

# Integrability of the Einstein-nonlinear $SU(2)$ $\sigma$ -model in a nontrivial topological sector

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**Abstract** The integrability of the  $\Lambda$ -Einstein-nonlinear  $SU(2)$   $\sigma$ -model with nonvanishing cosmological charge is studied. We apply the method of singularity analysis of differential equations and we show that the equations for the gravitational field are integrable. The first few terms of the solution are presented.

Nonlinear  $\sigma$ -models are important theoretical models in physics for the properties that they provide which are of special interest [1, 2]. The Einstein-nonlinear  $\sigma$ -models, in which the total action integral is the sum of the Einstein–Hilbert action integral and the action integral which corresponds to the nonlinear matter source, have provided different kinds of solutions for the gravitational equations. Specifically it has been shown that there exist black hole solutions with a regular event horizon which asymptotically approach the Schwarzschild spacetime, in the context of the Einstein–Skyrme model, which violates the “no hair” conjecture for black holes (see for instance [3] and the references therein).

The purpose of this work is to study the integrability of the field equations of the Einstein-nonlinear  $SU(2)$   $\sigma$ -model, which have been studied previously in [4]. We do this by using the method of singularity analysis of differential equations.<sup>1</sup> The application of singularity analysis in gravitational theories has been applied by many researchers in the past, for instance in the case of the mixmaster universe (Bianchi IX)

[5–7], in scalar field cosmology [8] and in modified theories of gravity [9–11].

Consider a Riemannian manifold  $M$  with metric  $g_{\mu\nu}$  of Lorentzian signature. The action integral of the field equations for the Einstein-nonlinear  $SU(2)$   $\sigma$ -model in a four-dimensional manifold is given by

$$S = S_{\text{EH}} + S_{(\sigma)}, \quad (1)$$

where  $S_{\text{EH}}$  is the Einstein–Hilbert action with the cosmological constant, i.e.,  $S_{\text{EH}} = \int dx^4 (R - 2\Lambda)$ , and  $S_{(\sigma)}$  is the action integral of the nonlinear sigma model [15]

$$S_{(\sigma)} = \frac{K}{2} \int dx^4 \sqrt{-g} \left( (U^{-1}U_{;\mu}) g^{\mu\nu} (U^{-1}U_{;\nu}) \right), \quad (2)$$

where  $U(x^\nu)$  is the  $SU(2)$ -valued scalar and  $K$  is a positive constant. The physical implication of the action, (2), is that it describes the dynamics of low energy pions. The gravitational field equations are derived by variation of the action integral (1) with respect to the metric tensor  $g_{\mu\nu}$ . This leads to the following set of equations:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu}, \quad (3)$$

in which the left hand side of (3) corresponds to the Einstein–Hilbert action where  $G_{\mu\nu}$  is the Einstein tensor and  $\Lambda$  is the cosmological constant. The right hand side of (3) is that of the nonlinear  $\sigma$ -model and provides the matter source. The explicit form of the energy-momentum tensor is

$$T_{\mu\nu} = -\frac{K}{2} (U^{-1}U_{;\mu}) (U^{-1}U_{;\nu}) + \frac{K}{4} g_{\mu\nu} (U^{-1}U_{;\kappa}) g^{\kappa\lambda} (U^{-1}U_{;\lambda}). \quad (4)$$

Furthermore variation with respect to the scalar-valued  $U(x^\nu)$  in (1) leads to the constraint equation  $(U^{-1}U_{;\mu})_{;\nu}$

<sup>1</sup> For a review of the singularity analysis see [12] and subsequent developments in [13, 14].

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$g^{\mu\nu} = 0$ , while the latter can follow from the application of the Bianchi identity in (3), that is,  $T^{\mu\nu}_{;\nu} = 0$ .

By following the Ansatz, which was proposed in [16–19] and its generalizations [20–23], for the parametrization of the  $SU(2)$  algebra in [4] Ayón-Beato, Canfora and Zanelli found that, for the four-dimensional spacetime,

$$ds^2 = -F(r) (dt + \cos \theta d\varphi)^2 + N(r)^2 dr^2 + \rho^2(r) (d\theta^2 + \sin^2 \theta d\varphi^2), \tag{5}$$

the conservation equation,  $T^{\mu\nu}_{;\nu} = 0$ , is satisfied always and the energy-momentum tensor, (4), is expressed only in terms of the fields  $F$ ,  $\rho$  and  $N$ . Specifically, the matter field equations  $(U^{-1}U_{;\mu})_{;\nu} g^{\mu\nu} = 0$ , are identically satisfied, so that one has only to deal with the Einstein field equations while the solution provides a nontrivial topological sector [4].

