

A new look at the Y tetraquarks and Ω_c baryons in the diquark model

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Abstract We analyze the hidden charm P -wave tetraquarks in the diquark model, using an effective Hamiltonian incorporating the dominant spin–spin, spin–orbit and tensor interactions. We compare with other P -wave systems such as P -wave charmonia and the newly discovered Ω_c baryons, analysed recently in this framework. Given the uncertain experimental situation on the Y states, we allow for different spectra and discuss the related parameters in the diquark model. In addition to the presently observed ones, we expect many more states in the supermultiplet of $L = 1$ diquarkonia, whose J^{PC} quantum numbers and masses are worked out, using the parameters from the currently preferred Y -states pattern. The existence of these new resonances would be a decisive footprint of the underlying diquark dynamics.

1 Introduction

The experimental discovery of exotic, hidden charm or beauty, states has opened a new field in hadron spectroscopy. The exotic states, called X , Y , Z and P_c , have been analysed in a number of theoretical models. They have been claimed to be hybrid charmonia, molecules, disguised charmonium

states, or just coupled-channel or threshold effects; see [1–6] for recent reviews and the references therein. We concentrate on the alternative diquark–antidiquark interpretation, tetraquark in brief, introduced in [7, 8] following the light pentaquark picture discussed in [9], which has the potential to include all exotic hadrons seen thus far in a single scheme.

The objects of our interest in this paper are the so called Y states, with $J^{PC} = 1^{--}$, described as tetraquarks with orbital angular momentum $L = 1$ and $L = 3$. Y -states have also been interpreted as hadron molecules in [10], with $Y(4008)$ not foreseen in that case. The state $Y(4260)$ has also been advocated as an example of a $(c\bar{c})_8g$ hybrid [11]. However, evidence of two resonant structures in $e^+e^- \rightarrow \pi^+\pi^-h_c$ in the $Y(4260)$ region has been presented by the BESIII collaboration [12]. This would imply in the hybrid picture the existence of two states, nearby in mass, having different $c\bar{c}$ -spins. Another analysis of the Y states in the diquark approach can be found in [13].

The univocal prediction of tetraquarks is that here must be only four Y states arising from the orbital angular momentum $L = 1$ and no radial excitation, as shown in Table 1. Parity requires L to be odd, and charge conjugation requires the spin structure of these states to be symmetric under the exchange diquark \leftrightarrow antidiquark, due to the factor $(-1)^L$ introduced by the exchange of coordinates. Besides those of Table 1, there are two other spin structures possible which are antisymmetric and in P wave they would give $J^{PC} = 1^{-+}$. One obtains $J^{PC} = 1^{--}$ also from $L = 3$, but this state is expected to be considerably heavier and we do not consider it.

In [14], the four basic $L = 1$ resonances with $J^{PC} = 1^{--}$ of the diquark–antidiquark spectrum were identified with

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Table 1 $J^{PC} = 1^{--}$ tetraquarks involving a diquark–antidiquark $Q\bar{Q}$ pair in P wave

Label	$ S_Q, S_{\bar{Q}}; S, L\rangle_J$
Y_1	$ 0, 0; 0, 1\rangle_1$
Y_2	$(1, 0; 1, 1\rangle_1 + 0, 1; 1, 1\rangle_1)/\sqrt{2}$
Y_3	$ 1, 1; 0, 1\rangle_1$
Y_4	$ 1, 1; 2, 1\rangle_1$

$Y(4008)$, $Y(4260)$, $Y(4290)$ (a broad structure in the h_c channel), or $Y(4220)$ (a narrow structure in the same channel) and a possible $\Lambda\bar{\Lambda}$ resonance around 4630 MeV. The $Y(4360)$ and the $Y(4660)$, also known at that time, were tentatively considered to be $n = 2$ radial excitations of $Y(4008)$ and $Y(4260)$, respectively. Since that paper appeared, the experimental situation has evolved considerably. The status of the $Y(4008)$ is no longer established, and the $Y(4260)$ is now claimed by BESIII as a double humped structure [15], which is resolved into two resonances: a lower component, $Y(4220)$, with observed decays into $h_c \pi^+\pi^-$ and $\chi_{c0} \omega$, and a higher component, $Y(4330)$, which decays into $J/\psi \pi^+\pi^-$. On the other hand, it was also observed that $Y(4630)$ and $Y(4660)$ could be fitted as a unique resonance, mainly decaying into $\Lambda\bar{\Lambda}$ [16].

In conclusion, there seem to be at present two favoured scenarios, SI and SII, both comprising four Y states and based essentially on the Belle, BaBar and BESIII data, namely:

- Scenario I: $Y(4008)$, $Y(4260)$, $Y(4360)$, $Y(4660)$, favoured in [1];
- Scenario II: $Y(4220)$, $Y(4330)$, $Y(4390)$, $Y(4660)$, favoured in [6].

SI assumes $Y(4008)$ to be a real resonance and it features $Y(4260)$ as a single state. Belle [17] finds that data are better fit with two resonances, $Y(4260)$ and $Y(4008)$, and the width of the $Y(4008)$ is found to be a factor 2 larger than that of the $Y(4260)$. However, the $Y(4008)$ has been seen so far by Belle only and current analysis of this resonance from BESIII is inconclusive [15]. On this basis, SII discards the $Y(4008)$ and it features the two lines resolving the $Y(4260)$, according to BESIII [15], $Y(4220)$ and $Y(4330)$, as lowest resonances. The $Y(4360)$ and $Y(4390)$ appearing in SI and SII, respectively, are considered as the same resonance seen in different experiments. Similarly, in both SI and SII, one considers $Y(4660)$ and the proposed $\Lambda - \bar{\Lambda}$ resonance at 4630 MeV to correspond to the same state.

The spectra of the two scenarios extend over 400–600 MeV, and one could wonder if this is consistent with these states belonging to a single, fine structure multiplet, given that P -wave charmonium states are all within an interval of about 100 MeV. However, the difference can be defended by the different composition of the tetraquarks w.r.t. the charmonia, as explained below.

First, in tetraquarks the total quark spin goes up to $S = 2$, which amplifies the range of the spin–orbit and tensor couplings, and the effect of the tensor force in tetraquarks has not been investigated so far. Secondly, the constituents of Y states are diquarks and antidiquarks with spin 0 and 1, the “good” and “bad” diquarks in Jaffe’s terminology; see [9]. For S -wave tetraquarks, the mass difference between $Z(4020)$ and $Z(3900)$ results in a mass difference of “bad” to “good” $[cq]$ diquarks of about 120 MeV, and the Y states contain from zero to two “bad” diquarks. This splits the masses of the different components of a tetraquark multiplet considerably more than in the conventional charmonia. Note that the diquark mass difference is related to the spin–spin coupling between the charm and the light quark in the diquark, which comes out to be 3–4 times the c – q coupling in the charmed baryons [14]. In QCD, these couplings are proportional to the quark overlap probability, $|\psi(0)|^2$, and the result is simply understood to indicate a closer packing of diquarks in the tetraquarks than in baryons.

The upper range of masses in the two scenarios goes into the region where radial excitations of the lowest P -wave tetraquarks are expected and one may wonder if the highest and, possibly, the next to highest Y states may be the radial excitations of the two lowest ones. For definiteness we assume this not to be the case.

It is still possible, however, that a better experimental resolution may substantiate the differences observed between the 4360–4390 and the 4630–4660 peaks so as to indicate the presence of one or more radial excitation in the region, as was assumed in [14]. We examine the issue of possible radial excitations in Sect. 4. A similar issue has been raised for the excited $\Omega_c = css$ states, whose mass spectrum has been measured by the LHCb collaboration [18], and confirmed recently by Belle [19], except perhaps the $\Omega_c(3119)$. The LHCb mass spectrum is discussed in Refs. [20–23], following the analysis of [20], in which all observed five states are assumed to be P -wave c quark and ss diquark. Also here, the highest mass states overlap with the $2S$, positive parity, radial excitations of the S -wave Ω_c [24, 25]. This issue will be clarified as and when the J^P quantum numbers of the excited Ω_c states are experimentally determined.

