

Thermodynamics and relativistic kinetic theory for q -generalized Bose–Einstein and Fermi–Dirac systems

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Abstract The thermodynamics and covariant kinetic theory are elaborately investigated in a non-extensive environment considering the non-extensive generalization of Bose–Einstein (BE) and Fermi–Dirac (FD) statistics. Starting with Tsallis’ entropy formula, the fundamental principles of thermostatics are established for a grand canonical system having q -generalized BE/FD degrees of freedom. Many particle kinetic theory is set up in terms of the relativistic transport equation with q -generalized Uehling–Uhlenbeck collision term. The conservation laws are realized in terms of appropriate moments of the transport equation. The thermodynamic quantities are obtained in a weak non-extensive environment for a massive pion–nucleon and a massless quark–gluon system with non-zero baryon chemical potential. In order to get an estimate of the impact of non-extensivity on the system dynamics, the q -modified Debye mass and hence the q -modified effective coupling are estimated for a quark–gluon system.

1 Introduction

Boltzmann–Gibbs (B–G) statistics has long been serving as the founding structure of a wide range of physical systems, especially the ones containing a large number of particles, in the presence of short-range correlations (exponentially decaying). Any understanding beyond such correlations, where the memory effects are significant and non-Markovian processes are likely to occur, requires theoretical modeling of the system under consideration in terms of a generalized statistical approach, such that in an appropriate limit case it yields the usual B–G statistics. In this context, the Tsallis approach, which is the non-extensive generaliza-

tion of B–G statistics [1–12], could serve the development of a description of such systems with long-range correlations.

In the context of high-energy collisions, the observables (such as particle spectra and transverse momentum fluctuations) in certain situations could be quantitatively explained better in terms of non-extensive statistics producing power-law distributions. In high-energy experiments, the momentum distribution provided by B–G statistics, being exponential in nature, gives a sensible description of particle production data only at low transverse momentum (below $p_T \sim 1$ GeV). However, for higher p_T ranges, the particle spectrum is observed to follow a power-law type tail. In the current literature, the requirement of a generalized statistics, leading to a power-law kind of distribution of emitted particles in high-energy collisions, has been extensively studied [13–20]. This could be a hint towards the presence of long-range correlations among the particles within the system. In particular, for hot QCD systems produced at relativistic heavy ion collision experiments, this condition looks more feasible. In experimental facilities like RHIC (Relativistic Heavy Ion Collider at BNL, USA) and LHC (Large hadron Collider at CERN, Geneva), due to large momentum transfer, the interaction strength among the quarks and gluons becomes weaker by the virtue of asymptotic freedom, which can only be realized at temperatures much beyond the transition temperature (T_c) from a hadron to a quark–gluon plasma (QGP) phase. Around T_c the interaction is considerably large, giving rise to a strongly coupled QGP where the partonic degrees of freedom could be deconfined just beyond their nucleonic volume. This phenomenon leads to long-range entanglements and consequently the memory effects of microscopic interactions are significant.

The two-particle, long-range correlations in the systems created in collider experiments already have been identified in recent work where the correlation is quantified by the “ridge structure”, observed in particle multiplic-

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ity distributions as a function of relative pseudo-rapidity [21–25]. Clearly the B–G statistics, which works wonderfully well for systems having close spatial connections and short-ranged time connections, fails to describe such exotic systems at a quantitative level. Further the “hypothesis of molecular chaos” from Boltzmann kinetic theory, assuming the interacting particles to be uncorrelated, does not hold in such a situation and non-Markovian processes become relevant where the consequences of particle collisions are no more independent of the past interaction memory. In Refs. [26,27], the ridge phenomenon in the pp collision at 7 TeV center of mass energy has been interpreted using a Tsallis distribution indicating a power-law behavior at large transverse momenta rather than an exponential one. The non-Markovian processes with long-range correlations have been precisely studied within the frame work of the Tsallis statistics in [28]. The complexity of the system where the space-time and consequently the phase space are non-fractal, proves to be beyond the scope of standard B–G statistics and hence for a non-trivial system like QGP the non-extensive generalization becomes quite relevant.

In the context of describing the strongly interacting system created in heavy ion collisions, Tsallis statistics has been applied to the observation of various phenomena. Apart from describing the transverse momentum spectra as already mentioned, a large number of observables and system properties have been analyzed in the light of non-extensive dynamics. The relativistic equation of state of hadronic matter and quark–gluon plasma at finite temperature and baryon density have been investigated in the framework of the non-extensive statistical mechanics in [29–31]. The non-extensive statistical effects in the hadron to quark–gluon phase transition have been studied in Refs. [32,33]. Strangeness production has been mentioned under the same statistical set up in [34]. Local deconfinement in relativistic systems including strong coupling diffusion and memory effects have been discussed in [35,36]. The kinetic freeze-out temperature and radial flow velocity have been extracted from an improved Tsallis distribution in [37]. A modified Hagedorn formula and also a limit temperature concept of a parton gas with power-law tailed distribution presented, respectively, in [38] and [39]. In [40–42], the effects of non-extensive statistics have been manifested in terms of the fluctuation of system variables such as temperature and chemical fluctuations in high-energy nuclear collisions. A non-extensive Boltzmann equation and associated hadronization of quark matter have been discussed in [43]. Recently a considerable amount of work has been carried out regarding the bulk observables of ultra-relativistic heavy ion collisions, such as the transverse momentum distribution, radial flow, evolution of fluctuation, nuclear modification factor, speed of

sound etc. under the Tsallis generalization of B–G statistics [44–53,62,63].

In view of the facts mentioned above, the Tsallis statistics has a vast applicability in strongly interacting systems related to heavy ion physics. Thus, one needs to set up a complete model involving relativistic kinetic theory and fluid dynamics under this generalized statistical scheme. In this context, there are a few attempts which provide macroscopic thermodynamic quantities such as particle number density, energy density, pressure, equation of state and transport parameters, starting from a relativistic microscopic theory [54–57]. In most of these studies the non-extensive generalization has been made over the Boltzmann distribution function while describing the single particle momentum distribution, but the quantum statistical effects of the Bose–Einstein (BE) and Fermi–Dirac (FD) distributions are missing. The quantum statistical factors (Bose enhancement for BE and Pauli blocking for FD systems, respectively) essential to describe a QCD (quantum chromodynamics) system, are not being included in the phase space integral of the collision term while setting up the relativistic transport equation. This fact sets the motivation for the present investigations.

In this work a complete thermostatistical model for a grand canonical system under non-extensive environment and hence a relativistic kinetic theory for a many particle system including B–E and F–D distributions for individual species of particles have been formulated in detail. Referring to earlier work, in [58,59] the q -generalized Fermi–Dirac distribution have been discussed and in [60] a non-extensive quantum H-theorem has been studied. In the current work the quantum statistical factors in the momentum distribution of final state particles, introduced by Uehling and Uhlenbeck in semiclassical transport theory, have been carefully included while developing the q -generalized theory. In the constructed formalism the thermodynamic macroscopic state variables and effective coupling for hot QCD system in non-extensive environment have been estimated.

The manuscript is organized as follows. Section 2 deals with the formalism, where firstly the non-extensive thermostatics for a grand canonical ensemble and then the non-extensive relativistic kinetic theory with quantum statistical effects are discussed in Sects. 2.1 and 2.2, respectively. In Sect. 2.3 the analytical expressions of the thermodynamic quantities are obtained and finally Sect. 2.4 estimates the effective coupling for a q -generalized QCD system. Section 3 contains the results displaying the temperature dependence of the evaluated quantities with finite baryon chemical potential, and the relevant discussions as well. The manuscript has been completed incorporating the concluding remarks and possible outlooks of the present work in Sect. 4.

2 Formalisms

Here, the salient features of the non-extensive thermostatistics for a grand canonical ensemble and the relativistic kinetic theory for a multi-component system with constituents obeying BE/FD distributions have been discussed. The entropy maximization technique in the first case using the method of Lagrange’s undetermined multipliers and in the second case by applying the laws of summation invariants have been shown to obtain the identical expressions of single particle BE/FD distribution function in a non-extensive environment. This consistency provides the ground for the microscopic definitions of the generalized entropy and collision integral including the quantum statistical effects of Bose enhancement and Pauli blocking. Afterwards, the distribution functions are employed to obtain a number of the thermodynamic quantities essential to specify the macroscopic properties of a system. In the present work these quantities have been estimated for a massive pion–nucleon and a massless quark–gluon system with finite baryon chemical potentials. Finally, the modification of QCD coupling describing the interaction dynamics of the system under the effects of non-extensivity has also been estimated in order to include the long-range interaction measures.