Spacetime, (5), is a locally rotational spacetime and for arbitrary functions,  $F$  and  $\rho$ , admits a four-dimensional Killing Algebra which comprises the autonomous symmetry,  $A_1 = \{\partial_t\}$ , and  $SO(3)$ , i.e., the Killing vectors form the  $A_1 \oplus SO(3)$  Lie algebra [24].

In the minisuperspace approach the gravitational field equations (3) can arise from the Euler–Lagrange equations of the singular Lagrangian,

$$L(F, F', \rho, \rho', N) = \frac{4}{N} \left( \sqrt{F} F'^2 + \frac{\rho}{\sqrt{F}} F' \rho' \right) + \Lambda \rho^2 N \sqrt{F} - \frac{K}{2} N F^{-\frac{1}{2}} (\rho^2 - 2F) - \rho^{-2} N \sqrt{F} (F + 4\rho^2), \tag{6}$$

where the equation

$$\frac{\partial L}{\partial N} = 0 \tag{7}$$

gives the constraint equation or the component

$$G'_r - T'_r = 0 \tag{8}$$

of (3). In (6) we can see the dynamical terms which correspond to the  $R^{(3)}$  curvature term of (5) of the cosmological constant and of the  $\sigma$ -model.

Because the field equations are singular there could exist a nonlocal conservation law which is generated by the conformal Killing vectors of the minisuperspace, for details see [26,27]. In our consideration, as the minisuperspace of (6) has dimension two, there exist an infinite number of conformal Killing vectors and an infinite number of nonlocal conservation laws. Hence with the use of a nonlocal conservation law the two second-order differential equations and the first-order differential equation, which describe the gravitational field equations, can be reduced to the second-order nonautonomous equation [25]

$$0 = ry \left( Kr^6 - 2r^3 (K - 4 + 4r^2 \Lambda) y + 2y^2 \right) y'' - \left( 6Kr^6 (y^2)' + Kr^7 (y')^2 - \frac{3}{2} (y^4)' - \frac{2}{3} r (y^3)' (8r^4 \Lambda + y') \right), \tag{9}$$

where  $y = y(r) = \rho F$ ,  $N(r)$  is

$$N(r) = \frac{(2y^2 - Kr^6)}{2y^2 (4r^3 + y - 4\Lambda r^5) + Ky (r^6 - 2r^3 y)} \frac{h'(r)}{V_{\text{eff}}}, \tag{10}$$

in which

$$\left( r^3 (y^2)' \right) (h(r))^2 = (2Kr^6 - 16\Lambda r^5) \tag{11}$$

$$+ 4yr^3 ((4 - K) + 4y), \tag{12}$$

and the new gauge has been selected to be so that  $\rho = r$ . The term  $V_{\text{eff}}$  in (10) includes all the potential terms of (6) so that the Lagrangian (6) is that of geodesic equations in a two-dimensional manifold.

The nonautonomous equation (9) can always be written in the form of an autonomous third-order differential equation by introducing the new variables  $r \rightarrow Y(x)$  and  $y \rightarrow Y_{,x}$ . The third-order equation is

$$0 = Y^6 Y_x \left( 8\Lambda (Y_{,xx})^2 - Y_{,x} (3KY_{,xx} + 8\Lambda Y_{,xxx}) \right) + Y^7 \left( KY_{,x} Y_{,xxx} - 2K (Y_{,xx})^2 \right) + 2(K - 4) Y^4 Y_{,x} \left( (Y_{,xx})^2 - Y_{,x} Y_{,xxx} \right) + 2Y (Y_{,x})^3 Y_{,xxx} + \left( 16\Lambda Y^5 + 6(Y_{,x}) \right) (Y_{,x})^3 Y_{,xx}, \tag{13}$$

for which reduction with the autonomous symmetry,  $\partial_x$ , leads to the original equation (9).

We found that (13), except the autonomous one, does not admit any other point symmetry vector for any value of the cosmological constant. That is an interesting result because it indicates that there exists a unique relation among Eqs. (9) and (13). On the other hand, it has been found in [25] that Eq. (9) admits a rescaling symmetry when the cosmological constant is zero. The application of this symmetry vector reduced Eq. (9) to a first-order Abel equation. However, for nonvanishing cosmological constant only special solutions of the form  $y(r) = \sigma_0 r^3$  have been derived, where  $\sigma_0 = -1, \frac{K}{4}$ . Those special solutions correspond to specific initial conditions which describe the asymptotic behaviour of the general evolution of the system.

