Eur. Phys. J. C (2018) 78:195
https://doi.org/10.1140/epjc/s10052-018-5669-4

THE EUROPEAN PHYSICAL JOURNAL C

Regular Article - Theoretical Physics

Dirac field and gravity in NC $SO(2,3)_\star$ model

Dragoljub Gočanin[a], **Voja Radovanović**[b]

Faculty of Physics, University of Belgrade, Studentski trg 12, 11000 Beograd, Serbia

Received: 27 November 2017 / Accepted: 24 February 2018 / Published online: 8 March 2018
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Abstract Action for the Dirac spinor field coupled to gravity on noncommutative (NC) Moyal-Weyl spacetime is obtained without prior knowledge of the metric tensor. We emphasize gauge origins of gravity and its interaction with fermions by demonstrating that a classical action invariant under $SO(2,3)$ gauge transformations can be exactly reduced to the Dirac action in curved spacetime after breaking the original symmetry down to the local Lorentz $SO(1,3)$ symmetry. The commutative $SO(2,3)$ invariant action can be straightforwardly deformed via Moyal-Weyl $\star$-product to its NC $SO(2,3)_\star$ invariant version which can be expanded perturbatively in powers of the deformation parameter using the Seiberg-Witten map. The NC gravity-matter couplings in the expansion arise as an effect of the gauge symmetry breaking. We calculate in detail the first order NC correction to the classical Dirac action in curved spacetime and show that it does not vanish. Moreover, linear NC effects are apparent even in flat spacetime. We analyse NC deformation of the Dirac equation, Feynman propagator and dispersion relation for electrons in Minkowski spacetime and conclude that constant NC background acts as a birefringent medium for electrons propagating in it.

1 Introduction

Quantum Field Theory (QFT) and General Relativity (GR) are two cornerstones of modern theoretical physics. Although these theories have been tested to an excellent degree of accuracy in their respective areas of applicability, occurrence of singularities in both of them strongly indicates that they are incomplete. GR, as a classical theory of gravity, describes large-scale geometric structure of spacetime and its relation to the distribution of matter. On the other hand, QFT, standing on the principles of Quantum Mechanics and Special Relativity, provides us with the Standard Model of elementary particles which successfully utilizes the idea of local symmetry to describe the fundamental particle interactions. Understanding quantum nature of spacetime and reconciling gravity with other fundamental interactions is considered to be one of the main goals of contemporary physics.

In order to obtain a consistent unified theory, certain modifications of the basic concepts of QFT and GR are necessary. Various approaches have been proposed so far, stemming from String Theory, Loop Quantum Gravity, Noncommutative (NC) Field Theory, etc. and all of them, in some radical way, change the notion of point particle and/or that of spacetime.

In the last twenty years, Noncommutative Field Theory has become a very important direction of investigation in theoretical high energy physics and gravity. Its basic insight is that the quantum nature of spacetime, at the microscopic level, should mean that even the spacetime coordinates are to be treated as mutually incompatible observables, satisfying some non trivial commutation relations. The simplest choice of noncommutativity is the so called canonical noncommutativity, defined by

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}\,, \tag{1.1}$$

where $\theta^{\mu\nu}$ are components of a constant antisymmetric matrix.

To establish canonical noncommutativity, instead of using abstract algebra of coordinates, i.e. noncommutative spacetime, one can equivalently introduce the noncommutative Moyal-Weyl $\star$-product,

$$f(x)\star g(x) = e^{\frac{i}{2}\theta^{\alpha\beta}\frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial y^\beta}} f(x)g(y)|_{y\to x}, \tag{1.2}$$

as a multiplication of functions (fields) defined on the usual, commutative (undeformed) spacetime. The quantity $\theta^{\mu\nu}$ is considered to be a small deformation parameter that has dimensions of $(length)^2$ (in natural units). It is a fundamental constant, like the Planck length or the speed of light.

[a] e-mail: dgocanin@ipb.ac.rs

[b] e-mail: rvoja@ipb.ac.rs

Recently, a lot of attention has been devoted to NC gravity, and many different approaches to this problem have been developed. In [1–3] a deformation of pure Einstein gravity based on the Seiberg-Witten map is proposed. Twist approach to noncommutative gravity was explored in [4–7]. Lorentz symmetry in NC field theories was considered in [8,9]. Some other proposals are given in [10–22]. The connection to supergravity was established in [23,24]. The extension of NC gauge theories to orthogonal and symplectic algebra was considered in [25,26]. Finally, in the previous papers of one of the authors [27–30] an approach based on the deformed Anti de Sitter (AdS) symmetry group, i.e. $SO(2,3)_\star$ group, and canonical noncommutativity was established. In this approach NC gravity is treated as a gauge theory. It becomes manifest only after the suitable symmetry breaking. The action was constructed without previous introduction of the metric tensor and the second order NC correction to the Einstein-Hilbert action was found explicitly. Special attention has been devoted to the meaning of the coordinates used. Namely, it was shown that coordinates in which we postulate canonical noncommutativity are the Fermi inertial coordinates, i.e. coordinates of an inertial observer along the geodesic. The commutator between arbitrary coordinates can be derived from the canonical noncommutativity as demonstrated in [27].

A next natural step is to consider coupling of matter fields and gravity in the framework of NC $SO(2,3)_\star$ model. In this paper we specifically focus on the NC coupling of the Dirac spinor field and gravity. Previously, noncommutative coupling of spinors and gravity has been treated by Aschieri and Castellani [31,32]. Their model, based on the local $SO(1,3)_\star$ symmetry, is significantly different from the one presented here. On the formal side, in their approach, the vierbein field e^a_μ is an adjoint field, i.e. it transforms in the adjoint representation of $SO(1,3)$ group, whereas here, the vierbein and the $SO(1,3)$ spin-connection are just different components of the total $SO(2,3)$ gauge potential , and they are both being treated on equal footing. Therefore, in our model, the vierbein field holds the same status and transforms in the same way as an ordinary gauge potential. More importantly, physical implications to which these two models lead are quite different. The mentioned authors have found that, in the case of massless Majorana spinors, the first non vanishing NC correction to the action in curved spacetime is at the second order in powers of $\theta^{\alpha\beta}$ (all odd-power corrections being equal to zero). Within the same framework, the coupling of the Dirac spinors and gravity has been treated in [33] and the first order NC action is obtained. Our objective was to construct a plausible theoretical model that would enable us to explore the behavior of matter in NC spacetime and to actually calculate how this noncommutativity modifies the potentially observable physics, e.g. the dispersion relation for an electron. We were specifically interested in effects linear in $\theta^{\alpha\beta}$. The model that we present here predicts a non vanishing linear correction to the Dirac action in curved spacetime that survives even in flat spacetime. This feature enables us to investigate potentially observable NC effects much more easily. It leads us to an important physical prediction of the linearly deformed dispersion relation for electrons in NC Minkowski spacetime along with the Zeeman-like splitting of the undeformed energy levels. Also, the energy levels become helicity-dependent due to noncommutativity of the background spacetime which behaves as a birefringent medium for electrons propagating in it. As an aside, we should also mention that the differences between these two models revealed themselves already in the case of pure gravity. Namely, we have showed that the deformation of Minkowski spacetime is obtained in NC $SO(2,3)_\star$ model [27].

The paper is organized as follows. In the next section we introduce the basic elements of AdS algebra and present a model of commutative action based on local AdS symmetry. In the third section we shortly review the theory of gauge fields on NC spacetime. In Sect. 4, we deform our commutative action using the Moyal-Weyl $\star$-product. We use the Seiberg-Witten map to expand the NC action perturbatively in powers of the deformation parameter $\theta^{\alpha\beta}$ and calculate in detail the first order NC correction to the Dirac action in curved spacetime. Finally, in Sect. 5, we consider a special case of flat spacetime and analyse the NC correction to the Dirac equation, Feynman propagator and dispersion relation for electrons. Section 6 contains discussion and conclusion.

2 Commutative model

Before introducing the model of commutative (i.e. undeformed) action based on $SO(2,3)$ gauge symmetry, we will present some basic definitions concerning the AdS algebra. Many more details can be found in our previous papers [27–30].

The generators of $SO(2,3)$ gauge group are denoted by M_{AB}, where the group indices $A, B, \ldots$ take values $0, 1, 2, 3, 5$. These generators satisfy the following commutation relations:

$$[M_{AB}, M_{CD}] = i(\eta_{AD}M_{BC} + \eta_{BC}M_{AD} - \eta_{AC}M_{BD} - \eta_{BD}M_{AC}), \tag{2.1}$$

where $\eta_{AB} = \mathrm{diag}(+,-,-,-,+)$ is the $5D$ internal space metric tensor. A realization of this algebra can be obtained from $5D$ gamma matrices. Namely, if Γ^A are $5D$ gamma matrices that satisfy anticommutation relations:

$$\{\Gamma^A, \Gamma^B\} = 2\eta^{AB}, \tag{2.2}$$

then generators are given by

$$M_{AB} = \frac{i}{4}[\Gamma_A, \Gamma_B] \,. \tag{2.3}$$

One choice of $5D$ gamma matrices is $\Gamma_A = (i\gamma_a\gamma_5, \gamma_5)$, where γ_a are the usual $4D$ gamma matrices. The local Lorentz indices $a, b, ...$ take values 0, 1, 2, 3. In this particular representation, $SO(2,3)$ generators are given by

$$M_{ab} = \frac{i}{4}[\gamma_a, \gamma_b] = \frac{1}{2}\sigma_{ab} \,,$$
$$M_{5a} = \frac{1}{2}\gamma_a. \tag{2.4}$$

The total $SO(2,3)$ gauge potential, $\omega_\mu = \frac{1}{2}\omega_\mu{}^{AB}M_{AB}$, can be decomposed as

$$\omega_\mu = \frac{1}{4}\omega_\mu{}^{ab}\sigma_{ab} - \frac{1}{2l}e^a_\mu\gamma_a, \tag{2.5}$$

where e^a_μ and $\omega_\mu{}^{ab}$ are the vierbein and the $SO(1,3)$ spin-connection, respectively, and l is a constant length scale. The world indices $\mu, \nu, ...$ take values 0, 1, 2, 3. We see that along with the spin-connection, which is naturally related to the $SO(1,3)$ gauge group and suitable for introducing fermionic spinor fields in curved spacetime, here we get additional gauge field - the vierbein, which is related to the metric tensor by

$$\eta_{ab}e^a_\mu e^b_\nu = g_{\mu\nu} \,, \quad e = \det(e^a_\mu) = \sqrt{-g}. \tag{2.6}$$

Thus, in this model, the vierbein and the $SO(1,3)$ spin-connection are just different components of the total $SO(2,3)$ gauge potential, and they are both being treated on equal footing.

