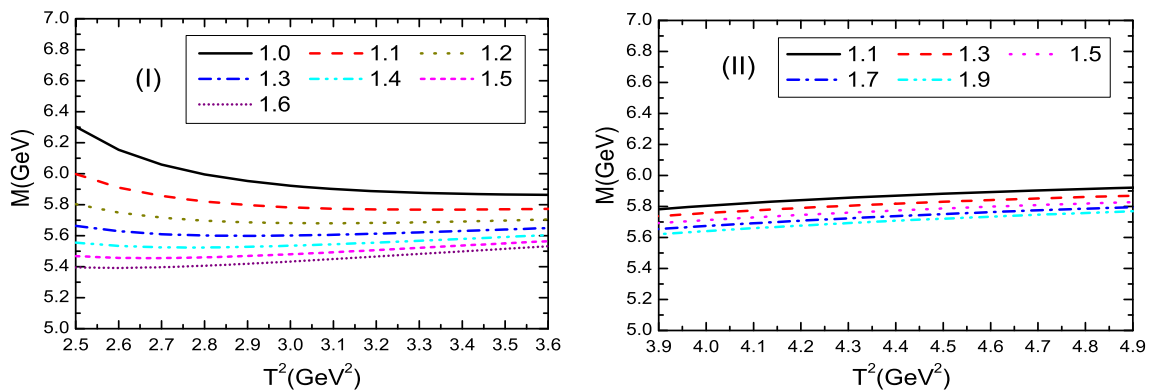


Fig. 4 The masses of the triply-charmed pentaquark states with variations of the Borel parameters T^2 and energy scales μ , where the (I) and (II) denote the negative parity and positive parity pentaquark states, respectively, the 1.0, 1.1, 1.2, ... denote the energy scales μ of the QCD spectral densities

mined by the energy scale formula. For the $J^P = \frac{1}{2}^+$ triply-charmed pentaquark state, no platform is more flat than others, we determine the optimal energy scale by the energy scale formula $\mu = \sqrt{M_P^2 - (3M_c)^2}$. In this article, we obtain the continuum threshold parameters $\sqrt{s_0} = M_P + (0.6 \sim 0.8) \text{ GeV}$. In previous works [31–35, 46–50], we study the heavy, doubly-heavy, triply heavy baryon states

and hidden-charm pentaquark states in a systematic way with the QCD sum rules, the continuum threshold parameters $\sqrt{s_0} = M_{gr} + (0.6 \sim 0.8) \text{ GeV}$ work well, where the gr denotes the ground states. We expect that the relation survives for the triply-charmed pentaquark states.

In the QCD sum rules for the M_+ , we choose much larger Borel parameter T^2 than that for the M_- , see Table 1, the

contributions of the higher dimensional vacuum condensates especially the terms associate with $\frac{1}{T^2}$ and $\frac{1}{T^4}$ in the QCD spectral density are greatly suppressed. Moreover, in the QCD sum rules for the M_+ , the higher dimensional vacuum condensates obtain additional suppression due to the special combination of the QCD spectral densities $\rho_{QCD}^1(s)$ and $\rho_{QCD}^0(s)$, $\sqrt{s}\rho_{QCD}^1(s) - m_c\rho_{QCD}^0(s)$, which also leads to a relation between the pole residues $\lambda_- \approx 1.5\lambda_+$. From Table 1 and Fig. 1, we can see that in the Borel window in the QCD sum rules for the M_+ , the dominant contributions come from the perturbative term plus the quark condensate term, the higher dimensional vacuum condensates play a minor important role, which is in contrast to the QCD sum rules for the M_- . So the extracted mass M_+ is less sensitive to the energy scale μ of the QCD spectral density than the extracted mass M_- , see Fig. 4.

The triply-charmed pentaquark states $P_{cc\bar{c}ud}$ can be produced in the pp collisions at the Large Hadron Collider,

$$\begin{aligned}
 pp &\rightarrow \Xi_{bcu}^+ X \rightarrow P_{cc\bar{c}ud}^+ K^- \pi^+ X, \\
 pp &\rightarrow \Xi_{bbu}^0 X \rightarrow P_{cc\bar{c}ud}^+ K^- \pi^+ \pi^- X,
 \end{aligned}
 \tag{39}$$

through the decays $b \rightarrow c\bar{c}s$ and $b \rightarrow c\bar{u}d$ at the quark level, where the superscript + of the $P_{cc\bar{c}ud}^+$ denotes the electronic charge. The triply-charmed pentaquark states $P_{cc\bar{c}ud}$ can also be produced in the $\Lambda_Q \Lambda_{Q'}$ fusions [60],

$$\begin{aligned}
 \Lambda_b \Lambda_c &\rightarrow \Xi_{bcu}^+ n \rightarrow P_{cc\bar{c}ud}^+ K^- \pi^+ n, \\
 \Lambda_b \Lambda_b &\rightarrow \Xi_{bbu}^0 n \rightarrow P_{cc\bar{c}ud}^+ K^- \pi^+ \pi^- n.
 \end{aligned}
 \tag{40}$$

We can search for the $P_{cc\bar{c}ud}$ states in their two-body strong decays in the future.