Below we assume the case of nonvanishing cosmological constant and we perform the singularity analysis. Note

that, if (13) is integrable, then (9) is also integrable which means that the gravitational field equations for the Einstein-nonlinear  $SU(2)$   $\sigma$ -model are also integrable, that is, the dynamical system which follows from the action integral, (1), is integrable.

We define the new variable  $\Phi(x) = Y(x)^2$  and we search for power-law solutions  $\Phi(x) = \alpha\chi^p$  in (13) from where we have the following possible sets  $(\chi = x - x_0)$ , where  $x_0$  is the location of the putative movable singularity):

$$p = -1 \quad \text{with } \alpha = -\frac{2}{K}, \quad \alpha = \frac{1}{2}, \tag{14}$$

and

$$p = -\frac{1}{2} \quad \text{with } \alpha = \pm \frac{i}{2} \sqrt{\frac{3}{2\Lambda}}. \tag{15}$$

We consider the values of (14) from which we can see that these are the power-law solutions of (9) which we described above. The next step, in order to test if (13) passes the singularity test, is to determine the resonances. Let  $\alpha = -\frac{2}{K}$ , which is the case in which the solution leads to a Lorentzian signature spacetime, for details see [25]. Then by substituting

$$\Phi(\chi) = \alpha\chi^{-1} + \gamma\chi^{-1+s} \tag{16}$$

into (13) and taking the terms linear in  $\gamma$ , we have  $s(2s^2 - s - 3) = 0$ , which gives the triple solution

$$s_0 = -1, \quad s_1 = 0, \quad s_2 = \frac{3}{2}. \tag{17}$$

Here we remark that the resonances are the same and for  $\alpha = \frac{1}{2}$ , from where we can say that (13) passes the singularity test and it is integrable.

As far as the solution is concerned, we can write it in a series form in which, from  $s_2$ , we see that the powers of  $\chi$  in the series increase by  $\frac{1}{2}$ . Therefore the solution is

$$\begin{aligned} \Phi(\chi) = & m_0\chi^{-1} + m_1\chi^{-\frac{1}{2}} + m_2 + m_3\chi^{\frac{1}{2}} \\ & + \sum_{l=+4}^{+\infty} m_l\chi^{-1+\frac{ml}{2}}, \end{aligned} \tag{18}$$

where  $m_3$  and  $m_0$  are arbitrary constants<sup>2</sup> and  $m_1, m_2, m_l$  have to be determined. In particular they are functions of  $m_0, m_3, K$  and  $\Lambda$ .

Hence we substitute (18) into (13) and find for the coefficient constants that  $m_1 = 0$ ,

$$m_2 = \frac{3}{8m_0\Lambda} (m_0(4 + K(2m_0 - 1)) - 2), \tag{19}$$

and

$$m_4 = -\frac{3(2m_0 - 1)}{64m_0^3\Lambda^2} (12 + 8(K - 1)m_0 + K^2m_0^2(1 + m_0)), \tag{20}$$

<sup>2</sup> Recall that there exists a resonance with value zero.

from which we can see that for  $m_0 = -\frac{2}{K}$  it follows that  $m_2 = 0$  and  $m_4 = 0$ . For values of  $\chi$  such that  $\chi^{-1} \gg \chi^2$  the solution takes the following form:

$$\Phi(\chi) \simeq -\frac{2}{K}\chi^{-1} + m_3\chi^{\frac{1}{2}}. \tag{21}$$

Now the spacetime, (5), has the following form:

$$\begin{aligned} ds^2 = & -\frac{\Phi(\chi),_{\chi}}{2\Phi(\chi)} (dt + \cos\theta d\varphi)^2 + \frac{N(\Phi, \Phi,_{\chi})^2\Phi(\chi),_{\chi}}{2\sqrt{\Phi(\chi)}} \\ & \times d\chi^2 + \Phi(\chi) (d\theta^2 + \sin^2\theta d\varphi^2). \end{aligned} \tag{22}$$

Furthermore from the second dominant behavior  $p = -\frac{1}{4}$  we find the resonances