The field strength tensor is built from the gauge potential in the usual way,

$$F_{\mu\nu} = \partial_\mu\omega_\nu - \partial_\nu\omega_\mu - i[\omega_\mu, \omega_\nu] = \frac{1}{2}F_{\mu\nu}{}^{AB}M_{AB}. \tag{2.7}$$

Its components are

$$F_{\mu\nu}{}^{ab} = R_{\mu\nu}{}^{ab} - \frac{1}{l^2}(e^a_\mu e^b_\nu - e^b_\mu e^a_\nu) \,,$$
$$F_{\mu\nu}{}^{a5} = \frac{1}{l}T_{\mu\nu}{}^a, \tag{2.8}$$

where we recognize

$$R_{\mu\nu}{}^{ab} = \partial_\mu\omega_\nu{}^{ab} - \partial_\nu\omega_\mu{}^{ab} + \omega_\mu{}^{ac}\omega_\nu{}^{cb} - \omega_\mu{}^{bc}\omega_\nu{}^{ca}, \tag{2.9}$$
$$T_{\mu\nu}{}^a = \nabla_\mu e^a_\nu - \nabla_\nu e^a_\mu, \tag{2.10}$$

as the curvature tensor and torsion, respectively.

In the papers of Stelle, West and Wilczek [34,35] and also MacDowell, Mansouri and Towsend [36,37] a commutative action for pure gravity with $SO(2,3)$ gauge symmetry was constructed. Also, in the papers of Chamseddine and Mukhanov GR is formulated by gauging $SO(1,4)$ or, more suitable for supergravity, $SO(2,3)$ group [38,39]. Proceeding within this general framework, which is motivated by the idea of constructing a unified symmetry setup for general relativity and gauge field theories, we show that it can also accommodate fermionic matter fields, specifically, the Dirac spinor field. We are going to do that by providing a commutative action for the Dirac spinors invariant under ordinary $SO(2,3)$ gauge transformations which exactly reduces to the standard Dirac action in curved spacetime after the suitable symmetry breaking. Its NC deformation can be represented as a perturbative expansion in powers of the deformation parameter $\theta^{\alpha\beta}$ via Seiberg-Witten map, each term being $SO(2,3)$ invariant. After the symmetry breaking down to the local Lorentz $SO(1,3)$ symmetry, we get NC deformation of the Dirac action in curved spacetime.

Let ψ be a Dirac spinor field which transforms in the fundamental representation of $SO(2,3)$ gauge group, i.e.

$$\delta_\epsilon\psi = i\epsilon\psi = \frac{i}{2}\epsilon^{AB}M_{AB}\psi, \tag{2.11}$$

where ϵ^{AB} are antisymmetric gauge parameters. The covariant derivative of a Dirac spinor is given by

$$D_\mu\psi = \partial_\mu\psi - \frac{i}{2}\omega_\mu{}^{AB}M_{AB}\psi. \tag{2.12}$$

Taking a hermitian conjugate of the previous expression we get

$$D_\mu\bar\psi = \partial_\mu\bar\psi + \frac{i}{2}\bar\psi\;\omega_\mu{}^{AB}M_{AB}. \tag{2.13}$$

In order to break $SO(2,3)$ gauge symmetry [35–37] we introduce an auxiliary field $\phi = \phi^A\Gamma_A$. This field is a spacetime scalar and an internal-space vector. It transforms in the adjoint representation of $SO(2,3)$, i.e.

$$\delta_\epsilon\phi = i[\epsilon, \phi], \tag{2.14}$$

and it is constrained by the condition $\phi^A\phi_A = l^2$. Note that this field has mass dimension -1. The covariant derivative of an adjoint field is given by

$$D_\mu\phi = \partial_\mu\phi - i[\omega_\mu, \phi]. \tag{2.15}$$

Consider the following kinetic-type "symmetric-phase" action:

$$S_{kin} = \frac{i}{12}\int d^4x\;\varepsilon^{\mu\nu\rho\sigma}\Big[\bar\psi D_\mu\phi D_\nu\phi D_\rho\phi D_\sigma\psi$$
$$-D_\sigma\bar\psi D_\mu\phi D_\nu\phi D_\rho\phi\psi\Big]. \tag{2.16}$$

This action is invariant under $SO(2,3)$ gauge transformations, and it is hermitian up to the surface term which vanishes.

It is straightforward to show that the total covariant derivative of a spinor field can be decomposed as

$$D_\sigma\psi = \nabla_\sigma\psi + \frac{i}{2l}e^a_\sigma\gamma_a\psi, \tag{2.17}$$

where

$$\nabla_\sigma \psi = \partial_\sigma \psi - \frac{i}{4}\omega_\sigma{}^{ab}\sigma_{ab}\psi, \tag{2.18}$$

is the usual $SO(1,3)$ covariant derivative.

We break the $SO(2,3)$ symmetry down to the local Lorentz $SO(1,3)$ symmetry by fixing the value of auxiliary field ϕ, specifically by taking $\phi^a = 0$ and $\phi^5 = l$. According to (2.15), the components of $D_\mu\phi$ become $(D_\mu\phi)^a = e^a_\mu$ and $(D_\mu\phi)^5 = 0$ and thus we obtain the action in the broken-symmetry phase,

$$S_{kin} = \frac{i}{2}\int d^4x\ e\ \left[\bar\psi\gamma^\sigma\nabla_\sigma\psi - \nabla_\sigma\bar\psi\gamma^\sigma\psi\right] \\ -\frac{2}{l}\int d^4x\ e\ \bar\psi\psi, \tag{2.19}$$

which is exactly the Dirac action in curved spacetime for spinors of mass $2/l$.

We want to be able to have fermions with an arbitrary mass m, not just the specific one $2/l$. To obtain the correct masses of fermionic particles we have to include additional terms in the action. There are five terms invariant under $SO(2,3)$ transformations that can be used to supplement the original action in order to obtain the correct Dirac mass term in curved spacetime after the symmetry breaking. These terms only differ in the position of the auxiliary field ϕ, and they are:

$$\bar\psi D_\mu\phi D_\nu\phi D_\rho\phi D_\sigma\phi\phi\psi\ ,\ \bar\psi D_\mu\phi D_\nu\phi D_\rho\phi\phi D_\sigma\phi\psi\ , \\ \bar\psi D_\mu\phi D_\nu\phi\phi D_\rho\phi D_\sigma\phi\psi,\ \bar\psi D_\mu\phi D_\nu\phi D_\rho\phi D_\sigma\phi\phi\psi\ , \\ \bar\psi D_\mu\phi\phi D_\nu\phi D_\rho\phi D_\sigma\phi\psi. \tag{2.20}$$

From them we can build only three independent hermitian "mass terms" (terms of the type $\bar\psi\ldots\psi$) denoted by $S_{m,i}$ $(i = 1,2,3)$:

$$S_{m,1} = \frac{i}{2}c_1\Big(\frac{m}{l} - \frac{2}{l^2}\Big)\int d^4x\ \varepsilon^{\mu\nu\rho\sigma} \\ \Big[\bar\psi D_\mu\phi D_\nu\phi D_\rho\phi D_\sigma\phi\phi\psi \\ +\bar\psi\phi D_\mu\phi D_\nu\phi D_\rho\phi D_\sigma\phi\psi\Big], \\ S_{m,2} = \frac{i}{2}c_2\Big(\frac{m}{l} - \frac{2}{l^2}\Big)\int d^4x\ \varepsilon^{\mu\nu\rho\sigma} \\ \Big[\bar\psi D_\mu\phi D_\nu\phi D_\rho\phi\phi D_\sigma\phi\psi \\ +\bar\psi D_\mu\phi\phi D_\nu\phi D_\rho\phi D_\sigma\phi\psi\Big], \\ S_{m,3} = i\ c_3\Big(\frac{m}{l} - \frac{2}{l^2}\Big)\int d^4x\ \varepsilon^{\mu\nu\rho\sigma} \\ \bar\psi D_\mu\phi D_\nu\phi\phi D_\rho\phi D_\sigma\phi\psi. \tag{2.21}$$

The undetermined dimensionless coefficients c_1, c_2 and c_3 are introduced for generality, and they will be fixed later.

After the symmetry breaking, the sum of the mass terms in (2.21), denoted by S_m, reduces to

$$S_m = \sum_{i=1}^{3} S_{m,i} = 24(c_2 - c_1 - c_3)\left(m - \frac{2}{l}\right)\int d^4x\ e\ \bar\psi\psi. \tag{2.22}$$

If we want to have the correct mass term in the total action after the symmetry breaking, the coefficients c_1, c_2, and c_3 must satisfy the following constraint:

$$c_2 - c_1 - c_3 = -\frac{1}{24}. \tag{2.23}$$

Then (2.22) becomes

$$S_m = -\left(m - \frac{2}{l}\right)\int d^4x\ e\ \bar\psi\psi, \tag{2.24}$$

Terms in (2.19) and (2.24) that have a factor of the cosmological mass $2/l$ in them cancel each other out, and therefore, the total symmetric-phase commutative action

$$S = S_{kin} + S_m, \tag{2.25}$$

where S_{kin} is given in (2.16) and S_m is the sum of the three mass terms in (2.21), exactly reduces, after the symmetry breaking, to the Dirac action for spinors of mass m in curved spacetime,

$$S = \frac{i}{2}\int d^4x\ e\ \left[\bar\psi\gamma^\sigma\nabla_\sigma\psi - \nabla_\sigma\bar\psi\gamma^\sigma\psi\right] - m\int d^4x e\bar\psi\psi. \tag{2.26}$$

Thus, by starting with a theory with $SO(2,3)$ gauge symmetry, by a suitable "gauge fixing", we have obtained the standard minimal coupling of the massive Dirac spinor field and gravity.

3 Gauge theories and the Seiberg-Witten map

Let us briefly review the theory of gauge fields on noncommutative spacetime by summarizing the most relevant results. Our approach is based on the enveloping algebra formalism and the use of the Seiberg-Witten (SW) map [40,41]. Following the steps of an ordinary (undeformed) gauge field theory, one introduces NC spinor field $\widehat\psi$ (which belongs to the fundamental representation), NC adjoint field $\widehat\phi$ and NC gauge potential $\widehat\omega_\mu$. We use this gauge potential to construct NC field strength,

$$\widehat F_{\mu\nu} = \partial_\mu\widehat\omega_\nu - \partial_\nu\widehat\omega_\mu - i[\widehat\omega_\mu \stackrel{\star}{,} \widehat\omega_\nu]. \tag{3.1}$$

The covariant derivatives of NC spinor and adjoint field are defined by

$$D_\mu\widehat\psi = \partial_\mu\widehat\psi - i\widehat\omega_\mu \star \widehat\psi, \tag{3.2}$$

$$D_\mu \widehat{\phi} = \partial_\mu \widehat{\phi} - i[\widehat{\omega}_\mu \stackrel{\star}{,} \widehat{\phi}]. \tag{3.3}$$

Note that the structure of the NC covariant derivatives, in both representations, is the same as in the undeformed field theory, the only difference being the use of the Moyal-Weyl $\star$-product instead of the ordinary commutative multiplication.