2.1 Non-extensive thermostatistics for a grand canonical ensemble

To start with, non-extensive generalization of the entropy proposed by Tsallis [1–8] is given now:

$$S_q = k \frac{1 - \sum_{i=1}^W p_i^q}{q - 1}, \quad [q \in \mathbb{R}]. \tag{1}$$

Here k is a positive constant. p_i is the probability associated with the i th state and $W \in \mathbb{N}$ is the total number of possible microscopic configurations of the system following the norm condition:

$$\sum_{i=1}^W p_i = 1. \tag{2}$$

It is quite straightforward to realize for the limiting condition of the entropic index $q \rightarrow 1$, and Eq. (1) reduces to the well-known form of the Boltzmann–Gibbs entropy, namely $\lim_{q \rightarrow 1} S_q = -k \sum_{i=1}^W p_i \ln p_i$. Equation (1) can alternatively also be defined with the help of a generalized differential operator as

$$S_q = -k \left\{ D_q \sum_{i=1}^W p_i^\alpha \right\}_{\alpha=1}, \tag{3}$$

following the definition of the operator $D_q f(x) \equiv \frac{f(qx) - f(x)}{qx - x}$, which for $q \rightarrow 1$ reduces to $\frac{df(x)}{dx}$. One important signature of the entropy definition given in (1) is the pseudo-additive rule for a combined system consisting of two individual systems A and B ,

$$\frac{S_q(A + B)}{k} = \frac{S_q(A)}{k} + \frac{S_q(B)}{k} + (1 - q) \frac{S_q(A)}{k} \frac{S_q(B)}{k}. \tag{4}$$

Now in order to obtain an equilibrium probability distribution in a grand canonical system, we need to extremize S_q in the presence of a set of appropriate constraints regarding the choice of internal energy and particle number [61]. So along with the entropy definition from (1) and norm constraint from (2), the choice of internal energy and particle number is addressed from the Tsallis original third choice of energy constraint for a canonical system [2], extending currently for a grand canonical one by the two following equations:

$$\frac{\sum_{i=1}^W p_i^q \epsilon_i}{\sum_{i=1}^W p_i^q} = \bar{E}, \tag{5}$$

$$\frac{\sum_{i=1}^W p_i^q n_i}{\sum_{i=1}^W p_i^q} = \bar{N}. \tag{6}$$

Here i labels the possible quantum states of the whole system where \bar{E} and \bar{N} are the average energy and average number of particles of the same. The best known way to get a solution of this variational problem is using the method of Lagrange’s undetermined multipliers, for which the following expression needs to be extremized:

$$Q = \frac{S_q}{k} + \alpha \sum_{i=1}^W p_i - \beta \frac{\sum_{i=1}^W p_i^q \epsilon_i}{\sum_{i=1}^W p_i^q} - \gamma \frac{\sum_{i=1}^W p_i^q n_i}{\sum_{i=1}^W p_i^q}, \tag{7}$$

where α , β and γ are the Lagrange undetermined multipliers. After extremization and following the standard thermodynamic definitions, such as $\beta = \frac{1}{kT}$ and $\gamma = -\beta\mu$ with T the temperature of the system and μ the chemical potential for each particle, we finally achieve the probability distribution of a grand canonical system in terms of fundamental thermodynamic quantities,

$$p_i = \frac{1}{Z_q} \left[1 - (1 - q) \frac{\left\{ \frac{1}{T}(\epsilon_i - \bar{E}) - \frac{\mu}{T}(n_i - \bar{N}) \right\}}{\left\{ \sum_{i=1}^W p_i^q \right\}} \right]^{\frac{1}{1-q}}, \tag{8}$$

with

$$Z_q = \sum_{i=1}^W \left[1 - (1 - q) \frac{\left\{ \frac{1}{T}(\epsilon_i - \bar{E}) - \frac{\mu}{T}(n_i - \bar{N}) \right\}}{\left\{ \sum_{i=1}^W p_i^q \right\}} \right]^{\frac{1}{1-q}}, \tag{9}$$

as the q -generalized grand canonical partition function. From Eqs. (8) and (9) a very useful identity can be achieved which is given below,

$$Z_q^{1-q} = \sum_{i=1}^W p_i^q. \quad (10)$$

To get the expressions compactified it is now the time to define the q -generalized exponential and logarithmic functions,

$$\exp_q(x) = \{1 + (1-q)x\}^{\frac{1}{1-q}}, \quad (11)$$

$$\ln_q x = \frac{x^{1-q} - 1}{1-q}, \quad (12)$$

which reduces to the conventional exponentials and logarithms as $q \rightarrow 1$. With the help of the q -exponential function the probability distribution as well as the partition function from Eqs. (8) and (9) can be redefined as

$$p_i = \frac{1}{Z_q} \exp_q \left[\frac{-\left\{ \frac{1}{T}(\epsilon_i - \bar{E}) - \tilde{\mu}(n_i - \bar{N}) \right\}}{\left\{ \sum_{i=1}^W p_i^q \right\}} \right], \quad (13)$$

$$Z_q = \sum_{i=1}^W \exp_q \left[\frac{-\left\{ \frac{1}{T}(\epsilon_i - \bar{E}) - \tilde{\mu}(n_i - \bar{N}) \right\}}{\left\{ \sum_{i=1}^W p_i^q \right\}} \right], \quad (14)$$

with $\tilde{\mu} = \mu/T$. The definition of the temperature and chemical potential can be derived in terms of the macroscopic quantities as the following:

$$\frac{\partial S_q}{\partial \bar{E}} = \frac{1}{T}, \quad (15)$$

$$\frac{\partial S_q}{\partial \bar{N}} = -\frac{\mu}{T} \quad (16)$$

which is consistent with situation for $q = 1$.

Now the choice of energy and particle number constraints from Eqs. (5) and (6), respectively, needs some discussions offered below. There are a number of reasons that this choice is an unique one which is free from the unfamiliar consequences with respect to the other choices proposed [2]. First, its invariance under the uniform translation of energy and particle number spectrum ($\{\epsilon_i\}$ and $\{n_i\}$, respectively) makes the thermostatical quantities independent of the choice of origin of energy and particle number densities. Secondly, the normalization condition is carefully preserved in this choice ($\ll 1 \gg_q = 1$, where $\ll O_i \gg_q \equiv \frac{\sum_{i=1}^W p_i^q O_i}{\sum_{i=1}^W p_i^q}$). Finally, the most important one is that it preserves the additive property of generalized internal energy ($\bar{E}(A+B) = \bar{E}(A) + \bar{E}(B)$) in the exact same form of standard thermodynamics ($q = 1$). In other words, the microscopic energy conservation is retained macroscopically as well. This is an extremely crucial property as regards describing the dynamics of the system.

Next, it can be trivially shown that Eqs. (13) and (14) can be presented by a set of simpler expressions in terms of renormalized temperature and chemical potential as follows:

$$p_i = \frac{1}{Z_q} \exp_q \left[- \left\{ \frac{\epsilon_i}{T'} - \tilde{\mu}' n_i \right\} \right], \quad (17)$$

$$Z_q = \sum_{i=1}^W \exp_q \left[- \left\{ \frac{\epsilon_i}{T'} - \tilde{\mu}' n_i \right\} \right], \quad (18)$$

with

$$T' = T Z_q^{1-q} + (1-q)\bar{E} - \mu(1-q)\bar{N}, \quad (19)$$

$$\tilde{\mu}' = \tilde{\mu} \frac{1}{Z_q^{1-q} + \frac{1}{T}(1-q)\bar{E} - \tilde{\mu}(1-q)\bar{N}}. \quad (20)$$

Systems using some arbitrary finite temperature rather than specific temperature dependence of the involved thermostatical quantities (and so including the chemical potential), can conveniently use the definitions provided by Eqs. (17) and (18). The redefined expressions of the temperature (T') and chemical potential (μ') will be denoted by the arbitrary temperature (T) and chemical potential (μ) hereafter.

We will now proceed to obtain the single particle distribution functions for the bosons and fermions. The quantum mechanical state of an entire system is uniquely specified only when the occupation numbers of the single particle states are explicitly provided. Thus the total energy and particle number of the system when it is in the state i , with n_1 number of particles in state $k = 1$, n_2 number of particles in state $k = 2$ and so on, is given by

$$\epsilon_i = \sum_k n_k e_k, \quad n_i = \sum_k n_k. \quad (21)$$

The sum extends over all the possible states of a particle and e_k is the energy of a particle in a single particle state k . Hence in terms of the single particle occupation states the expression for partition function from (18) becomes [61]

$$Z_q = \prod_{k=1}^{\infty} \sum_{n_k=0}^{\infty} \left[1 - (1-q) \frac{1}{T} (e_k - \mu) n_k \right]^{\frac{1}{1-q}} \quad (22)$$

From Eq. (22) it is trivial to obtain from the standard technology of statistical mechanics, the single particle distribution function for Bose–Einstein and Fermi–Dirac systems, respectively, as the follows:

$$f_q = \frac{1}{\left[1 - (1-q) \left\{ \frac{e}{T} - \frac{\mu}{T} \right\} \right]^{\frac{1}{q-1}}} \mp 1 \quad (23)$$

Here e and μ denote the single particle energy and chemical potential along with T as the bulk temperature of the system

under consideration. Equation (23) gives us the expression for q -generalized Bose–Einstein (BE) and Fermi–Dirac (FD) single particle distribution function in a non-extensive environment.

