Some families of relativistic anisotropic compact stellar models embedded in pseudo-Euclidean space E^5 : an algorithm

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Abstract This work presents an algorithm for modeling a static spherically symmetric compact object satisfying the Eiesland condition (Eiesland in Am Math Soc 27:213, 1925), a necessary and sufficient condition that a general centro-symmetric space shall be of the class 1. The model parameters are obtained accordingly by employing the boundary conditions to the interior solutions. The equation of state has been extracted from the solution and one found that it is *almost* linear. These models are free of physical and geometric singularities and satisfy the necessary physical conditions to have astrophysical significance. The central and surface densities, and pressures of some compact stars like PSR J1614-2230, Vela X-1 have been calculated from these models. Detailed analyses of these models have also been made with the help of numerical and graphical studies.

1 Introduction

The study of static spherically symmetric perfect fluid solution of Einstein's field equations to model isolated systems like massive compact astrophysical stellar object has been a key issue in relativistic astrophysics since the work of Schwarzschild [2,3], Tolman [4], and Oppenheimer and Volkoff [5]. Two traditional approaches usually are followed to obtain a realistic stellar model. In one approach one needs to solve Einstein's gravitational field equations, an under-

determined system of nonlinear ordinary differential equations of second order. For the special case of a static isotropic perfect fluid the field equations can be reduced to a set of three coupled ordinary differential equations in four unknowns. In arriving at exact solutions, one can solve the field equations by making an ad hoc assumption for one of the metric functions or for the energy density (the Tolman [4] method). Hence the equation of state can be computed from the resulting metric. As might be expected with Tolman's method, unphysical pressure–density configurations are found more frequently than physical ones.

In the other approach one can start with an explicit equation of state and the integration starts at the center of the star with a prescribed central pressure (Oppenheimer–Volkoff [5] method). The integrations are iterated until the pressure decreases to zero, indicating the surface of the star has been reached. Such input equations of state do not normally yield closed-form solutions.

Compact astrophysical objects may not be necessarily entirely composed of a perfect fluid (i.e., the principal stresses are equal). The central energy density of such compact objects could be of the order of $10^{15} \text{ g cm}^{-3}$, several times higher than the normal nuclear matter density. Since the theoretical investigations [6–13] the pressure anisotropy has become one of the most important factors in the study of compact stellar objects. The realistic stellar models show that the nuclear matter may be anisotropic at least in certain very high density ranges ($\rho > 10^{15} \text{ g cm}^{-3}$), where the nuclear interactions must be treated relativistically. According to these views, in such massive stellar objects the radial pressure may not be equal to the tangential pressure. Bowers and Liang [14] first generalized the equation of hydrostatic equilibrium for the case local anisotropy. Since their pioneering work there has been published an extensive literature devoted to studying anisotropic spherically symmetric static general relativistic neutral/charged configurations, showing that the assumption of local isotropy is a too stringent condi-

Few years back in one of my papers ([15] published by EPJ C) I acknowledged the encouragement, enthusiasm, support, and keen interest that Prof. A. A. Z. Ahmad, former Chairperson, Department of Mathematics and Natural Sciences, BRAC University, had always shown me.

This work is respectfully dedicated to the memory of Prof. Ahmad (1938–2018), our beloved professor of physics, who passed away during the review of this manuscript. The publication of this paper would definitely brought him a lot of happiness.

With his death we have lost a great mentor and a creative, thoughtful and active member of the physics community.

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tion which may excessively constrain the modeling of self-gravitating objects.

Therefore, it is always interesting to relax the condition that matter within the star is a perfect fluid and explore the consequences produced by deviations from local isotropy under a variety of circumstances which gives rise to observable and measurable properties of the stellar configuration. Over the years continuously growing interests enabled researchers to develop mathematically simple, exact analytical models of self-bound strange stars within the framework of a linear equation of state based on the MIT bag model together with a particular choice of metric potentials/mass function. References may be found in [15]. Some recent references include [16–31].

As stated, the principal motivation of this work is to develop some new analytical relativistic stellar models by obtaining closed-form solutions of Einstein field equations. The solutions obtained by satisfying applicable physical boundary conditions provide a mathematically simple family of compact stars.

Our plan is to organize the paper as follows. In Sect. 2, we present the Einstein field equations for the static spherically symmetric fluid distribution. This section also contains the necessary and sufficient conditions for a Riemannian metric to be class 1. In Sect. 3 we assume a particular form of one of the metric potentials and solve Einstein field equations explicitly. In Sect. 4 we present the elementary criteria to be satisfied by the interior solution so as to present a realistic stellar model. The interior spacetime will be matched to the exterior spacetime described by the unique Schwarzschild metric in Sect. 5. Physically realistic fluid models will be constructed and a stability analysis will be made on the models obtained in Sects. 6 and 7. In Sect. 8 we apply the model solutions to estimate the range of values of some physical quantities such as central density, surface density, and central pressure of various compact objects having mass and radius similar to PSR J1614-2230 and Vela X-1. Finally, Sect. 9 discusses and concludes the work with physical analysis and results.

2 The Einstein field equations

The interior of a static spherically symmetric object is described by the line element

$$ds^2 = e^{v(r)} dt^2 - e^{\lambda(r)} dr^2 - r^2 d\Omega^2 \quad (2.0.1)$$

where $d\Omega^2 = d\theta^2 + \sin^2 \theta d\varphi^2$ is the usual *standard metric* on a 2-sphere S^2 . Let us assume that the matter within the star is locally anisotropic in nature and correspondingly the energy-momentum tensor in a locally instantaneous rest frame is described by

$$T_{\hat{\nu}}^{\hat{\mu}} = \text{diag}(\rho, -P_r, -P_t, -P_t) \quad (2.0.2)$$

where ρ , P_r and P_t are the energy density, radial pressure and tangential pressure, respectively. The Einstein field equations for the line element (2.0.1) and the energy-momentum tensor (2.0.2) are given by [14],

$$\kappa\rho = \frac{(1 - e^{-\lambda})}{r^2} + \frac{\lambda' e^{-\lambda}}{r}, \quad (2.0.3)$$

$$\kappa P_r = \frac{v' e^{-\lambda}}{r} - \frac{(1 - e^{-\lambda})}{r^2}, \quad (2.0.4)$$

$$\kappa P_t = \frac{e^{-\lambda}}{4} \left(2v'' + v'^2 - v'\lambda' + \frac{2v'}{r} - \frac{2\lambda'}{r} \right), \quad (2.0.5)$$

where $\kappa = 8\pi$ in geometrized units $c = G = 1$, and primes represent differentiation with respect to the radial coordinate r .

Using Eqs. (2.0.4) and (2.0.5), one can write

$$\begin{aligned} \Delta &= \kappa(P_t - P_r) \\ &= e^{-\lambda} \left[\frac{v''}{2} - \frac{\lambda' v'}{4} + \frac{v'^2}{4} - \frac{v' + \lambda'}{2r} + \frac{e^{\lambda} - 1}{r^2} \right] \end{aligned} \quad (2.0.6)$$

where Δ is called the anisotropic factor; it measures the pressure anisotropy inside the fluid sphere, and $2\Delta/r$ is called the anisotropic force; it is repulsive in nature if $P_t > P_r$ and is attractive if the inequality is reversed.

2.1 Voss–Gauss and Mainardi–Codazzi equations: Eiesland condition

Any Riemannian n -manifold V^n can be immersed (isometrically embedded) in an Euclidean m -space E^m , where $m = \frac{1}{2}n(n+1)$,¹ known as Schlaefli's² conjecture [33]. It was later proved by Janet [34], Cartan [35], and Burstin [36] and now is known as the *Janet–Cartan–Burstin theorem* in differential geometry. A particular V^n may be immersed in an Euclidean space of dimension lower than m . The least possible value of h for which V^n can be embedded into E^{n+h} is called the *class* of V^n [37,38]³ and $0 \leq h \leq \frac{1}{2}n(n-1)$. The necessary and sufficient conditions for a Riemannian metric to be class of $h = 1$ is that if there exists a symmetric tensor $b_{\mu\nu}$ that satisfies the Voss–Gauss–Codazzi equations given as [40–42]

$$R_{\mu\nu\alpha\beta} = \varepsilon(b_{\mu\alpha}b_{\nu\beta} - b_{\mu\beta}b_{\nu\alpha}), \quad (2.1.1)$$

$$b_{\mu\nu;\alpha} - b_{\mu\alpha;\nu} = 0, \quad (2.1.2)$$

¹ $m \geq \frac{1}{2}n(n+1)$. See [32].

² Schlaefli in modern literature.

³ For the English translation of [38] see [39].

where $b_{\mu\nu}$ are the coefficients of the second fundamental form and $\varepsilon = +1$ or -1 , and the semicolon ; represents covariant derivatives.

Nearly a century ago Kasner [43] showed that it is impossible to embed any vacuum solution of $R_{\mu\nu} = 0$ in a pseudo-Euclidean space of dimension 5. The approach and proof of Kasner was further corrected and improved by Szekeres [44].

The case of Riemannian manifolds with indefinite metrics was first considered by Friedman [45] who extended the Janet–Cartan–Burstin theorem and proved that a Riemannian manifold $V^n(p, q)$ with p positive and q negative eigenvalues can be analytically and isometrically embedded in a pseudo-Euclidean space $E^m(r, s)$, where $m = \frac{1}{2}n(n+1)$ and $r \geq p, s \geq q, p+q = n$. For an overview of this topic, see [46–51].

A theorem due to Eiesland [1] states that a necessary and sufficient condition that a general centro-symmetric space

$$ds^2 = \phi_1 dt^2 - \phi_2 dr^2 - \phi_3 d\Omega^2, \quad (2.1.3)$$

ϕ_1, ϕ_2 , and ϕ_3 being arbitrary functions of r and t , shall be of the first class is

$$R_{0202}R_{1313} = R_{0101}R_{2323} + R_{1202}R_{1303}. \quad (2.1.4)$$

Equation (2.1.4) will be referred to as the Eiesland condition. Much later Karmarkar [52] and Kohler and Chao [53] rederived this condition.

If the space (2.1.3) is static and reduced to the form

$$ds^2 = \phi_1 dt^2 - (1 + \phi_2)dr^2 - r^2 d\Omega^2, \quad (2.1.5)$$

then the condition (2.1.4) becomes

$$\phi_1'' = \frac{1}{2} \left[\frac{\phi_1' \phi_2'}{\phi_2} + \frac{(\phi_1')^2}{\phi_1} \right]. \quad (2.1.6)$$

Equation (2.1.6) is a differential equation of first order linear in ϕ_2 but of second order nonlinear in ϕ_1 , which when integrated gives

$$\phi_2 = K \frac{\phi_1'^2}{\phi_1}, \quad (2.1.7)$$

where $K > 0$ is a constant of integration. Further integration leads to the following:

$$\sqrt{\phi_1} = A + B \int \sqrt{\phi_2} dr, \quad (2.1.8)$$

where A and B are arbitrary constants of integration.

