Stationary bound-state massive scalar field configurations supported by spherically symmetric compact reflecting stars

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Abstract It has recently been demonstrated that asymptotically flat neutral reflecting stars are characterized by an intriguing no-hair property. In particular, it has been proved that these horizonless compact objects cannot support spatially regular static matter configurations made of scalar (spin-0) fields, vector (spin-1) fields and tensor (spin-2) fields. In the present paper we shall explicitly prove that spherically symmetric compact reflecting stars can support stationary (rather than static) bound-state massive scalar fields in their exterior spacetime regions. To this end, we solve analytically the Klein–Gordon wave equation for a linearized scalar field of mass μ and proper frequency ω in the curved background of a spherically symmetric compact reflecting star of mass M and radius R_s . It is proved that the regime of existence of these stationary composed star-field configurations is characterized by the simple inequalities $1 - 2M/R_s < (\omega/\mu)^2 < 1$. Interestingly, in the regime $M/R_s \ll 1$ of weakly self-gravitating stars we derive a remarkably compact analytical equation for the discrete spectrum $\{\omega(M, R_s, \mu)\}_{n=1}^{n=\infty}$ of resonant oscillation frequencies which characterize the stationary composed compact-reflecting-star-linearized-massive-scalar-field configurations. Finally, we verify the accuracy of the analytically derived resonance formula of the composed star-field configurations with direct numerical computations.

1 Introduction

In their physically interesting and mathematically elegant no-hair theorems, Bekenstein [1] and Teitelboim [2] (see also [3–9] and references therein) have explicitly proved that static non-linear matter configurations which are made of spatially regular massive scalar fields cannot be supported in the exte-

rior spacetime regions of asymptotically flat black holes¹ (see [10–39] for the case of stationary black-hole spacetimes). This intriguing physical property of the static sector of the non-linearly coupled Einstein-scalar field equations is often attributed to the fact that classical black-hole spacetimes are characterized by one-way membranes (event horizons) that irreversibly absorb matter and radiation fields.

Interestingly, it has recently been demonstrated that the presence of an absorbing (attractive) horizon is not a necessary condition for the validity of the no-hair property for compact objects in general relativity. Specifically, it has been explicitly proved in [40–42] that asymptotically flat horizonless compact stars with reflecting (that is, repulsive rather than attractive) boundary conditions are also characterized by an intriguing no-hair property. In particular, the no-hair theorems presented in [40–43] have revealed the interesting fact that spatially regular static configurations made of scalar (spin-0) fields, vector (spin-1) fields and tensor (spin-2) fields cannot be supported by spherically symmetric compact stars with reflecting^{2,3} boundary conditions.

¹ It is worth mentioning that, thanks to the intriguing physical phenomenon of rotational superradiance in spinning spacetimes, stationary (rather than static) massive scalar field configurations (and, in general, massive bosonic field configurations) can be supported in the exterior spacetime regions of rotating black holes.

² The reflecting (repulsive) boundary conditions of these compact objects should be contrasted with the more familiar absorbing (attractive) boundary conditions which characterize the event horizons of classical black-hole spacetimes.

³ It is important to note that the original no-hair theorem presented in [40] is valid for compact reflecting stars with Dirichlet boundary conditions [that is, $\psi(r = R_s) = 0$ at the surface $r = R_s$ of the compact object]. This theorem relies on the characteristic asymptotic behavior $\psi(r \rightarrow \infty) \rightarrow 0$ of the scalar eigenfunction for finite-mass field configurations [see Eqs. (7) and (8) of [40]] which, together with the Dirichlet boundary condition, implies that the spatially regular static scalar fields must have an extremum point, $r = r_{\text{peak}} \geq R_s$, outside the compact star. In particular, as explicitly proved in [40], the scalar Klein–Gordon equation is violated at this extremum point, a fact which rules out the

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It is of physical interest to test the general validity of the no-hair property revealed in [40–42] for horizonless compact reflecting objects in general relativity. In particular, in the present paper we raise the following physically intriguing question: Can compact reflecting stars⁴ support stationary (rather than static) scalar fields in their exterior spacetime regions?

In order to address this physically interesting question, we shall analyze below the Klein–Gordon wave equation for a stationary linearized scalar field of mass μ and proper frequency ω in the background of a spherically symmetric compact reflecting star of mass M and radius R_s . We shall first prove that the stationary (that is, non-decaying in time) massive scalar field configurations, if they exist, must be characterized by the compact relation $1 - 2M/R_s < (\omega/\mu)^2 < 1$.⁵ Interestingly, we shall then prove explicitly that spherically symmetric compact reflecting stars can support stationary bound-state configurations made of spatially regular linearized massive scalar fields. In particular, we shall derive a remarkably compact analytical formula [see Eq. (34) below] for the discrete resonant oscillation spectrum $\{\omega(M, R_s, \mu; n)\}_{n=1}^{n=\infty}$ ⁶ which characterizes the stationary composed reflecting-star-massive-scalar-field configurations in the physical regime $M/R_s \ll 1$.