Fields $\widehat{\psi}$ and $\widehat{\phi}$, along with their covariant derivatives (3.2) and (3.3), transform in the fundamental and adjoint representation, respectively, under NC infinitesimal gauge transformations, i.e.

$$\begin{aligned}
&\delta^\star_\epsilon \widehat{\psi} = i\widehat{\Lambda}_\epsilon \star \widehat{\psi}\,, \quad \delta^\star_\epsilon D_\mu \widehat{\psi} = i\widehat{\Lambda}_\epsilon \star D_\mu \widehat{\psi},\\
&\delta^\star_\epsilon \widehat{\phi} = i[\widehat{\Lambda}_\epsilon \stackrel{\star}{,} \widehat{\phi}], \quad \delta^\star_\epsilon D_\mu \widehat{\phi} = i[\widehat{\Lambda}_\epsilon \stackrel{\star}{,} D_\mu \widehat{\phi}].
\end{aligned} \tag{3.4}$$

The transformation laws for NC gauge potential and field strength are

$$\delta^\star_\epsilon \widehat{\omega}_\mu = \partial_\mu \widehat{\Lambda}_\epsilon - i[\widehat{\omega}_\mu \stackrel{\star}{,} \widehat{\Lambda}_\epsilon], \tag{3.5}$$

$$\delta^\star_\epsilon \widehat{F}_{\mu\nu} = i[\widehat{\Lambda}_\epsilon \stackrel{\star}{,} \widehat{F}_{\mu\nu}]. \tag{3.6}$$

We see that NC field strength $\widehat{F}_{\mu\nu}$ transforms in the adjoint representation of the deformed gauge group $SO(2,3)_\star$ just as ordinary field strength $F_{\mu\nu}$ transforms in the adjoint representation of $SO(2,3)$. In the previous transformation rules, $\widehat{\Lambda}_\epsilon$ is a NC gauge parameter, and ϵ a commutative gauge parameter.

Because of noncommutativity of the $\star$-product, NC adjoint fields, say $\widehat{F}_{\mu\nu}$, do not belong to the basic Lie algebra of a gauge group, since the deformed commutation relations do not close in the Lie algebra itself. These fields actually belong to the enveloping algebra of the gauge group. The closure condition for a gauge transformation algebra becomes a set of differential equations, which are solved by iteration order by order in NC parameter $\theta^{\alpha\beta}$. Seiberg-Witten map provides a solution to these equations. It also ensures that no additional degrees of freedom are included when we make a NC deformation of an ordinary gauge field theory. The NC quantities can be represented as power series in the deformation parameter $\theta^{\alpha\beta}$, with expansion coefficients built out of the commutative quantities: ϵ, ϕ, ψ and ω_μ.

$$\widehat{\omega}_\mu = \omega_\mu - \frac{1}{4}\theta^{\alpha\beta}\{\omega_\alpha, \partial_\beta \omega_\mu + F_{\beta\mu}\} + \mathcal{O}(\theta^2), \tag{3.7}$$

$$\widehat{\phi} = \phi - \frac{1}{4}\theta^{\alpha\beta}\{\omega_\alpha, (\partial_\beta + D_\beta)\phi\} + \mathcal{O}(\theta^2), \tag{3.8}$$

$$\widehat{\psi} = \psi - \frac{1}{4}\theta^{\alpha\beta}\omega_\alpha(\partial_\beta + D_\beta)\psi + \mathcal{O}(\theta^2), \tag{3.9}$$

$$\widehat{\bar{\psi}} = \bar{\psi} - \frac{1}{4}\theta^{\alpha\beta}(\partial_\beta + D_\beta)\bar{\psi}\omega_\alpha + \mathcal{O}(\theta^2), \tag{3.10}$$

$$\widehat{\Lambda}_\epsilon = \epsilon - \frac{1}{4}\theta^{\alpha\beta}\{\omega_\alpha, \partial_\beta \epsilon\} + \mathcal{O}(\theta^2). \tag{3.11}$$

Using the SW map, we can derive the first order NC corrections to the field strength, and the covariant derivatives of adjoint and spinor field. They are given by

$$\begin{aligned}
\widehat{F}_{\mu\nu} = F_{\mu\nu} &- \frac{1}{4}\theta^{\alpha\beta}\{\omega_\alpha, (\partial_\beta + D_\beta)F_{\mu\nu}\}\\
&+ \frac{1}{2}\theta^{\alpha\beta}\{F_{\alpha\mu}, F_{\beta\nu}\} + \mathcal{O}(\theta^2),
\end{aligned} \tag{3.12}$$

$$\begin{aligned}
D_\mu \widehat{\phi} = D_\mu \phi &- \frac{1}{4}\theta^{\alpha\beta}\{\omega_\alpha, (\partial_\beta + D_\beta)D_\mu \phi\}\\
&+ \frac{1}{2}\theta^{\alpha\beta}\{F_{\alpha\mu}, D_\beta \phi\} + \mathcal{O}(\theta^2),
\end{aligned} \tag{3.13}$$

$$\begin{aligned}
D_\mu \widehat{\psi} = D_\mu \psi &- \frac{1}{4}\theta^{\alpha\beta}\omega_\alpha(\partial_\beta + D_\beta)D_\mu \psi\\
&+ \frac{1}{2}\theta^{\alpha\beta}F_{\alpha\mu}D_\beta \psi + \mathcal{O}(\theta^2).
\end{aligned} \tag{3.14}$$

All these results will be put into use in the next section where we turn to the NC version of the symmetric-phase action for the Dirac spinor field and calculate its perturbative expansion in powers of the deformation parameter.

4 NC action

Now we are going to deform the commutative symmetric-phase action (2.25) by replacing ordinary commutative fields, ϕ and ψ, with their noncommutative counterparts, $\widehat{\phi}$ and $\widehat{\psi}$, and by applying the Moyal-Weyl $\star$-product defined in (1.2) instead of the usual commutative multiplication. This NC action can be expanded perturbatively in powers of the deformation parameter $\theta^{\alpha\beta}$, assuming it to be small. We will investigate the first order NC correction to the kinetic and the mass terms separately. It will be demonstrated by explicit calculation that the first order NC correction after the symmetry breaking does not vanish.

4.1 NC deformation of the kinetic term

The noncommutative version of the kinetic action (2.16) will be denoted by a "hat" symbol and it is given by

$$\begin{aligned}
\widehat{S}_{kin} = \frac{i}{12}\int d^4x\, \varepsilon^{\mu\nu\rho\sigma}
&\Big[\widehat{\bar{\psi}} \star (D_\mu \widehat{\phi}) \star (D_\nu \widehat{\phi}) \star (D_\rho \widehat{\phi}) \star (D_\sigma \widehat{\psi})\\
&- (D_\sigma \widehat{\bar{\psi}}) \star (D_\mu \widehat{\phi}) \star (D_\nu \widehat{\phi}) \star (D_\rho \widehat{\phi}) \star \widehat{\psi}\Big].
\end{aligned} \tag{4.1}$$

Using the infinitesimal transformation rules (3.4) one can readily check that the action (4.1) is invariant under deformed $SO(2,3)_\star$ gauge transformations. Moreover, this action is hermitian up to the surface term which vanishes.

Let us now expand the action (4.1) up to the first order in the deformation parameter $\theta^{\alpha\beta}$ using the SW map. Generally, for any two NC fields $\widehat{A}$ and $\widehat{B}$, the first order NC correction to their product is given by

$$\big(\widehat{A}\star\widehat{B}\big)^{(1)}=\widehat{A}^{(1)}B+A\widehat{B}^{(1)}+\frac{i}{2}\theta^{\alpha\beta}\partial_\alpha A\partial_\beta B\,. \tag{4.2}$$

If both of these two fields transform in the adjoint representation, the last formula takes on the specific form, namely,

$$\begin{aligned}\big(\widehat{A}\star\widehat{B}\big)^{(1)}=&-\frac{1}{4}\theta^{\alpha\beta}\{\omega_\alpha,(\partial_\beta+D_\beta)AB\}+\frac{i}{2}\theta^{\alpha\beta}D_\alpha AD_\beta B\\&+cov(\widehat{A}^{(1)})B+Acov(\widehat{B}^{(1)})\,,\end{aligned}\tag{4.3}$$

where $cov(\widehat{A}^{(1)})$ is the covariant part of $A's$ first order NC correction, and $cov(\widehat{B}^{(1)})$, the covariant part of $B's$ first order NC correction. Applying the rule (4.3) twice, and using the expansion (3.13) for the covariant derivative of the adjoint field ϕ, we can obtain the first order NC correction to the product $D_\mu\widehat{\phi}\star D_\nu\widehat{\phi}\star D_\rho\widehat{\phi}$:

$$\begin{aligned}&\big(D_\mu\widehat{\phi}\star D_\nu\widehat{\phi}\star D_\rho\widehat{\phi}\big)^{(1)}\\&\quad=-\frac{1}{4}\theta^{\alpha\beta}\{\omega_\alpha,(\partial_\beta+D_\beta)(D_\mu\phi D_\nu\phi D_\rho\phi)\}\\&\qquad+\frac{i}{2}\theta^{\alpha\beta}D_\alpha(D_\mu\phi D_\nu\phi)(D_\beta D_\rho\phi)\\&\qquad+\frac{i}{2}\theta^{\alpha\beta}(D_\alpha D_\mu\phi)(D_\beta D_\nu\phi)D_\rho\phi\\&\qquad+\frac{1}{2}\theta^{\alpha\beta}\{F_{\alpha\mu},D_\beta\phi\}D_\nu\phi D_\rho\phi\\&\qquad+\frac{1}{2}\theta^{\alpha\beta}D_\mu\phi\{F_{\alpha\nu},D_\beta\phi\}D_\rho\phi\\&\qquad+\frac{1}{2}\theta^{\alpha\beta}D_\mu\phi D_\nu\phi\{F_{\alpha\rho},D_\beta\phi\}.\end{aligned}\tag{4.4}$$

Note that the composite field $D_\mu\widehat{\phi}\star D_\nu\widehat{\phi}\star D_\rho\widehat{\phi}$ also transforms in the adjoint representation of $SO(2,3)_\star$ since it is a product of the fields that transform in the adjoint representation. Thus, according to the rule (4.3), we could immediately say, without explicit calculation, what is the non-covariant part in the first order NC correction to $D_\mu\widehat{\phi}\star D_\nu\widehat{\phi}\star D_\rho\widehat{\phi}$, i.e. what is the first term in (4.4). It is non-covariant because of the way in which it incorporates the gauge potential ω_α and the partial derivative ∂_β. The other terms appearing in (4.4) are manifestly covariant. The use of the rule (4.3) significantly simplifies the calculation. Non-covariant part of any composite field that transforms in the adjoint representation has the same form as the second term in (3.8).