Within the two scenarios given above, we work out the mass spectrum derived from the spin–orbit, spin–spin and tensor coupling interactions, the latter was not included in [14]. The principal aim is to investigate whether the tetraquark picture may provide a satisfactory description of the presently determined Y -states in the $c\bar{c}$ sector, eventually distinguishing between SI and SII. For instance, since parameters are obtained from the solution of a second order equation, the mass formulae could produce complex parameters. In the case that there is no real-parameter solution possible, the underlying theoretical framework, namely the diquark picture, can be excluded as a physical template for

the Y states. We use the reality condition on the parameters to eliminate some alternative assignments, as discussed below.

On the positive side, one would expect the value of the chromomagnetic, spin–spin coupling inside the diquark, $[\kappa_{cq}]_P$, to be close to the analogous parameter derived for the S -wave tetraquarks, which is $[\kappa_{cq}]_S \simeq 67$ MeV as discussed in [14]. For tight diquarks, this parameter should not be too much affected by the addition of one unit of orbital angular momentum. In addition, comparison with the S -wave tetraquark masses can give the energy for the excitation of one unit of orbital momentum.

The diquark mass being very similar to the charm quark mass, one expects the excitation energy of the tightly bound Y states to be similar to the one obtained from the comparison of the P and S -wave charmonia and for the P -wave Ω_c states. This expectation is indeed satisfied by one solution in each of the two scenarios.

The tetraquark scheme predicts several other negative parity states with different J , and another 1^{--} resonance arising from $L = 3$. Ignoring the $L = 3$ state, which is presumably rather heavy, we comment in the end on the composition of the full $L = 1$ supermultiplet and give an estimate of the expected masses.

The plan of the paper is as follows. In Sect. 2, we repeat the analysis of the five $L = 1$ charmed baryons Ω_c , whose mass spectrum has recently been measured by the LHCb collaboration [18], following the analysis of [20]. In this connection, we offer an alternative calculation of the tensor couplings in terms of the Wigner’s $6j$ symbols.

In Sect. 3, we introduce the two scenarios compatible with the present data, derive the mass formulae for the Y states and obtain the parameters of the Hamiltonian from the mass spectra. The role of radial excitations is discussed in Sect. 4. Error analysis and correlations among the parameters are presented in Sect. 5. Results are discussed in Sect. 6.

A picture of the full $L = 1$ multiplet is reported in Sect. 7 and conclusions are presented in Sect. 8.

In Appendix A we derive the tensor couplings for the Ω_c baryons and diquarkonium, using Wigner’s $6j$ symbols. Correlation matrices in the analysis of the data on the Y states are given in Appendix B.

2 Effective Hamiltonian for the Ω_c baryons

In the diquark–quark description, the Hamiltonian for the Ω_c states can be written as [20]

$$H_{\text{eff}} = m_c + 2m_s + 2\kappa_{ss} \mathbf{S}_{s_1} \cdot \mathbf{S}_{s_2} + \frac{B_Q}{2} \mathbf{L}^2 + V_{SD},$$

$$V_{SD} = a_1 \mathbf{L} \cdot \mathbf{S}_{[ss]} + a_2 \mathbf{L} \cdot \mathbf{S}_c + b \frac{S_{12}}{4} + c \mathbf{S}_{[ss]} \cdot \mathbf{S}_c. \quad (1)$$

In Eq. (1), m_c and m_s are the masses of the c and the s quarks, respectively, κ_{ss} is the spin–spin coupling of the quarks in the diquark, and \mathbf{L} is the orbital angular momentum of the diquark–quark system. The coefficients a_1 and a_2 are the strengths of the spin–orbit terms involving the spin of the diquark $\mathbf{S}_{[ss]}$ and the charm–quark spin \mathbf{S}_c , respectively, c is the strength of the spin–spin interaction between the diquark and the charm quark, and $S_{12}/4$ represents the tensor interaction, defined by

$$\frac{S_{12}}{4} = Q(\mathbf{S}_1, \mathbf{S}_2) = 3(\mathbf{S}_1 \cdot \mathbf{n})(\mathbf{S}_2 \cdot \mathbf{n}) - (\mathbf{S}_1 \cdot \mathbf{S}_2), \quad (2)$$

where \mathbf{S}_1 and \mathbf{S}_2 are the spins of the diquark and the charm quark, respectively, and $\mathbf{n} = \mathbf{r}/r$ is the unit vector along the radius vector of a particle.

The scalar operator of Eq. (2) can be expressed as the product $3S_1^i S_2^j N_{ij}$, where the tensor operator is

$$N_{ij} = n_i n_j - \frac{1}{3} \delta_{ij}. \quad (3)$$

To compute the matrix elements of this operator between states with the same fixed value $L = 1$ one uses the identity from Landau and Lifshitz [26] (see also [27]):

$$\langle N_{ij} \rangle = a(L)(L_i L_j + L_j L_i - \frac{2}{3} \delta_{ij} L(L+1));$$

$$a(L) = \frac{-1}{(2L-1)(2L+3)}. \quad (4)$$

One finds

$$\left\langle \frac{S_{12}}{4} \right\rangle_{(L=1)} = 3 a(L) \times \left\langle (\mathbf{L} \cdot \mathbf{S}_1)(\mathbf{L} \cdot \mathbf{S}_2) + (\mathbf{L} \cdot \mathbf{S}_2)(\mathbf{L} \cdot \mathbf{S}_1) - \frac{4}{3} (\mathbf{S}_1 \cdot \mathbf{S}_2) \right\rangle. \quad (5)$$

which requires the matrix elements

$$\langle S', L; J | \mathbf{L} \cdot \mathbf{S}_{1,2} | S, L; J \rangle. \quad (6)$$

S and S' are the baryon spins.

The latter can be computed by applying the operators $\mathbf{L} \cdot \mathbf{S}_{1,2}$ to the products of three angular momentum states; see [20]. More effectively, one can use Wigner’s $6j$ symbols (now implemented in computer codes), as is customary for analogous cases in atomic and nuclear physics and is explained in Appendix A.

In either way, one obtains

$$J = 1/2 : \quad \frac{1}{4} \langle S_{12} \rangle = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -1 \end{pmatrix},$$

$$J = 3/2 : \quad \frac{1}{4} \langle S_{12} \rangle = \begin{pmatrix} 0 & -\frac{1}{2\sqrt{5}} \\ -\frac{1}{2\sqrt{5}} & \frac{4}{5} \end{pmatrix}, \quad (7)$$

$$J = 5/2 : \quad \frac{1}{4} \langle S_{12} \rangle = -\frac{1}{5}.$$

Table 2 Values of the parameters a_1, a_2, b, c and M_0 (in MeV), determined from the masses of the Ω_c baryons given in [18] and the spin assignments from [20]

a_1	a_2	b	c	M_0
26.95	25.75	13.52	4.07	3079.94

After diagonalisation, we get the mass corrections arising from the Hamiltonian (1). We remind the reader that in all the five states, there is the common mass term

$$M_0 \equiv m_c + 2m_s + \frac{1}{2}\kappa_{ss} + B_Q. \tag{8}$$

In order to determine the parameters a_1, a_2, b and c , Karliner and Rosner [20] have used the spin averaged mass and have worked with the mass differences of the five Ω_c states. We reproduce their values, summarised in Table 2, where we have also given the value of M_0 .

3 Effective Hamiltonian for Y tetraquarks

Y -states have the quark content $[cq]_3[\bar{c}\bar{q}]_3$, where the subscripts denote colour representations. Tetraquarks with $J^{PC} = 1^{--}$ are obtained for $L = 1, 3$. Spin wave functions are given in Table 1, in the basis $S_Q = S_{[cq]}, S_{\bar{Q}} = S_{[\bar{c}\bar{q}]}, L$ with $S = S_Q + S_{\bar{Q}}$ and $J = S + L$.

We extend the Hamiltonian of P -wave tetraquarks given in [14] by including the tensor coupling contribution

$$H_{\text{eff}} = 2m_Q + \frac{B_Q}{2}L^2 - 3\kappa_{cq} + 2a_Y L \cdot S + b_Y \frac{S_{12}}{4} + \kappa_{cq}[2(S_q \cdot S_c + S_{\bar{q}} \cdot S_{\bar{c}}) + 3], \tag{9}$$

S_{12} is defined as in Eq. (2), with $S_{1,2}$ representing the spins of the diquark and antidiquark, and

$$2(S_q \cdot S_c + S_{\bar{q}} \cdot S_{\bar{c}}) + 3 = 2N_1, \tag{10}$$

where N_1 is the number of spin 1, ‘‘bad’’, diquarks. Comparing to (1), we see that in this case the coefficients a_1 and a_2 are $a_1 = a_2 \equiv 2a_Y$ due to charge conjugation invariance. The spin–spin interaction between diquark and antidiquark is neglected here since in P -wave the overlap probability is suppressed [14]. In the Ω_c case, the spin–spin interaction, represented by c , Table 2, is similarly suppressed and the same happens in P -wave charmonia.