For the spacetime (2.0.1), Eq. (2.1.8) takes the following form:

$$e^{v/2} = A + B \int \sqrt{e^\lambda - 1} dr. \quad (2.1.9)$$

By using Eq. (2.1.9) we can rewrite Eqs. (2.0.3)–(2.0.6) as

$$\kappa \rho = \frac{(1 - e^{-\lambda})}{r^2} - \frac{(e^{-\lambda})'}{r}, \quad (2.1.10)$$

$$\kappa P_r = \frac{2B}{e^\lambda e^{v/2}} \frac{\sqrt{e^\lambda - 1}}{r} - \frac{(1 - e^{-\lambda})}{r^2}, \quad (2.1.11)$$

$$\Delta = \frac{B\sqrt{e^\lambda - 1}}{2e^\lambda e^{v/2}} \frac{d}{dr} \ln \left(\frac{r^2}{1 - e^{-\lambda}} \right) \times \left(\frac{e^{v/2} \sqrt{e^\lambda - 1}}{rB} - 1 \right), \quad (2.1.12)$$

$$\kappa P_t = \kappa P_r + \Delta. \quad (2.1.13)$$

3 Solutions of field equations: generating new metric potentials e^λ

To express e^v explicitly as a function of r , one can assume the following form of e^λ :

$$e^\lambda = 1 + Cr^2 f(r) \quad (3.1)$$

where $C > 0$ has dimension length⁻² and f is a continuous function of r with $f(r) > 0$. Using Eq. (3.1), Eq. (2.1.9) becomes

$$e^{v/2} = A + B\sqrt{C}F(r), \quad (3.2)$$

where

$$F(r) = \int r\sqrt{f(r)} dr. \quad (3.3)$$

Equations (3.1) and (3.2) have been extensively studied by various authors with some particular explicit forms of e^λ or e^v as a function of the radial coordinate r ; for details see Table 1, which contains a few choices made for $f(r)$.

In this work we consider the following forms:

$$f(r) = \frac{(a + br^2)^p}{(1 + cr^2)^q}, \quad (3.4)$$

$$f(r) = \frac{(a + b^2r^4)^{2m}}{(1 + c^2r^4)^q}, \quad (3.5)$$

$$f(r) = \frac{(a + br^2)^{2m}}{(1 + c^2r^4)^q}, \quad (3.6)$$

where $a, b, c \in \mathbb{R}, m \in \mathbb{N}$ and $p, q \in \mathbb{Z}$, for which Eq. (3.3) leads to the following integrals:

$$F(r) = \int r \frac{(\sqrt{a + br^2})^p}{(\sqrt{1 + cr^2})^q} dr, \quad (3.7)$$

Table 1 List of particular form of generating function f that generates exact static spherically symmetric class 1 solutions satisfying the Eiesland condition

e^λ	$e^{v/2}$	Generated by one of the present models	References
$\frac{2(1+ar^2)}{(2-ar^2)}$	$A - \frac{\sqrt{3}B}{\sqrt{a}}\sqrt{2-ar^2}$ ^a	Y	[54]
$1 + \frac{ar^2}{(1+br^2)^2}$	$A + \frac{\sqrt{a}B}{2b}\ln(1+br^2)$	Y	[55]
$1 + ar^2(1+br^2)^n, n \neq -2$ ^b	$A + \frac{\sqrt{a}B}{b(n+2)}(1+br^2)^{n/2+1}$	Y	[56]
$1 + ar^2 + br^4$	$A + \frac{B}{3b}(a+br^2)^{3/2}$	Y ^c	[57]
$(1+br^2)^2$	$A + \frac{B}{3\sqrt{b}}(2+br^2)^{3/2}$	Y ^c	[58]
$1 + ar^2 \cos^2(br^2)$	$A + \frac{\sqrt{a}B}{2b}\sin(br^2)$	N	[59]
$\frac{4+cr^2\left(e^{(ar^2+b)} - e^{-(ar^2+b)}\right)^2}{4}$ ^d	$A + B\left(\frac{e^{(ar^2+b)} + e^{-(ar^2+b)}}{2}\right)$	N	[60]
$\frac{1+2cr^2+\cosh(2(ar^2+b))}{1+\cosh(2(ar^2+b))}$ ^e	$A\left(B + \arctan\left(\sinh(ar^2+b)\right)\right)$	N	[61]
$1 + ar^2 \tan^2(br^2+c)$	$A - \frac{\sqrt{a}B}{2b}\ln\left(\cos(br^2+c)\right)$	N	[62]
$1 + a^2r^2 \tanh^2(br^2+c)$	$A + B \ln \cosh(br^2+c)$	N	[63]
$\frac{4(1+ar^2)^2}{(2-ar^2)^2}$	$A + \frac{B}{\sqrt{a}}\sqrt{12+3ar^2} - \frac{3\sqrt{2}B}{\sqrt{a}}\operatorname{arctanh}\left(\frac{\sqrt{4+ar^2}}{\sqrt{6}}\right)$	Y	[64]
$1 + \frac{ar^2}{1+br^4}$	$A + \frac{\sqrt{a}B}{2\sqrt{b}}\operatorname{arcsinh}\left(\sqrt{br^2}\right)$	Y	[65]
$1 + \frac{a^2r^2}{(1+b^2r^4)^2}$	$A + \frac{aB}{2b}\arctan\left(br^2\right)$	Y	[66]
$1 + ar^2 \sin^2(br^2+c)$	$A - \frac{\sqrt{a}B}{2b}\cos(br^2+c)$	N	[67]
$1 + \frac{a^2r^2}{(1+b^2r^4)^n}$	$A + \frac{1}{2}aBr^2 {}_2F_1\left(\frac{1}{2}, \frac{n}{2}; \frac{3}{2}; -b^2r^4\right)$ ^f	Y	[68]
$1 + ar^2 \left(\sec^2(br^2+c)\right)^2$	$A + \frac{B\sqrt{a}}{2b}\tan(br^2+c)$	N	[69]
$1 + ar^2 \left(1 + \sin^2(br^2+c)\right)^2$	$A + \frac{B\sqrt{a}}{2}r^2 - \frac{B\sqrt{a}}{2b}\cos(br^2+c)$	N	[70]

^a Equation (13) in [54] is incorrectly typed as $e^{v/2} = A - \frac{\sqrt{3}B}{\sqrt{a}\sqrt{2-ar^2}}$.

^b Some particular cases ($n = 1$, $n = 2$ and $n = -1$) of this model have also been found in [57,58], Bhar et al. [26] and Maurya et al. [71]

^c These two are the same models

^d $e^x - e^{-x} = 2 \sinh x$ and $e^x + e^{-x} = 2 \cosh x$. Also see the model IIIb in Table 2 of [72]

^e $\frac{1}{\sqrt{1+\cosh(2x)}} = \frac{1}{\sqrt{2}} \operatorname{sech} x$. Also see the model VII in Table 2 of [72].

^f ${}_2F_1(a, b; c; x)$ is the hypergeometric function. For further details see Appendix F

$$F(r) = \int r \frac{(a+b^2r^4)^m}{(\sqrt{1+c^2r^4})^q} dr, \quad (3.8)$$

$$F(r) = \int r \frac{(a+br^2)^m}{(\sqrt{1+c^2r^4})^q} dr. \quad (3.9)$$

function [73–76]. The complete closed-form integral (3.7) is given in Appendices A–C. The complete closed-form integrals (3.8) and (3.9) are given in Appendix D and Appendix E.

For some particular choices of p and q in Eqs. (3.4) and (3.7), we get the following models.

If p , q are both even integers then the integrand is a rational algebraic function; otherwise it is an irrational algebraic

Model I: $p \neq 0, q = 0, b \neq 0$.

$$e^\lambda = 1 + Cr^2(a + br^2)^p \quad (3.10)$$

$$e^{v/2} = A + B\sqrt{C}F(r), \quad (3.11)$$

where

$$F(r) = \begin{cases} \frac{1}{b(p+2)} \left(\sqrt{a + br^2} \right)^{p+2}, & p \neq -2, \\ \frac{1}{2b} \ln(a + br^2), & p = -2. \end{cases}$$

Particular model Ia: Plugging the value $C = a$, $a = 1$, $b = 0$, $c = b$, $p = n$ into Eqs. (3.10) and (3.11) we rediscover the model of Singh and Pant [56],

$$e^\lambda = 1 + ar^2(1 + br^2)^n,$$

$$e^{v/2} = A + \frac{B\sqrt{a}}{(n+2)b}(1 + br^2)^{n/2+1}.$$

Model II: $p = 2, q = 2$

$$e^\lambda = 1 + \frac{Cr^2(a + br^2)^2}{(1 + cr^2)^2}, \quad (3.12)$$

$$e^{v/2} = A + B\sqrt{C}F(r), \quad (3.13)$$

where

$$F(r) = \begin{cases} \frac{b}{2c^2}(1 + cr^2) + \left(\frac{ac - b}{2c^2} \right) \ln(1 + cr^2), & c \neq 0, \\ \frac{1}{4b}(a + br^2)^2, & c = 0, b \neq 0, \\ \frac{a}{2}r^2, & c = 0, b = 0. \end{cases}$$

Particular model IIa: Plugging the value $a = 1$, $c = 0$ into Eqs. (3.12) and (3.13) we rediscover the model of Bhar et al. [26].

$$e^\lambda = 1 + Cr^2(1 + br^2)^2,$$

$$e^{v/2} = A + \frac{B\sqrt{C}}{4b}(1 + br^2)^2.$$

Particular model IIb: Plugging the value $C = a$, $a = 1$, $b = 0$, $c = b$ into Eqs. (3.12) and (3.13) we rediscover the model of Singh and Pant [55],

$$e^\lambda = 1 + \frac{ar^2}{(1 + br^2)^2},$$

$$e^{v/2} = A + \frac{B\sqrt{a}}{2b} \ln(1 + br^2).$$

Model III: $p = 2, q = 1$

Case I: $c \neq 0$.

$$e^\lambda = 1 + \frac{Cr^2(a + br^2)^2}{(1 + cr^2)^2}, \quad (3.14)$$

$$e^{v/2} = A + B\sqrt{C}F(r), \quad (3.15)$$

where

$$F(r) = \frac{1}{3c^2}(3ac - 2b + bcr^2)\sqrt{1 + cr^2}.$$

Particular model IIIa: Plugging the value $C = 3a/2$, $a = 1$, $b = 0$, $c = -a/2$ into Eqs. (3.14) and (3.15), after a little algebra, we rediscover the model of Singh et al. [54],

$$e^\lambda = \frac{2(1 + ar^2)}{2 - ar^2},$$

$$e^{v/2} = A - \frac{B\sqrt{3}}{\sqrt{a}}\sqrt{2 - ar^2}.$$

Model IV: $p = 1, q = 2$.