2.2 Non-extensive relativistic kinetic theory with quantum statistical effects

In this section, the basic macroscopic thermodynamic variables will be defined in the frame work of a relativistic kinetic theory in a non-extensive environment. For this purpose we first need to provide the microscopic definition for the q -generalized entropy and then set up a kinetic or transport equation describing the space-time behavior of one particle distribution function.

In terms of the single particle distribution function the entropy 4-current is defined in the following integral form for a multicomponent system with N number of species [62, 63]:

$$S_q^\mu(x, q) = - \sum_{k=1}^N \int \frac{d^3 p_k}{(2\pi)^3 p_k^0} p_k^\mu \times \left\{ (f_q^k)^q \ln_q f_q^k + \frac{1}{z} (1 - z f_q^k)^q \ln_q (1 - z f_q^k) \right\}, \quad (24)$$

with $z = 1$ for FD and $z = -1$ for BE case. For $z \rightarrow 0$, Eq. (24) reduces to q -generalized Boltzmann entropy, $S_q^\mu(x, q) = - \sum_{k=1}^N \int \frac{d^3 p_k}{(2\pi)^3 p_k^0} p_k^\mu (f_q^k)^q \{ \ln_q f_q^k - 1 \}$. $f_q^k(x, p_k, q)$ is the notation for the single particle distribution function belonging to the k th species in a non-extensive environment depending upon the particle 4-momentum p_k^μ , space-time coordinate x and the entropic index q . The q th power over particle distribution function defining the thermodynamic quantities is justified from the discussions of the last section.

Next the relativistic transport equation for a non-extensive system is presented where the distribution function belonging to each component of the system satisfy the equation of motion in the following manner:

$$p_k^\mu \partial_\mu (f_q^k)^q = \sum_{l=1}^N C_q^{kl} \quad [k = 1, \dots, N]. \quad (25)$$

Here I introduce the microscopic collision integral for a BE–FD system under the non-extensive environment in terms of single particle momentum distributions and the entropic index- q , generalizing the definition used in [55, 57]. The detailed q -generalized collision term C_q^{kl} including the quantum statistical effects (Uehling–Uhlenbeck terms [65]), describing the binary collision $k + l \rightarrow i + j$, is presented now:

$$C_q^{kl} = \frac{1}{2} \sum_{i,j=1}^N \int \frac{d^3 p_l}{(2\pi)^3 p_l^0} \int \frac{d^3 p_i}{(2\pi)^3 p_i^0} \int \frac{d^3 p_j}{(2\pi)^3 p_j^0} \times \left\{ h_q[f_q^i, f_q^j, (1 \pm f_q^k), (1 \pm f_q^l)] W_{i,j|k,l} - h_q[f_q^k, f_q^l, (1 \pm f_q^i), (1 \pm f_q^j)] W_{k,l|i,j} \right\}, \quad (26)$$

where the h_q factors are defined in terms of the particle distribution functions in the following manner:

$$h_q[f_q^k, f_q^l, (1 \pm f_q^i), (1 \pm f_q^j)] = \exp_q \left[\ln_q f_q^k + \ln_q f_q^l + \frac{(1 \pm f_q^i)^{q-1}}{(f_q^i)^{q-1}} \ln_q (1 \pm f_q^i) + \frac{(1 \pm f_q^j)^{q-1}}{(f_q^j)^{q-1}} \ln_q (1 \pm f_q^j) \right]. \quad (27)$$

Here $W_{k,l|i,j}$ is the interaction rate for the binary collision process $k + l \rightarrow i + j$. The 1/2 factor in the right hand side of Eq. (26) takes care of the indistinguishability of the final state particles if their momenta are exchanged from p^i, p^j to p^j, p^i . In Boltzmann limit, the collision integral (26) reproduces the q -generalized Boltzmann collision term as given in [55, 57]. As $q \rightarrow 1$ Eq. (26) simply reduces to the usual Uehling–Uhlenbeck collision term.

Now an important remark has to be made at this point. In the standard Boltzmann transport equation, the ‘‘Hypothesis of molecular chaos’’ or ‘‘Stosszahlansatz’’ postulates that in binary collisions the colliding particles are uncorrelated. This means any correlation present in an early time must be ignored. Thus in Boltzmann transport equation the statistical assumption as regards the number of binary collisions occurring is proportional to the simple product of the distribution functions of colliding particles along the quantum corrections of the final state distribution functions multiplied with the interaction rate [64]. However, with systems where long-range correlations are present, the memory effects are significant which lead to non-Markovian processes. Thus the q -generalized complex looking behavior of the transport equation expressed by Eqs. (25), (26) and (27) is explained in the present case.

From the expression for entropy 4-current (24), we can obtain the expression for local entropy production as follows:

$$\sigma(x, q) = \partial_\mu S_q^\mu(x, q), \quad (28)$$

which for the present case turns out to be

$$\sigma = - \sum_{k=1}^N \int \frac{d^3 p_k}{(2\pi)^3 p_k^0} p_k^\mu \partial_\mu (f_q^k)^q \{ \Psi_q^k \}, \quad (29)$$

with

$$\Psi_q^k = \ln_q f_q^k - \left\{ \frac{1 \pm f_q^k}{f_q^k} \right\}^{q-1} \ln_q(1 \pm f_q^k). \tag{30}$$

Substituting Eq. (25) into (29), we achieve the entropy production in terms of the collision term,

$$\begin{aligned} \sigma &= - \sum_{k,l=1}^N \int \frac{d^3 p_k}{(2\pi)^3 p_k^0} C_q^{kl} \{ \Psi_q^k \} \\ &= -\frac{1}{2} \sum_{k,l,i,j=1}^N \int d\Gamma_k d\Gamma_l d\Gamma_i d\Gamma_j \{ \Psi_q^k \} \\ &\quad \times \left\{ h_q[f_q^i, f_q^j, (1 \pm f_q^k), (1 \pm f_q^l)] W_{i,j|k,l} \right. \\ &\quad \left. - h_q[f_q^k, f_q^l, (1 \pm f_q^i), (1 \pm f_q^j)] W_{k,l|i,j} \right\}, \tag{31} \end{aligned}$$

with the phase space factors now on denoted by $d\Gamma_i = \frac{d^3 p_i}{(2\pi)^3 p_i^0}$. Interchanging the initial and final integration variables in the last term of Eq. (31) and noting the transition rate to be symmetric under the exchange of index pair (k, l) and (i, j) , the expression for entropy production finally reduces to

$$\begin{aligned} \sigma &= -\frac{1}{4} \sum_{k,l,i,j=1}^N \int d\Gamma_k d\Gamma_l d\Gamma_i d\Gamma_j \\ &\quad \times \{ \Psi_q^k + \Psi_q^l - \Psi_q^i - \Psi_q^j \} \\ &\quad \times \left\{ h_q[f_q^i, f_q^j, (1 \pm f_q^k), (1 \pm f_q^l)] W_{i,j|k,l} \right\}. \tag{32} \end{aligned}$$

From the bilateral normalization property of the transition rate we have

$$\sum_{i,j=1}^N \int d\Gamma_i d\Gamma_j W_{k,l|i,j} = \sum_{i,j=1}^N \int d\Gamma_i d\Gamma_j W_{i,j|k,l}. \tag{33}$$

Multiplying both sides of Eq. (33) by $h_q[f_q^k, f_q^l, (1 \pm f_q^i), (1 \pm f_q^j)]$ and integrating over $d\Gamma_k$ and $d\Gamma_l$ and then interchanging (k, i) and (l, j) in the left hand side after summing over k and l , we end with the following relation:

$$\begin{aligned} \sum_{k,l,i,j=1}^N \int d\Gamma_k d\Gamma_l d\Gamma_i d\Gamma_j \left\{ \frac{h_q[f_q^k, f_q^l, (1 \pm f_q^i), (1 \pm f_q^j)]}{h_q[f_q^i, f_q^j, (1 \pm f_q^k), (1 \pm f_q^l)]} \right. \\ \left. - 1 \right\} h_q[f_q^i, f_q^j, (1 \pm f_q^k), (1 \pm f_q^l)] W_{i,j|k,l} = 0. \tag{34} \end{aligned}$$