Footnote 3 continued

existence of spatially regular static scalar hair outside the compact reflecting stars. The no-hair theorem for the compact reflecting objects can easily be extended to the case of Neumann boundary conditions with $d\psi/dr = 0$ at the surface $r = R_s$ of the star, in which case the scalar Klein–Gordon differential equation is violated at the surface itself. Interestingly, the regime of validity of the no-hair property for spherically symmetric asymptotically flat compact reflecting objects can further be extended to the more generic case of Robin boundary conditions with $\psi + b \cdot d\psi/dr = 0$ and $b < 0$ at the surface $r = R_s$ of the star. In this more generic case, the radial scalar eigenfunction is characterized by the functional relation $\psi \cdot d\psi/dr > 0$ at $r = R_s$, a property which, together with the characteristic asymptotic behavior $\psi(r \rightarrow \infty) \rightarrow 0$ of finite mass matter configurations [see Eqs. (7) and (8) of [40]], imply that the scalar eigenfunction must have an extremum point, $r = r_{\text{peak}} \geq R_s$, outside the reflecting star. As explicitly proved in [40], the Klein–Gordon equation for the static scalar field is violated at this extremum point, a fact which rules out the existence of static spatially regular scalar hairy configurations outside the compact reflecting objects.

⁴ The term ‘reflecting star’ is used here to describe an object with a well defined compact surface on which the scalar field vanishes [see Eq. (7) below].

⁵ It is worth noting that, as expected, this result agrees with the no-scalar-hair theorem presented in [40], according to which static ($\omega = 0$) field configurations cannot be supported by compact reflecting stars.

⁶ Here the resonance parameter n is an integer which characterizes the stationary bound-state massive scalar field configurations in the spacetime of the spherically symmetric compact reflecting star [see Eq. (30) below].

2 Description of the system

We consider a physical system which is composed of a spherically symmetric compact reflecting star which is linearly coupled to a stationary bound-state massive scalar field configuration. The curved spacetime outside the compact star of mass M is characterized by the spherically symmetric line element [44]⁷

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad \text{for } r > R_s, \quad (1)$$

where R_s is the radius of the star and

$$f(r) = 1 - \frac{2M}{r}. \quad (2)$$

The dynamics of the linearized massive scalar field $\Psi(t, r, \theta, \phi)$ in the curved spacetime of the compact star is governed by the familiar Klein–Gordon wave equation [45–52]⁸

$$(\nabla^\nu \nabla_\nu - \mu^2)\Psi = 0. \quad (3)$$

It is convenient to decompose the scalar field eigenfunction in the form⁹

$$\Psi(t, r, \theta, \phi) = \int \sum_{lm} e^{im\phi} S_{lm}(\theta) R_{lm}(r; \omega) e^{-i\omega t} d\omega. \quad (4)$$

Substituting the scalar eigenfunction (4) and the metric components of the curved line element (1) into the Klein–Gordon wave equation (3), one finds that the spatial behavior of the radial scalar function $R_{lm}(r)$ is determined by the ordinary differential equation [45–54]

$$\frac{d}{dr} \left[r^2 f(r) \frac{dR_{lm}}{dr} \right] + \left[\frac{(\omega r)^2}{f(r)} - (\mu r)^2 - K_l \right] R_{lm} = 0, \quad (5)$$

where $K_l = l(l+1)$ (with $l \geq |m|$) is the familiar angular eigenvalue of the spatially regular angular scalar eigenfunction $S_{lm}(\theta)$. Clearly, the radial scalar equation (5) is invariant under the symmetry transformation $\omega \rightarrow -\omega$. We shall henceforth assume, without loss of generality, that

$$\omega > 0. \quad (6)$$

⁷ We shall use natural units in which $G = c = \hbar = 1$.

⁸ Note that the physical parameter μ of the massive scalar field stands for $\mu/\hbar$. It therefore has the dimensions of $(\text{length})^{-1}$.

⁹ Here the integers l and m , which are characterized by the relation $l \geq |m|$, are the spherical and azimuthal harmonic indices of the stationary scalar field modes, respectively.

We are interested in stationary (that is, non-decaying in time) bound-state configurations of the massive scalar fields which are supported by the central compact reflecting star. These field configurations are characterized by the following two boundary conditions: (1) an inner boundary condition of a vanishing scalar eigenfunction [40, 55, 56]

$$\Psi(r = R_s) = 0 \quad (7)$$

at the reflecting surface of the spherically symmetric compact star, and (2) the asymptotic large- r boundary condition

$$\Psi(r \rightarrow \infty) \sim r^{\kappa-1} e^{-\sqrt{\mu^2 - \omega^2} r} \quad \text{with } \omega^2 < \mu^2 \quad (8)$$

of an exponentially decaying (normalizable) scalar eigenfunction at spatial infinity [here $\kappa \equiv M(2\omega^2 - \mu^2)/\sqrt{\mu^2 - \omega^2}$, see Eqs. (21)–(22) below].