Case I: $c \neq 0, a - b/c \neq 0$.

$$e^\lambda = 1 + \frac{Cr^2(a + br^2)}{(1 + cr^2)^2}, \quad (3.16)$$

$$e^{v/2} = A + B\sqrt{C}F(r), \quad (3.17)$$

where

$$F(r) = \begin{cases} \frac{1}{c}\sqrt{a + br^2} - \frac{1}{c}\sqrt{a - \frac{b}{c}} \operatorname{arctanh}\left(\frac{\sqrt{a + br^2}}{\sqrt{a - \frac{b}{c}}}\right), & a - \frac{b}{c} > 0, \frac{b}{c} \neq 0, \\ \frac{1}{c}\sqrt{a + br^2} - \frac{1}{c}\sqrt{\left|a - \frac{b}{c}\right|} \arctan\left(\frac{\sqrt{a + br^2}}{\sqrt{\left|a - \frac{b}{c}\right|}}\right), & a - \frac{b}{c} < 0, \frac{b}{c} > 0. \end{cases}$$

Particular Model IVa: Plugging the value $C = a - b$, $a = 1$, $b = a$, $c = b$ into Eqs. (3.16) and (3.17), we obtain

$$e^\lambda = \frac{(1 + ar^2)^2}{(1 + br^2)^2}, \quad a > b, \quad (3.18)$$

$$e^{v/2} = A + \frac{B\sqrt{a - b}}{b}\sqrt{2 + (a + b)r^2} - \frac{B(a - b)}{b}G(r), \quad (3.19)$$

where

$$G(r) = \begin{cases} \sqrt{1 - \frac{a}{b}} \operatorname{arctanh}\left(\frac{\sqrt{2 + (a + b)r^2}}{\sqrt{1 - \frac{a}{b}}}\right), & a > 0, b < 0, \\ \sqrt{\left|1 - \frac{a}{b}\right|} \arctan\left(\frac{\sqrt{2 + (a + b)r^2}}{\sqrt{\left|1 - \frac{a}{b}\right|}}\right); & a > b > 0. \end{cases}$$

Particular case: Plugging the value $b = -a/2$ into Eqs. (3.18) and (3.19) we rediscover the model of Singh et al. [64].

$$e^\lambda = \frac{4(1+ar^2)^2}{(2-ar^2)^2},$$

$$e^{v/2} = A - \frac{B}{\sqrt{a}}\sqrt{12+3ar^2} + \frac{3B\sqrt{2}}{\sqrt{a}} \operatorname{arctanh}\left(\frac{\sqrt{4+ar^2}}{\sqrt{6}}\right).$$

Case II: $c \neq 0, a = b/c > 0$.

$$e^{v/2} = A + \frac{B\sqrt{C}}{c}\sqrt{\frac{b}{c}}\sqrt{1+cr^2}. \quad (3.20)$$

Case III: $c = 0, b \neq 0$.

$$e^{v/2} = A + \frac{B\sqrt{C}}{3b}(a+br^2)^{3/2}. \quad (3.21)$$

Particular model IVb: Plugging the values $C = 1, c = 0$ into Eqs. (3.16) and (3.21) we rediscover the model of Singh and Pant [57].

$$e^\lambda = 1 + ar^2 + br^4,$$

$$e^{v/2} = A + \frac{B}{3b}(a+br^2)^{3/2}.$$

Case IV: $c = 0, b = 0, a > 0$.

$$e^{v/2} = A + \frac{B\sqrt{a}\sqrt{C}}{2}r^2, \quad (3.22)$$

Model V: $p = q = 1$.

$$e^\lambda = 1 + \frac{Cr^2(a+br^2)}{(1+cr^2)}, \quad (3.23)$$

$$e^{v/2} = A + B\sqrt{C}F(r), \quad (3.24)$$

where

$$F(r) = \begin{cases} \frac{1}{2c}\sqrt{a+br^2}\sqrt{1+cr^2} + \operatorname{sgn}(c)\frac{1}{2\sqrt{bc}}\left(a - \frac{b}{c}\right) \\ \times \operatorname{arcsinh}\left(\sqrt{\frac{b}{ac-b}}\sqrt{1+cr^2}\right), \\ a - \frac{b}{c} > 0, \frac{b}{c} > 0, \\ \frac{1}{2c}\sqrt{a+br^2}\sqrt{1+cr^2} - \operatorname{sgn}(c)\frac{1}{2\sqrt{bc}}\left|a - \frac{b}{c}\right| \\ \times \operatorname{arcsinh}\left(\frac{\sqrt{a+br^2}}{\left|a - \frac{b}{c}\right|}\right), a - \frac{b}{c} < 0, \frac{b}{c} > 0, \\ \frac{1}{2c}\sqrt{a+br^2}\sqrt{1+cr^2} + \operatorname{sgn}(c)\frac{1}{2\sqrt{|bc|}}\left(a - \frac{b}{c}\right) \\ \times \operatorname{arcsin}\left(\sqrt{\left|\frac{b}{ac-b}\right|}\sqrt{1+cr^2}\right), a - \frac{b}{c} > 0, \frac{b}{c} < 0. \end{cases}$$

where $\operatorname{sgn}(x)$ is the signum function. See Appendix B.

Model VI: $p = 1, q = -1$.

$$e^\lambda = 1 + Cr^2(a+br^2)(1+cr^2), \quad (3.25)$$

$$e^{v/2} = A + B\sqrt{C}F(r), \quad (3.26)$$

where

$$F(r) = \begin{cases} \frac{(ac+b+2bcr^2)}{8bc}\sqrt{a+br^2}\sqrt{1+cr^2} - \frac{1}{8b}\frac{\left(a - \frac{b}{c}\right)^2}{\sqrt{\frac{b}{c}}} \\ \times \operatorname{arcsinh}\left(\sqrt{\frac{b}{ac-b}}\sqrt{1+cr^2}\right), a - \frac{b}{c} > 0, \frac{b}{c} > 0, \\ \frac{(ac+b+2bcr^2)}{8bc}\sqrt{a+br^2}\sqrt{1+cr^2} - \frac{1}{8b}\frac{\left(a - \frac{b}{c}\right)^2}{\sqrt{\frac{b}{c}}} \\ \times \operatorname{arcsinh}\left(\frac{\sqrt{a+br^2}}{\sqrt{\left|a - \frac{b}{c}\right|}}\right), a - \frac{b}{c} < 0, \frac{b}{c} > 0, \\ \frac{(ac+b+2bcr^2)}{8bc}\sqrt{a+br^2}\sqrt{1+cr^2} - \frac{1}{8b}\frac{\left(a - \frac{b}{c}\right)^2}{\sqrt{\left|\frac{b}{c}\right|}} \\ \times \operatorname{arcsin}\left(\sqrt{\left|\frac{b}{ac-b}\right|}\sqrt{1+cr^2}\right), a - \frac{b}{c} > 0, \frac{b}{c} < 0. \end{cases}$$

Model VII: $p = -1, q = 1$.

$$e^\lambda = 1 + \frac{Cr^2}{(a+br^2)(1+cr^2)}, \quad (3.27)$$

$$e^{v/2} = A + B\sqrt{C}F(r), \quad (3.28)$$

where

$$F(r) = \begin{cases} \frac{\operatorname{sgn}(c)}{\sqrt{bc}} \operatorname{arcsinh}\left(\sqrt{\frac{b}{ac-b}}\sqrt{1+cr^2}\right), a - \frac{b}{c} > 0, \frac{b}{c} > 0, \\ \frac{\operatorname{sgn}(c)}{\sqrt{bc}} \operatorname{arcsinh}\left(\frac{\sqrt{a+br^2}}{\sqrt{\left|a - \frac{b}{c}\right|}}\right), a - \frac{b}{c} < 0, \frac{b}{c} > 0, \\ \frac{\operatorname{sgn}(c)}{\sqrt{|bc|}} \operatorname{arcsin}\left(\sqrt{\left|\frac{b}{ac-b}\right|}\sqrt{1+cr^2}\right), a - \frac{b}{c} > 0, \frac{b}{c} < 0. \end{cases}$$

For some particular choices of m and q in Eqs. (3.5) and (3.8), we get the following models.

Model VIII: $m = 1, q = 1$ and $c \neq 0$.

$$e^\lambda = 1 + Cr^2 \frac{(a+b^2r^4)^2}{1+c^2r^4}, \quad (3.29)$$

$$e^{v/2} = A + B\sqrt{C}\left[\frac{b^2}{4c^2}r^2\sqrt{1+c^2r^4} + \frac{1}{2c}\left(a - \frac{b^2}{2c^2}\right)\operatorname{arcsinh}(cr^2)\right]. \quad (3.30)$$

Model IX: $m = 1$, $n = 2$, and $c \neq 0$.

$$e^\lambda = 1 + Cr^2 \frac{(a + b^2 r^4)^2}{(1 + c^2 r^4)^2}, \quad (3.31)$$

$$e^{v/2} = A + B\sqrt{C} \left[\frac{b^2}{2c^2} r^2 + \frac{1}{2c} \left(a - \frac{b^2}{c^2} \right) \arctan(cr^2) \right]. \quad (3.32)$$

By plugging the values $a = 1$, $b = 0$ into Eqs. (3.29), (3.30), (3.31) and (3.32) we rediscover the models [65,66], respectively.

For some particular choices of m and q in Eqs. (3.6) and (3.9), we get the following models:

Model X: $m = 1$, $q = -1$, and $c \neq 0$.

$$e^\lambda = 1 + Cr^2(a + br^2)^2(1 + c^2 r^4), \quad (3.33)$$

$$e^{v/2} = A + B\sqrt{C} \left[\frac{1}{12c^2} \left[3ac^2 r^2 + 2b(1 + c^2 r^4) \right] \times \sqrt{1 + c^2 r^4} + \frac{a}{4c} \operatorname{arcsinh}(cr^2) \right]. \quad (3.34)$$

Model XI: $m = 1$, $q = 1$, and $c \neq 0$.

$$e^\lambda = 1 + Cr^2 \frac{(a + br^2)^2}{1 + c^2 r^4}, \quad (3.35)$$

$$e^{v/2} = A + B\sqrt{C} \left[\frac{b}{2c^2} \sqrt{1 + c^2 r^4} + \frac{a}{2c} \operatorname{arcsinh}(cr^2) \right]. \quad (3.36)$$

Model XII: $m = 1$, $q = 2$, and $c \neq 0$.

$$e^\lambda = 1 + Cr^2 \frac{(a + br^2)^2}{(1 + c^2 r^4)^2}, \quad (3.37)$$

$$e^{v/2} = A + B\sqrt{C} \left[\frac{b}{4c^2} \ln(1 + c^2 r^4) + \frac{a}{2c} \arctan(cr^2) \right]. \quad (3.38)$$

On using (3.1), we can rewrite Eqs. (2.0.3) and (2.0.6) as

$$\kappa \rho = f_1(r) - f_2(r), \quad (3.39)$$

$$\kappa P_r = f_3(r) - f_1(r), \quad (3.40)$$

$$\Delta = f_4(r) f_5(r) f_6(r), \quad (3.41)$$

$$\kappa P_t = \kappa P_r + \Delta, \quad (3.42)$$

where

$$f_1(r) = \frac{Cf(r)}{1 + Cr^2 f(r)},$$

$$f_2(r) = -C \frac{2f(r) + rf'(r)}{(1 + Cr^2 f(r))^2},$$

$$f_3(r) = \frac{2B\sqrt{C}\sqrt{f(r)}}{e^{v/2}(1 + Cr^2 f(r))},$$

$$f_4(r) = \frac{1}{4} r f_3(r),$$

$$f_5(r) = \frac{Cr(2f(r) + rf'(r))}{1 + Cr^2 f(r)} - \frac{f'(r)}{f(r)},$$

$$f_6(r) = \frac{\sqrt{C}}{B} \sqrt{f(r)} e^{v/2} - 1.$$

Equations (3.39), (3.40) comprise the equation of state (EoS) of these models.