Multiplying Eq. (34) with 1/4 and adding it to the right hand side of Eq. (32), we are left with the final expression for entropy production:

$$\begin{aligned} \sigma &= \frac{1}{4} \sum_{i,j,k,l=1}^N \int d\Gamma_k d\Gamma_l d\Gamma_i d\Gamma_j [A - \ln_q A - 1] \\ &\quad \times h_q[f_q^i, f_q^j, (1 \pm f_q^k), (1 \pm f_q^l)] W_{i,j|k,l}, \tag{35} \end{aligned}$$

with

$$\begin{aligned} A &= \frac{h_q[f_q^k, f_q^l, (1 \pm f_q^i), (1 \pm f_q^j)]}{h_q[f_q^i, f_q^j, (1 \pm f_q^k), (1 \pm f_q^l)]} \\ &= \exp_q[\Psi_q^k + \Psi_q^l - \Psi_q^i - \Psi_q^j]. \tag{36} \end{aligned}$$

The function $A - \ln_q A - 1$ is always positive for positive A and positive q . It vanishes if and only if A is equal to 1, i.e., production rate of q -generalized entropy is always positive. So at equilibrium, $\sigma = 0$, using $A = 1$ we obtain $\ln_q A = 0$, i.e., $\Psi_q^k + \Psi_q^l = \Psi_q^i + \Psi_q^j$. This clearly reveals Ψ_q is a summation invariant. Now following the basic definition of the summation invariant, Ψ_q is constructed as a linear combination of a constant and the 4-momenta p_k^μ . This condition provides the structural equation defining space-time dependence of distribution function in the following way:

$$\ln_q f_q^k - \left\{ \frac{1 \pm f_q^k}{f_q^k} \right\}^{q-1} \ln_q(1 \pm f_q^k) = a_k(x) + b_\mu(x) p_k^\mu. \tag{37}$$

Here a_k and b^μ are space-time dependent arbitrary quantities except the constraint that the function $a_k(x)$ must be additively conserved, i.e., $a_k(x) + a_l(x) = a_i(x) + a_j(x)$. The 4-momentum conservation is always implied $p_k^\mu + p_l^\mu = p_i^\mu + p_j^\mu$. Equation (37) after a few steps of simplification gives the expression for a q -generalized BE-FD single particle distribution function belonging to k th species for a multicomponent relativistic system by the following equation:

$$f_q^k = \frac{1}{[1 + (q - 1) \{ \frac{p_k^\mu u_{q\mu}}{T_q} - \frac{\mu_{kq}}{T_q} \}]^{\frac{1}{q-1}} \mp 1}, \tag{38}$$

where we can make the identification

$$a_k = \frac{\mu_{kq}}{T_q}, \quad b^\mu = -\frac{u_q^\mu}{T_q}, \tag{39}$$

with μ_{kq} is the chemical potential for each particle of the k th species, $T_q(x, q)$ is the bulk temperature and $u_q^\mu(x, q)$ is the hydrodynamic 4-velocity of the fluid system under non-extensive environment. Hence T_q, u_q^μ and μ_{kq} are the intrinsic parameters of a q -equilibrated thermodynamic system. One can readily notice Eq. (38) invariably leads to Eq. (23) with particle species k for a co-moving frame of the fluid ($u_q^\mu(x) = (1, 0, 0, 0)$). This identification provides the necessary conformation about the form of entropy and collision integral defined within the scope of covariant kinetic theory.

Hence Eq. (26) can be prescribed as the explicit definition of the collision term for a q -generalized system including the necessary quantum corrections.

On the foundation of the formalism developed so far it is the time to define the basic macroscopic variables in the language of relativistic kinetic theory. For this purpose we start with the transport equation (25) itself. First Eq. (25) is integrated over $\frac{d^3 p_k}{(2\pi)^3 p_k^0}$ and then summed over $k = 0, N$. By virtue of summation invariance the zeroth moment of the collision term vanishes on the right hand side while the left hand side gives the conservation law as

$$\partial_\mu N_q^\mu(x, q) = 0, \tag{40}$$

where

$$N_q^\mu = \sum_{k=1}^N g_k \int \frac{d^3 p_k}{(2\pi)^3 p_k^0} p_k^\mu \{f_q^k(x, p_k, q)\}^q, \tag{41}$$

is defined as the q -generalized total particle 4-flow with g_k as the degeneracy number. The particle 4-flow N_q^μ must be proportional to the hydrodynamic 4-velocity u_q^μ since the equilibrium distribution function f_q^k singles out this particular direction in space-time. Thus the macroscopic definition of particle 4-flow is given by

$$N_q^\mu(x, q) = n_q u_q^\mu. \tag{42}$$

The proportionality factor is the q -generalized particle number density which can be expressed as

$$\begin{aligned} n_q(x, q) &= N_q^\mu u_{q\mu} \\ &= \sum_{k=1}^N g_k \int \frac{d^3 p_k}{(2\pi)^3 p_k^0} p_k^\mu u_{q\mu} (f_q^k)^q. \end{aligned} \tag{43}$$

Applying the conservation law (40) to the expression (42) we finally obtain the equation of continuity for a q -equilibrated system as follows:

$$Dn_q = -n_q \partial_\mu u_q^\mu, \tag{44}$$

with $D = u_q^\mu \partial_\mu$ as the covariant time derivative in the local rest frame.

Next, the same interaction and summation is taken for Eq. (25) but after multiplying with particle 4-momentum p_k^μ , which again reduces the right hand side to zero since the first moment of the collision term vanishes as well under the principle of summation invariance. The resulting equation gives the energy-momentum conservation law in the following manner:

$$\partial_\mu T_q^{\mu\nu}(x, q) = 0, \tag{45}$$

where

$$T_q^{\mu\nu} = \sum_{k=1}^N g_k \int \frac{d^3 p_k}{(2\pi)^3 p_k^0} p_k^\mu p_k^\nu \{f_q^k(x, p_k, q)\}^q, \tag{46}$$

is defined as the q -generalized energy-momentum tensor. Noting it is a rank-2 tensor, its macroscopic definition is expressed in terms of the available rank-2 tensors at our disposal, i.e., $u_q^\mu u_q^\nu$ and $g^{\mu\nu}$ in the following way:

$$T_q^{\mu\nu} = \epsilon_q u_q^\mu u_q^\nu - P_q \Delta_q^{\mu\nu}, \tag{47}$$

with $\Delta_q^{\mu\nu} = g^{\mu\nu} - u_q^\mu u_q^\nu$ as the projection operator. The metric $g^{\mu\nu}$ of the system is defined here as $g^{\mu\nu} = (1, -1, -1, -1)$. The q -generalized energy density ϵ_q and pressure P_q are defined as

$$\begin{aligned} \epsilon_q &= T_q^{\mu\nu} u_{q\mu} u_{q\nu} \\ &= \sum_{k=1}^N g_k \int \frac{d^3 p_k}{(2\pi)^3 p_k^0} \{p_k^\mu u_{q\mu}\}^2 (f_q^k)^q, \end{aligned} \tag{48}$$

$$\begin{aligned} P_q &= -\frac{1}{3} T_q^{\mu\nu} \Delta_{q\mu\nu} \\ &= -\frac{1}{3} \sum_{k=1}^N g_k \int \frac{d^3 p_k}{(2\pi)^3 p_k^0} p_k^\mu p_k^\nu \Delta_{q\mu\nu} (f_q^k)^q. \end{aligned} \tag{49}$$

Applying the conservation law (45) to the expression (47) and then contracting with u_q^μ and $\Delta_q^{\mu\nu}$ from the left we finally find the result of the equation of energy and equation of motion, respectively, as follows:

$$De_q = -\frac{P_q}{n_q} \partial_\mu u_q^\mu, \tag{50}$$

$$Du_q^\mu = \frac{1}{n_q h_q} \nabla_\nu P_q^\mu, \tag{51}$$

where $e_q = \frac{\epsilon_q}{n_q}$ and $h_q = e_q + \frac{P_q}{n_q}$ are the total energy and total enthalpy per particle, respectively. For a multi-component system they are defined as $e_q = \sum_{k=1}^N x_{kq} e_{kq}$ and $h_q = \sum_{k=1}^N x_{kq} h_{kq}$, where e_{kq} and h_{kq} are the same for k th species along with $x_{kq} = \frac{n_{kq}}{n_q}$ as the particle fraction for the k th species. Finally, putting the expression of q -distribution function from (38) into the entropy expression (24) and contracting with u_q^μ , we obtain definition of the entropy density in terms of macroscopic thermodynamic quantities defined so far,

$$\begin{aligned} S_q &= S_q^\mu u_{q\mu} \\ &= \frac{\epsilon_q + P_q}{T_q} - \frac{n_q \mu_q}{T_q}, \end{aligned} \tag{52}$$

with $\mu_q = \sum_{k=1}^N x_{kq} \mu_{kq}$ as the total chemical potential per particle of the system. It is interesting that similar conservation laws and equations of motion have also been observed in [55,57] discussing the non-extensive hydrodynamics for relativistic systems in Boltzmann generalization.