If we have a field $\widehat{A}$ that transforms in the adjoint representation, and field $\widehat{B}$ that transforms in the fundamental representation, then rule (4.2) again takes on the specific form, namely,

$$\begin{aligned}\big(\widehat{A}\star\widehat{B}\big)^{(1)}=&-\frac{1}{4}\theta^{\alpha\beta}\omega_\alpha(\partial_\beta+D_\beta)(AB)+\frac{i}{2}\theta^{\alpha\beta}D_\alpha AD_\beta B\\&+cov(\widehat{A}^{(1)})B+Acov(\widehat{B}^{(1)}).\end{aligned}\tag{4.5}$$

Similar recursive relation can be found in [42]. The non-covariant part of any composite field that transforms in the fundamental representation has the same form as the second term in (3.9).

Using the result (4.4) and the expansion (3.14) for the covariant derivative of a spinor field, we can obtain the first order NC correction to the noncommutative product $D_\mu\widehat{\phi}\star D_\nu\widehat{\phi}\star D_\rho\widehat{\phi}\star D_\rho\widehat{\psi}$. Applying the rule (4.5), and setting $\widehat{A}:=D_\mu\widehat{\phi}\star D_\nu\widehat{\phi}\star D_\rho\widehat{\phi}$ and $\widehat{B}:=D_\sigma\widehat{\psi}$, we get

$$\begin{aligned}&(D_\mu\widehat{\phi}\star D_\nu\widehat{\phi}\star D_\rho\widehat{\phi}\star D_\sigma\widehat{\psi})^{(1)}\\&\quad=-\frac{1}{4}\theta^{\alpha\beta}\omega_\alpha(\partial_\beta+D_\beta)(D_\mu\phi D_\nu\phi D_\rho\phi D_\sigma\psi)\\&\qquad+\frac{i}{2}\theta^{\alpha\beta}D_\alpha(D_\mu\phi D_\nu\phi D_\rho\phi)(D_\beta D_\sigma\psi)\\&\qquad+\frac{i}{2}\theta^{\alpha\beta}D_\alpha(D_\mu\phi D_\nu\phi)(D_\beta D_\rho\phi)D_\sigma\psi\\&\qquad+\frac{i}{2}\theta^{\alpha\beta}(D_\alpha D_\mu\phi)(D_\beta D_\nu\phi)D_\rho\phi D_\sigma\psi\\&\qquad+\frac{1}{2}\theta^{\alpha\beta}\{F_{\alpha\mu},D_\beta\phi\}D_\nu\phi D_\rho\phi D_\sigma\psi\\&\qquad+\frac{1}{2}\theta^{\alpha\beta}D_\mu\phi\{F_{\alpha\nu},D_\beta\phi\}D_\rho\phi D_\sigma\psi\\&\qquad+\frac{1}{2}\theta^{\alpha\beta}D_\mu\phi D_\nu\phi\{F_{\alpha\rho},D_\beta\phi\}D_\sigma\psi\\&\qquad-\frac{1}{2}\theta^{\alpha\beta}D_\mu\phi D_\nu\phi D_\rho\phi F_{\sigma\alpha}D_\beta\psi.\end{aligned}\tag{4.6}$$

The composite field $D_\mu\widehat{\phi}\star D_\nu\widehat{\phi}\star D_\rho\widehat{\phi}\star D_\sigma\widehat{\psi}$ transforms in the fundamental representation since it is a product of the field $D_\mu\widehat{\phi}\star D_\nu\widehat{\phi}\star D_\rho\widehat{\phi}$ that transforms in the adjoint representation, and the field $D_\sigma\widehat{\psi}$ that transforms in the fundamental representation, and for that reason the first term in (4.6), i.e. the non-covariant term, has the same form as the corresponding non-covariant term in (3.9). Again, we knew that from the general result (4.5). The other terms in (4.6) are manifestly covariant.

Using the NC expansion of the Dirac adjoint field (3.10), setting $\widehat{A}:=\widehat{\bar{\psi}}$ and $\widehat{B}:=D_\mu\widehat{\phi}\star D_\nu\widehat{\phi}\star D_\rho\widehat{\phi}\star D_\sigma\widehat{\psi}$, the general rule (4.2) gives us the first order correction to the total product $\widehat{\bar{\psi}}\star D_\mu\widehat{\phi}\star D_\nu\widehat{\phi}\star D_\rho\widehat{\phi}\star D_\sigma\widehat{\psi}$ which is a scalar of $SO(2,3)_\star$ group:

$$\begin{aligned}&(\widehat{\bar{\psi}}\star D_\mu\widehat{\phi}\star D_\nu\widehat{\phi}\star D_\rho\widehat{\phi}\star D_\sigma\widehat{\psi})^{(1)}\\&\quad=-\frac{1}{4}\theta^{\alpha\beta}\bar{\psi}F_{\alpha\beta}D_\mu\phi D_\nu\phi D_\rho\phi D_\sigma\psi\\&\qquad+\frac{i}{2}\theta^{\alpha\beta}\bar{\psi}D_\alpha(D_\mu\phi D_\nu\phi D_\rho\phi)(D_\beta D_\sigma\psi)\\&\qquad+\frac{i}{2}\theta^{\alpha\beta}\bar{\psi}D_\alpha(D_\mu\phi D_\nu\phi)(D_\beta D_\rho\phi)D_\sigma\psi\\&\qquad+\frac{i}{2}\theta^{\alpha\beta}\bar{\psi}(D_\alpha D_\mu\phi)(D_\beta D_\nu\phi)D_\rho\phi D_\sigma\psi\\&\qquad+\frac{1}{2}\theta^{\alpha\beta}\bar{\psi}\{F_{\alpha\mu},D_\beta\phi\}D_\nu\phi D_\rho\phi D_\sigma\psi\end{aligned}$$

$$+\frac{1}{2}\theta^{\alpha\beta}\bar{\psi}D_\mu\phi\{F_{\alpha\nu}, D_\beta\phi\}D_\rho\phi D_\sigma\psi$$
$$+\frac{1}{2}\theta^{\alpha\beta}\bar{\psi}D_\mu\phi D_\nu\phi\{F_{\alpha\rho}, D_\beta\phi\}D_\sigma\psi$$
$$-\frac{1}{2}\theta^{\alpha\beta}\bar{\psi}D_\mu\phi D_\nu\phi D_\rho\phi F_{\sigma\alpha}D_\beta\psi. \tag{4.7}$$

Finally, we present the first order NC correction to the commutative kinetic action in the symmetric phase, i.e. the $n=1$ term in the perturbative expansion of the full NC kinetic action $\widehat{S}_{kin}=\sum_n\widehat{S}^{(n)}_{kin}$:

$$\widehat{S}^{(1)}_{kin}=\frac{i}{12}\,\theta^{\alpha\beta}\int d^4x\ \varepsilon^{\mu\nu\rho\sigma}\Big[-\frac{1}{4}\bar{\psi}F_{\alpha\beta}D_\mu\phi D_\nu\phi D_\rho\phi D_\sigma\psi$$
$$+\frac{i}{2}\bar{\psi}D_\alpha(D_\mu\phi D_\nu\phi D_\rho\phi)(D_\beta D_\sigma\psi)$$
$$+\frac{i}{2}\bar{\psi}D_\alpha(D_\mu\phi D_\nu\phi)(D_\beta D_\rho\phi)D_\sigma\psi$$
$$+\frac{i}{2}\bar{\psi}(D_\alpha D_\mu\phi)(D_\beta D_\nu\phi)D_\rho\phi D_\sigma\psi$$
$$+\frac{1}{2}\bar{\psi}\{F_{\alpha\mu}, D_\beta\phi\}D_\nu\phi D_\rho\phi D_\sigma\psi$$
$$+\frac{1}{2}\bar{\psi}D_\mu\phi\{F_{\alpha\nu}, D_\beta\phi\}D_\rho\phi D_\sigma\psi$$
$$+\frac{1}{2}\bar{\psi}D_\mu\phi D_\nu\phi\{F_{\alpha\rho}, D_\beta\phi\}D_\sigma\psi$$
$$-\frac{1}{2}\bar{\psi}D_\mu\phi D_\nu\phi D_\rho\phi F_{\sigma\alpha}D_\beta\psi\Big]+h.c.\,. \tag{4.8}$$

This action possesses ordinary $SO(2,3)$, i.e. AdS symmetry, and this was to be expected by the virtue of the SW map. Namely, we started with the NC action (4.1) invariant under the deformed $SO(2,3)_\star$ gauge transformations and expanded it perturbatively in powers of the deformation parameter $\theta^{\alpha\beta}$ (up to the first order, but we could straightforwardly proceed further). By using the SW map we ensure that the obtained perturbative corrections are invariant, in each order, under the ordinary $SO(2,3)$ gauge transformations. Our result (4.8) explicitly confirms that.

In order to break the $SO(2,3)$ symmetry of the action (4.8) down to the local Lorentz $SO(1,3)$ symmetry, we set $\phi^a=0$ and $\phi^5=l$. The action reduces to