In the following, we perform Fierz re-arrangement to the current $J(x)$ both in the color and Dirac-spinor spaces to obtain the result,

$$\begin{aligned}
 J &= -\frac{1}{4}S\gamma_5c\bar{c}d + \frac{1}{4}S\gamma_\lambda\gamma_5c\bar{c}\gamma^\lambda d \\
 &+ \frac{1}{8}S\sigma_{\lambda\tau}\gamma_5c\bar{c}\sigma^{\lambda\tau}d + \frac{1}{4}S\gamma_\lambda c\bar{c}\gamma^\lambda\gamma_5d \\
 &+ \frac{1}{4}iSc\bar{c}i\gamma_5d \\
 &+ \frac{1}{4}S\gamma_5d\bar{c}c - \frac{1}{4}S\gamma_\lambda\gamma_5d\bar{c}\gamma^\lambda c \\
 &- \frac{1}{8}S\sigma_{\lambda\tau}\gamma_5d\bar{c}\sigma^{\lambda\tau}c \\
 &- \frac{1}{4}S\gamma_\lambda d\bar{c}\gamma^\lambda\gamma_5c - \frac{1}{4}iSd\bar{c}i\gamma_5c,
 \end{aligned}
 \tag{41}$$

and introduce the notations $S\Gamma c = \varepsilon^{ijk}u_i^T C\gamma_5c_j\Gamma c_k$ and $S\Gamma d = \varepsilon^{ijk}u_i^T C\gamma_5c_j\Gamma d_k$ for simplicity, here the Γ denotes the Dirac matrixes.

The components $S(x)\Gamma c(x)\bar{c}(x)\Gamma'd(x)$ and $S(x)\Gamma d(x)\bar{c}(x)\Gamma'c(x)$ couple potentially to the baryon-meson pairs. The revelent thresholds are $M_{\eta_c\Lambda_c^+} = 5.270$ GeV, $M_{\eta_c\Lambda_c^+(2595)} =$

5.576 GeV, $M_{\eta_c\Sigma_c^+(2455)} = 5.436$ GeV, $M_{\eta_c\Sigma_c^+(2520)} = 5.501$ GeV, $M_{J/\psi\Lambda_c^+} = 5.383$ GeV, $M_{J/\psi\Lambda_c^+(2595)} = 5.689$ GeV, $M_{J/\psi\Sigma_c^+(2455)} = 5.550$ GeV, $M_{J/\psi\Sigma_c^+(2520)} = 5.614$ GeV [53], $M_{\Xi_{cc}^{++}D^-} = 5.491$ GeV, $M_{\Xi_{cc}^{++}D^{*-}} = 5.632$ GeV [37]. After taking into account the currents-hadrons duality, we obtain the Okubo-Zweig-Iizuka super-allowed decays,

$$\begin{aligned}
 P\left(\frac{1^-}{2}\right) &\rightarrow \eta_c\Lambda_c^+, \eta_c\Lambda_c^+(2595), \eta_c\Sigma_c^+(2455), \eta_c\Sigma_c^+(2520), \\
 &J/\psi\Lambda_c^+, J/\psi\Sigma_c^+(2455), \Xi_{cc}^{++}D^-,
 \end{aligned}
 \tag{42}$$

$$\begin{aligned}
 P\left(\frac{1^+}{2}\right) &\rightarrow \eta_c\Lambda_c^+, \eta_c\Lambda_c^+(2595), \eta_c\Sigma_c^+(2455), \eta_c\Sigma_c^+(2520), J/ \\
 &\psi\Lambda_c^+, J/\psi\Sigma_c^+(2455), \\
 &J/\psi\Lambda_c^+(2595), J/\psi\Sigma_c^+(2520), \Xi_{cc}^{++}D^-, \Xi_{cc}^{++}D^{*-},
 \end{aligned}
 \tag{43}$$

We can search for the triply-charmed pentaquark states $P_{cc\bar{c}ud}$ in those decays in the future. The LHCb collaboration observed the $P_c(4380)$ and $P_c(4450)$ in the process,

$$pp \rightarrow \Lambda_b X \rightarrow P_c(4380/4450)^+ K^- \rightarrow J/\psi p K^- X, \tag{44}$$

in the $J/\psi p$ invariant mass-spectrum [15]. The triply-charmed pentaquark states can be observed analogously, for example, in the process

$$pp \rightarrow \Xi_{bcu}^+ X \rightarrow P_{cc\bar{c}ud}^+ K^- \pi^+ X \rightarrow J/\psi\Lambda_c^+ K^- \pi^+ X, \tag{45}$$

in the $J/\psi\Lambda_c^+$ invariant mass-spectrum. The LHCb collaboration have observed the doubly-charmed baryon state Ξ_{cc}^{++} in the $\Lambda_c^+ K^- \pi^+ \pi^+$ invariant mass-spectrum [37], the triply-charmed pentaquark states $P_{cc\bar{c}ud}$ may be observed in the future.

4 Conclusion

In this article, we construct the scalar-diquark-scalar-diquark-antiquark type current to interpolate the ground state triply-charmed pentaquark states with $J^P = \frac{1}{2}^\pm$, and carry out the operator product expansion up to the vacuum condensates of dimension 10 consistently. We obtain the QCD spectral densities through dispersion relation and separate the contributions of the negative-parity and positive parity triply-charmed pentaquark states explicitly. Then we extract the masses and pole residues in the Borel windows at the optimal energy scales of the QCD spectral densities, which are determined by the energy scale formula $\mu = \sqrt{M_p^2 - (3M_c)^2}$. Experimentally, there are no candidates for the triply-charmed pentaquark states, we can search for the triply-charmed pentaquark states in the Okubo-Zweig-Iizuka super-allowed strong decays in the future.

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