4 Physical acceptability conditions

Only physically acceptable fluid sphere is of astrophysical interest. A physically acceptable interior solution of the gravitational field equations must comply with the certain (not necessarily mutually independent) physical conditions [77–80]:

1. The radius of the fluid distribution in an isolated system is determined by the vanishing of the radial pressure at the boundary $r = R$, i.e., $P_r(r = R) = 0$. However, the energy density and tangential pressure may follow $\rho(r = R) \geq 0$ and $P_t(r = R) \geq 0$.
2. The positive definiteness of radial and tangential pressures and density, i.e., $P_r, P_t, \rho \geq 0$.
3. The absence of singularities. The solution should be free of physical and geometric singularities. This requires that $e^v > 0$, $e^\lambda > 0$ and $0 \leq P_r(r) < \infty$, $0 \leq P_t(r) < \infty$, $0 \leq \rho(r) < \infty$ are finite in the range $0 \leq r \leq R$.
4. In order to have an equilibrium configuration the matter must be stable against the collapse of local regions. This requires that the radial pressure P_r must be a monotonically non-decreasing function of ρ ,

$$\frac{dP_r}{d\rho} \geq 0.$$

5. Subluminal adiabatic sound speed. The inequalities $0 \leq \sqrt{dP_r/d\rho} \leq 1$, $0 \leq \sqrt{dP_t/d\rho} \leq 1$ are the conditions that the speed of sound would not exceed that of light.
6. A physically reasonable energy-momentum tensor has to obey either:

- the dominant energy condition (DEC) $\rho \geq |P_r|$ and $\rho \geq |P_t|$, or
- the strong energy condition (SEC) $\rho + P_r + 2P_t \geq 0$, $\rho + P_r \geq 0$, $\rho + P_t \geq 0$.

7. The energy-momentum tensor has to obey trace condition $T^\mu_\mu = T = \rho - P_r - 2P_t \geq 0$.

8. Pressure and density should be maximal at the center and monotonically decreasing towards the pressure free interface (i.e., boundary of the fluid sphere). Mathematically,

$$\frac{dP_r}{dr} \leq 0, \quad \frac{dP_t}{dr} \leq 0, \quad \frac{d\rho}{dr} \leq 0, \quad 0 \leq r \leq R.$$

Table 2 Some values of the adjustable parameters a , b , and c that generate some compact objects having mass and radii similar to PSR J1614-2230 and Vela X-1. The observed mass and predicted radii of PSR J1614-2230 and Vela X-1 are taken from [82]

Compact objects	a	b (km ⁻²)	c (km ⁻²)	Radius R (km)	Mass $M(M_\odot)$	C (km ⁻²)	B	A
Model III ($p = 2$, $q = 1$)								
PSR J1614-2230	0.0695	0.0001	0.001	9.69	1.97	2.804874	0.039960	0.215259
Vela X-1	0.0538	0.0001	0.001	9.56	1.77	3.628078	0.038652	1.428979
Model II ($p = 2$, $q = 2$)								
PSR J1614-2230	0.05089	0.0001	0.001	9.69	1.97	5.255187	0.039960	0.404461
Vela X-1	0.04191	0.0001	0.001	9.56	1.77	6.018968	0.038969	0.481201
Model IV ($p = 1$, $q = 2$)								
PSR J1614-2230	0.0321	0.0001	0.001	9.69	1.97	0.420739	0.039960	-0.165775
Vela X-1	0.0256	0.0001	0.001	9.56	1.77	0.413713	0.038653	0.104861
Model VIII ($m = 1$, $q = 1$)								
PSR J1614-2230	1.335	0.001	0.0001	9.69	1.97	0.118756	0.039960	0.359130
Vela X-1	1.843	0.001	0.0001	9.56	1.77	0.007113	0.039960	0.398696

9. The pressure anisotropy vanishes at the center, i.e., $\Delta(0) = 0$ [14,81].

5 Boundary conditions

The interior solution should match continuously with the exterior Schwarzschild solution given by

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 d\Omega^2, \quad r \geq R, \quad (5.1)$$

where $M = m(R)$ is the total mass of the fluid sphere. By matching the interior solution (2.0.1) and the exterior solution (5.1) at the boundary $r = R$, we get

$$e^{\nu_{\Sigma}/2} = \sqrt{1 - \frac{2M}{R}}, \quad (5.2)$$

$$e^{-\lambda_{\Sigma}} = 1 - \frac{2M}{R}, \quad (5.3)$$

$$P_r(R) = 0. \quad (5.4)$$

Using the boundary conditions (5.2) and (5.4), the constants A , B , and C can be determined as follows:

$$C = \frac{1}{R^2 f(R)} \left[\left(1 - \frac{2M}{R}\right)^{-1} - 1 \right], \quad (5.5)$$

$$B = \frac{1}{2R} \sqrt{1 - \frac{2M}{R}} \sqrt{\left(1 - \frac{2M}{R}\right)^{-1} - 1}, \quad (5.6)$$

$$A = \sqrt{1 - \frac{2M}{R}} - BF(R). \quad (5.7)$$

The constants a , b , c , and M , R are chosen as free parameters. The values of the other constants A , B , C for the compact stars having mass and radii similar to PSR J1614-2230 and Vela X-1 are obtained in Table 2.

6 Central values of pressure, energy density and the mass function

The central values of P_r , P_t , ρ are obtained:

$$\kappa \rho_c = 3Cf(0) > 0, \quad (6.1)$$

$$\kappa P_r(0) = \kappa P_t(0) = \frac{2B\sqrt{C}\sqrt{f(0)}}{\exp(\frac{1}{2}\nu(0))} - Cf(0) > 0. \quad (6.2)$$

Using the relationship between e^λ and the mass function $m(r)$,

$$e^{-\lambda} = 1 - \frac{2m(r)}{r}, \quad (6.3)$$

together with the help of Eq. (3.1) we get the expression of the mass function:

$$m(r) = \frac{Cr^3 f(r)}{2(1 + Cr^2 f(r))}, \quad (6.4)$$

and the total mass M becomes

$$M = m(R) = \frac{CR^3 f(R)}{2(1 + CR^2 f(R))}. \quad (6.5)$$

The gravitational redshift at the surface of the star is given by

$$z_s = z(R) = e^{-\nu(R)/2} - 1. \quad (6.6)$$

7 Stability and equilibrium conditions

7.1 Static stability criterion

A necessary but not sufficient condition for a non-rotating spherically symmetric equilibrium stellar models to be stable under small radial pulsation is that its mass M increases with growing central density ρ_c ,

$$\frac{dM}{d\rho_c} > 0.$$

This so-called *static stability criterion* [83,84] is widely used in the literature.

By using Eqs. (6.1) and (6.5) the mass can be expressed in terms of the central density as

$$M(\rho_c) = \frac{\kappa}{2} \frac{\rho_c R^3 f(R)}{3a^{2m} + \kappa \rho_c R^2 f(R)}$$

and

$$\frac{\partial M}{\partial \rho_c} = \frac{3\kappa}{2} \frac{a^{2m} R^3 f(R)}{(3a^{2m} + \kappa \rho_c R^2 f(R))^2} > 0,$$

which shows that the model (3.1) satisfies the static stability criterion.

7.2 Relativistic adiabatic index and causality conditions

For a relativistic anisotropic sphere the stability is related to the adiabatic index Γ , the ratio of two specific heats, defined by Chan et al. [85],

$$\Gamma = \frac{\rho + P_r}{P_r} \frac{dP_r}{d\rho}. \quad (7.2.1)$$

Now $\Gamma > 4/3$ gives the condition for the stability of a Newtonian sphere and $\Gamma = 4/3$ being the condition for a neutral equilibrium proposed by Bondi [86]. This condition changes for a relativistic isotropic sphere due to the regenerative effect of pressure, which renders the sphere more unstable. For an anisotropic general relativistic sphere the situation becomes more complicated, because the stability will depend on the type of anisotropy. For an anisotropic relativistic sphere the stability condition is given by Chan et al. [85],

$$\Gamma > \frac{4}{3} + \left[\frac{4}{3} \frac{(P_{t0} - P_{r0})}{|P'_{r0}|r} + \frac{1}{2} \kappa \frac{\rho_0 P_{r0}}{|P'_{r0}|r} \right], \quad (7.2.2)$$

where P_{r0} , P_{t0} , and ρ_0 are the initial radial, tangential, and energy density in static equilibrium. The first and last term inside the square brackets represent the anisotropic and relativistic corrections, respectively, and both quantities are positive, which increases the instability range of Γ [85,87,88].

The radial and tangential speeds of sound of a compact star model are given by

$$v_r^2 = \frac{dP_r}{d\rho} = \frac{dP_r/dr}{d\rho/dr}, \quad v_t^2 = \frac{dP_t}{d\rho} = \frac{dP_t/dr}{d\rho/dr}. \quad (7.2.3)$$

The stability of anisotropic stars under the radial perturbations is studied by using the concept known as “cracking” [87]. Using this concept Abreu et al. [80] showed that the region of the anisotropic fluid sphere where $-1 \leq v_t^2 - v_r^2 \leq 0$ is potentially stable but the region where $0 < v_t^2 - v_r^2 \leq 1$ is potentially unstable, where

$$v_t^2 - v_r^2 = \frac{d\Delta/dr}{d\rho/dr} \quad (7.2.4)$$

and

$$\kappa \frac{d\rho}{dr} = f'_1(r) - f'_2(r), \quad (7.2.5)$$

$$\kappa \frac{dP_r}{dr} = f'_3(r) - f'_1(r), \quad (7.2.6)$$

$$\frac{d\Delta}{dr} = f'_4(r)f_5(r)f_6(r) + f_4(r)f'_5(r)f_6(r) + f_4(r)f_5(r)f'_6(r), \quad (7.2.7)$$

$$\kappa \frac{dP_t}{dr} = \kappa \frac{dP_r}{dr} + \frac{d\Delta}{dr}. \quad (7.2.8)$$

Table 3 Values of the physical quantities such as central density, surface density, and central pressure of various compact objects listed in Table 2

Compact objects	Central density ρ_c (g cm ⁻³)	Surface density ρ_s (g cm ⁻³)	Central pressure P_c (dyne cm ⁻²)	$2M/R$	z_s
Model III ($p = 2$, $q = 1$)					
PSR J1614-2230	2.17×10^{15}	6.58×10^{14}	4.54×10^{35}	0.5997	0.5806
Vela X-1	1.68×10^{15}	6.71×10^{14}	2.86×10^{35}	0.5462	0.4844
Model II ($p = 2$, $q = 2$)					
PSR J1614-2230	2.19×10^{15}	6.55×10^{14}	4.55×10^{35}	0.5997	0.5806
Vela X-1	1.69×10^{15}	6.66×10^{14}	2.86×10^{35}	0.5462	0.4844
Model IV ($p = 1$, $q = 2$)					
PSR J1614-2230	2.17×10^{15}	6.55×10^{14}	4.41×10^{35}	0.5997	0.5806
Vela X-1	1.70×10^{15}	6.63×10^{14}	2.79×10^{35}	0.5462	0.4844
Model VIII ($m = 1$, $q = 1$)					
PSR J1614-2230	2.55×10^{15}	6.20×10^{14}	5.86×10^{35}	0.5997	0.5806
Vela X-1	2.11×10^{15}	6.14×10^{14}	4.38×10^{35}	0.5462	0.4844

For the details of $f'_i(r)$ see Eqs. (G.1)–(G.6) in Appendix G. Now to keep $-1 \leq v_t^2 - v_r^2 \leq 0$ throughout the fluid distribution, following (7.2.4), it is required that

$$0 \leq \frac{d\Delta}{dr} < -\frac{d\rho}{dr},$$

as we have $d\rho/dr < 0$. Therefore, Δ is an increasing function of r .