2.3 Thermodynamics quantities in q -generalized BE and FD system

The very basic technique of determining the thermodynamic quantities given in Eqs. (43), (48) and (49), lies in performing the moment integral $\int \frac{d^3 p_k}{(2\pi)^3 p_k^0} p_k^\mu \cdots \{f_q^k\}^q$ over the q th power of the non-extensive distribution function (38). To obtain a convenient way to execute it, here an useful identity is being provided. First Eq. (38) can be simply expressed in the following way:

$$f_q^k = \frac{1}{\exp\left[\frac{1}{(q-1)} \ln\{1 + (q-1)y_k\}\right] \mp 1}, \tag{53}$$

denoting $y_k = \frac{p_k \cdot u_q - \mu_{kq}}{T_q}$. Equation (53) readily leads to its derivative in the following form:

$$\frac{\partial f_q^k}{\partial y_k} = -f_q^k (1 \pm f_q^k) \frac{1}{\{1 + (q-1)y_k\}}. \tag{54}$$

Again from (53), the argument can be extracted in the following way:

$$\left\{ \frac{f_q^k}{1 \pm f_q^k} \right\}^{(1-q)} = \{1 + (q-1)y_k\}. \tag{55}$$

Comparing Eqs. (54) and (55), we obtain the following identity:

$$\{f_q^k\}^q = \mp \frac{\partial}{\partial y_k} \left\{ \frac{(1 \pm f_q^k)^{q-1}}{(q-1)} \right\}. \tag{56}$$

Now the $(1 \pm f_q^k)^{q-1}$ term can be expressed in an infinite series for small $(q-1)$ values, (such that quadratic terms $\sim (q-1)^2$ are being neglected) in the following manner:

$$(1 \pm f_q^k)^{q-1} = 1 + (q-1) \sum_{l=1}^{\infty} (\pm)^l \frac{\{F_k \exp(-y_k)\}^l}{l}, \tag{57}$$

with $F_k = 1 - \frac{1}{2}(1-q)y_k^2$. Upon taking the derivative of Eq. (57) and with the virtue of identity (56), for a constant value of μ_{kq} we obtain $\{f_q^k\}^q$ in an infinite series as follows:

$$\begin{aligned} \{f_q^k\}^q &= \sum_{l=1}^{\infty} (\pm)^{l-1} e^{l\tilde{\mu}_k} e^{-l\tau_k} + (q-1) \\ &\times \sum_{l=1}^{\infty} (\pm)^{l-1} e^{l\tilde{\mu}_k} e^{-l\tau_k} \left\{ \frac{l}{2} \tilde{\mu}_k^2 + \tilde{\mu}_k - l\tilde{\mu}_k \tau_k - \tau_k + \frac{l}{2} \tau_k^2 \right\}. \end{aligned} \tag{58}$$

Here, $\tau_k = \frac{p_k \cdot u_q}{T_q}$. From now on wards we will denote T_q and μ_{kq} simply by T and μ_k for convenience. Details of the derivations are given in the appendix. For the mathematical properties of q -logarithm and q -exponential functions Ref. [66] has been essentially helpful. So one can see Eq. (58) contains a series term contributing in the usual BE–FD integrals and another series term proportional to $(q-1)$, which contributes in the non-extensivity while determining the thermodynamic quantities. In the Boltzmann limit Eq. (56) simply becomes

$$\{f_q^k\}_B^q = -\frac{\partial}{\partial y_k} \{f_q^k\}_B, \tag{59}$$

which finally can be expressed as

$$\begin{aligned} \{f_q^k\}_B^q &= e^{\tilde{\mu}_k} e^{-\tau_k} \\ &+ (q-1) \left[\frac{1}{2} \tilde{\mu}_k^2 + \tilde{\mu}_k - \tilde{\mu}_k \tau_k - \tau_k + \frac{1}{2} \tau_k^2 \right], \end{aligned} \tag{60}$$

with the q -generalized Boltzmann distribution function as

$$\begin{aligned} \{f_q^k\}_B &= \exp_q(-(\tau_k - \tilde{\mu}_k)), \\ &= \{1 + (q-1)(\tau_k - \tilde{\mu}_k)\}. \end{aligned} \tag{61}$$

Now one can readily note that Eq. (58) reduces to Eq. (60), if the series truncates at the leading term ($l=1$) only. However, the higher order terms resulting from the quantum correlations will be observed to have significant effect (at least at quantitative level) on the thermodynamic variables in the next section.

Hence putting the form of $\{f_q^k\}^q$ from Eq. (58) into the momentum integral (43), (48) and (49), the expressions for particle number density, energy density and pressure can be achieved. While for massive hadron gas the integrals reduced to infinite series over modified Bessel function of second kind, for a massless quark–gluon gas it reduces to PolyLog functions. The complete analytical expressions for the two cases are given below.

2.3.1 Massive hadron gas with non-zero baryon chemical potential

For this case first the infinite series has been defined for the BE and FD case respectively as follows:

$$S_{kn}^\alpha = \sum_{l=1}^\infty (\pm)^{l-1} \exp(l\tilde{\mu}_k) \frac{1}{l^\alpha} K_n(lz_k), \tag{62}$$

with $z_k = \frac{m_k}{T_q}$, m_k being the mass of the k th hadron and $K_n(lz_k)$ is the modified Bessel function of the second kind with order n and argument lz_k defined as

$$K_n(z) = \frac{2^n n!}{(2n)!} \frac{1}{z^n} \int_z^\infty d\tau (\tau^2 - z^2)^{n-\frac{1}{2}} \exp(-\tau). \tag{63}$$

The \pm signs in Eq. (62), respectively, indicate bosonic and fermionic hadrons. Following these notations, the analytical results for macroscopic thermodynamic quantities of a massive hadron gas in a non-extensive environment have been given.

The expression for the particle number density is

$$\begin{aligned} \frac{n_q}{\left\{\frac{T^3}{2\pi^2}\right\}} &= \sum_{k=1}^\infty g_k \left[(z_k^2 S_{k2}^1) \right. \\ &+ (q-1) \left\{ \tilde{\mu}_k^2 \left(\frac{1}{2} z_k^2 S_{k2}^0 \right) - \tilde{\mu}_k (2z_k^2 S_{k2}^1 + z_k^3 S_{k1}^0) \right. \\ &\left. \left. + \left(3z_k^2 S_{k2}^2 + \frac{3}{2} z_k^3 S_{k1}^1 + \frac{1}{2} z_k^4 S_{k0}^0 \right) \right\} \right]. \tag{64} \end{aligned}$$

The expression for the energy density is

$$\begin{aligned} \frac{\epsilon_q}{\left\{\frac{T^4}{2\pi^2}\right\}} &= \sum_{k=1}^\infty g_k \left[\left(3z_k^2 S_{k2}^2 + z_k^3 S_{k1}^1 \right) \right. \\ &+ (q-1) \left\{ \tilde{\mu}_k^2 \left(\frac{3}{2} z_k^2 S_{k2}^1 + \frac{1}{2} z_k^3 S_{k1}^0 \right) \right. \\ &+ \tilde{\mu}_k \left(-9z_k^2 S_{k2}^2 - 4z_k^3 S_{k1}^1 - z_k^4 S_{k0}^0 \right) \\ &+ \left(18z_k^2 S_{k2}^3 + \frac{15}{2} z_k^3 S_{k1}^2 + \frac{3}{2} z_k^4 S_{k0}^1 \right. \\ &\left. \left. + \frac{1}{2} z_k^4 S_{k2}^1 + \frac{1}{2} z_k^5 S_{k1}^0 \right) \right\} \right]. \tag{65} \end{aligned}$$

The expression for the pressure is

$$\begin{aligned} \frac{P_q}{\left\{\frac{T^4}{2\pi^2}\right\}} &= \sum_{k=1}^\infty g_k \left[\left(z_k^2 S_{k2}^2 \right) \right. \\ &+ (q-1) \left\{ \tilde{\mu}_k^2 \left(\frac{1}{2} z_k^2 S_{k2}^1 \right) - \tilde{\mu}_k \left(3z_k^2 S_{k2}^2 + z_k^3 S_{k1}^1 \right) \right. \\ &\left. \left. + \left(6z_k^2 S_{k2}^3 + \frac{5}{2} z_k^3 S_{k1}^2 + \frac{1}{2} z_k^4 S_{k0}^1 \right) \right\} \right]. \tag{66} \end{aligned}$$

2.3.2 Massless QGP with non-zero quark chemical potential

For this case first the PolyLog function is defined as follows:

$$\sum_{k=1}^\infty (\pm)^{k-1} \frac{e^{\pm k\tilde{\mu}}}{k^a} = \pm \text{PolyLog}[a, \pm e^{\pm\tilde{\mu}}]. \tag{67}$$

Furthermore, for small quark chemical potential μ , keeping only terms up to the order of $\tilde{\mu}^2$, the PolyLogs satisfy the following identity:

$$\begin{aligned} \text{PolyLog}[a, -e^{\pm\tilde{\mu}}] &= \text{PolyLog}[a, -1] \\ &+ \tilde{\mu} \text{PolyLog}[a-1, -1] \\ &+ \frac{\tilde{\mu}^2}{2} \text{PolyLog}[a-2, -1]. \tag{68} \end{aligned}$$

With the help of Eqs. (67) and (68), for a system with massless quarks, antiquarks and gluons, the macroscopic thermodynamic quantities for a non-extensive environment are given in this section.