The calculation of the matrix elements of the $L \cdot S_X$ operator, with $S_X = S_{[cq]}$ and $S_{[\bar{c}\bar{q}]}$, is described in Appendix A, Eqs. (A23) and (A25). We note here that:

- tensor couplings are non-vanishing only for the states with $S_Q = S_{\bar{Q}} = 1$;
- the operator $L \cdot S_Q$ is not invariant under charge conjugation and it does mix the states Y_3 and Y_4 , with a

$J^{PC} = 1^{-+}$ state with the composition (quantum numbers follow the entries in Table 1):

$$Y^{(+)} = |1, 1; 1, 1\rangle_1. \tag{11}$$

- $Y^{(+)}$ appears as an intermediate state in the products $(L \cdot S_Q)(L \cdot S_{\bar{Q}})$ and $(L \cdot S_{\bar{Q}})(L \cdot S_Q)$, giving contribution to both diagonal and non-diagonal terms; charge conjugation invariance is, of course, restored when making the sum of the two products, which is block diagonal in the basis (Y_3, Y_4) and $Y^{(+)}$.

In conclusion, we have to consider the full (3×3) matrix $\langle L \cdot S_{[cq]} \rangle$. Using (A10) and (A19) we find

$$\begin{aligned} \langle L \cdot S_{[cq]} \rangle_{J=1} &= \langle 1, S'; 1 | L \cdot S_{[cq]} | 1, S; 1 \rangle \\ &= \sqrt{(2S+1)(2S'+1)} \sum_{j=0}^2 (2j+1) \\ &\quad \times \frac{1}{2} [j(j+1) - 4] \begin{Bmatrix} 1 & 1 & j \\ 1 & 1 & S' \end{Bmatrix} \begin{Bmatrix} 1 & 1 & j \\ 1 & 1 & S \end{Bmatrix}, \end{aligned} \tag{12}$$

where $S, S' = 0, 1, 2$ and the curly brackets denote Wigner’s 6j symbols [30]. After Eqs. (5) and (9), tensor couplings over the Y_3 – Y_4 states are represented by the non-diagonal matrix:

$$\frac{\langle S_{12} \rangle}{4} = \begin{pmatrix} 0 & 2/\sqrt{5} \\ 2/\sqrt{5} & -7/5 \end{pmatrix}. \tag{13}$$

The eigenvalues of the mass matrix of Y -states derived from Eqs. (9) and (13), are written as:

$$\begin{aligned} M_1 &= M(Y_1) = M_{00} - 3\kappa_{cq} \equiv \tilde{M}_{00}, \\ M_2 &= M(Y_2) = \tilde{M}_{00} - 2a_Y + 2\kappa_{cq}, \\ M_3 &= \tilde{M}_{00} + 4\kappa_{cq} + E_+, \\ M_4 &= \tilde{M}_{00} + 4\kappa_{cq} + E_-, \end{aligned} \tag{14}$$

and $M_{00} = 2m_Q + B_Q$. We have made explicit that the states $Y_{1,2}$ in Table 1 are eigenstates of the mass matrix, while $M_{3,4}$ are the eigenvalues of the matrix

$$2a_Y \langle L \cdot S \rangle + b_Y \langle S_{12} \rangle / 4, \tag{15}$$

with

$$\begin{aligned} E_{\pm} &= \frac{1}{10} \\ &\quad \times \left(-30a_Y - 7b_Y \mp \sqrt{3} \sqrt{300a_Y^2 + 140a_Y b_Y + 43b_Y^2} \right), \\ M_3 + M_4 &= 2(\tilde{M}_{00} + 4\kappa_{cq}) + \frac{1}{5}(-30a_Y - 7b_Y) \\ &= 2(\tilde{M}_{00} + 4\kappa_{cq}) + E_+ + E_-, \\ M_4 - M_3 &= \frac{\sqrt{3}}{5} \sqrt{300a_Y^2 + 140a_Y b_Y + 43b_Y^2} \\ &= E_- - E_+ \geq 0. \end{aligned} \tag{16}$$

In scenario SI, we take the four $J^{PC} = 1^{--} Y$ states to be $Y(4008)$, $Y(4260)$, $Y(4360)$ and $Y(4660)$, with masses (all in MeV)

$$M_1 = 4008 \pm 40_{-28}^{+114}, \quad M_2 = 4230 \pm 8, \\ M_3 = 4341 \pm 8, \quad M_4 = 4643 \pm 9. \tag{17}$$

Masses are taken from PDG [28], except for the $Y(4008)$, which is from Belle [17].

In scenario SII the masses of $Y(4220)$, $Y(4330)$, $Y(4390)$ and $Y(4660)$ are (all in MeV):

$$M_1 = 4219.6 \pm 3.3 \pm 5.1, \quad M_2 = 4333.2 \pm 19.9, \\ M_3 = 4391.5 \pm 6.3, \quad M_4 = 4643 \pm 9, \tag{18}$$

i.e. the state with the mass M_4 is the same as in SI.

For S -waves [14], the spin–spin interaction gives a larger mass to $S = 1$ diquarks with respect to $S = 0$ ones and the same for antidiquarks. For this reason, it is natural to associate Y_1 and Y_2 with the two lightest particles in increasing mass order, $M_1 < M_2$. This association is forced by the fact that if we exchange the role of Y_1 and Y_2 , we obtain only complex solutions for the parameters of the Hamiltonian.

In the case of SII, the association agrees with the fact that Y_1 has a sizeable component over the state with vanishing $c-\bar{c}$ spin, in agreement with the observed decays of $Y(4220)$ into h_c , while Y_2 has pure $c-\bar{c}$ spin equal one, in line with the observed decays of $Y(4330)$ into J/ψ . The assignment allows also to describe the decay $Y(4330) \rightarrow X(3872) + \gamma$ as an allowed electric-dipole transition, given that Y_2 has the same spin structure as one attributes in the model to $X(3872)$.

On the other hand, for Y_3 and Y_4 , both containing two spin 1 diquarks, we shall allow for both possibilities: $Y_3 \leftrightarrow M_3$; $Y_4 \leftrightarrow M_4$, with $M_3 < M_4$, or the other way round.

Before proceeding to the estimate of the values of the parameters M_{00} , a_Y , κ_{cq} and b_Y , we first note their possible interdependence on each other. From Eq. (16) for $M_4 - M_3$ follows that this mass difference is invariant under the simultaneous sign change $(a_Y, b_Y) \rightarrow (-a_Y, -b_Y)$. Hence, from this mass difference alone, we have two solutions: $a_Y < 0$ and $a_Y > 0$. We shall call them case 1 and case 2, respectively. In line with the analysis for the Ω_c states, given in Table 2, only $a_Y > 0$ should be kept. This is also the choice suggested by the natural mass ordering, in which the $J = 3$ state should have a higher mass than the $J = 1$ states. So, the only physically acceptable solution is the one which has positive value of a_Y irrespective of the sign of the value of b_Y .

However, as the errors on some of the masses are large, we shall see below that, including the errors, solutions whose central values have $a_Y < 0$, are also allowed.

In addition to Eq. (16), the mass difference $M_2 - M_1$ provides a constraint on the parameters a_Y and κ_{cq} :

$$M_2 - M_1 = 2(\kappa_{cq} - a_Y) > 0. \tag{19}$$

Table 3 Values of the parameters in scenarios I (SI) and II (SII) and $\pm 1\sigma$ errors (all in MeV). Here, $c1$ and $c2$ refer to the two solutions of the secular equation for $Y_{3,4}$

	a_Y	b_Y	κ_{cq}	M_{00}
SI (c1)	-22 ± 32	-89 ± 77	89 ± 11	4275 ± 54
SI (c2)	48 ± 23	11 ± 91	159 ± 20	4484 ± 26
SII (c1)	-3 ± 18	-105 ± 32	54 ± 8	4380 ± 25
SII (c2)	48 ± 8	-32 ± 47	105 ± 4	4535 ± 10

Thus, in both scenarios for the Y_i masses, $\kappa_{cq} > a_Y$, with the two approaching each other as this mass difference decreases.