8 An application of the model for some well-known strange star candidates

The analysis of very compact astrophysical objects has been a key issue in relativistic astrophysics for the last few decades. Recent observations show that the estimated mass and radius of several compact objects such as X-ray pulsar Her X-1, X-ray burster 4U 1820-30, millisecond pulsar SAX J 1808.4-3658, X-ray sources 4U 1728-34, PSR 0943+10, and RX J185635-3754 are not compatible with the standard neutron star models [89, 90]. For a recent review the reader is referred to [91].

Based on the analytic models developed so far, to get an estimate of the range of various physical parameters of some potential strange star candidates, we have calculated the values of the relevant physical quantities, such as the central pressure, central and surface densities, by using the refined mass and predicted radius of PSR J1614-2230 and Vela X-1 recently reported in Gangopadhyay et al. [82]. The values are reported in Table 3.

Fig. 1 Plot of pressure anisotropy Δ (Eq. (3.41)) for a compact object like PSR J1614-2230 using the model III with $p = 2$, $q = 1$ and the values of parameters $a = 0.0695$, $b = 0.0001$, and $c = 0.001$. For details see Table 2

Fig. 2 Plot of energy density ρ in MeV fm^{-3} for the parameter values used in Fig. 1

Fig. 3 Plot of radial and tangential pressures P_r , P_t in MeV fm^{-3} for the parameter values used in Fig. 1. The solid (blue) line corresponds to P_r , and the dashed (red) line corresponds to P_t

9 Physical analysis and results

For the symbolic and numerical computations, we have used several mathematical software packages such as MATLAB (version R2017b), Maple (version 2017.3), Maxima (version 5.41.0), SageMath, Mathematica (version 11.2.0), and GeoGebra (version 5.0.425).

To generate a particular anisotropic fluid sphere by using the particular model III, we set $a = 0.0695$, $b = 0.0001$, and $c = 0.001$. And the global parameters have been set as $M = 1.97M_\odot$ and $R = 9.69$ km, similar to the PSR J1614-2230 [82]. For this choice the constants are calculated as $C = 2.804874$, $B = 0.039960$, and $A = 0.215259$. The

Fig. 4 Plot of radial and tangential pressure gradients for the parameter values used in Fig. 1. The solid (blue) line corresponds to $\kappa dP_r/dr$ (Eq. (7.2.6)), and the dashed (red) line corresponds to $\kappa dP_t/dr$ (Eq. (7.2.8))

Fig. 5 Plot of energy density gradient $\kappa d\rho/dr$ (Eq. (7.2.5)) for the parameter values used in Fig. 1

central and surface densities are found to be $\rho_c = 2.17 \times 10^{15} \text{ g cm}^{-3}$ and $\rho_s = 6.58 \times 10^{14} \text{ g cm}^{-3}$ and the central pressure is obtained as $P_c = 4.54 \times 10^{35} \text{ dyne cm}^{-2}$.

The behavior of pressure anisotropy for a compact object like PSR J1614-2230 is demonstrated in Fig. 1 which is found to be increasing with r . The behaviors of the energy density, and the radial and tangential pressures inside the star are presented in Figs. 2 and 3. The radial pressure P_r vanishes at the boundary of the star. On the other hand the matter density and tangential pressure are positive at the boundary, $r = R$. The pressure and energy density gradients are presented in Figs. 4 and 5. From these figures it is clear that the gradients remain strictly negative throughout the distri-

Fig. 6 Plot of equation of state $P_r(\rho)$ for the parameter values used in Fig. 1

Fig. 7 Plot of the speeds of adiabatic sound for the parameter values used in Fig. 1. The solid (blue) line corresponds to $v_r = \sqrt{dP_r/d\rho}$, and the dashed (red) line corresponds to $v_t = \sqrt{dP_t/d\rho}$. The solid (blue) line indicates that $dP_r/d\rho \approx \text{constant}$, and the equation of state (Fig. 6) is *almost* linear

bution and hence P_r , P_t , ρ are monotonically decreasing. The pressure–density profile for this star is plotted in Fig. 6. It seems that the radial pressure and the energy density follow a linear relationship. The adiabatic speeds of sound v_r and v_t are shown in Fig. 7 from which it is found that the speeds are monotonically decreasing in nature. It has also been observed from Fig. 7 that $dP_r/d\rho \approx \text{constant}$ which indicates that the equation of state (Fig. 6) is *almost* linear. Figure 8 shows that the condition $-1 \leq v_t^2 - v_r^2 \leq 0$ is satisfied throughout the fluid configuration and hence the fluid

Fig. 8 Plot of the $v_t^2 - v_r^2$ for the parameter values used in Fig. 1

sphere generated by the particular choice of parameters may be considered potentially stable.

In Table 2 we report some values of adjustable parameters for which the fluid sphere satisfies the elementary criteria stated in Sect. 4. The values of ρ_c , ρ_s , P_c and z_s for different compact stars using the models we developed in this paper have been tabulated in Table 3.

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Appendix A: $\int \frac{r(\sqrt{a+br^2})^p}{(\sqrt{1+cr^2})^q} dr$, $p, q \in \mathbb{N}$.

Case I: $p = 2m$, $q = 2n$, $m, n \in \mathbb{N}$.

$$\int \frac{r(\sqrt{a+br^2})^{2m}}{(\sqrt{1+cr^2})^{2n}} dr = \begin{cases} \frac{1}{2c} \left(\frac{b}{c}\right)^{n-1} \sum_{i=0}^m \binom{m}{i} \left(a - \frac{b}{c}\right)^{m-i} \Lambda_{i,n}(r), \\ a - \frac{b}{c} \neq 0, \frac{b}{c} > 0, \\ \frac{1}{2c} \left|\frac{b}{c}\right|^{n-1} \sum_{i=0}^m \binom{m}{i} (-1)^i \left(a - \frac{b}{c}\right)^{m-i} \Lambda_{i,n}(r), \\ a - \frac{b}{c} > 0, \frac{b}{c} < 0, \\ \frac{1}{2c} \left(\frac{b}{c}\right)^{n-1} \Lambda_{m,n}, \\ a - \frac{b}{c} = 0, \frac{b}{c} > 0, \end{cases}$$

where

$$\Lambda_{i,n}(r) = \begin{cases} \frac{\left|\frac{b}{c}\right|^{i-n+1}}{i-n+1} (1+cr^2)^{i-n+1}, & i \neq n-1, \\ \ln(1+cr^2), & i = n-1. \end{cases}$$

Case II: $p = 2m$, $q = 2n+1$, $m \in \mathbb{N}$, $n \in \mathbb{N} \cup \{0\}$.

$$\int \frac{r(\sqrt{a+br^2})^{2m}}{(\sqrt{1+cr^2})^{2n+1}} dr = \begin{cases} \frac{1}{c} \sum_{i=0}^m \binom{m}{i} \frac{\left(a - \frac{b}{c}\right)^{m-i} \left(\frac{b}{c}\right)^i}{2(i-n)+1} (\sqrt{1+cr^2})^{2(i-n)+1}, \\ a - \frac{b}{c} > 0, \frac{b}{c} > 0, \\ \frac{1}{c} \sum_{i=0}^m \binom{m}{i} (-1)^{m-i} \frac{\left|a - \frac{b}{c}\right|^{m-i} \left(\frac{b}{c}\right)^i}{2(i-n)+1} (\sqrt{1+cr^2})^{2(i-n)+1}, \\ a - \frac{b}{c} < 0, \frac{b}{c} > 0, \\ \frac{1}{c} \sum_{i=0}^m \binom{m}{i} (-1)^i \frac{\left(a - \frac{b}{c}\right)^{m-i} \left|\frac{b}{c}\right|^i}{2(i-n)+1} (\sqrt{1+cr^2})^{2(i-n)+1}, \\ a - \frac{b}{c} > 0, \frac{b}{c} < 0, \\ \frac{1}{b} \left(\frac{b}{c}\right)^{m+1} \frac{(\sqrt{1+cr^2})^{2m-2n+1}}{2m-2n+1}, \\ a - \frac{b}{c} = 0, \frac{b}{c} > 0. \end{cases}$$

Case III: $p = 2m + 1$, $q = 2n$, $m \in \mathbb{N} \cup \{0\}$, $n \in \mathbb{N}$.

$$\int \frac{r(\sqrt{a+br^2})^{2m+1}}{(\sqrt{1+cr^2})^{2n}} dr$$

$$= \begin{cases} \frac{1}{c} \left(\frac{b}{c}\right)^{n-1} \left[\sum_{i=0}^{n-1} \binom{m+1}{i} (-1)^{n-i} \left(\sqrt{a-\frac{b}{c}}\right)^{2m-2n+3} \right. \\ \quad \times I_{2n-2i-1} + H(m+1-n) \sum_{i=n}^{m+1} \binom{m+1}{i} (-1)^{i-n} \\ \quad \times \left(a-\frac{b}{c}\right)^{m-n+1} \sum_{j=0}^{i-n} \binom{i-n}{j} \frac{(-1)^j}{\left(a-\frac{b}{c}\right)^j} \\ \quad \times \left.\frac{(\sqrt{a+br^2})^{2j+1}}{2j+1} \right], a-\frac{b}{c} > 0, \frac{b}{c} \neq 0, \\ \frac{1}{c} \left(\frac{b}{c}\right)^{n-1} \left[\sum_{i=0}^{n-1} \binom{m+1}{i} (-1)^{m+1-i} \left(\sqrt{\left|a-\frac{b}{c}\right|}\right)^{2m-2n+3} \right. \\ \quad \times J_{2n-2i-2} + H(m+1-n) \sum_{i=n}^{m+1} \binom{m+1}{i} (-1)^{m+1-i} \\ \quad \times \left|a-\frac{b}{c}\right|^{m-n+1} \sum_{j=0}^{i-n} \binom{i-n}{j} \frac{1}{\left|a-\frac{b}{c}\right|^j} \\ \quad \times \left.\frac{(\sqrt{a+br^2})^{2j+1}}{2j+1} \right], a-\frac{b}{c} < 0, \frac{b}{c} > 0, \\ \frac{1}{c} \left(\sqrt{\frac{b}{c}}\right)^{2m+1} \frac{(\sqrt{1+cr^2})^{2m-2n+3}}{2m-2n+3}, a-\frac{b}{c} = 0, \frac{b}{c} > 0, \end{cases}$$

where H is the Heaviside step function or, unit step function, defined by

$$H(x-a) = \begin{cases} 0, & x < a, \\ 1, & x \geq a, \end{cases}$$

and

$$I_{2l-1} = \begin{cases} \operatorname{arctanh} \left(\frac{\sqrt{a+br^2}}{\sqrt{a-\frac{b}{c}}} \right), & l = 1, \\ (-1)^{l-1} \frac{\left(\sqrt{a-\frac{b}{c}}\right)^{2l-3}}{(2l-3) \left(\frac{b}{c}\right)^{l-1}} \frac{\sqrt{a+br^2}}{(\sqrt{1+cr^2})^{l-1}} \\ \quad + \frac{2l-3}{2l-2} I_{2l-3}, l \in \mathbb{N}, & l \geq 2, \end{cases}$$

$$J_l = \begin{cases} \operatorname{arctan} \left(\frac{\sqrt{a+br^2}}{\sqrt{\left|a-\frac{b}{c}\right|}} \right), & l = 0, \\ \left(\sqrt{a-\frac{b}{c}}\right)^{l-1} \frac{\sqrt{a+br^2}}{l \left(\sqrt{\frac{b}{c}}\right)^{l-1}} \frac{1}{(\sqrt{1+cr^2})^l} + \frac{l-1}{l} J_{l-2}, l \geq 2. \end{cases}$$

Case IV: $p = 2m + 1$, $q = 2n + 1$, $m, n \in \mathbb{N} \cup \{0\}$.