$$\widehat{S}^{(1)}_{kin}=\theta^{\alpha\beta}\Big[-\frac{1}{8}\int d^4x\ e\ R_{\alpha\mu}{}^{ab}e^\mu_a\ \bar{\psi}\gamma_b\nabla_\beta\psi$$
$$+\frac{1}{16}\int d^4x\ e\ R_{\alpha\beta}{}^{ab}e^\sigma_b\ \bar{\psi}\gamma_a\nabla_\sigma\psi$$
$$-\frac{i}{32}\int d^4x\ e\ R_{\alpha\beta}{}^{ab}\varepsilon_{abc}{}^{d}e^\sigma_d\ \bar{\psi}\gamma^c\gamma^5\nabla_\sigma\psi$$
$$-\frac{i}{16}\int d^4x\ e\ R_{\alpha\mu}{}^{bc}e^\mu_a\varepsilon^a{}_{bcm}\ \bar{\psi}\gamma^m\gamma^5\nabla_\beta\psi$$
$$-\frac{i}{24}\int d^4x\ e\ R_{\alpha\mu}{}^{ab}\varepsilon_{abc}{}^{d}e^c_\beta$$
$$(e^\mu_d e^\sigma_s-e^\mu_s e^\sigma_d)\ \bar{\psi}\gamma^s\gamma^5\nabla_\sigma\psi$$
$$-\frac{i}{8l}\int d^4x\ e\ T_{\alpha\beta}{}^{a}e^\sigma_a\ \bar{\psi}\nabla_\sigma\psi$$
$$+\frac{i}{8l}\int d^4x\ e\ T_{\alpha\mu}{}^{a}e^\mu_a\ \bar{\psi}\nabla_\beta\psi$$
$$+\frac{1}{16l}\int d^4x\ e\ T_{\alpha\beta}{}^{a}e^\mu_a\ \bar{\psi}\sigma_\mu{}^\sigma\nabla_\sigma\psi$$
$$+\frac{1}{8l}\int d^4x\ e\ T_{\alpha\mu}{}^{a}e^\mu_b\ \bar{\psi}\sigma_a{}^b\nabla_\beta\psi$$
$$-\frac{1}{12l}\int d^4x\ e\ T_{\alpha\mu}{}^{a}\varepsilon_{ab}{}^{cd}e^b_\beta e^\mu_c e^\sigma_d\ \bar{\psi}\gamma^5\nabla_\sigma\psi$$
$$+\frac{7i}{48l^2}\int d^4x\ e\ \varepsilon_{abc}{}^{d}e^a_\alpha e^b_\beta e^\sigma_d\ \bar{\psi}\gamma^c\gamma^5\nabla_\sigma\psi$$
$$-\frac{1}{4}\int d^4x\ e\ (\nabla_\alpha e^a_\mu)(e^\mu_a e^\sigma_b-e^\sigma_a e^\mu_b)\ \bar{\psi}\gamma^b\nabla_\beta\nabla_\sigma\psi$$
$$-\frac{1}{4l}\int d^4x\ e\ \bar{\psi}\sigma_\alpha{}^\sigma\nabla_\beta\nabla_\sigma\psi$$
$$-\frac{i}{8}\int d^4x\ e\ \eta_{ab}(\nabla_\alpha e^a_\mu)(\nabla_\beta e^b_\nu)\varepsilon^{cdrs}e^\mu_c$$
$$e^\nu_d e^\sigma_s\ \bar{\psi}\gamma_r\gamma_5\nabla_\sigma\psi$$
$$+\frac{i}{12}\int d^4x\ e\ (\nabla_\alpha e^a_\mu)(\nabla_\beta e^b_\nu)\varepsilon_b{}^{cds}e^\mu_c$$
$$e^\nu_d e^\sigma_s\ \bar{\psi}\gamma_a\gamma_5\nabla_\sigma\psi$$
$$-\frac{1}{12l}\int d^4x\ e\ e^c_\alpha(\nabla_\beta e^b_\nu)\varepsilon_{bc}{}^{ds}e^\nu_d e^\sigma_s\ \bar{\psi}\gamma_5\nabla_\sigma\psi$$
$$-\frac{1}{8l}\int d^4x\ e\ (\nabla_\alpha e^a_\mu)(e^\mu_a e^\sigma_b-e^\sigma_a e^\mu_b)e^c_\beta\ \bar{\psi}\sigma^b{}_c\nabla_\sigma\psi$$
$$-\frac{i}{2l}\int d^4x\ e\ (\nabla_\alpha e^a_\mu)e^\mu_a\ \bar{\psi}\nabla_\beta\psi$$
$$-\frac{1}{8l}\int d^4x\ e\ (\nabla_\alpha e^a_\mu)e^\mu_b\ \bar{\psi}\sigma_a{}^b\nabla_\beta\psi$$
$$+\frac{1}{96l}\int d^4x\ e\ R_{\alpha\beta}{}^{ab}\ \bar{\psi}\sigma_{ab}\psi$$
$$-\frac{5}{48l}\int d^4x\ e\ R_{\alpha\mu}{}^{ab}e^\mu_a e^c_\beta\ \bar{\psi}\sigma_{bc}\psi$$
$$-\frac{1}{16l}\int d^4x\ e\ R_{\alpha\mu}{}^{ab}e_{\beta a}e^\mu_c\ \bar{\psi}\sigma_b{}^c\psi$$
$$-\frac{3}{32l^2}\int d^4x\ e\ T_{\alpha\beta}{}^{a}\ \bar{\psi}\gamma_a\psi$$
$$-\frac{1}{16l^2}\int d^4x\ e\ T_{\alpha\mu}{}^{a}e^\mu_a\ \bar{\psi}\gamma_\beta\psi$$
$$+\frac{1}{16l^2}\int d^4x\ e\ T_{\alpha\mu}{}^{a}e_{\beta a}\ \bar{\psi}\gamma^\mu\psi$$
$$+\frac{1}{12l}\int d^4x\ e\ \eta_{ab}(\nabla_\alpha e^a_\mu)(\nabla_\beta e^b_\nu)\ \bar{\psi}\sigma^{\mu\nu}\psi$$
$$-\frac{1}{6l}\int d^4x\ e\ (\nabla_\alpha e^a_\mu)(\nabla_\beta e^b_\nu)(e^\mu_a e^\nu_c-e^\mu_c e^\nu_a)\ \bar{\psi}\sigma^c{}_b\psi$$
$$-\frac{3}{16l^2}\int d^4x\ e\ (\nabla_\alpha e^a_\mu)e^\mu_a\ \bar{\psi}\gamma_\beta\psi$$
$$+\frac{1}{16l^2}\int d^4x\ e\ (\nabla_\alpha e^a_\mu)e_{\beta a}\ \bar{\psi}\gamma^\mu\psi$$

$$-\frac{1}{3l^3}\int d^4x\; e\;\bar{\psi}\sigma_{\alpha\beta}\psi\quad\Big]+h.c.\,. \tag{4.9}$$

This result exhibits the type of couplings between Dirac spinors and gravity that emerge due to noncommutativity. Some of them pertain even in flat spacetime and this causes some interesting new physical effects.

4.2 NC deformation of the mass terms

In this section we consider a noncommutative deformation of the mass terms. It is obtained by replacing the ordinary commutative product with the NC Moyal-Weyl $\star$-product in the mass terms (2.21):

$$\begin{aligned}\widehat{S}_m =&\frac{i}{2}\Big(\frac{m}{l}-\frac{2}{l^2}\Big)\int d^4x\;\varepsilon^{\mu\nu\rho\sigma}\\ &\times\Big[\,c_1\widehat{\bar{\psi}}\star D_\mu\widehat{\phi}\star D_\nu\widehat{\phi}\star D_\rho\widehat{\phi}\star D_\sigma\widehat{\phi}\star\widehat{\phi}\star\widehat{\psi}\\ &+c_2\widehat{\bar{\psi}}\star D_\mu\widehat{\phi}\star D_\nu\widehat{\phi}\star D_\rho\widehat{\phi}\star\widehat{\phi}\star D_\sigma\widehat{\phi}\star\widehat{\psi}\\ &+c_3\widehat{\bar{\psi}}\star D_\mu\widehat{\phi}\star D_\nu\widehat{\phi}\star\widehat{\phi}\star D_\rho\widehat{\phi}\star D_\sigma\widehat{\phi}\star\widehat{\psi}\,\Big]+h.c.\,.\end{aligned} \tag{4.10}$$

Again, by using the Seiberg-Witten map we can represent this action as a perturbation series in powers of the deformation parameter $\theta^{\alpha\beta}$, taking only the first order term into account. We present below the result of this operation for each of the three mass terms, denoted by $\widehat{S}^{(1)}_{m,i}$ $(i=1,2,3)$, separately:

$$\begin{aligned}\widehat{S}^{(1)}_{m,1} =&\frac{ic_1}{2}\Big(\frac{m}{l}-\frac{2}{l^2}\Big)\,\theta^{\alpha\beta}\int d^4x\;\varepsilon^{\mu\nu\rho\sigma}\\ &\times\Big[+\frac{i}{2}\bar{\psi}D_\alpha(D_\mu\phi D_\nu\phi D_\rho\phi D_\sigma\phi\phi)D_\beta\psi\\ &-\frac{1}{4}\bar{\psi}F_{\alpha\beta}D_\mu\phi D_\nu\phi D_\rho\phi D_\sigma\phi\phi\psi\\ &+\frac{i}{2}\bar{\psi}D_\alpha(D_\mu\phi D_\nu\phi D_\rho\phi D_\sigma\phi)D_\beta\phi\psi\\ &+\frac{i}{2}\bar{\psi}D_\alpha(D_\mu\phi D_\nu\phi)D_\beta(D_\rho\phi D_\sigma\phi)\phi\psi\\ &+\frac{i}{2}\bar{\psi}D_\mu\phi D_\nu\phi(D_\alpha D_\rho\phi)(D_\beta D_\sigma\phi)\phi\psi\\ &+\frac{i}{2}\bar{\psi}(D_\alpha D_\mu\phi)(D_\beta D_\nu\phi)D_\rho\phi D_\sigma\phi\phi\psi\\ &+\frac{1}{2}\bar{\psi}\{F_{\alpha\mu},D_\beta\phi\}D_\nu\phi D_\rho\phi D_\sigma\phi\phi\psi\\ &+\frac{1}{2}\bar{\psi}D_\mu\phi\{F_{\alpha\nu},D_\beta\phi\}D_\rho\phi D_\sigma\phi\phi\psi\\ &+\frac{1}{2}\bar{\psi}D_\mu\phi D_\nu\phi\{F_{\alpha\rho},D_\beta\phi\}D_\sigma\phi\phi\psi\\ &+\frac{1}{2}\bar{\psi}D_\mu\phi D_\nu\phi D_\rho\phi\{F_{\alpha\sigma},D_\beta\phi\}\phi\psi\,\Big].\end{aligned} \tag{4.11}$$

$$\begin{aligned}\widehat{S}^{(1)}_{m,2} =&\frac{ic_2}{2}\Big(\frac{m}{l}-\frac{2}{l^2}\Big)\,\theta^{\alpha\beta}\int d^4x\;\varepsilon^{\mu\nu\rho\sigma}\\ &\times\Big[+\frac{i}{2}\bar{\psi}D_\alpha(D_\mu\phi D_\nu\phi D_\rho\phi\phi D_\sigma\phi)D_\beta\psi\\ &-\frac{1}{4}\bar{\psi}F_{\alpha\beta}D_\mu\phi D_\nu\phi D_\rho\phi\phi D_\sigma\phi\psi\\ &+\frac{i}{2}\bar{\psi}D_\alpha(D_\mu\phi D_\nu\phi D_\rho\phi\phi)(D_\beta D_\sigma\phi)\psi\\ &+\frac{i}{2}\bar{\psi}D_\alpha(D_\mu\phi D_\nu\phi D_\rho\phi)D_\beta\phi D_\sigma\phi\psi\\ &+\frac{i}{2}\bar{\psi}D_\alpha(D_\mu\phi D_\nu\phi)(D_\beta D_\rho\phi)\phi D_\sigma\phi\psi\\ &+\frac{i}{2}\bar{\psi}(D_\alpha D_\mu\phi)(D_\beta D_\nu\phi)D_\rho\phi\phi D_\sigma\phi\psi\\ &+\frac{1}{2}\bar{\psi}\{F_{\alpha\mu},D_\beta\phi\}D_\nu\phi D_\rho\phi\phi D_\sigma\phi\psi\\ &+\frac{1}{2}\bar{\psi}D_\mu\phi\{F_{\alpha\nu},D_\beta\phi\}D_\rho\phi\phi D_\sigma\phi\psi\\ &+\frac{1}{2}\bar{\psi}D_\mu\phi D_\nu\phi\{F_{\alpha\rho},D_\beta\phi\}\phi D_\sigma\phi\psi\\ &+\frac{1}{2}\bar{\psi}D_\mu\phi D_\nu\phi D_\rho\phi\phi\{F_{\alpha\sigma},D_\beta\phi\}\psi\,\Big].\end{aligned} \tag{4.12}$$