The central values of the parameters a_Y , b_Y , κ_{cq} , and M_{00} are determined from the masses given in Eq. (17) for SI and in Eq. (18) for SII and presented in Table 3. In each scenario, we indicate with (c1) and (c2) the two solutions obtained from the secular equation for Y_3 and Y_4 .

4 Variations on the theme

We briefly comment on radial excitations, considering scenario III, proposed in [5], which envisages the confirmation of $Y(4008)$ and the doubling of $Y(4260)$:

- Scenario III [5]: $Y(4008)$, $Y(4230)$, $Y(4330)$, $Y(4390)$, $Y(4660)$.

Given the masses of the first three states, we obtain the parameters of the Hamiltonian as functions of the mass of the fourth ground state, M_4 . The parameters are real if this mass is such as to make positive the radicand in Eq. (16). Numerically, this implies

$$M_4 \geq 4450 \text{ MeV}. \tag{20}$$

This is consistent with $Y(4660)$ being the fourth ground state, with the parameters very similar to those of scenario I, but not with $Y(4390)$, which has to be the radial excitation of $Y(4008)$.

Yet another possibility is to take the first three states of scenario II and leave undetermined M_4 . The reality condition gives a result close to (20); however, with a parameter κ_{cq} a bit smaller than expected from the S -wave masses. Assuming the fourth state to be the $Y(4660)$, we go back to scenario II with $\kappa_{cq} \simeq 54 \text{ MeV}$, which is in the acceptable range. In both alternatives considered, the full range of masses in scenarios I and II is acceptable for P -wave ground states.

5 Errors and correlations

To work out errors and correlations among the parameters, we have used the method of least squares to determine the

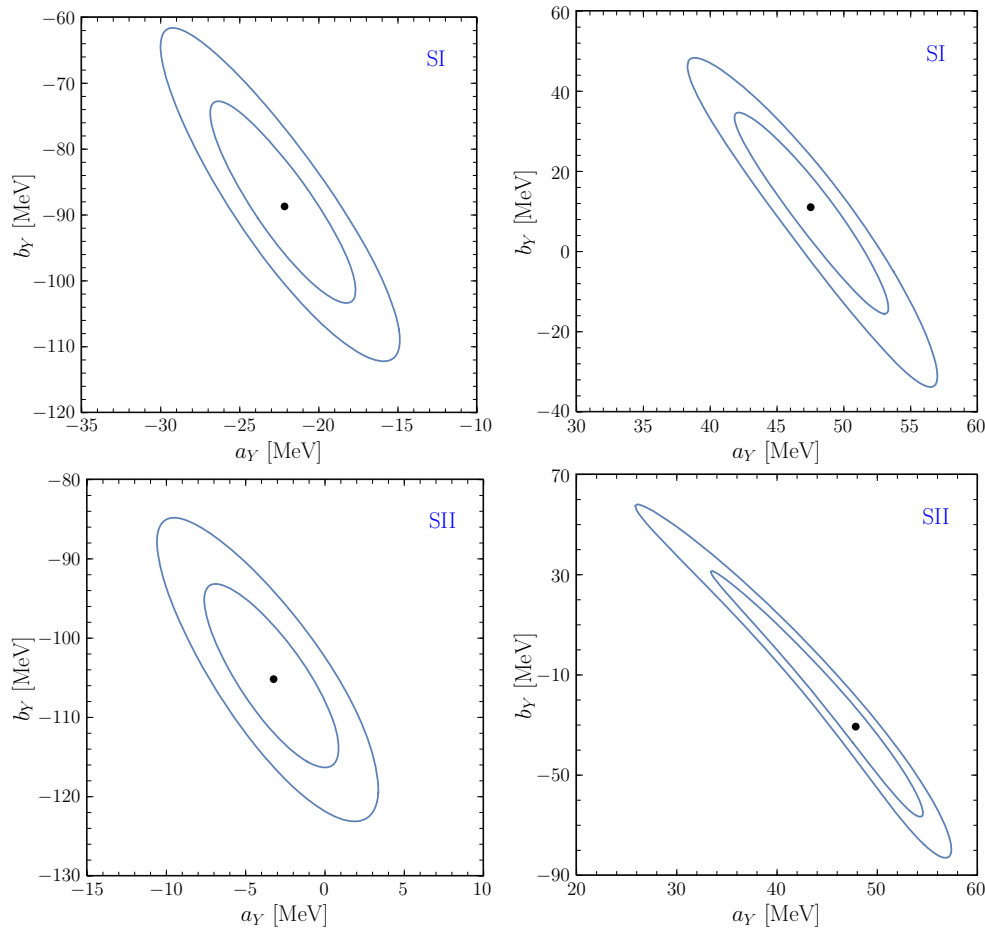


Fig. 1 1σ- and 2σ-contours in the $a_Y - b_Y$ parameter plane corresponding to 68.3 and 95.4% C.L. for the scenario I (SI) in the top two frames and scenario II (SII) in the bottom two frames. The dot in each

frame shows the position of the best-fit value which is the minimum of the χ^2 -function. The best-fit value of a_Y is negative (case 1) in the left panels and positive (case 2) in the right panels

best-fit values and the covariance matrices. For this purpose, the χ^2 -function is calculated. In general [28],

$$\chi^2(\vec{\theta}) = \sum_{i=1}^N \frac{(y_i - \mu_i(\vec{\theta}))^2}{\Delta y_i^2}, \tag{21}$$

where $\vec{y} = (y_1, \dots, y_N)$ is the set of the experimentally measured values which are assumed to be independent and Δy_i are their variances. The quantities $\mu_i(\vec{\theta})$ are dependent on the unknown parameters which are collected as the vector $\vec{\theta} = (\theta_1, \dots, \theta_m)$, where $m \leq N$. For the problem at hand, we take the parameter-dependent functions from Eq. (14), $\mu_i(\vec{\theta}) = M_i$, where $i = 1, \dots, 4$, and

$$\vec{\theta} = (\theta_1, \theta_2, \theta_3, \theta_4) \equiv (M_{00}, \kappa_{cq}, a_Y, b_Y). \tag{22}$$

The best-fit estimations of the parameters θ_k , obtained after minimising the χ^2 -function, are presented in Table 3, as central values. Note that each scenario results in two solu-

tions which differ by the sign of the best-fit value of a_Y , in line with the discussion above. The variances of the parameters are also shown in Table 3, while the correlation matrices are collected in Appendix B. The parameters (22) are strongly correlated, as all the correlation moments in the corresponding matrices (B1)–(B4) are close in magnitude to unity. To show this, we plot two-dimensional confidence level (C.L.) contours involving some of the coefficients.

The correlations among the parameters a_Y and b_Y in terms of the 68.3% ($\chi^2 = \chi_{\min}^2 + 2.3$ for two degrees of freedom) and 95.4% ($\chi^2 = \chi_{\min}^2 + 6.18$) C.L. contours are presented in Fig. 1. Similar contours demonstrating correlations among a_Y and κ_{cq} are shown in Fig. 2. The first and the second rows in these figures correspond to scenarios I and II, respectively, and in each row, the left panels are plotted for the negative best-fit value of a_Y (c1), while the right panels are for the positive best-fit value (c2). Our analysis shows that the scenario I (c1) is not tenable, as, within 95.4% C.L. and even higher, a_Y remains negative. Thus, the requirement of