Subcase IVa: $a - \frac{b}{c} > 0$, $\frac{b}{c} > 0$.

$$\int \frac{r(\sqrt{a+br^2})^{2m+1}}{(\sqrt{1+cr^2})^{2n+1}} dr$$

$$= \frac{1}{c} \left(\frac{b}{c}\right)^{n-1} \left[\sum_{i=0}^n \binom{m+1}{i} \left(a-\frac{b}{c}\right)^{m+1-i} \Psi_{i,n}(r) \right. \\ \quad + H(m-n) \sum_{i=n+1}^{m+1} \binom{m+1}{i} \left(a-\frac{b}{c}\right)^{m+1-i} \Xi_{i,n}(r) \\ \quad \times \left. \sum_{j=0}^{i-n} \binom{i-n}{j} \frac{(-1)^j}{\left(a-\frac{b}{c}\right)^j} \frac{(\sqrt{a+br^2})^{2j+1}}{2j+1} \right]$$

where

$$\Psi_{i,n}(r) = \begin{cases} \frac{1}{\left(a-\frac{b}{c}\right)^{n-i}} \sum_{j=0}^{n-i} \binom{n-i}{j} \frac{(-1)^j}{2(j-n+i)+1} \\ \quad \times \left(\frac{1}{\sqrt{\frac{b}{c}}} \frac{\sqrt{a+br^2}}{\sqrt{1+cr^2}} \right)^{2(n-i-j)-1}, & i \leq n-1, \\ \operatorname{arcsinh} \left(\sqrt{\left| \frac{b}{ac-b} \right|} \sqrt{1+cr^2} \right), & i = n \end{cases}$$

$$\Xi_{i,n}(r) = \left(a-\frac{b}{c}\right)^{i-n} \sum_{j=0}^{i-n} (-1)^{i-n-j} K_{2j+1},$$

$$K_l = \begin{cases} \operatorname{arcsinh} \left(\sqrt{\left| \frac{b}{ac-b} \right|} \sqrt{1+cr^2} \right), & l = 1, \\ \frac{\sqrt{\frac{b}{c}}}{(l-1) \left(\sqrt{a-\frac{b}{c}}\right)^{l-1}} \left(\sqrt{a+br^2}\right)^{l-2} \sqrt{1+cr^2} \\ \quad + \frac{l-2}{l-1} K_{l-2}, l \in \mathbb{N}, l \geq 2. \end{cases}$$

Subcase IVb: $a - \frac{b}{c} < 0$, $\frac{b}{c} > 0$.

$$\int \frac{r(\sqrt{a+br^2})^{2m+1}}{(\sqrt{1+cr^2})^{2n+1}} dr$$

$$= \frac{1}{b} \left(\frac{b}{c}\right)^{n+\frac{1}{2}} \left[\sum_{i=0}^n \binom{m+1}{i} (-1)^{m+1-i} \right. \\ \quad \times \left. \left|a-\frac{b}{c}\right|^{m+1-i} \Theta_{i,n}(r) \right]$$

$$+H(m-n) \sum_{i=n+1}^{m+1} \binom{m+1}{i} (-1)^{m+1-i} \\ \times \left| a - \frac{b}{c} \right|^{m+1-i} \Omega_{i,n}(r) \Bigg]$$

where

$$\Theta_{i,n}(r) = \begin{cases} \frac{1}{\left| a - \frac{b}{c} \right|^{n-i}} L_{2n-2i-1}, & i \leq n-1, \\ \operatorname{arcsinh} \left(\frac{\sqrt{a+br^2}}{\sqrt{\left| a - \frac{b}{c} \right|}} \right), & i = n \end{cases}$$

$$\Omega_{i,n}(r) = \frac{\left(\sqrt{\frac{b}{c}} \right)^{2i-2n-1}}{2(i-n)} \sqrt{a+br^2} (\sqrt{1+cr^2})^{2i-2n-1} \\ + \frac{2i-2n-1}{2i-2n} \left| a - \frac{b}{c} \right|^{i-n} M_{2i-2n-1},$$

$$L_l = \begin{cases} \frac{1}{\sqrt{\frac{b}{c}}} \frac{\sqrt{a+br^2}}{\sqrt{1+cr^2}}, & l = 1, \\ \frac{\left(\sqrt{\left| a - \frac{b}{c} \right|} \right)^{l-1}}{l \left(\sqrt{\frac{b}{c}} \right)^l} \frac{\sqrt{a+br^2}}{(\sqrt{1+cr^2})^l} + \frac{l-1}{l} L_{l-2}, & l \geq 2, \end{cases}$$

$$M_l = \begin{cases} \operatorname{arcsinh} \left(\frac{\sqrt{a+br^2}}{\sqrt{\left| a - \frac{b}{c} \right|}} \right), & l = 1, \\ \frac{\left(\sqrt{\frac{b}{c}} \right)^{l-2}}{(l-1) \left(\sqrt{\left| a - \frac{b}{c} \right|} \right)^{l-2}} \sqrt{a+br^2} (\sqrt{1+cr^2})^{l-2} \\ + \frac{l-2}{l-1} M_{l-2}, & l \geq 2. \end{cases}$$

Subcase IVc: $a - \frac{b}{c} > 0$, $\frac{b}{c} < 0$.

$$\int \frac{r(\sqrt{a+br^2})^{2m+1}}{(\sqrt{1+cr^2})^{2n+1}} dr \\ = \frac{1}{b} \left| \frac{b}{c} \right|^{n+\frac{1}{2}} \left[\sum_{i=0}^n \binom{m+1}{i} (-1)^i \left(a - \frac{b}{c} \right)^{m+1-i} \Upsilon_{i,n}(r) \right. \\ \left. + H(m-n) \sum_{i=n+1}^{m+1} \binom{m+1}{i} (-1)^i \left(a - \frac{b}{c} \right)^{m+1-i} \Phi_{i,n}(r) \right].$$

where

$$\Upsilon_{i,n}(r) = \frac{1}{\left(a - \frac{b}{c} \right)^{n-i}} N_{2n-2i},$$

$$\Phi_{i,n}(r) = \left(a - \frac{b}{c} \right)^{i-n+1} O_{2i-2n+1},$$

$$N_l = \frac{\left(\sqrt{a - \frac{b}{c}} \right)^{l-2}}{(l-1) \left(\sqrt{\left| \frac{b}{c} \right|} \right)^{l-2}} \frac{\sqrt{a+br^2}}{(\sqrt{1+cr^2})^{l-2}} \\ + \frac{l-2}{l-1} N_{l-2}, \quad l \geq 2,$$

$$O_l = \frac{\left(\sqrt{\left| \frac{b}{c} \right|} \right)^{l-1}}{l \sqrt{a - \frac{b}{c}}} \sqrt{a+br^2} (\sqrt{1+cr^2})^{l-1} \\ + \frac{l-1}{l} O_{l-2}, \quad l \geq 2,$$

$$N_0 = O_0 = \operatorname{arcsin} \left(\frac{\sqrt{a+br^2}}{\sqrt{a - \frac{b}{c}}} \right).$$

Subcase IVd: $a - \frac{b}{c} = 0$, $\frac{b}{c} > 0$.

$$\int \frac{r(\sqrt{a+br^2})^{2m+1}}{(\sqrt{1+cr^2})^{2n+1}} dr = \frac{1}{2b} \sqrt{\frac{b}{c}} \left(\frac{b}{c} \right)^{m+1} \bar{\Lambda}_{m,n},$$

where

$$\bar{\Lambda}_{m,n}(r) = \begin{cases} \frac{(1+cr^2)^{m-n+1}}{m-n+1}, & m \neq n-1, \\ \ln(1+cr^2), & m = n-1. \end{cases}$$

Appendix B: $\int \frac{r(\sqrt{a+br^2})^p}{(\sqrt{1+cr^2})^q} dr$, $p = m$, $q = -n$, m , $n \in \mathbb{N}$

$$\int \frac{r(\sqrt{a+br^2})^p}{(\sqrt{1+cr^2})^q} dr = \int r (\sqrt{a+br^2})^m (\sqrt{1+cr^2})^n dr$$

Case I: $a - \frac{b}{c} > 0$, $\frac{b}{c} > 0$.

$$\int r (\sqrt{a+br^2})^m (\sqrt{1+cr^2})^n dr \\ = \begin{cases} \frac{\left(\sqrt{a - \frac{b}{c}} \right)^{m+n+2}}{b \left(\sqrt{\frac{b}{c}} \right)^n} \sum_{i=0}^{\frac{n+1}{2}} \binom{\frac{n+1}{2}}{i} (-1)^{\frac{n+1}{2}-i} P_{m+2+2i}, & n \text{ is odd,} \\ \frac{\left(\sqrt{a - \frac{b}{c}} \right)^n}{b \left(\sqrt{\frac{b}{c}} \right)^n} \sum_{i=0}^{\frac{n}{2}} \binom{\frac{n}{2}}{i} (-1)^{\frac{n}{2}-i} \frac{(\sqrt{a+br^2})^{m+2+2i}}{(m+2+2i) \left(a - \frac{b}{c} \right)^i}, & n \text{ is even.} \end{cases}$$

where

$$P_l = \begin{cases} \operatorname{arcsinh} \left(\sqrt{\frac{b}{ac-b}} \sqrt{1+cr^2} \right), & l = 1, \\ \frac{\sqrt{\frac{b}{c}}}{(l-1) \left(\sqrt{a-\frac{b}{c}} \right)^{l-1}} \left(\sqrt{a+br^2} \right)^{l-2} \sqrt{1+cr^2} \\ + \frac{l-2}{l-1} P_{l-2}, & l \geq 2 \end{cases}$$

Case II: $a - \frac{b}{c} < 0$, $\frac{b}{c} > 0$.