The expression for the particle number density is

$$\begin{aligned} \frac{n_q}{\left\{\frac{T^3}{2\pi^2}\right\}} &= 2 \left[g_g \zeta(3) - g_q \{ 2\text{PolyLog}[3, -1] - \tilde{\mu}^2 \ln 2 \} \right] \\ &+ (q-1) \left[6 \{ g_g \zeta(4) - 2g_q \text{PolyLog}[4, -1] \} \right. \\ &\left. + 8\tilde{\mu} g_q \text{PolyLog}[3, -1] - 8\tilde{\mu}^2 g_q \text{PolyLog}[2, -1] \right]. \tag{69} \end{aligned}$$

The expression for the energy density and pressure is

$$\begin{aligned} \frac{\epsilon_q}{\left\{\frac{T^4}{2\pi^2}\right\}} &= \frac{3P_q}{\left\{\frac{T^4}{2\pi^2}\right\}} \\ &= 6 \left[g_g \zeta(4) - g_q \{ 2\text{PolyLog}[4, -1] + \tilde{\mu}^2 \text{PolyLog}[2, -1] \} \right] \\ &+ (q-1) \left[36 \{ g_g \zeta(5) - 2g_q \text{PolyLog}[5, -1] \} \right. \\ &\left. + 36\tilde{\mu} g_q \text{PolyLog}[4, -1] - 42\tilde{\mu}^2 g_q \text{PolyLog}[3, -1] \right]. \tag{70} \end{aligned}$$

$\zeta(n)$ are the Riemann zeta functions defined as $\zeta(n) = \text{PolyLog}[n, 1] = \sum_{k=1}^\infty \frac{1}{k^n}$, and $\text{PolyLog}[1, -1] = -\ln 2$. The gluon and quark/antiquark degeneracies are respectively given by $g_g = 16$ and $g_q = 2N_c N_f$, where $N_c = 3$ is the color and $N_f = 2$ is the flavor degrees of freedom for the quarks.

2.4 Effective coupling in q generalized hot QCD system

After defining all the required thermodynamic state variables in a non-extensive environment, it is instructive to look for the Debye mass and the effective coupling of the system under the same. Previously the Debye shielding has been discussed with the non-extensive effects for an electron–ion plasma in [67,68]. Here the effects of non-extensivity are being observed on the Debye mass and the effective coupling for an interacting quark–gluon plasma system.

Following the definition of the Debye mass from semi-classical transport theory [69–72] the same has been defined for a non-extensive system as follows:

$$m_D^2|_q = -4\pi\alpha_s(T, \mu) \sum_{k=1}^N 2C_k \int \frac{d^3 p_k}{(2\pi)^3} \frac{\partial}{\partial |\vec{p}_k|} \{f_q^k\}^q, \quad (71)$$

with $C_k = N_c$ for gluons and $C_k = N_f$ for quark/antiquarks. Applying Eq. (58) into (71), we obtain the q -generalized Debye mass value by the following expression:

$$m_D^2|_q = 4\pi\alpha_s \frac{T^2}{\pi^2} \left[2N_c \zeta(2) - 2N_f \text{PolyLog}[2, -1] + \frac{N_f}{2} \tilde{\mu}^2 + (q-1) \left\{ 2N_c \zeta(3) - 2N_f \text{PolyLog}[3, -1] \right\} \right]. \quad (72)$$

The first part of Eq. (72) is simply the leading order HTL estimation of the Debye mas/s,

$$m_D^2|_{q \rightarrow 1} = 4\pi\alpha_s T^2 \left[\left\{ \frac{N_c}{3} + \frac{N_f}{6} \right\} + \tilde{\mu}^2 \frac{N_f}{2\pi^2} \right]. \quad (73)$$

From Eqs. (72) and (73), it is instructive to obtain the q -generalized effective coupling for a non-extensive system by the following expression:

$$\alpha_q(T, \mu, q) = \alpha_s(T, \mu) \times \left[1 + (q-1) \frac{\left\{ \frac{2N_c}{\pi^2} \zeta(3) - \frac{2N_f}{\pi^2} \text{PolyLog}[3, -1] \right\}}{\left\{ \frac{N_c}{3} + \frac{N_f}{6} \right\} + \tilde{\mu}^2 \left\{ \frac{N_f}{2\pi^2} \right\}} \right]. \quad (74)$$

Here $\alpha_s(T, \mu)$ is the QCD running coupling constant at finite temperature and quark chemical potential. Here its value has been set from the 2-loop QCD gauge coupling calculation at finite temperature from Ref. ([73]).

However, Eq. (74) presents only an indicative way of obtaining the effective coupling of a q -generalized system and is not derived from the interaction dynamics of the system in a field theoretical approach. More fundamental studies are required to obtain an estimation of the QCD coupling that obeys Tsallis statistics for a relativistic system of quantum fields. All that can be concluded is that Eq. (74) provides an

effective way to observe the effect of the Tsallis distribution on the coupling constant of the system, employing semiclassical transport theory.

3 Results and discussions

In this section, we proceed to analyze the quantitative impact of non-extensivity in small $(q - 1)$ limit along with the quantum corrections in terms of BE/FD statistics on various thermodynamic quantities, compared with the same under a Boltzmann generalization. The results for massive and massless systems are presented in two separate subsections.

Before proceeding for providing the results, it is essential to specify the value of the entropic parameter q for systems likely to be created in relativistic heavy ion collisions. We must stick to the small $(q - 1)$ limit in order to neglect the quadratic terms $\sim (q - 1)^2$. In the literature the value of q has been attempted to obtain from the fluctuation of the system parameters like temperature or number concentration [17, 18,40,74,75]. In [75] this value has been reported by a range $1.0 < q < 1.5$ for high-energy nuclear collisions. In [46], an upper limit for q is given by $q < \frac{4}{3}$ in order to get a non-divergent particle momentum distribution $E \frac{dN}{d^3p}$. Apart from that, in the literature the q parameter has been extracted by fitting the transverse momentum spectra of final state hadrons with the experimental data from ALICE, ATLAS and CMS [44,45,62,63]. These fitted values of the q parameter ranges from 1.110 ± 0.218 to 1.158 ± 0.142 . In the light of the above observations, the q parameter has been set at $q = 1.15$ and $q = 1.3$ as an intermediate and an extremal value, apart from $q = 1$, which corresponds to the ideal BE/FD case.

Results of massive pion–nucleon system with non-zero baryon chemical potential

In this section, first the effects of non-extensivity on the macroscopic thermodynamic quantities have been shown for a massive pion–nucleon gas with chemical potential for nucleons $\mu_N = 0.1$ GeV. The temperature dependence of $(n_q - n_1)/n_1$, $(\epsilon_q - \epsilon_1)/\epsilon_1$ and $(P_q - P_1)/P_1$ have been plotted for $q = 1.15$ and $q = 1.3$ in three separate panels in Fig. 1 as obtained from Eqs. (64), (65) and (66). Clearly $A_q (A \equiv n, \epsilon, P)$ being the q generalized thermodynamic quantities (with quantum corrections) and A_1 are the same as with $q = 1$, i.e., the ideal BE/FD quantities; the plotted ratios express the relative significance of the non-extensive generalization of macroscopic quantities (with BE/FD distributions) with respect to the ideal ones. From Fig. 1 the q -corrections appear to be comparable with the quantities with ideal BE/FD distributions, which further increase with increasing q values. These corrections are observed to be

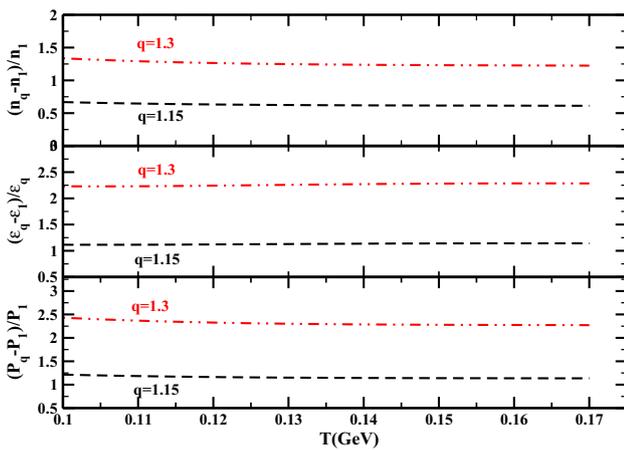


Fig. 1 Relative ratios of the non-extensive correction to the $q = 1$ values of the thermodynamic quantities

more pronounced for energy density and pressure than the particle number density. Hence the effect of non-extensive generalization of single particle Bose–Einstein and Fermi–Dirac distributions, is observed to produce significant effect while determining the thermodynamic quantities in a multi-component system, which is a massive pion–nucleon gas in the present case.