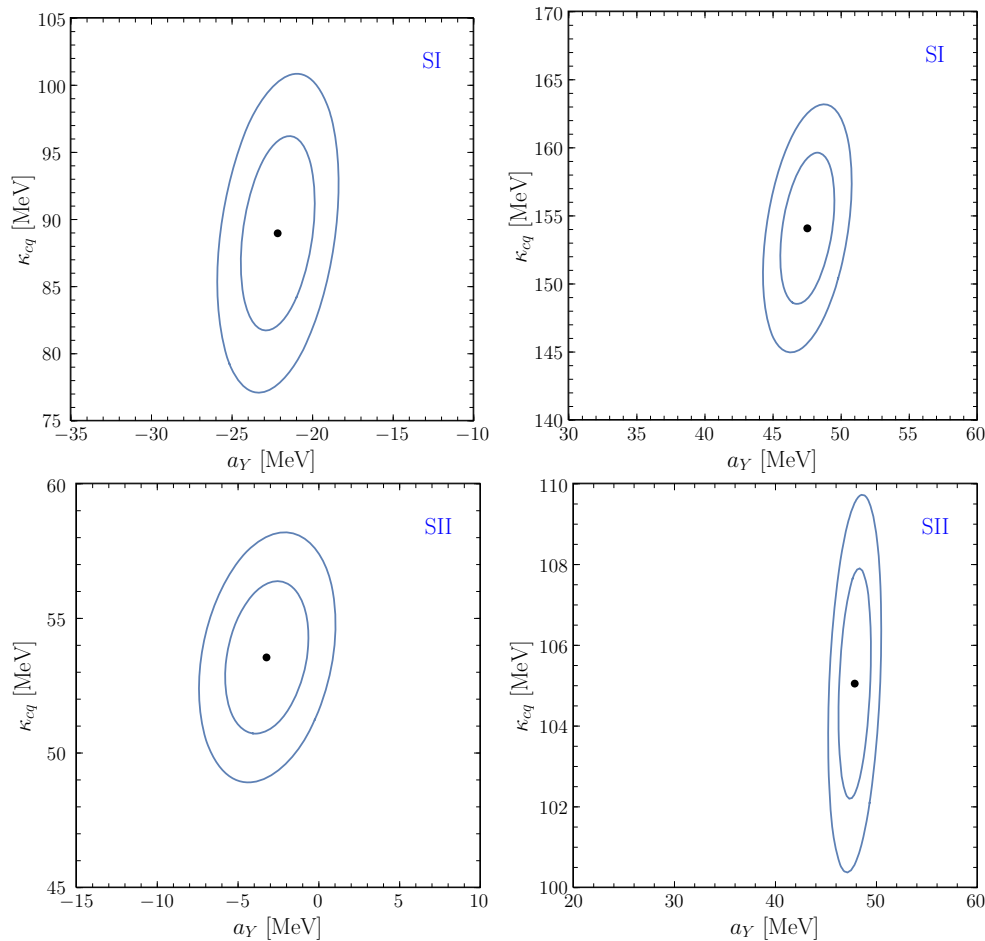


Fig. 2 1σ - and 2σ -contours in the $a_Y - \kappa_{cq}$ parameter plane corresponding to 68.3 and 95.4% C.L. for scenario I (SI) in the top two frames and scenario II (SII) in the bottom two frames. The dot in each

frame shows the position of the best-fit value which is the minimum of the χ^2 -function. The best-fit value of a_Y is negative (case 1) in the left panels and positive (case 2) in the right panels

positive a_Y disfavors case 1 in scenario I. In scenario II (c1), small positive values of a_Y are allowed with a relatively large probability. In (c2), large positive values of a_Y are predicted for both scenarios I and II.

6 Discussion

With the current uncertainty of the experimental scenarios and many parameters one cannot draw quantitative conclusions, except observing that the values of the parameters are qualitatively similar to those derived in the P -wave Ω_c -states in three of the four solutions. One can, however, underline two criteria that lead to some preference for scenario II.

The first is the value of the chromomagnetic coupling κ_{cq} . We expect the fitted parameter to be close to the analogous parameter derived for the S -wave tetraquarks, which is $[\kappa_{cq}]_S \simeq 67$ MeV as discussed in [14]. Indeed, there are

no reasons to believe that the chromomagnetic coupling κ_{cq} in the diquark should change with the addition of one unit of orbital angular momentum. At 95% C.L., the allowed value of κ_{cq} from the Y states in scenario II (c1) comes out somewhat smaller than anticipated, while it is somewhat larger in scenario II (c2). (See the lower two frames in Fig. 2.) Thus, this criterion would favour scenario II.

A second expectation is for the Hamiltonian in Eq. (9) to describe both S and P -wave states, with the same value of the diquark mass. As commented in [14], Y_2 , which in SII corresponds to $Y(4330)$, is in the same spin state as the $X(3872)$ except that there is a gap in mass between the two, which here is fully accounted for by B_Q and by the spin-orbit interaction. If this is the case, one can derive the excitation energy of one unit of orbital momentum from the equation

$$M_2 - M[X(3872)] = B_Q - 2a_Y - [\kappa_{cq}]_P + [\kappa_{cq}]_S. \tag{23}$$

Table 4 Values of the parameters M_{00} , a_Y , b_Y (all in MeV), and $\chi^2_{\min}/\text{n.d.f.}$ resulting from the χ^2 analysis with fixing $\kappa_{cq} = 67$ MeV

Scenario	M_{00}	a_Y	b_Y	$\chi^2_{\min}/\text{n.d.f.}$
SI	4321 ± 79	2 ± 41	-141 ± 63	12.8/1
SII	4421 ± 6	22 ± 3	-136 ± 6	1.3/1

Using the input from Table 3 and $[\kappa_{cq}]_S = 67$ MeV, we obtain

$$B_Q = \begin{cases} 336 \text{ MeV, SI(c1)} \\ 545 \text{ MeV, SI(c2)} \\ 441 \text{ MeV, SII(c1)} \\ 596 \text{ MeV, SII(c2)}. \end{cases} \quad (24)$$

We may see what happens if we force the diquark spin-spin couplings to be equal, $[\kappa_{cq}]_P = [\kappa_{cq}]_S = 67$ MeV. The χ^2 analysis is redone with four experimental input values and three unknown variables $\theta_k = (M_{00}, a_Y, b_Y)$. We have one degree of freedom and can discriminate minima according to χ^2_{\min} . The best-fit values and variances of the parameters M_{00} , a_Y , and b_Y (all in MeV) corresponding to the minima with the lowest χ^2_{\min} in each scenario are reported in Table 4.

There are other minima in both scenarios, but their χ^2_{\min} are larger, and hence we do not discuss the resulting parameters.

With the parameters from Table 4 we obtain

$$B_Q = B_Q(Y) = \begin{cases} 362 \text{ MeV SI,} \\ 505 \text{ MeV SII.} \end{cases} \quad (25)$$

The value for B_Q from the analysis of the Ω_c -baryon resonances in the diquark model can be obtained from the expression for M_0 , given in Eq. (8). Using the input values of the quark masses and κ_{ss} from [29] yields

$$B_Q(\Omega_c) = 325 \text{ MeV.} \quad (26)$$

The values obtained for B_Q in Eqs. (25) and (26) can be compared with the orbital angular momentum excitation energy in charmonium, given by the analogous formula

$$B_Q(c\bar{c}) = M(h_c) - \frac{1}{4}[3M(J/\psi) + M(\eta_c)] = 457 \text{ MeV.} \quad (27)$$

The combination of J/ψ - and η_c -meson masses eliminates the contribution of the S -wave spin-spin interaction in the J/ψ -meson, absent in the h_c , which has $S_{c\bar{c}} = 0$.

The similarity of the results for different P -wave systems with subcomponents in colour 3 and $\bar{3}$ is interesting and may suggest more precise calculations for the Y states, with the potential methods applied successfully to charmonia.

The states with masses M_3 and M_4 are linear combinations of Y_3 ($S = 0$) and Y_4 ($S = 2$). We note that in both SI and SII, the eigenvectors corresponding to M_3 and M_4 in c1

Table 5 First two columns: components of the eigenvector v_4 belonging to the highest eigenvalue, M_4 , in the basis Y_3 , Y_4 . Third column, probability of $S_{c\bar{c}} = 1$ in v_4

	$Y_3, S = 0$	$Y_4, S = 2$	Prob.($S_{c\bar{c}}=1$) in v_4
SI (c1)	-0.27	0.96	0.94
SI (c2)	0.99	0.03	0.25
SII(c1)	-0.41	0.91	0.87
SII (c2)	-0.99	0.11	0.26

are close to $S = 0$ and $S = 2$, respectively, while in c2, it is the opposite, i.e., they are close to $S = 2$ and $S = 0$, respectively. Table 5 (columns 1 and 2) gives the components of the eigenvector associated with M_4 , which is called v_4 , for different scenarios and solutions. The orthogonal vector v_3 is not shown. The eigenvectors carry interesting information; the projection of the eigenvector on the state with $c\bar{c}$ spin = 1 is related to the probability of this state to decay into a J/ψ ($S_{c\bar{c}} = 1$) rather than in h_c ($S_{c\bar{c}} = 0$). The third column gives the probability of finding $S_{c\bar{c}} = 1$ in v_4 . The table indicates that $Y(4660)$ in solutions c2 should have a good probability to decay into h_c while in c1 the J/ψ should dominate. This is quantified in the entries in Table 5 (third column).