$$\begin{aligned} & \int r \left(\sqrt{a+br^2} \right)^m \left(\sqrt{1+cr^2} \right)^n dr \\ &= \begin{cases} \frac{\left(\sqrt{a-\frac{b}{c}} \right)^{m+n+2}}{b \left(\sqrt{\frac{b}{c}} \right)^n} \sum_{i=0}^{\frac{m+1}{2}} \binom{\frac{m+1}{2}}{i} (-1)^{\frac{m+1}{2}-i} Q_{n+2+2i}, & n \text{ is odd,} \\ \frac{\left(\sqrt{a-\frac{b}{c}} \right)^m}{c} \sum_{i=0}^{\frac{m}{2}} \binom{\frac{m}{2}}{i} \frac{(-1)^{\frac{m}{2}-i} \left(\frac{b}{c} \right)^i}{(n+2+2i) \left| a-\frac{b}{c} \right|^i} \\ \times \left(\sqrt{1+cr^2} \right)^{n+2+2i}, & n \text{ is even} \end{cases} \end{aligned}$$

where

$$Q_l = \begin{cases} \operatorname{arcsinh} \left(\frac{\sqrt{a+br^2}}{\sqrt{a-\frac{b}{c}}} \right), & l = 1, \\ \frac{\left(\sqrt{\frac{b}{c}} \right)^{l-2}}{(l-1) \left(\sqrt{a-\frac{b}{c}} \right)^{l-1}} \sqrt{a+br^2} \left(\sqrt{1+cr^2} \right)^{l-2} \\ + \frac{l-1}{l} Q_{l-2} : & l \geq 2. \end{cases}$$

Case III: $a - \frac{b}{c} > 0$, $\frac{b}{c} < 0$

$$\begin{aligned} & \int r \left(\sqrt{a+br^2} \right)^m \left(\sqrt{1+cr^2} \right)^n dr \\ &= \begin{cases} \frac{\left(\sqrt{a-\frac{b}{c}} \right)^{m+n+2}}{\operatorname{sgn}(c)|b| \left(\sqrt{\frac{b}{c}} \right)^n} \sum_{i=0}^{\frac{n+1}{2}} \binom{\frac{n+1}{2}}{i} (-1)^i K_{m+1+2i}, & n \text{ is odd,} \\ -\frac{\left(\sqrt{a-\frac{b}{c}} \right)^n}{\operatorname{sgn}(c)|b| \left(\sqrt{\frac{b}{c}} \right)^n} \sum_{i=0}^{\frac{n}{2}} \binom{\frac{n}{2}}{i} (-1)^i \frac{\left(\sqrt{a+br^2} \right)^{m+2+2i}}{(m+2+2i) \left(a-\frac{b}{c} \right)^i}, & n \text{ is even,} \end{cases} \end{aligned}$$

where

$$K_l = \begin{cases} \operatorname{arcsin} \left(\sqrt{\frac{b}{ac-b}} \sqrt{1+cr^2} \right), & l = 0, \\ \sqrt{\frac{b}{ac-b}} \sqrt{1+cr^2}, & l = 1, \\ \frac{\sqrt{\frac{b}{c}}}{l \left(\sqrt{a-\frac{b}{c}} \right)^l} \left(\sqrt{a+br^2} \right)^{l-1} \sqrt{1+cr^2} \\ + \frac{l-1}{l} J_{l-2}, & l \geq 2, \end{cases}$$

and $\operatorname{sgn}(x)$ is the *signum function* defined by

$$\operatorname{sgn}(x) = \begin{cases} -1, & x < 0, \\ 0, & x = 0, \\ 1, & x > 0. \end{cases}$$

Case IV: $a - \frac{b}{c} = 0$, $\frac{b}{c} > 0$

$$\begin{aligned} & \int r \left(\sqrt{a+br^2} \right)^m \left(\sqrt{1+cr^2} \right)^n dr \\ &= \frac{\left(\sqrt{\frac{b}{c}} \right)^m}{c(m+n+2)} \left(\sqrt{1+cr^2} \right)^{m+n+2} \end{aligned}$$

Appendix C: $\int \frac{r(\sqrt{a+br^2})^p}{(\sqrt{1+cr^2})^q} dr$, $p = -m$, $q = n$, $m, n \in \mathbb{N}$

$$\int \frac{r(\sqrt{a+br^2})^p}{(\sqrt{1+cr^2})^q} dr = \int \frac{r}{\left(\sqrt{a+br^2} \right)^m \left(\sqrt{1+cr^2} \right)^n} dr$$

Case I: $a - \frac{b}{c} > 0$, $\frac{b}{c} > 0$.

$$\int \frac{r}{\left(\sqrt{a+br^2} \right)^m \left(\sqrt{1+cr^2} \right)^n} dr = \frac{1}{c} \left(\sqrt{\frac{b}{c}} \right)^{n-2} I_{n-1,m}$$

where

$$\begin{aligned} I_{n,m} = & -\frac{1}{(n-1) \left(\sqrt{\frac{b}{c}} \right)^{n-1}} \frac{1}{\left(\sqrt{a+br^2} \right)^m \left(\sqrt{1+cr^2} \right)^{n-1}} \\ & - \frac{m}{n-1} I_{n-2,m+2}, \quad n \geq 2, \end{aligned}$$

$$I_{0,l} = \begin{cases} \operatorname{arcsinh} \left(\sqrt{\left| \frac{b}{ac-b} \right|} \sqrt{1+cr^2} \right), & l=1, \\ \frac{1}{\left(\sqrt{a-\frac{b}{c}} \right)^{l-1}} I_{l-2}, & l \in \mathbb{N}, l \geq 2 \end{cases}$$

$$I_l = \begin{cases} \arctan \left(\sqrt{\left| \frac{b}{ac-b} \right|} \sqrt{1+cr^2} \right), & l=0, \\ \sqrt{\frac{b}{c}} \frac{\sqrt{a+br^2}}{\sqrt{1+cr^2}}, & l=1, \\ \frac{\left(\sqrt{a-\frac{b}{c}} \right)^{l-1} \sqrt{\frac{b}{c}}}{l} \frac{1}{\left(\sqrt{a+br^2} \right)^l} + \frac{l-1}{l} I_{l-2}, & l \geq 2 \end{cases}$$

$$I_{1,l} = \begin{cases} \frac{1}{\left(\sqrt{a-\frac{b}{c}} \right)^l} \ln \left(\frac{\sqrt{\frac{b}{c}} \sqrt{1+cr^2}}{\sqrt{a-\frac{b}{c}} + \sqrt{a+br^2}} \right) \\ + H \left(\frac{l-3}{2} \right) \frac{1}{\left(\sqrt{a-\frac{b}{c}} \right)^l} \\ \times \sum_{i=1}^{\frac{l-1}{2}} \binom{\frac{l-1}{2}}{i} (-1)^i \bar{I}_{2i-1}, & l \text{ is odd}, \\ \frac{1}{\left(\sqrt{a-\frac{b}{c}} \right)^l} \ln \left(\frac{\sqrt{\frac{b}{c}} \sqrt{1+cr^2}}{\sqrt{a+br^2}} \right) \\ + H \left(\frac{l}{2} - 2 \right) \frac{1}{\left(\sqrt{a-\frac{b}{c}} \right)^l} \\ \times \sum_{i=1}^{\frac{l}{2}-1} \binom{\frac{l}{2}-1}{i} \frac{(-1)^i}{2i} \left(\frac{1+cr^2}{a+br^2} \right)^i, & l \text{ is nonzero even} \end{cases}$$

$$\bar{I}_l = \begin{cases} -\frac{\sqrt{a-\frac{b}{c}}}{\sqrt{a+br^2}}, & l=1, \\ \frac{\sqrt{a-\frac{b}{c}} \left(\sqrt{\frac{b}{c}} \right)^{l-1}}{l} \frac{(\sqrt{1+cr^2})^{l-1}}{\left(\sqrt{a+br^2} \right)^l} \\ + \frac{l-1}{l} \bar{I}_{l-2}, & l \geq 2. \end{cases}$$

Case II: $a - \frac{b}{c} < 0$, $\frac{b}{c} > 0$.

$$\int \frac{r}{\left(\sqrt{a+br^2} \right)^m \left(\sqrt{1+cr^2} \right)^n} dr = \frac{1}{c} \left(\sqrt{\frac{b}{c}} \right)^{n-2} J_{n-1,m}$$

where

$$J_{n,m} = -\frac{1}{(n-1) \left(\sqrt{\left| \frac{b}{c} \right|} \right)^{n-1}} \times \frac{1}{\left(\sqrt{a+br^2} \right)^m \left(\sqrt{1+cr^2} \right)^{n-1}} - \frac{m}{n-1} J_{n-2,m-2},$$

$$J_{0,m} = \begin{cases} \operatorname{arcsinh} \left(\frac{\sqrt{a+br^2}}{\sqrt{\left| a-\frac{b}{c} \right|}} \right), & m=1, \\ \frac{1}{\left(\sqrt{\left| a-\frac{b}{c} \right|} \right)^{m-1}} \sum_{i=0}^{\frac{m-3}{2}} \binom{\frac{m-3}{2}}{i} \frac{(-1)^i}{2i-m+2} \\ \times \left(\frac{\sqrt{\frac{b}{c}} \sqrt{1+cr^2}}{\sqrt{a+br^2}} \right)^{m-2-2i}, & m \text{ is odd}, m \geq 3, \\ \frac{1}{\left(\sqrt{\left| a-\frac{b}{c} \right|} \right)^{m-1}} \sum_{i=0}^{\frac{m}{2}-1} \binom{\frac{m}{2}-1}{i} (-1)^i J_{m-1-2i}, & m \text{ is even}, m \geq 2, \end{cases}$$

$$J_l = \begin{cases} \ln \left(\frac{\sqrt{a+br^2}}{\sqrt{\left| a-\frac{b}{c} \right|} + \sqrt{\frac{b}{c}} \sqrt{1+cr^2}} \right), & l=1, \\ -\frac{\sqrt{\left| a-\frac{b}{c} \right|} \left(\sqrt{\frac{b}{c}} \right)^{l-2}}{l-1} \frac{(\sqrt{1+cr^2})^{l-2}}{\left(\sqrt{a+br^2} \right)^{l-1}} + \frac{l-2}{l-1} J_{l-2}, & l \geq 2. \end{cases}$$

Case III: $a - \frac{b}{c} > 0$, $\frac{b}{c} < 0$.