$$\begin{aligned}\widehat{S}^{(1)}_{m,3} =&\frac{ic_3}{2}\Big(\frac{m}{l}-\frac{2}{l^2}\Big)\,\theta^{\alpha\beta}\int d^4x\;\varepsilon^{\mu\nu\rho\sigma}\\ &\times\Big[+\frac{i}{2}\bar{\psi}D_\alpha(D_\mu\phi D_\nu\phi\phi D_\rho\phi D_\sigma\phi)D_\beta\psi\\ &-\frac{1}{4}\bar{\psi}F_{\alpha\beta}D_\mu\phi D_\nu\phi\phi D_\rho\phi D_\sigma\phi\psi\\ &+\frac{i}{2}\bar{\psi}D_\alpha(D_\mu\phi D_\nu\phi\phi)D_\beta(D_\rho\phi D_\sigma\phi)\psi\\ &+\frac{i}{2}\bar{\psi}D_\alpha(D_\mu\phi D_\nu\phi)D_\beta\phi D_\rho\phi D_\sigma\phi\psi\\ &+\frac{i}{2}\bar{\psi}(D_\alpha D_\mu\phi)(D_\beta D_\nu\phi)\phi D_\rho\phi D_\sigma\phi\psi\\ &+\frac{i}{2}\bar{\psi}D_\mu\phi D_\nu\phi\phi(D_\alpha D_\rho\phi)(D_\beta D_\sigma\phi)\psi\\ &+\frac{1}{2}\bar{\psi}\{F_{\alpha\mu},D_\beta\phi\}D_\nu\phi\phi D_\rho\phi D_\sigma\phi\psi\\ &+\frac{1}{2}\bar{\psi}D_\mu\phi\{F_{\alpha\nu},D_\beta\phi\}\phi D_\rho\phi D_\sigma\phi\psi\\ &+\frac{1}{2}\bar{\psi}D_\mu\phi D_\nu\phi\phi\{F_{\alpha\rho},D_\beta\phi\}D_\sigma\phi\psi\\ &+\frac{1}{2}\bar{\psi}D_\mu\phi D_\nu\phi\phi D_\rho\phi\{F_{\alpha\sigma},D_\beta\phi\}\psi\,\Big].\end{aligned} \tag{4.13}$$

None of the three mass terms in (4.10) is more preferable than the others, and so we will treat all three of them on equal footing. According to (2.22), we should assume that $c_1=-c_2=c_3$ if they are to contribute equally, at the commutative level, to the Dirac mass term after the symmetry breaking. The coefficients must also satisfy the constraint (2.23), and so we will set $c_1=-c_2=c_3=\frac{1}{72}$.

After the symmetry breaking, the first order NC correction to the sum of the three mass terms (4.11), (4.12) and (4.13) becomes

$$\begin{aligned}\widehat{S}_m^{(1)} = \theta^{\alpha\beta}\Bigg[&-\frac{i}{4}\left(m-\frac{2}{l}\right)\int d^4x\ e\ (\nabla_\alpha e_\mu^a)e_a^\mu\ \bar{\psi}\nabla_\beta\psi \\ &+\frac{1}{24}\left(m-\frac{2}{l}\right)\int d^4x\ e\ \eta_{ab}(\nabla_\alpha e_\mu^a)(\nabla_\beta e_\nu^b)\ \bar{\psi}\sigma^{\mu\nu}\psi \\ &-\frac{1}{12}\left(m-\frac{2}{l}\right)\int d^4x\ e\ (\nabla_\alpha e_\mu^a)(\nabla_\beta e_\nu^b)(e_a^\mu e_c^\nu \\ &-e_c^\mu e_a^\nu)\ \bar{\psi}\sigma^c{}_b\psi \\ &-\frac{1}{36}\left(\frac{m}{l}-\frac{2}{l^2}\right)\int d^4x\ e\ (\nabla_\alpha e_\mu^a)e_a^\mu\ \bar{\psi}\gamma_\beta\psi \\ &-\frac{1}{96}\left(m-\frac{2}{l}\right)\int d^4x\ e\ R_{\alpha\beta}{}^{ab}\ \bar{\psi}\sigma_{ab}\psi \\ &-\frac{1}{12}\left(m-\frac{2}{l}\right)\int d^4x\ e\ R_{\alpha\mu}{}^{ab}e_a^\mu e_\beta^c\ \bar{\psi}\sigma_{bc}\psi \\ &-\frac{1}{72}\left(\frac{m}{l}-\frac{2}{l^2}\right)\int d^4x\ e\ T_{\alpha\beta}{}^a\ \bar{\psi}\gamma_a\psi \\ &-\frac{7}{72}\left(\frac{m}{l}-\frac{2}{l^2}\right)\int d^4x\ e\ T_{\alpha\mu}{}^a e_a^\mu\ \bar{\psi}\gamma_\beta\psi \\ &-\frac{1}{8}\left(\frac{m}{l^2}-\frac{2}{l^3}\right)\int d^4x\ e\ \bar{\psi}\sigma_{\alpha\beta}\psi\ \Bigg] + h.c.\,. \end{aligned} \tag{4.14}$$

The full NC action at the first order in $\theta^{\alpha\beta}$ after the symmetry breaking is the sum of the kinetic term (4.9) and the mass term (4.14),

$$\widehat{S}^{(1)} = \widehat{S}_{kin}^{(1)} + \widehat{S}_m^{(1)}\,. \tag{4.15}$$

The result (4.15) is the sought first order noncommutative correction to the Dirac action in curved spacetime. This action couples Dirac spinors to the geometrical quantities like curvature, torsion, etc. It is manifestly $SO(1,3)$ gauge invariant and also, as we shall elaborate below, charge-conjugation invariant. The non-vanishing of the first order NC correction to the Dirac action in curved spacetime is a significant result since it enables us to extract potentially observable NC effects already at the lowest perturbative order. We will see in the next section that the first order NC corrections to the Dirac action pertains also in flat spacetime, making it even easier to investigate modifications of e.g. the Feynman propagator or dispersion relation for electrons. Note that the first non-vanishing NC correction to the Einstein-Hilbert action is at the second order in $\theta^{\alpha\beta}$. This result is confirmed in many papers [1–3,27–32].

4.3 C-conjugation

Let us analyse the behavior of the action (4.15), which is the sum of (4.9) and (4.14), under charge conjugation transformation. The charge conjugation operator is a unitary operator denoted by $\mathcal{C}$. The undeformed Dirac field (now treated as an operator-valued function) and its adjoint transform in the following way:

$$\begin{aligned}\mathcal{C}\psi(x)\mathcal{C}^{-1} &= -\bar{\psi}(x)C\,,\\ \mathcal{C}\bar{\psi}(x)\mathcal{C}^{-1} &= -\psi^T(x)C^{-1}\,,\end{aligned} \tag{4.16}$$

where C is a matrix defined by $C\gamma_a C^{-1} = -\gamma_a^T$. In the representation we use, it holds that $C^{-1} = C^\dagger = C^T = -C$. The identity $C\Gamma_A C^{-1} = \Gamma_A^T$ for $5D$ gamma-matrices also holds. The $SO(2,3)$ gauge potential ω_μ is invariant under C-conjugation, i.e. $\mathcal{C}\omega_\mu{}^{AB}\mathcal{C}^{-1} = \omega_\mu{}^{AB}$. This relation entails the invariance of the vierbein and the spin-connection,

$$\begin{aligned}\mathcal{C}e_\mu^a\mathcal{C}^{-1} &= e_\mu^a,\\ \mathcal{C}\omega_\mu{}^{ab}\mathcal{C}^{-1} &= \omega_\mu{}^{ab},\end{aligned} \tag{4.17}$$

and, by extension, the invariance of the curvature tensor and torsion which are built out of these quantities.

Using the transformation properties (4.16) and (4.17), one can readily verify that the undeformed (zeroth order) action after the symmetry breaking, given in (2.26), is invariant under charge conjugation. The invariance of the kinetic part follows from

$$\mathcal{C}\left(\bar{\psi}\gamma^a e_a^\sigma\nabla_\sigma\psi\right)\mathcal{C}^{-1} = -\left(\nabla_\sigma\bar{\psi}\right)\gamma^a e_a^\sigma\psi. \tag{4.18}$$

The undeformed action before symmetry breaking is also invariant since the auxiliary field ϕ is not effected by C-conjugation.

Now consider the first order NC corrections (4.9) and (4.14). They have the form $\theta^{\alpha\beta}\mathcal{L}_{\alpha\beta}^{(1)}$ and all terms in $\mathcal{L}_{\alpha\beta}^{(1)}$ pick up a minus sign under charge conjugation, for example,

$$\mathcal{C}\bar{\psi}\sigma_{\alpha\beta}\psi\mathcal{C}^{-1} = -\bar{\psi}\sigma_{\alpha\beta}\psi. \tag{4.19}$$

This, together with the transformation law for the deformation parameter [43,44],

$$\mathcal{C}\theta^{\alpha\beta}\mathcal{C}^{-1} = -\theta^{\alpha\beta}, \tag{4.20}$$

leads to the conclusion that action (4.15) is indeed invariant under charge conjugation. Such a behavior of the deformation parameter $\theta^{\alpha\beta}$ under charge conjugation transformation can be justified in several ways. Consider the transformation law for the NC Dirac spinor field $\widehat{\psi}$ and its adjoint. These fields can be expanded via SW map as in (3.9) and (3.10), and it must be ensured that they have the same sort of behaviour under charge conjugation as their undeformed counterparts, i.e. we demand that

$$\begin{aligned}\mathcal{C}\widehat{\psi}(x)\mathcal{C}^{-1} &= -\widehat{\bar{\psi}}(x)C\,,\\ \mathcal{C}\widehat{\bar{\psi}}(x)\mathcal{C}^{-1} &= -\widehat{\psi}^T(x)C^{-1}\,.\end{aligned} \tag{4.21}$$

Using the identity $C\omega_\alpha C^{-1} = -\omega_\alpha^T$, we can readily verify that

$$\mathcal{C}(\omega_\alpha\partial_\beta\psi)\mathcal{C}^{-1} = \partial_\beta\bar{\psi}(\omega_\alpha C), \tag{4.22}$$

$$\mathcal{C}(\omega_\alpha D_\beta \psi)\mathcal{C}^{-1} = D_\beta \bar{\psi}(\omega_\alpha C), \tag{4.23}$$

which are the transformation properties of the terms appearing in (3.9), and similarly for the corresponding terms in (3.10). If we want (4.21) to hold, $\theta^{\alpha\beta}$ must change its sign under C-conjugation. Aside from this formal argument based on symmetry considerations, there is a heuristic argument for assuming the transformation law (4.20) given in [44], stemming from string theory. It is explained there that an electric dipole moment of an open string is proportional to $\theta^{\alpha\beta}$. This motivates the conclusion that $\theta^{\alpha\beta}$ goes to $-\theta^{\alpha\beta}$ under charge conjugation. Because of the property of the Moyal-Weyl $\star$-product under C-conjugation, the algebra of coordinates remains unaffected.