7 The full $L = 1$ supermultiplet

We expect many particles in the supermultiplet of $L = 1$ diquarkonia, analogous to the χ -states of charmonia and bottomonia. We find (in parenthesis the multiplicity of the states is given)

$$\begin{aligned} &3^{--} (1); \\ &2^{--} (2); 2^{-+} (2); \\ &1^{--} (4); 1^{-+} (2); \\ &0^{--} (1); 0^{-+} (2). \end{aligned}$$

The total number of states coincides with the total number of quark spin and orbital momentum states, i.e., $2^4 \times 3 = 48$, as one verifies easily. Spin compositions are given in Tables 6 and 1, and tentative masses are presented in Table 6.

Indications exist for two 0^{-+} states. However, in the same channel there should appear two conventional radially excited charmonia, $\eta_c(3S)$ and $\eta_c(4S)$, for a total of four states, with possible mixing and corresponding distortions of the spectrum.

8 Conclusions

We have analysed the masses of the four lightest Y states, using two experimental scenarios proposed in [1,6] and the

Table 6 Spin composition, couplings and tentative masses (in MeV) of the particles in the P -wave supermultiplet, in addition to the states in Table 1. Mass formulae are derived from Eq. (9), with M_{00}, a_Y, b_Y taken from the second line of Table 4 and $\kappa_{cq} = 67$ MeV. N_1 is the number of spin-1 “bad” diquarks, defined in Eq. (10)

J^{PC}	$ S_{\mathcal{Q}}, S_{\bar{\mathcal{Q}}}; S, L\rangle_J$	N_1	$2L \cdot S$	$S_{12}/4$	Mass (MeV) best fit Table 4
3^{--}	$ 1, 1; 2, 1\rangle_3$	2	4	$-2/5$	4630
2^{--}	$ 1, 1; 2, 1\rangle_2$	2	-2	$+7/5$	4254
2_a^{--}	$ \frac{(1,0)+(0,1)}{\sqrt{2}}; 1, 1\rangle_2$	1	$+2$	0	4398
2^{+-}	$ 1, 1; 1, 1\rangle_2$	2	$+2$	$-1/5$	4559
2_b^{+-}	$ \frac{(1,0)-(0,1)}{\sqrt{2}}; 1, 1\rangle_2$	1	$+2$	0	4398
1^{-+}	$ 1, 1; 1, 1\rangle_1$	2	-2	$+1$	4308
1_b^{-+}	$ \frac{(1,0)-(0,1)}{\sqrt{2}}; 1, 1\rangle_1$	1	-2	0	4310
0^{-+}	$ 1, 1; 1, 1\rangle_0$	2	-4	-2	4672
0_b^{-+}	$ \frac{(1,0)-(0,1)}{\sqrt{2}}; 1, 1\rangle_0$	1	-4	0	4266
0_a^{--}	$ \frac{(1,0)+(0,1)}{\sqrt{2}}; 1, 1\rangle_0$	1	-4	0	4266

effective Hamiltonian appropriate for $L = 1, J^{PC} = 1^{--}$ tetraquarks, already introduced for the P -wave charmonia and for the excited Ω_c states.

The current uncertainties on the spectrum of Y -states hinder us to reach a completely quantitative conclusion. However, we find (i) the coefficient of the spin-orbit interaction to be positive, within errors, and comparable to that found for the Ω_c states, (ii) the mass difference of the “bad” and “good” diquarks to be similar to what was found previously for the S -wave tetraquarks, (iii) the energy of the orbital excitation is found to be quite comparable to the values for charmonia and Ω_c , (iv) at variance with the latter cases, the coefficient of the tensor coupling turns out to be large and negative. The scenario with five Y states proposed in [5], including $Y(4008)$ and the two components of the previous $Y(4260)$, is also consistent if one assumes $Y(4390)$ to be the radial excitation of $Y(4008)$.

Features (i) to (iii) are coherent with our a priori expectations, while we have no particular objection to or explanation of (iv). A slight preference for scenario II results, if one insists on enforcing the exact equality of the mass difference of the “bad” and “good” diquarks in S and P wave states.

Hopefully, some clarification on the composition of the Y states will be provided by BESIII, Belle II, and LHCb. With precise measurements, the parameters of the effective Hamiltonian can be determined more accurately, providing a quantitative test of the underlying diquark model. Tetraquarks require many more states in P wave other than the Y states reported in Table 1, which we have listed in Table 6. Tentative mass values are derived from the parameters reported in the second row in Table 4.

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Appendix A: Spin-orbit, tensor coupling and Wigner’s $6j$ symbols

Combining three angular momenta, j_1, j_2, j_3 to a given J , one may follow two paths, characterised by the values of the intermediate angular momentum to which the first two are combined, e.g. j_1 and j_2 to j_{12} or j_2 and j_3 to j_{23} , each path corresponding to different base vectors. In the formulae given below, these two bases are characterised as follows:

$$|(j_1, j_2)_{j_{12}}, j_3; J\rangle, \quad |j_1, (j_2, j_3)_{j_{23}}; J\rangle, \tag{A1}$$

or, with a shorter notation

$$|j_{12}, j_3; J\rangle, \quad |j_1, j_{23}; J\rangle, \tag{A2}$$

where it is understood that j_1, j_2, j_3 and J are held fixed.

Vectors in the two bases are, of course, related by a unitary transformation:

$$|j_1, j_{23}; J\rangle = \sum_{j_{12}} C_{j_{23}, j_{12}} |j_{12}, j_3; J\rangle. \tag{A3}$$

Besides j_{12} and j_{23} , the Clebsch–Gordon coefficients C depend upon the angular momenta that are being held fixed,

j_1, j_2, j_3 and J , that is, the C s depend on *six angular momenta*. To maximise the symmetry properties, one rewrites Eq. (A3) as [30]

$$|j_1, j_{23}; J\rangle = \sum_{j_{12}} (-1)^{j_1+j_2+j_3+J} \sqrt{(2j_{12}+1)(2j_{23}+1)} \times \left\{ \begin{matrix} j_1 & j_2 & j_{12} \\ j_3 & J & j_{23} \end{matrix} \right\} |j_{12}, j_3; J\rangle. \tag{A4}$$

Wigner’s 6j symbols are represented by the curly brackets. They appear in the calculation of the matrix elements of the spin–orbit Hamiltonian or the tensor coupling for two particles with spins S_1 and S_2 and different masses in the orbital angular momentum L . Examples are the P -wave Ω_c baryons and the diquark–antidiquark tetraquarks in P -wave, considered in the present paper.

In these cases, to classify states it is convenient to couple S_1 and S_2 to a total spin S and couple S to L to obtain the total J , that is,

$$j_1 = L, \quad j_2 = S_1, \quad j_3 = S_2, \quad j_{23} = S_1 + S_2 = S. \tag{A5}$$

In this basis the matrix elements of the total spin–orbit operator are easily computed according to the formula:

$$\langle \mathbf{L} \cdot \mathbf{S} \rangle = \frac{1}{2} [J(J+1) - L(L+1) - S(S+1)]. \tag{A6}$$

In the spin–orbit interaction and in the tensor coupling, however, one encounters the matrix elements of the operator $\mathbf{L} \cdot \mathbf{S}_1 = \mathbf{j}_1 \cdot \mathbf{j}_2$, which would require a complicated calculation based on writing explicitly the states as products of three angular momentum states and applying the operator $\mathbf{L} \cdot \mathbf{S}_1$ to them.

A more convenient way to proceed is to use Eq. (A4) and set

$$j_1 = L, \quad j_2 = S_1, \quad j_{12} = L + S_1, \quad j_3 = S_2. \tag{A7}$$

In this basis,

$$\langle \mathbf{L} \cdot \mathbf{S}_1 \rangle = \frac{1}{2} [j_{12}(j_{12}+1) - L(L+1) - S_1(S_1+1)], \tag{A8}$$

is diagonal on the basis vectors.