Next, in order to visualize the impact of non-extensivity on the quantum corrections, the non-extensive corrections ($A_q - A_1$) of particle number density, energy density and pressure obtained by generalizing the BE/FD distribution, relative to the same by generalizing Boltzmann distribution have been given in Fig. 2. δA_{QC} denotes the terms proportional to $(q - 1)$ in Eqs. (64), (65) and (66) and δA_B denotes the same with q -generalized Boltzmann distribution given in Eq. (61). The relative ratio shows the quantum correction taken in the non-extensive terms of the thermodynamic quantities, makes how much difference compared to the ones without the quantum corrections. The relative change is 2–3% for energy density and pressure, which displays maximum impetus for a particle number density up to 6% for the massive π - N system.

Finally, both the significance of the non-extensivity (for $q = 1.15$ and $q = 1.3$) and the quantum corrections of the entropy density for the π - N system following from Eq. (52) are depicted in Fig. 3 in two separate panels. The effect of non-extensive terms in the entropy appears to be of the same order of the leading term itself. The quantum correction in the non-extensive terms shows $\sim 3\%$ increment compared with the Boltzmann generalization.

After discussing the basic thermodynamic quantities, the square of sound velocity (c_s^2) has been plotted as a function of temperature for a massive pion–nucleon gas in a non-extensive environment along with the quantum corrections,

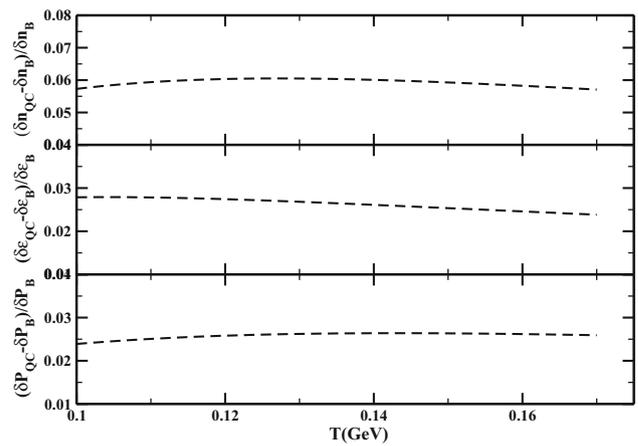


Fig. 2 Relative correction of non-extensive part using BE/FD distribution over Boltzmann statistics

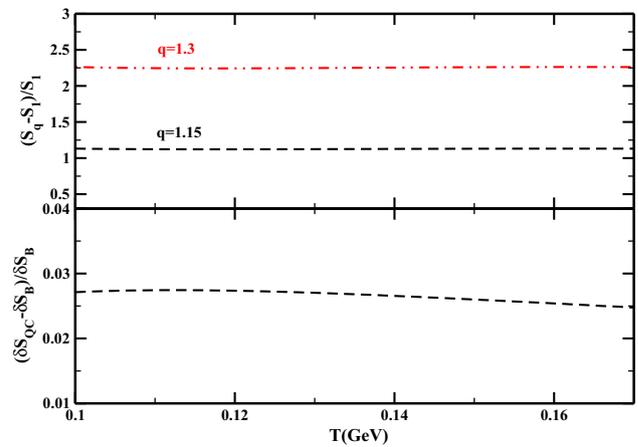


Fig. 3 The non-extensive correction and the quantum correction over non-extensivity of the entropy of the system

for $q = 1$, $q = 1.15$ and $q = 1.3$ in Fig. 4. For a system composed of massless particles the value of c_s^2 simply becomes $1/3$ (which is the Stefan–Boltzmann (SB) limit) irrespective of the value of q , which is depicted by the dotted straight line. For a massive pion–nucleon gas, c_s^2 shows the usual increasing trend with increasing temperature for all values of q , which tends to approach the SB limit at considerably high temperatures. With higher q values the speed of sound appears to become larger which is in accordance with Ref. [56] and faster approaches the SB limit. This increment is much expected due to the significant increase in the system pressure in a non-extensive environment with respect to the ideal ($q = 1$) one. The kink in the temperature dependence of c_s^2 (the minimum most point) below 0.2 GeV agrees with the lattice results, where with increasing q values the shift of the kink towards lower temperatures (0.145–0.150 GeV) indicates the fact that the minimum of the speed of sound lies on the low temperature side of the crossover region [76–79].

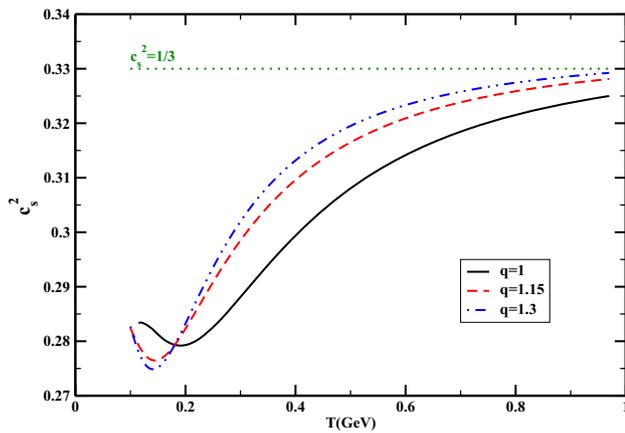


Fig. 4 Velocity of sound as a function of the temperature in non-extensive environment

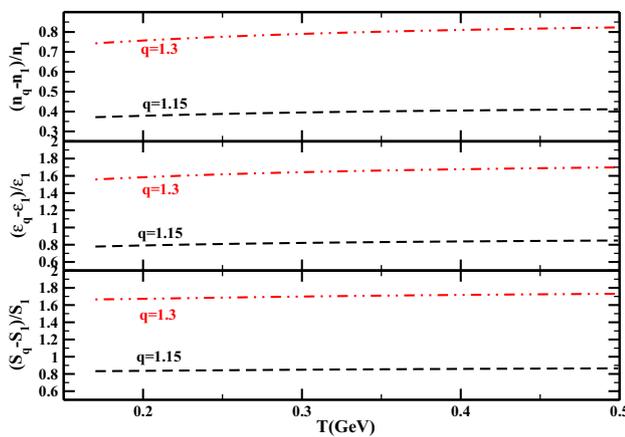


Fig. 5 Relative ratios of the non-extensive correction to the $q = 1$ values of the thermodynamic quantities

Results of massless quark–gluon system with non-zero quark chemical potential

After having presented the results for the massive system, now we turn to the same for a massless quark–gluon system with quark flavor $N_f = 2$ and quark chemical potential $\mu = 0.1$ GeV. Figure 5 depicts the effect of q -parameters on the thermodynamic quantities (particle number density, energy density and entropy as given in Eqs. (69), (70) and (52)) estimated by generalizing BE/FD distributions under a non-extensive environment. The q corrections are comparable to the leading terms in this case as well, however, the increments are little less compared to that of the massive case. Clearly the effect of a finite mass introduces larger contributions to the terms proportional to $(q - 1)$ for $q = 1.15$ and $q = 1.3$. However, for the massless quark–gluon case the change in the thermodynamic quantities is quite significant as observed from Fig. 5 and hence the effect of non-extensivity is proved to be quite relevant in the quark–gluon sector as well.