$$s_0 = -1, \quad s_1 = -\frac{1}{2}, \quad s_2 = \frac{3}{2}, \tag{23}$$

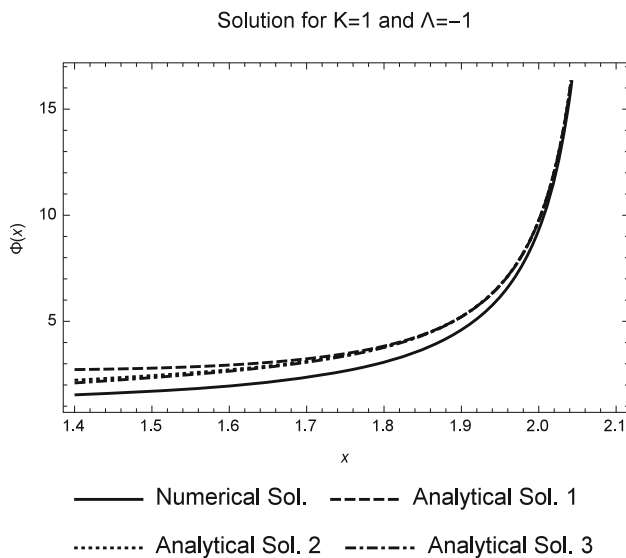
which leaves us with the second solution,

$$\begin{aligned} \Phi(\chi) = & \sum_{l=-4}^{-\infty} n_l\chi^{-\frac{1+l}{2}} + n_{-2}\chi^{-\frac{3}{2}} + n_{-1}\chi^{-1} \\ & + n_0\chi^{-\frac{1}{2}} + n_1 + n_2 + n_3\chi + \sum_{l=+4}^{+\infty} n_l\chi^{-\frac{1+l}{2}}, \end{aligned} \tag{24}$$

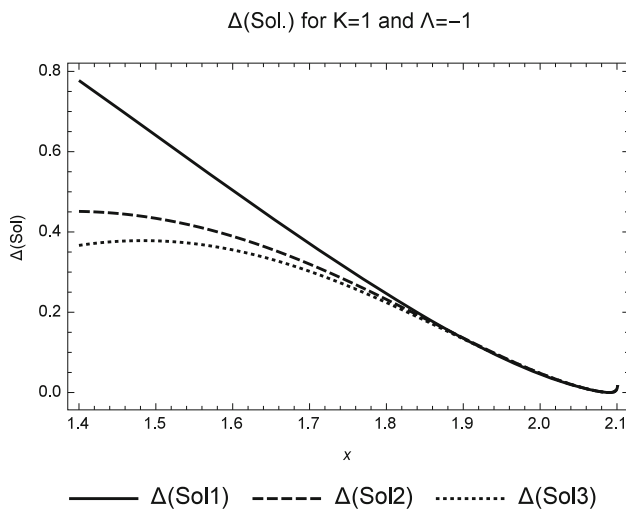
which is a right and left Laurent expansion. The free parameters of the solution (24) are  $n_{-1}$ , and  $n_4$ .

One important issue of the general solution of (22) is the number of constants: two as is seen from the solution (18);  $\{m_0, m_3\}$ ,  $\{n_{-1}, n_4\}$  and the constant  $K$ , which corresponds to the matter source. One would expect three integration constants for the system (3). The latter is hidden in  $\chi$ , as for the singularity analysis we move to the complex plane in which  $\chi = x - x_0$  and  $x_0$  is the position of the singularity. However, that is not an essential constant because it can always be absorbed with the transformation of the coordinate,  $x$ , which means that in the end there are only two constants.

On the other hand, as we discussed above, the asymptotic behaviour of the general solution (18) is  $\Phi(\chi) \simeq \chi^{-1}$ , which is the dominant term around the singularity of (13). If we start far from that solution, that is, from different initial conditions, then in the asymptotic behavior of the solution only the terms of lower powers of  $\chi$  contribute in the solution. Hence the solution can be described well from the first terms of the Laurent series. We demonstrate that in Fig. 1 where the numerical solution of (13) (using Mathematica's NDSolve routine) is given and compared with analytical solutions from the Laurent expansion (18) where we considered the first four terms of (18),  $(m_0 - m_3)$  (Analytic Sol. 1), the six first terms  $(m_0 - m_5)$  (Analytic Sol. 2) and the seven first terms  $(m_0 - m_6)$  (Analytic Sol. 3). Figure 2 presents the absolute error between the analytical solutions and the numerical solution. We observe that for those specific initial



**Fig. 1** Numerical simulation and analytical approximations close to the singularity for the master equation



**Fig. 2** Relative error of the analytical approximations with numerical simulation for the master equation close to the singularity

conditions, where the singularity  $x_0$  is at  $x_0 \simeq 2.10$ , that are very close in the region of the singularity the approximation works well. Of course eventually the error becomes big and other terms of the Laurent expansion have to be considered.

We remark that in the case of vanishing cosmological constant the singularity analysis shows that the resonances depend on the parameter  $K$ . Last but not least an important observation which we can extract from the singularity analysis as regards the stability of the leading order term is that the solution is unstable. The reason for that is the existence of the terms given by the right Laurent expansion which dominate as far as we are moving from the singularity; see also [28] and the references therein.

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