Using Eq. (A4), with Eq. (A5) on the LHS and Eq. (A7) on the RHS, one gets

$$\begin{aligned} & \mathbf{L} \cdot \mathbf{S}_1 |S, L; J\rangle \\ &= \sum_{j_{LS_1}} (-1)^{L+S_1+S_2+J} \sqrt{(2j_{LS_1}+1)(2S+1)} \\ & \times \frac{1}{2} [j_{LS_1}(j_{LS_1}+1) - L(L+1) - S_1(S_1+1)] \\ & \times \left\{ \begin{matrix} L & S_1 & j_{LS_1} \\ S_2 & J & S \end{matrix} \right\} |j_{LS_1}, S_2; J\rangle. \end{aligned} \tag{A9}$$

Here, we have used the symbol $j_{12} = j_{LS_1}$, whereas $j_{23} = S$ on the lhs, according to (A5). It follows that

$$\begin{aligned} \langle S', L; J | \mathbf{L} \cdot \mathbf{S}_1 | S, L; J \rangle &= \sqrt{(2S+1)(2S'+1)} \\ & \times \sum_{j_{LS_1}} \frac{1}{2} [j_{LS_1}(j_{LS_1}+1) - L(L+1) - S_1(S_1+1)] \\ & \times (2j_{LS_1}+1) \left\{ \begin{matrix} L & S_1 & j_{LS_1} \\ S_2 & J & S' \end{matrix} \right\} \left\{ \begin{matrix} L & S_1 & j_{LS_1} \\ S_2 & J & S \end{matrix} \right\}, \end{aligned} \tag{A10}$$

since by definition

$$\langle j_{12}, j_3; J | j_1, j_{23}; J \rangle = \langle j_1, j_{23}; J | j_{12}, j_3; J \rangle = C_{j_{23}, j_{12}}, \tag{A11}$$

is the coefficient given explicitly in Eq. (A4).

Tables of the 6j symbols can easily be implemented in a computer code and they are already available, making use of the command `SiXJSymbol[{j1, j2, j3}, {j4, j5, j6}]`, in the symbolic computer algebra system Mathematica [31]. Therefore the result in Eq. (A10) can be obtained with a program of a few lines [32]. In the following, we give the explicit formulae for the cases considered in the paper.

Ω_c baryons in P -wave: The constituents of the states are the $[ss]$ -diquark and the charmed quark c with

$$j_1 = L = 1, \quad j_2 = S_{[ss]} = 1, \quad j_3 = S_c = 1/2. \tag{A12}$$

We will call $j_{12} = j_{LS_{[ss]}}$ and $j_{23} = S = 1/2, 3/2$. We have to consider the matrix $\mathbf{L} \cdot \mathbf{S}_{[ss]}$ in the two cases: $J = 1/2$ and $J = 3/2$. In the $J = 1/2$ case, $j_{LS_{[ss]}}$ can take the values 0, 1 and Eq. (A10) reads

$$\begin{aligned} \langle \mathbf{L} \cdot \mathbf{S}_{[ss]} \rangle_{J=1/2} &\equiv \langle S', 1; 1/2 | \mathbf{L} \cdot \mathbf{S}_{[ss]} | S, 1; 1/2 \rangle \\ &= \sqrt{(2S+1)(2S'+1)} \sum_{j_{LS_{[ss]}}=0}^1 (2j_{LS_{[ss]}}+1) \\ & \times \frac{1}{2} [j_{LS_{[ss]}}(j_{LS_{[ss]}}+1) - 4] \\ & \times \left\{ \begin{matrix} 1 & 1 & j_{LS_{[ss]}} \\ 1/2 & 1/2 & S' \end{matrix} \right\} \left\{ \begin{matrix} 1 & 1 & j_{LS_{[ss]}} \\ 1/2 & 1/2 & S \end{matrix} \right\}, \end{aligned} \tag{A13}$$

where $S, S' = 1/2, 3/2$. This sum can be calculated easily when the required values of the 6j symbols are known:

$$\begin{aligned} \left\{ \begin{matrix} 1 & 1 & 0 \\ 1/2 & 1/2 & 1/2 \end{matrix} \right\} &= - \left\{ \begin{matrix} 1 & 1 & 0 \\ 1/2 & 1/2 & 3/2 \end{matrix} \right\} = \frac{1}{\sqrt{6}}, \\ \left\{ \begin{matrix} 1 & 1 & 1 \\ 1/2 & 1/2 & 1/2 \end{matrix} \right\} &= -\frac{1}{3}, \quad \left\{ \begin{matrix} 1 & 1 & 1 \\ 1/2 & 1/2 & 3/2 \end{matrix} \right\} = -\frac{1}{6}. \end{aligned} \tag{A14}$$

Therefore, in the basis of states (${}^4P_{1/2}, {}^2P_{1/2}$) we have (the notation ${}^{2S+1}P_J$ is the same as in [33]):

$$\langle \mathbf{L} \cdot \mathbf{S}_{[ss]} \rangle_{J=1/2} = \begin{pmatrix} -5/3 & \sqrt{2}/3 \\ \sqrt{2}/3 & -4/3 \end{pmatrix}. \tag{A15}$$

In the same way

$$\begin{aligned} \langle \mathbf{L} \cdot \mathbf{S}_{[ss]} \rangle_{J=3/2} &\equiv \langle S', 1; 3/2 | \mathbf{L} \cdot \mathbf{S}_{[ss]} | S, 1; 3/2 \rangle \\ &= \sqrt{(2S+1)(2S'+1)} \sum_{jLS_{[ss]}=1}^2 (2jL S_{[ss]} + 1) \\ &\quad \times \frac{1}{2} [jL S_{[ss]}(jL S_{[ss]} + 1) - 4] \\ &\quad \times \left\{ \begin{matrix} 1 & 1 & jL S_{[ss]} \\ 1/2 & 3/2 & S' \end{matrix} \right\} \left\{ \begin{matrix} 1 & 1 & jL S_{[ss]} \\ 1/2 & 3/2 & S \end{matrix} \right\}, \end{aligned} \tag{A16}$$

with the $6j$ symbol values:

$$\begin{aligned} \left\{ \begin{matrix} 1 & 1 & 1 \\ 1/2 & 3/2 & 3/2 \end{matrix} \right\} &= \frac{\sqrt{10}}{12}, & \left\{ \begin{matrix} 1 & 1 & 2 \\ 1/2 & 3/2 & 1/2 \end{matrix} \right\} &= \frac{1}{2\sqrt{3}}, \\ \left\{ \begin{matrix} 1 & 1 & 2 \\ 1/2 & 3/2 & 3/2 \end{matrix} \right\} &= \frac{1}{2\sqrt{30}}, \end{aligned} \tag{A17}$$

giving in the basis (${}^4P_{3/2}, {}^2P_{3/2}$)

$$\langle \mathbf{L} \cdot \mathbf{S}_{[ss]} \rangle_{J=3/2} = \begin{pmatrix} -2/3 & \sqrt{5}/3 \\ \sqrt{5}/3 & 2/3 \end{pmatrix}. \tag{A18}$$

Both results agree with [20].

Using the relation $(\mathbf{L} \cdot \mathbf{S}_c) = (\mathbf{L} \cdot \mathbf{S}) - (\mathbf{L} \cdot \mathbf{S}_{[ss]})$, it is easy to get the matrices $(\mathbf{L} \cdot \mathbf{S}_c)_{J=1/2}$ and $(\mathbf{L} \cdot \mathbf{S}_c)_{J=3/2}$.

Diquarkonium in P -wave: The constituents are the $[cq]$ diquark and the $[\bar{c}\bar{q}]$ antidiquark.

$$j_1 = L = 1, \quad j_2 = S_{[cq]} = 1, \quad j_3 = S_{[\bar{c}\bar{q}]} = 1. \tag{A19}$$

Here, $J = 1$ and $j_{23} = S = 0, 1, 2$.

Note that the state with $S = L = 1$ has positive charge conjugation, C , opposite to the value of C of the other two states and of the Y states.

The spin-orbit coupling must be even under C and, therefore, it is represented by

$$\mathbf{L} \cdot (\mathbf{S}_{[cq]} + \mathbf{S}_{[\bar{c}\bar{q}]}) = \mathbf{L} \cdot \mathbf{S}, \tag{A20}$$

which is diagonal on the states with $S = 0, 2$.