5 NC Dirac equation in flat spacetime

In this last section, we study the special case of flat spacetime in order to investigate the influence of noncommutativity (which survives in this limit) on the energy-momentum relation for electrons. In the flat spacetime limit, the action (4.15) becomes

$$\widehat{S}^{(1)} = \theta^{\alpha\beta}\int d^4x \left[-\frac{1}{2l}\bar{\psi}\sigma_\alpha{}^\sigma \partial_\beta\partial_\sigma\psi + \frac{7i}{24l^2}\varepsilon_{\alpha\beta}{}^{\rho\sigma}\bar{\psi}\gamma_\rho\gamma_5\partial_\sigma\psi - M\bar{\psi}\sigma_{\alpha\beta}\psi\right], \tag{5.1}$$

where we introduced the notation $M := \frac{m}{4l^2} + \frac{1}{6l^3}$. Therefore, the effect of noncommutativity, in the form of new couplings in the action, is relevant even in flat spacetime.

The total NC action in flat spacetime to the first order is

$$\widehat{S} = \widehat{S}^{(0)} + \widehat{S}^{(1)} = \int d^4x\ \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi + \theta^{\alpha\beta}\int d^4x \left[-\frac{1}{2l}\bar{\psi}\sigma_\alpha{}^\sigma \partial_\beta\partial_\sigma\psi + \frac{7i}{24l^2}\varepsilon_{\alpha\beta}{}^{\rho\sigma}\bar{\psi}\gamma_\rho\gamma_5\partial_\sigma\psi - M\bar{\psi}\sigma_{\alpha\beta}\psi\right]. \tag{5.2}$$

The existence of the first order NC correction to the Dirac action is a non trivial consequence of this model. We can easily derive the Feynman propagator for the Dirac field from the action (5.2). The result (in momentum space) is given by

$$iS_F(p) = \int d^4x\ \langle\Omega|T\psi(x)\bar{\psi}(0)|\Omega\rangle e^{ipx} = \frac{i}{\not{p} - m + i\epsilon} + \frac{i}{\not{p} - m + i\epsilon}(i\theta^{\alpha\beta}D_{\alpha\beta})\frac{i}{\not{p} - m + i\epsilon} + \dots, \tag{5.3}$$

where

$$D_{\alpha\beta} := \frac{1}{2l}\sigma_\alpha{}^\sigma p_\beta p_\sigma + \frac{7}{24l^2}\varepsilon_{\alpha\beta}{}^{\rho\sigma}\gamma_\rho\gamma_5 p_\sigma - M\sigma_{\alpha\beta}. \tag{5.4}$$

The Feynman propagator is modified due to the spacetime noncommutativity. Thus, we see that an electron effectively interacts with the NC background itself. In this respect, we may say that NC background acts like a background electromagnetic field.

By varying action (5.2) with respect to $\bar{\psi}$ we derive the modified Dirac equation in Minkowski spacetime:

$$\left[i\not{\partial} - m - \frac{1}{2l}\theta^{\alpha\beta}\sigma_\alpha{}^\sigma\partial_\beta\partial_\sigma + \frac{7i}{24l^2}\theta^{\alpha\beta}\varepsilon_{\alpha\beta}{}^{\rho\sigma}\gamma_\rho\gamma_5\partial_\sigma - \theta^{\alpha\beta}M\sigma_{\alpha\beta}\right]\psi = 0. \tag{5.5}$$

To simplify further analysis, we will assume that only two spatial dimensions are mutually incompatible, e.g. $[x^1, x^2] = i\theta^{12}$. Thus, we have $\theta^{12} = -\theta^{21} =: \theta \neq 0$ and all other components of $\theta^{\mu\nu}$ equal to zero.

The Eq. (5.5) reduces to

$$\left[i\not{\partial} - m - \frac{\theta}{2l}(\sigma_1{}^\sigma\partial_2\partial_\sigma - \sigma_2{}^\sigma\partial_1\partial_\sigma) + \frac{7i\theta}{12l^2}(\gamma_0\gamma_5\partial_3 - \gamma_3\gamma_5\partial_0) - 2\theta M\sigma_{12}\right]\psi = 0, \tag{5.6}$$

where we assumed the convention in which $\varepsilon^{0123} = 1$.

Let us now find the dispersion relation, i.e. energy-momentum relation, for the Dirac fermions. Since hamiltonian commutes with the whole momentum operator, we can assume the plane wave ansatz $\psi(x) = u(\mathbf{p})e^{-ip\cdot x}$ where $u(\mathbf{p})$ stands for a yet undetermined spinor amplitude

$$u(\mathbf{p}) = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}. \tag{5.7}$$

With this choice, Eq. (5.6) can be represented in the momentum space as

$$\left(\begin{pmatrix} E - m & -\boldsymbol{\sigma}\cdot\mathbf{p} \\ \boldsymbol{\sigma}\cdot\mathbf{p} & -E - m \end{pmatrix} + \theta\mathcal{M}\right)u(\mathbf{p}) = 0, \tag{5.8}$$

where the matrix $\mathcal{M}$ is given by

$$\mathcal{M} = \begin{pmatrix} A & \frac{1}{2l}p_z p_- & -\frac{7}{12l^2}p_z & \frac{1}{2l}Ep_- \\ \frac{1}{2l}p_z p_+ & -A & -\frac{1}{2l}Ep_+ & -\frac{7}{12l^2}p_z \\ \frac{7}{12l^2}p_z & \frac{1}{2l}Ep_- & B & \frac{1}{2l}p_z p_- \\ -\frac{1}{2l}Ep_+ & \frac{7}{12l^2}p_z & \frac{1}{2l}p_z p_+ & -B \end{pmatrix}. \tag{5.9}$$

Quantities E and $\mathbf{p}$ denote energy and momentum of a particle, respectively, and the matrix elements A and B are given by

$$A := -\frac{1}{2l}(p_x^2 + p_y^2) + \frac{7E}{12l^2} - 2M,$$
$$B := -\frac{1}{2l}(p_x^2 + p_y^2) - \frac{7E}{12l^2} - 2M, \tag{5.10}$$

with $p_\pm = p_x \pm i p_y$. We use the Dirac representation of $\gamma-$matrices.

Non trivial solutions of the homogeneous matrix equation (5.8) which, when written explicitly, states that

$$\begin{pmatrix} E - m + \theta A & \frac{\theta}{2l} p_z p_- & -p_z - \frac{7\theta}{12l^2} p_z & -p_- + \frac{\theta}{2l} E p_- \\ \frac{\theta}{2l} p_z p_+ & E - m - \theta A & -p_+ - \frac{\theta}{2l} E p_+ & p_z - \frac{7\theta}{12l^2} p_z \\ p_z + \frac{7\theta}{12l^2} p_z & p_- + \frac{\theta}{2l} E p_- & -E - m + \theta B & \frac{\theta}{2l} p_z p_- \\ p_+ - \frac{\theta}{2l} E p_+ & -p_z + \frac{7\theta}{12l^2} p_z & \frac{\theta}{2l} p_z p_+ & -E - m - \theta B \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = 0, \tag{5.11}$$

exist, if and only if, the determinant of the matrix $\not{p} - m + \theta \mathcal{M}$ (which is the matrix appearing in (5.11)) equals zero. This condition will give us the dispersion relation. The determinant depends on the energy which is also represented as a perturbative expansion in powers of θ,

$$E = \sum_{n=0}^{+\infty} E^{(n)}; \quad \text{where} \quad E^{(n)} \sim \frac{\theta^n}{(length)^{2n+1}}. \tag{5.12}$$

If the determinant is equal to zero, it is equal to zero order by order in θ, and we can derive the momentum dependence of $E^{(1)}$ term in the energy expansion, which is enough to see how noncommutativity influences the dispersion relation for the Dirac fermions. To get higher order energy terms we need higher order perturbative corrections to the Dirac action.

First we will consider an electron moving along the z-direction, i.e. in the direction orthogonal to the noncommutative x, y-plane. The matrix equation (5.11) reduces to

$$\begin{pmatrix} E - m + \theta A(0) & 0 & -p_z - \frac{7\theta}{12l^2} p_z & 0 \\ 0 & E - m - \theta A(0) & 0 & p_z - \frac{7\theta}{12l^2} p_z \\ p_z + \frac{7\theta}{12l^2} p_z & 0 & -E - m + \theta B(0) & 0 \\ 0 & -p_z + \frac{7\theta}{12l^2} p_z & 0 & -E - m - \theta B(0) \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = 0, \tag{5.13}$$

where $A(0) = A(p_x = p_y = 0)$ and likewise $B(0) = B(p_x = p_y = 0)$.

Non trivial solution for spinor components a, b, c, and d exist if at least one of the following two conditions is satisfied:

$$\left[E - m \pm \left(\frac{7E}{12l^2} - 2M\right)\theta\right]\left[E + m \pm \left(\frac{7E}{12l^2} + 2M\right)\theta\right]$$
$$= \left[p_z \pm \frac{7p_z}{12l^2}\theta\right]^2. \tag{5.14}$$

Four different solutions for the energy (to the first order in θ, i.e. $E = E^{(0)} + E^{(1)}$) are

$$E_{1,2} = E_{\mathbf{p}} \mp \left[\frac{m^2}{12l^2} - \frac{m}{3l^3}\right]\frac{\theta}{E_{\mathbf{p}}} + \mathcal{O}(\theta^2),$$
$$E_{3,4} = -E_{\mathbf{p}} \pm \left[\frac{m^2}{12l^2} - \frac{m}{3l^3}\right]\frac{\theta}{E_{\mathbf{p}}} + \mathcal{O}(\theta^2), \tag{5.15}$$

with $E_{\mathbf{p}} = \sqrt{m^2 + p_z^2}$. This is reminiscent of the well known Zeeman effect. The deformation parameter θ plays the role of a constant background magnetic field that causes the splitting of atomic energy levels.

In the rest frame ($\mathbf{p} = 0$) the energies reduce to:

$$E_{1,2}(0) = m \mp \left[\frac{m}{12l^2} - \frac{1}{3l^3}\right]\theta + \mathcal{O}(\theta^2),$$
$$E_{3,4}(0) = -m \pm \left[\frac{m}{12l^2} - \frac{1}{3l^3}\right]\theta + \mathcal{O}(\theta^2). \tag{5.16}$$

From (5.16) we see that the electron's mass gets renormalised due to the noncommutativity of the background spacetime and the correction is linear in the deformation parameter.