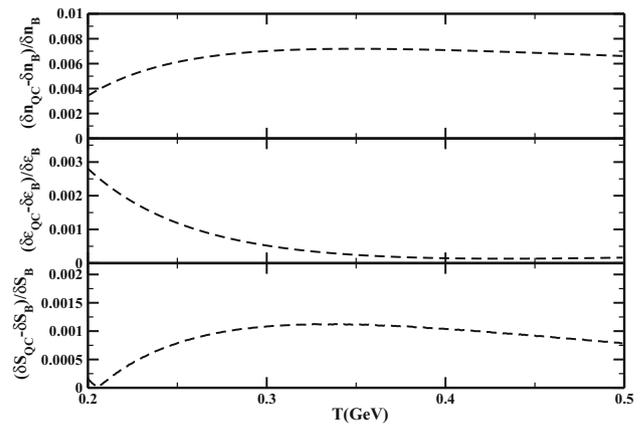


Fig. 6 Relative correction of non-extensive part using BE/FD distribution over Boltzmann statistics

Secondly, the significance of the quantum corrections taken in the terms proportional to $(q - 1)$ (other than $q = 1$) while determining the thermodynamic quantities has been shown in Fig. 6. The relative change in non-extensive terms, taking the quantum corrections, as compared to the same by a Boltzmann q -generalization is depicted by these ratios. However, the quantum corrections in the high temperature, massless case is observed to be significantly smaller than the massive case. This is quite anticipated because of the fact that at sufficiently high temperature and low density, the quantum statistics reaches its classical limit and consequently the quantum distribution laws, whether BE or FD, reduce to the Boltzmann distribution. Hence the reduction in the amount of change created by the quantum correction in the massless case, where the temperature is well above 0.2 GeV, is justified by the high temperature behavior of BE/FD statistics where they scale down to Boltzmann statistics.

Finally, it is interesting to understand the impact of q -generalization on the hot QCD coupling in the quark–gluon system and phenomenon such as Debye screening there. To that end, we proceed to investigate the QCD effective coupling in a non-extensive QGP environment. Following Eq. (74) the effective coupling α_q has been plotted as a function of the temperature in the upper panel of Fig. 7 for three values of q . The $q = 1$ case simply gives the running coupling constant α_s , while $q = 1.15$ and $q = 1.3$ are showing the effects of non-extensivity which enhances the temperature dependence of α_q . This enhancement is most prominent around the phase transition temperature (0.17–0.2 GeV) while in high temperature regions it tends to have reduced effects. In the lower panel of Fig. 7 the relative increment of α_q with respect to α_s has been shown, which shows 7–25% increment in the QCD coupling due to non-extensive effects. This significant increment is expected to improve the quantitative estimates of the thermodynamic quantities, which include the dynamical interactions and hence the coupling

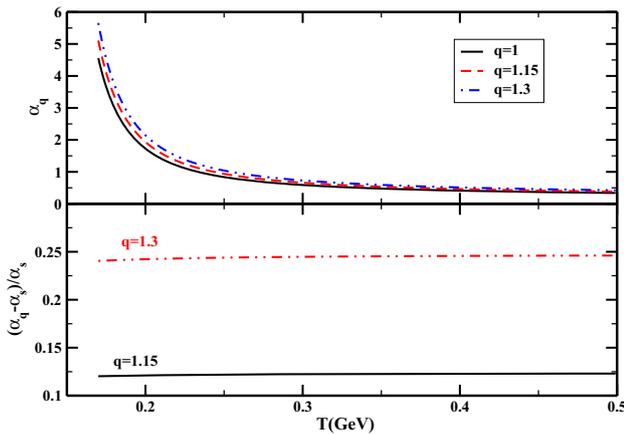


Fig. 7 The QCD coupling in a non-extensive environment as a function of the temperature and its increment with respect to running coupling

as the computational inputs, such as transport parameters of the system. Therefore we can conclude that this dynamical modification due to non-extensivity has far reaching effects on the viscosities and conductivities of the system and also on their application in hydrodynamic simulations, in order to describe the space-time evolution of the system.

4 Conclusion and outlook

In the present work, the relativistic kinetic theory and the thermodynamic properties have been obtained in detail by generalizing the Bose–Einstein and Fermi–Dirac distributions in an non-extensive environment following the prescription introduced by Tsallis. Following the non-extensive definition of the entropy provided by Tsallis, first, the q -generalized BE/FD distribution functions have been achieved from a grand canonical ensemble employing a number of constraints, namely, norm constraint and constraints of internal energy and particle number, using the method of Lagrange’s undetermined multipliers.

Next, the relativistic kinetic theory for a multi-component system has been explored under the non-extensive dynamics. Setting up the microscopic definition of q -entropy in terms of the single particle distribution function for a BE/FD system and defining a proper collision term under the same conditions, we finally achieve again the expression for the q -generalized BE/FD distribution functions by using the techniques of the entropy maximization and summation invariants. In a co-moving frame with the hydrodynamic velocity of the system, the obtained expression of the q -generalized BE/FD distribution function from relativistic kinetic theory reduces to the same obtained for grand canonical thermostatics, proving the congruity of the microscopic definitions of the entropy and the collision term that have been

employed here. Hence the definition of the proposed collision term is justified, which is the key finding of the current investigation.

After setting up the theoretical framework, the macroscopic state variables such as particle number density, energy density, pressure, entropy density and the velocity of sound, have been determined with these single particle distribution functions for a massive pion–nucleon and a massless quark–gluon system with non-zero baryon chemical potential in small $(q - 1)$ limit. Furthermore, the Debye mass and the effective coupling for an interacting QCD system have been estimated indicating the dynamical behavior of the system under the non-extensive generalization. The macroscopic thermodynamic quantities show significant increment due to the inclusion of non-extensive term for both systems, which seems to be more dominant in the massive case. The relative change in the non-extensive terms due the BE/FD generalization over the Boltzmann distribution ranges from 2–6% in hadronic system, which reduces to less than 1% for a quark–gluon system at higher temperature. For larger q values c_s^2 enhances, subsequently reaching the SB limit faster. Due to the non-extensive generalization, the temperature dependence of QCD coupling is observed to enhance significantly over the running coupling constant, which becomes ever larger for higher q values.

The present work opens up a number of possible horizons to be explored under the non-extensive generalization of the system properties concerning heavy ion physics. One immediate future project to be investigated, is the transport parameters of the system using the current formalism. Transport coefficients being crucial inputs in the hydrodynamic equations describing system’s space-time evolution, their response to a non-extensive medium where long-range correlations and memory effects are significant, is a highly interesting topic to venture in the near future. Anticipating the non-extensive formalism to provide a closer look to the systems created in heavy ion collisions, setting up the hydrodynamics equations under this construction and their solution to obtain the space-time behavior of the temperature of the system and hydrodynamic velocity are also extremely essential. A few studies in this regard have been done in [56,57]. More extensive studies including the proper dynamics is an essential future task in this line of work.

Finally, an extended study concerning the microscopic dynamics of the system under the application of non-extensivity from a first principles approach is extremely necessary. In [80] an effective field theory has been discussed to describe the nuclear and quark matter at high temperature by extending the Boltzmann–Gibbs canonical view to the Tsallis approach. A perturbation treatment of relativistic quantum field systems within the framework of Tsallis statistics have been studied in [81]. Following this line of work a complete study of the relativistic field theory under the non-extensive

framework, in order to describe the dynamical properties of QGP at finite temperature and baryon chemical potential, is the next urgent thing to look for. In [82] an effective theory has been modeled describing the interplay between the non-extensivity and the QCD strong interaction dynamics in terms of a quasi-particle model. So developing the complete generalization of the hot QCD medium including long-range interactions is most promising. Inspired by that, such a generalization with proper equations of state encoding the finite temperature medium effects from latest perturbative HTL calculations and lattice simulations are the immediate projects to be explored in the near future.

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Appendix A: Details of the $\{f_q^k\}^q$ derivation

Following the definition of f_q^k from (38) first we obtain

$$\begin{aligned} (1 \pm f_q^k)^{q-1} &= [1 \mp \{1 + (q-1)y_k\}^{\frac{1}{1-q}}]^{1-q} \\ &= \exp\left[(1-q)\ln[1 \mp X_k]\right], \end{aligned} \quad (\text{A1})$$

with

$$\begin{aligned} X_k &= [1 + (q-1)y_k]^{\frac{1}{1-q}} \\ &\times \exp\left[\frac{1}{1-q}\ln\{1 + (q-1)y_k\}\right]. \end{aligned} \quad (\text{A2})$$

Expanding the logarithm and exponential up to linear order of $(q-1)$ and ignoring terms $\sim [(q-1)^2]$, X_k becomes

$$X_k = \exp(-y_k)F_k, \quad (\text{A3})$$

with $F_k = 1 - \frac{1}{2}(1-q)y_k^2$. Putting (A3) into (A1) and again expanding the logarithm in an infinite power series we get

$$(1 \pm f_q^k)^{q-1} = \exp\left[(q-1)\sum_{l=1}^{\infty} (\pm)^l \frac{1}{l} e^{-ly_k} F_k^l\right]. \quad (\text{A4})$$

By further expansion of the exponential term, up to linearity in $(q-1)$, we get Eq. (57).

Taking its derivative and again keeping terms up to order of $(q-1)$ in F_k^l , we obtain the final expression of $\{f_q^k\}^q$ given by Eq. (58).

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