However, the C -even combination of the spin-orbit couplings appearing in the tensor coupling is

$$(\mathbf{L} \cdot \mathbf{S}_{[cq]})(\mathbf{L} \cdot \mathbf{S}_{[\bar{c}\bar{q}]}) + (\mathbf{L} \cdot \mathbf{S}_{[\bar{c}\bar{q}]})(\mathbf{L} \cdot \mathbf{S}_{[cq]}). \tag{A21}$$

$\mathbf{L} \cdot \mathbf{S}_{[cq]}$ is not C -invariant and it will mix the states with $S = 0, 2$ with the other state with $S = 1$. The states we have

denoted by $|Y_4\rangle, |Y_3\rangle$ have $S = 2, 0$ respectively and $C = -1$, and we denote by $|Y^{(+)}\rangle$ the state with $S = 1$ and $C = +1$; see Eq. (11). The state $|Y^{(+)}\rangle$ appears as intermediate state in the products in Eq. (A21), giving a contribution to diagonal terms and to non-diagonal terms which mix opposite charge conjugations. The latter, of course, cancel when the sum is taken in Eq. (A21).

In conclusion, we have to consider the full (3×3) matrix $\mathbf{L} \cdot \mathbf{S}_{[cq]}$. Using Eqs. (A10) and (A19) we find:

$$\begin{aligned} \langle \mathbf{L} \cdot \mathbf{S}_{[cq]} \rangle_{J=1} &= \langle S', 1; 1 | \mathbf{L} \cdot \mathbf{S}_{[cq]} | S, 1; 1 \rangle \\ &= \sqrt{(2S+1)(2S'+1)} \sum_{jLS_{[cq]}=0}^2 (2jL S_{[cq]} + 1) \\ &\quad \times \frac{1}{2} [jL S_{[cq]}(jL S_{[cq]} + 1) - 4] \\ &\quad \times \left\{ \begin{matrix} 1 & 1 & jL S_{[cq]} \\ 1 & 1 & S' \end{matrix} \right\} \left\{ \begin{matrix} 1 & 1 & jL S_{[cq]} \\ 1 & 1 & S \end{matrix} \right\}, \end{aligned} \tag{A22}$$

where $S, S' = 0, 1, 2$, leading (for $J = 1$) to

$$\langle \mathbf{L} \cdot \mathbf{S}_{[cq]} \rangle = \begin{pmatrix} -3/2 & 0 & 1/2\sqrt{5/3} \\ 0 & 0 & 2/\sqrt{3} \\ 1/2\sqrt{5/3} & 2/\sqrt{3} & -1/2 \end{pmatrix}, \tag{A23}$$

(states are ordered as $Y_4, Y_3, Y^{(+)}$) in agreement with the result obtained with the direct method of applying the operators $\mathbf{L} \cdot \mathbf{S}_{[cq]}$ to the products of angular momentum vectors. Here, the following values of the $6j$ symbols are required:

$$\begin{aligned} \left\{ \begin{matrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{matrix} \right\} &= - \left\{ \begin{matrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{matrix} \right\} = \left\{ \begin{matrix} 1 & 1 & 0 \\ 1 & 1 & 2 \end{matrix} \right\} = \frac{1}{3}, \\ \left\{ \begin{matrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{matrix} \right\} &= \left\{ \begin{matrix} 1 & 1 & 1 \\ 1 & 1 & 2 \end{matrix} \right\} = \frac{1}{6}, & \left\{ \begin{matrix} 1 & 1 & 2 \\ 1 & 1 & 2 \end{matrix} \right\} &= \frac{1}{30}, \end{aligned} \tag{A24}$$

and the rest can be obtained with the help of the $6j$ symbol symmetry under a permutation of columns and interchange of the upper and lower arguments in each of any two columns [30].

Using the relation $(\mathbf{L} \cdot \mathbf{S}_{[\bar{c}\bar{q}']}) = (\mathbf{L} \cdot \mathbf{S}) - (\mathbf{L} \cdot \mathbf{S}_{[cq]})$, we also get

$$\langle \mathbf{L} \cdot \mathbf{S}_{[\bar{c}\bar{q}']} \rangle = \begin{pmatrix} -3/2 & 0 & -1/2\sqrt{5/3} \\ 0 & 0 & -2/\sqrt{3} \\ -1/2\sqrt{5/3} & -2/\sqrt{3} & -1/2 \end{pmatrix}, \tag{A25}$$

again in agreement with the result obtained with the direct method.

The states 2^{-+} and $2^{- -}$ with $J = 2$ are also mixed by the operators $(\mathbf{L} \cdot \mathbf{S}_{[cq]})$ and $(\mathbf{L} \cdot \mathbf{S}_{[\bar{c}\bar{q}']})$. Let us start from the (2×2) matrix $(\mathbf{L} \cdot \mathbf{S}_{[cq]})$, which can be obtained as follows:

$$\langle \mathbf{L} \cdot \mathbf{S}_{[cq]} \rangle_{J=2} = \langle S', 1; 2 | \mathbf{L} \cdot \mathbf{S}_{[cq]} | S, 1; 2 \rangle$$

$$\begin{aligned}
 &= \sqrt{(2S+1)(2S'+1)} \sum_{jL S_{[cq]}=1}^2 (2jL S_{[cq]} + 1) \\
 &\quad \times \frac{1}{2} [jL S_{[cq]} (jL S_{[cq]} + 1) - 4] \\
 &\quad \times \begin{Bmatrix} 1 & 1 & jL S_{[cq]} \\ 1 & 2 & S' \end{Bmatrix} \begin{Bmatrix} 1 & 1 & jL S_{[cq]} \\ 1 & 2 & S \end{Bmatrix}, \tag{A26}
 \end{aligned}$$

where $S, S' = 1, 2$. For the matrices we obtain

$$\langle \mathbf{L} \cdot \mathbf{S}_{[cq]} \rangle_{J=2} = \begin{pmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}, \tag{A27}$$

and

$$\langle \mathbf{L} \cdot \mathbf{S}_{[\bar{c}\bar{q}]} \rangle_{J=2} = \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}, \tag{A28}$$

with the states ordered as 2^{-+} and $2^{- -}$.

The averages of the operators $(\mathbf{L} \cdot \mathbf{S}_{[cq]})$ and $(\mathbf{L} \cdot \mathbf{S}_{[\bar{c}\bar{q}]})$ over the states with $L = 1$ and $J = 0, 3$ are required to get the tensor contributions for the masses. They can be expressed in terms of the $6j$ symbols as follows:

$$\begin{aligned}
 \langle \mathbf{L} \cdot \mathbf{S}_{[cq]} \rangle_{J=0} &= \langle \mathbf{L} \cdot \mathbf{S}_{[\bar{c}\bar{q}]} \rangle_{J=0} \\
 &= -9 \begin{Bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{Bmatrix}^2 = -1, \tag{A29}
 \end{aligned}$$

$$\begin{aligned}
 \langle \mathbf{L} \cdot \mathbf{S}_{[cq]} \rangle_{J=3} &= \langle \mathbf{L} \cdot \mathbf{S}_{[\bar{c}\bar{q}]} \rangle_{J=3} \\
 &= 25 \begin{Bmatrix} 1 & 1 & 2 \\ 1 & 3 & 2 \end{Bmatrix}^2 = 1. \tag{A30}
 \end{aligned}$$

In addition to the values of the $6j$ symbols presented in (A24), for the derivation of (A27) and (A30) one needs to have two more:

$$\begin{Bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \end{Bmatrix} = -\frac{1}{2\sqrt{5}}, \quad \begin{Bmatrix} 1 & 1 & 2 \\ 1 & 3 & 2 \end{Bmatrix} = \frac{1}{5}. \tag{A31}$$

Appendix B: Correlation matrices

In this appendix the correlation matrices in the analysis of the data on Y states are collected. We label them in accordance with the notations used in Table 3.

SI (c1):

$$R = \begin{pmatrix} 1 - 0.890 & 0.995 & -0.990 \\ & 1 - 0.888 & 0.896 \\ & & 1 - 0.997 \\ & & & 1 \end{pmatrix}. \tag{B1}$$

SI (c2):

$$R = \begin{pmatrix} 1 & -0.927 & 0.974 & -0.960 \\ & 1 & -0.958 & 0.967 \\ & & 1 & -0.996 \\ & & & 1 \end{pmatrix}. \tag{B2}$$

SII (c1):

$$R = \begin{pmatrix} 1 & 0.971 & 0.986 & -0.968 \\ & 1 & 0.970 & -0.952 \\ & & 1 & -0.989 \\ & & & 1 \end{pmatrix}. \tag{B3}$$

SII (c2):

$$R = \begin{pmatrix} 1 & 0.838 & -0.528 & 0.686 \\ & 1 & -0.534 & 0.674 \\ & & 1 & -0.972 \\ & & & 1 \end{pmatrix}. \tag{B4}$$

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