By solving the matrix equation (5.13) for each of the four energy functions in (5.15), we get the following four linearly independent solutions of the NC Dirac equation (up to a normalization factor):

$$\psi_1 \sim \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E_{\mathbf{p}}+m}\left[1 + \left(\frac{m}{12l^2} - \frac{1}{3l^3}\right)\frac{\theta}{E_{\mathbf{p}}}\right] \\ 0 \end{pmatrix} e^{-iE_1 t + i p_z z},$$
$$\psi_2 \sim \begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{p_z}{E_{\mathbf{p}}+m}\left[1 - \left(\frac{m}{12l^2} - \frac{1}{3l^3}\right)\frac{\theta}{E_{\mathbf{p}}}\right] \end{pmatrix} e^{-iE_2 t - i p_z z},$$
$$\psi_3 \sim \begin{pmatrix} \frac{p_z}{E_{\mathbf{p}}+m}\left[1 + \left(\frac{m}{12l^2} - \frac{1}{3l^3}\right)\frac{\theta}{E_{\mathbf{p}}}\right] \\ 0 \\ 1 \\ 0 \end{pmatrix} e^{-iE_3 t - i p_z z},$$
$$\psi_4 \sim \begin{pmatrix} 0 \\ \frac{p_z}{E_{\mathbf{p}}+m}\left[1 - \left(\frac{m}{12l^2} - \frac{1}{3l^3}\right)\frac{\theta}{E_{\mathbf{p}}}\right] \\ 0 \\ 1 \end{pmatrix} e^{-iE_4 t + i p_z z}. \tag{5.17}$$

Spinors ψ_1 and ψ_2 (ψ_3, and ψ_4) correspond to positive (negative) energy solutions of the NC Dirac equation. Note that in commutative case the opposite helicity ($\pm\frac{1}{2}$) solutions have the same energy. However, in noncommutative case, as we can see, the solutions with opposite helicity have different energies. The noncommutativity of space, here taken to be confined in x, y-plane, causes the undeformed energy levels $\pm E_{\mathbf{p}}$ to split. The energy gap between the new levels is the same for $\pm E_{\mathbf{p}}$ and it equals

$$2\left[\frac{m^2}{12l^2} - \frac{m}{3l^3}\right]\frac{\theta}{E_{\mathbf{p}}}. \tag{5.18}$$

From the dispersion relations (5.15) we can easily find the (group) velocity of an electron. This velocity is defined by

$$\mathbf{v} \equiv \frac{\partial E}{\partial \mathbf{p}}. \tag{5.19}$$

For positive (negative) helicity solution ψ_1 (ψ_2) we get

$$\mathbf{v}_{1,2} = \frac{\mathbf{p}}{E_{\mathbf{p}}}\left[1 \pm \left(\frac{m^2}{12l^2} - \frac{m}{3l^3}\right)\frac{\theta}{E_{\mathbf{p}}^2} + \mathcal{O}(\theta^2)\right]. \tag{5.20}$$

These velocities can be rewritten in the following way:

$$\mathbf{v}_{1,2} = \frac{\mathbf{p}}{E_{1,2}} + \mathcal{O}(\theta^2). \tag{5.21}$$

Thus, we conclude that velocity of an electron moving in z-direction depends on its helicity. This is analogues to the birefringence effect, i.e. the optical property of a material having a refractive index that depends on the polarization and propagation direction of light. NC background acts as a birefringent medium for electrons propagating in it.

The Dirac spinor ψ_1 can be represented as

$$\psi_1 \sim \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E_1 + E_1(0)} \\ 0 \end{pmatrix} e^{-iE_1 t + ip_z z}. \tag{5.22}$$

and the corresponding Dirac spinor in the rest frame is

$$\psi_1(0) \sim \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{-iE_1(0)t}, \tag{5.23}$$

where

$$E_1(0) = m - \left[\frac{m}{12l^2} - \frac{1}{3l^3}\right]\theta. \tag{5.24}$$

The boost along z-direction in spinor representation is given by

$$S(\varphi) = \cosh\left(\frac{\varphi}{2}\right) I - \sinh\left(\frac{\varphi}{2}\right)\begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}, \tag{5.25}$$

where $v = \tanh(\varphi)$. If we take $v = -v_1 = -\frac{p_z}{E_1}$ we can construct the boost matrix that transforms the rest frame solution $\psi_1(p_z = 0)$ into the solution $\psi_1(p_z)$. It is given by

$$S(-p_z) = \sqrt{\frac{E_1(p_z) + E_1(0)}{2E_1(0)}}\, I + \sqrt{\frac{E_1(p_z) - E_1(0)}{2E_1(0)}}\begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}, \tag{5.26}$$

and we have

$$S(-p_z)\psi_1(0) = \psi_1(p_z). \tag{5.27}$$

This result shows that constant noncommutativity in x, y-plane is compatible with a Lorentz boost along z-direction. Similar statement holds for the other solutions.

For an electron moving in noncommutative x, y-plane, i.e. an electron whose momentum is $\mathbf{p} = (p_x, p_y, 0)$, by using the same procedure, we get the deformed energy levels:

$$\begin{aligned} E_{1,4} &= \pm E_{\mathbf{p}} - \left[\frac{m}{12l^2} - \frac{1}{3l^3}\right]\theta, \\ E_{2,3} &= \pm E_{\mathbf{p}} + \left[\frac{m}{12l^2} - \frac{1}{3l^3}\right]\theta, \end{aligned} \tag{5.28}$$

with $E_{\mathbf{p}} = \sqrt{m^2 + p_x^2 + p_y^2}$. It is interesting to note that, in this case, NC corrections do not depend on the momentum, as opposed to the NC corrections of the energy levels of an electron moving along z-direction, i.e. in the direction in which it does not feel the noncommutativity. Again, these energy levels exactly reduces to (5.16) when $\mathbf{p} = 0$.

The four independent Dirac spinors are:

$$\begin{aligned} \psi_1 &\sim \begin{pmatrix} 1 \\ 0 \\ 0 \\ \frac{p_+}{E_{\mathbf{p}}+m}\left[1 + \left(\frac{7}{12l^2} - \frac{m}{12l}\right)\theta\right] \end{pmatrix} e^{-iE_1 t + ip_x x + ip_y y}, \\ \psi_2 &\sim \begin{pmatrix} 0 \\ 1 \\ \frac{p_-}{E_{\mathbf{p}}+m}\left[1 - \left(\frac{7}{12l^2} - \frac{m}{12l}\right)\theta\right] \\ 0 \end{pmatrix} e^{-iE_2 t + ip_x x + ip_y y}, \\ \psi_3 &\sim \begin{pmatrix} 0 \\ \frac{p_+}{E_{\mathbf{p}}+m}\left[1 + \left(\frac{7}{12l^2} - \frac{m}{12l}\right)\theta\right] \\ 1 \\ 0 \end{pmatrix} e^{-iE_3 t - ip_x x - ip_y y}, \\ \psi_4 &\sim \begin{pmatrix} \frac{p_-}{E_{\mathbf{p}}+m}\left[1 - \left(\frac{7}{12l^2} - \frac{m}{12l}\right)\theta\right] \\ 0 \\ 0 \\ 1 \end{pmatrix} e^{-iE_4 t - ip_x x - ip_y y}. \end{aligned} \tag{5.29}$$

It turns out that these solutions cannot be obtained by boosting the corresponding rest frame solutions. This was to be expected since, as we have already mentioned, by choosing the canonical noncommutativity we have effectively fixed the coordinate system. In other words, we work in a preferred coordinate system in which only boosts along z-axis and rotations around z-axis are preserved.

6 Conclusion

We studied the coupling of the Dirac spinor field and gravity on noncommutative Moyal-Weyl spacetime starting from a commutative theory with AdS gauge symmetry. After its NC deformation one can perform a perturbative expansion of the $SO(2,3)_\star$ invariant action in powers of the deformation parameter $\theta^{\alpha\beta}$, assuming it to be small, by using the Seiberg-Witten map. In this way we ensure that the expansion has the ordinary $SO(2,3)$ symmetry, order by order in $\theta^{\alpha\beta}$. Breaking the symmetry down to the local Lorentz $SO(1,3)$ symmetry reduces the action to the NC Dirac action in curved spacetime. Explicit calculation of the first order NC correction is presented. It is invariant under local Lorentz transformations and charge conjugation, and we showed that it does not vanish even in flat spacetime. This significant feature enables us to study how linear NC effects influence the properties of a free electron in Minkowski spacetime. There is a linear deformation of the Dirac equation and the Feynman propagator due to noncommutativity. The dispersion relation for electrons is also modified. The undeformed energy levels of the commutative theory get split in the constant background NC spacetime - a phenomenon analogues to the Zeeman splitting of atomic energy levels in background magnetic field. We also found the explicit solutions of the NC Dirac equation in flat spacetime and demonstrated that, by introducing constant noncommutativity in flat spacetime, we are effectively working in the preferred class of coordinate systems. The helicity dependence of the deformed energy levels means that NC background acts as a birefringent medium for electrons propagating in it.

This could not be achieved by directly introducing noncommutativity into the free Dirac action (minimal substitution) giving

$$S = \int d^4x\ \widehat{\bar{\psi}} \star (i\gamma^\mu \partial_\mu - m)\widehat{\psi}. \tag{6.1}$$

Since

$$\int d^4x\ \widehat{f} \star \widehat{g} = \int d^4x\ fg, \tag{6.2}$$

the first order NC correction to the free Dirac action (6.1) vanishes.

Let us also mention the appearance of the term $\theta^{\alpha\beta}\bar{\psi}\sigma_{\alpha\beta}\psi$ in the NC Lagrangian density. It resembles the magnetic moment term in Electrodynamics with electromagnetic field strength tensor replaced by $\theta^{\alpha\beta}$. If we interpret the parameter of noncommutativity as a constant "electric/magnetic" background field the analogy becomes obvious. This is in accord with the behaviour of the deformation parameter under charge-conjugation and the upper mentioned Zeeman-like splitting of the energy levels.

In future work we plan to include electromagnetic field in our NC $SO(2,3)_\star$ model. This will lead us to a theory of NC electrodynamics, with potentially new phenomenology, that can be compared to the one established by the standard approach based on the minimal substitution. Minimal NC electrodynamics is not a renormalisabile theory because of the fermionic loop contributions [45–47]. It would be interesting to analyse the renormalisability of the presented model and to extend this approach to scalar and non-Abelian gauge fields in order to establish a complete theory concerning the behaviour of matter in noncommutative spacetime.

Acknowledgements We would like to thank Maja Burić, Marija Dimitrijević-Ćirić and Aleksandra Dimić for fruitful discussion and useful comments. The work is supported by project ON171031 of the Serbian Ministry of Education and Science and partially supported by the Action MP1405 QSPACE from the Europe an Cooperation in Science and Technology (COST). V.R. would like to thank ESI (Vienna) for hospitality during his stay.

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Funded by SCOAP^3.

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