$$\int \frac{r}{\left(\sqrt{a+br^2} \right)^m \left(\sqrt{1+cr^2} \right)^n} dr = \frac{1}{c} \left(\sqrt{\left| \frac{b}{c} \right|} \right)^{n-2} K_{n-1,m}$$

where

$$K_{n,m} = -\frac{1}{(n-1) \left(\sqrt{\left| \frac{b}{c} \right|} \right)^{n-1}} \frac{1}{\left(\sqrt{a+br^2} \right)^m \left(\sqrt{1+cr^2} \right)^{n-1}} + \frac{m}{n-1} K_{n-2,m-2}$$

$$K_{0,m} = \frac{1}{\left(\sqrt{a-\frac{b}{c}} \right)^{m-1}} K_{m-1},$$

$$K_l = \begin{cases} \operatorname{arcsinh} \left(\sqrt{\frac{b}{ac-b}} \sqrt{1+cr^2} \right), & l=0, \\ \operatorname{arctanh} \left(\sqrt{\frac{b}{ac-b}} \sqrt{1+cr^2} \right), & l=1, \\ \frac{\left(\sqrt{a-\frac{b}{c}} \right)^{l-2} \sqrt{\left| \frac{b}{c} \right|} \sqrt{1+cr^2}}{l-1} \frac{\sqrt{1+cr^2}}{\left(\sqrt{a+br^2} \right)^{l-1}} \\ + \frac{l-2}{l-1} K_{l-2}, & l \geq 2, \end{cases}$$

$$K_{1,l} = \begin{cases} \frac{1}{\sqrt{a-\frac{b}{c}}} \ln \left(\frac{\sqrt{\left| \frac{b}{c} \right|} \sqrt{1+cr^2}}{\sqrt{a-\frac{b}{c}} + \sqrt{a+br^2}} \right), & l=1, \\ \frac{1}{\left(a-\frac{b}{c} \right)} \ln \left(\frac{\sqrt{\left| \frac{b}{c} \right|} \sqrt{1+cr^2}}{\sqrt{a+br^2}} \right), & l=2, \\ K_{1,l-2} + \frac{1}{(l-2) \left(a-\frac{b}{c} \right) \left(\sqrt{a+br^2} \right)^{l-2}}, & l \geq 3. \end{cases}$$

Case IV: $a - \frac{b}{c} = 0$, $\frac{b}{c} > 0$.

$$\int \frac{r}{\left(\sqrt{a+br^2} \right)^m \left(\sqrt{1+cr^2} \right)^n} dr$$

$$= \frac{1}{c \left(\sqrt{\frac{b}{c}} \right)^m} \begin{cases} \frac{\left(\sqrt{1+cr^2} \right)^{-m-n+2}}{\ln \left(\sqrt{1+cr^2} \right)}, & m+n \neq 2, \\ \frac{-m-n+2}{\ln \left(\sqrt{1+cr^2} \right)}, & m+n = 2. \end{cases}$$

Appendix D: $\int r \frac{(a+b^2r^4)^m}{\left(\sqrt{1+c^2r^4} \right)^q} dr$, $m \in \mathbb{N}$, $q \in \mathbb{Z}$

Case I: $q = n$, $n \in \mathbb{N}$.

$$\int \frac{r(a+b^2r^4)^m}{\left(\sqrt{1+c^2r^4} \right)^n} dr$$

$$= \begin{cases} \frac{1}{2c} \sum_{i=0}^m \binom{m}{i} a^{m-i} \left(\frac{b^2}{c^2} \right)^i \left[\sum_{j=0}^i (-1)^{i-j} \binom{i}{j} J_{2j+1} \right], & c \neq 0, n=1, \\ \frac{1}{2c} \sum_{p=1}^m \binom{m}{p} a^{m-p} \left(\frac{b^2}{c^2} \right)^p \left[\sum_{i=0}^{\left[\frac{n-2}{2} \right]} (-1)^{p-i} \binom{p}{i} I_{n-2-2i} \right. \\ \left. + \frac{a^m}{2c} I_{n-2}, c \neq 0, n \geq 2, \right. \\ \left. \frac{1}{2} \sum_{i=0}^m \binom{m}{i} a^{m-i} b^{2i} \frac{r^{4i+2}}{2i+1}, c=0, \right] \end{cases}$$

where $\lfloor x \rfloor$ denotes the floor of x , the greatest integer less than or equal to x , H is the Heaviside step function defined in case III of Appendix A, and $\binom{n}{k}$ is the binomial coefficient, defined by [92, 93]

$$\binom{n}{k} = \begin{cases} \frac{n!}{k!(n-k)!}, & 0 \leq k \leq n, \\ 0, & k > n, \end{cases}$$

and I_l and J_l are defined by the following recurrence relations:

$$I_l = \begin{cases} \arctan(cr^2), & l=0, \\ \frac{cr^2}{\sqrt{1+c^2r^4}}, & l=1, \\ \frac{1}{l} \frac{cr^2}{(\sqrt{1+c^2r^4})^l} + \frac{l-1}{l} I_{l-2}, & l \geq 2. \end{cases}$$

$$J_l = \begin{cases} I_0, & l=0, \\ \operatorname{arcsinh}(cr^2), & l=1, \\ \frac{1}{l-1} cr^2 (\sqrt{1+c^2r^4})^{l-2} + \frac{l-2}{l-1} J_{l-2}, & l \geq 2. \end{cases}$$

Case II: $q = -n$, $n \in \mathbb{N}$.

$$\int r(a+b^2r^4)^m \left(\sqrt{1+c^2r^4} \right)^n dr$$

$$= \frac{1}{2c} \sum_{i=0}^m \left[\binom{m}{i} a^{m-i} \left(\frac{b^2}{c^2} \right)^i \sum_{j=0}^i \binom{i}{j} (-1)^{i-j} J_{2j+n+2} \right]$$

where $l \in \mathbb{N} \cup \{0\}$ and J_l is defined in case I of Appendix D.

Case III: $q = 0$.

$$\int r(a+b^2r^4)^m dr = \frac{1}{2b} \sum_{i=0}^m \binom{m}{i} \frac{a^{m-i} b^{2i+1}}{2i+1} r^{4i+2}.$$

Appendix E: $\int r \frac{(a+br^2)^m}{\left(\sqrt{1+c^2r^4} \right)^q} dr$, $m \in \mathbb{N}$, $q \in \mathbb{Z}$

Case I: $q = n$, $n \in \mathbb{N}$.

$$\int \frac{r(a+br^2)^m}{\left(\sqrt{1+c^2r^4} \right)^n} dr = \frac{1}{2c} \sum_{i=0}^m \binom{m}{i} a^{m-i} \left(\frac{b}{c} \right)^i J_{i,n},$$

where

$$J_{l,n} = -\frac{1}{n-2} \frac{(cr^2)^{l-1}}{\left(\sqrt{1+c^2r^4} \right)^{n-2}} + \frac{l-1}{n-2} J_{l-2,n-2}, \quad l \geq 2, \quad n \geq 3,$$

$$J_{l,0} = \frac{(cr^2)^{l+1}}{l+1}, \quad l \in \mathbb{N} \cup \{0\},$$

$$J_{l,1} = \begin{cases} \sum_{i=0}^{\frac{l-1}{2}} \binom{\frac{l-1}{2}}{i} \frac{(-1)^{\frac{l-1}{2}-i}}{2i+1} \left(\sqrt{1+c^2r^4}\right)^{2i+1}, & l \text{ is odd,} \\ \sum_{i=0}^{\frac{l}{2}} \binom{\frac{l}{2}}{i} (-1)^{\frac{l}{2}-i} J_{2i+1}, & l \text{ is even,} \end{cases}$$

J_l is defined in case I of Appendix D,

$$J_{l,2} = \begin{cases} \arctan(cr^2), & l = 0, \\ \frac{1}{2} \ln(1+cr^2), & l = 1, \\ \frac{(cr^2)^{l-1}}{l-1} - J_{l-2,2}, & l \geq 2. \end{cases}$$

$$J_{0,l} = \begin{cases} \operatorname{arcsinh}(cr^2), & l = 1, \\ \arctan(cr^2), & l = 2, \\ K_{l-2}, & l \geq 3. \end{cases}$$

$$K_l = \begin{cases} \arctan(cr^2), & l = 0, \\ \frac{cr^2}{\sqrt{1+c^2r^4}}, & l = 1, \\ \frac{1}{l} \frac{cr^2}{(\sqrt{1+c^2r^4})^l} + \frac{l-1}{l} K_{l-2}, & l \geq 3, \end{cases}$$

$$J_{1,l} = \begin{cases} -\frac{1}{(l-2)(\sqrt{1+c^2r^4})^{l-2}}, & l \neq 2, \\ \frac{1}{2} \ln(1+c^2r^4), & l = 2. \end{cases}$$

Case II: $q = -n$, $n \in \mathbb{N}$

$$\int r(a+br^2)^m \left(\sqrt{1+c^2r^4}\right)^n dr$$

$$= \frac{1}{2c} \sum_{i=0}^m \binom{m}{i} a^{m-i} \left(\frac{b}{c}\right)^i I_{i,n},$$

where

$$I_{l,n} = \begin{cases} \sum_{j=0}^{\frac{l-1}{2}} \binom{\frac{l-1}{2}}{j} \frac{(-1)^{\frac{l-1}{2}-j}}{2j+n+2} \left(\sqrt{1+c^2r^4}\right)^{2j+n+2}, & l \text{ is odd,} \\ \sum_{j=0}^{\frac{l}{2}} \binom{\frac{l}{2}}{j} (-1)^{\frac{l}{2}-j} J_{2j+n+2}, & l \text{ is even,} \end{cases}$$

where $l \in \mathbb{N} \cup \{0\}$ and J_l is defined in case I of Appendix D.

Case III: $q = 0$.

$$\int r(a+br^2)^m dr = \frac{1}{2b(m+1)} \left(a+br^2\right)^{m+1}.$$

Appendix F: Integral representation of ${}_2F_1(a, b; c; x)$

The Euler integral representation of ${}_2F_1(a, b; c; x)$ is [94]

$${}_2F_1(a, b; c; x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \times \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-xt)^{-a} dt,$$

and one can show that

$$\int_0^x (1+t^p)^{-1/q} dt = \frac{x}{p} \int_0^1 u^{\frac{1}{p}-1} (1+x^p u)^{-1/q} du$$

$$= x {}_2F_1\left(\frac{1}{q}, \frac{1}{p}; \frac{1}{p}+1; -x^p\right),$$

and hence

$$\int_0^r \frac{u}{(1+b^2u^4)^{n/2}} du = \frac{1}{2b} \int_0^{br^2} (1+t^2)^{-n/2} dt$$

$$= \frac{r^2}{2} {}_2F_1\left(\frac{n}{2}, \frac{1}{2}; \frac{3}{2}; -b^2r^4\right).$$

Appendix G: $f'_i(r)$, $i = 1, 2, \dots, 6$

$$f'_1(r) = -\frac{2C^2rf^2 + Cf'}{(1+Cr^2f)^2}, \quad (\text{G.1})$$

$$f'_2(r) = \frac{-C(Cr^3ff'' + rf'' - 2Cr^3f'^2 - 5Cr^2ff' - 8Cr^2f^2 + 3f')}{(1+Cr^2f)^3}, \quad (\text{G.2})$$

$$f'_3(r) = \frac{2B\sqrt{C}}{e^v(1+Cr^2f)^2} \left[\frac{f'}{2\sqrt{f}} e^{v/2} (1+Cr^2f) \right. \\ \left. - \sqrt{f} (B\sqrt{C}r\sqrt{f}(1+Cr^2f) + Cre^{v/2}(2f+rf')) \right], \quad (\text{G.3})$$

$$f'_4(r) = \frac{1}{4} (f_3 + rf'_3), \quad (\text{G.4})$$

$$f'_5(r) = \frac{C(Cr^4ff'' + r^2f'' - Cr^4f'^2 - 2Cr^2f^2 + 4rf' + 2f)}{(1+Cr^2f)^2} \\ - \frac{ff'' - f'^2}{f^2}, \quad (\text{G.5})$$

$$f'_6(r) = \frac{\sqrt{C}}{2B} \frac{f'}{\sqrt{f}} e^{v/2} + Crf. \quad (\text{G.6})$$

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