The ordinary differential equation (5) for the radial scalar eigenfunction, supplemented by the boundary conditions (7) and (8) at the surface of the spherically symmetric compact reflecting star and at spatial infinity, determines the discrete spectrum $\{\omega(M, R_s, \mu, l; n)\}$ (see footnote 6) of resonant oscillation frequencies which characterize the stationary bound-state massive scalar field configurations in the curved spacetime of the central supporting star. Below we shall use analytical techniques in order to determine this characteristic star–field resonance spectrum.

3 Lower bound on the resonant oscillation frequencies of the stationary bound-state massive scalar field configurations

In the present section we shall use simple analytical techniques in order to prove that the stationary composed compact-reflecting-star-massive-scalar-field configurations, if they exist, are characterized by resonant oscillation frequencies which, for given values of the physical parameters $\{M, R_s, \mu\}$, cannot be arbitrarily small. To this end, it proves useful to define the new radial function¹⁰

$$\Phi = r^{-\delta} R, \quad (9)$$

in terms of which the scalar differential equation (5) takes the form

$$r^2 f(r) \frac{d^2 \Phi}{dr^2} + [2\delta(r - 2M) + 2(r - M)] \frac{d\Phi}{dr} + \left[\delta(\delta - 1)f(r) + 2\delta\left(1 - \frac{M}{r}\right) \right] \Phi = 0.$$

¹⁰ For brevity, we shall henceforth omit the harmonic indices $\{l, m\}$ of the stationary linearized scalar field modes.

$$+ \frac{(\omega r)^2}{f(r)} - (\mu r)^2 - l(l + 1) \Big] \Phi = 0. \quad (10)$$

Taking cognizance of Eqs. (7), (8), and (9), one deduces that the radial eigenfunction $\Phi(r)$, which characterizes the spatial behavior of the stationary bound-state massive scalar field configurations in the curved spacetime (1) of the spherically symmetric compact reflecting star, must have (at least) one extremum point in the interval

$$r_{\text{ext}} \in (R_s, \infty). \quad (11)$$

In particular, at this extremum point the scalar eigenfunction $\Phi(r)$ is characterized by the simple relations

$$\left\{ \frac{d\Phi}{dr} = 0 \quad \text{and} \quad \Phi \cdot \frac{d^2 \Phi}{dr^2} < 0 \right\} \quad \text{for } r = r_{\text{ext}}. \quad (12)$$

Substituting the characteristic functional relations (12) into the scalar differential equation (10), one deduces that the composed compact-reflecting-star-stationary-bound-state-massive-scalar-field configurations, if they exist, are characterized by the (rather cumbersome) relation

$$\delta(\delta - 1)f(r_{\text{ext}}) + 2\delta \left(1 - \frac{M}{r_{\text{ext}}} \right) + \frac{(\omega r_{\text{ext}})^2}{f(r_{\text{ext}})} - (\mu r_{\text{ext}})^2 - l(l + 1) > 0 \quad (13)$$

at the extremum point $r = r_{\text{ext}}$. Taking cognizance of Eq. (2), one finds that the inequality (13) yields the upper bound

$$\frac{(\omega r_{\text{ext}})^2}{f(r_{\text{ext}})} > (\mu r_{\text{ext}})^2 + l(l + 1) - \delta^2 f(r_{\text{ext}}) - \delta \quad (14)$$

on the resonant oscillation frequencies of the stationary bound-state massive scalar field configurations. The r.h.s. of (14) is maximized for the simple choice $\delta = -[2f(r_{\text{ext}})]^{-1}$, in which case one obtains from (14) the simple lower bound

$$\omega^2 > \left(1 - \frac{2M}{r_{\text{ext}}} \right) \left[\mu^2 + \frac{l(l + 1)}{r_{\text{ext}}^2} \right] + \frac{1}{4r_{\text{ext}}^2}. \quad (15)$$

Finally, using the inequality $r_{\text{ext}} > R_s$ in (15) and taking cognizance of equation (8), one finds that the characteristic resonant oscillation frequencies of the stationary bound-state massive scalar field configurations in the curved spacetime of the spherically symmetric compact reflecting star are restricted to the dimensionless frequency interval (see footnote 5)

$$1 - \frac{2M}{R_s} < \left(\frac{\omega}{\mu} \right)^2 < 1. \quad (16)$$

4 The characteristic resonance condition of the stationary composed compact-reflecting-star-linearized-massive-scalar-field configurations

Interestingly, in the present section we shall explicitly show that the ordinary differential equation (5), which determines the spatial behavior of the stationary bound-state massive scalar fields in the spacetime of the compact reflecting star, is amenable to an analytical treatment in the large-radii regime¹¹

$$R_s \gg M. \quad (17)$$

In particular, we shall now derive the fundamental resonance condition which determines the discrete spectrum $\{\omega(M, R_s, \mu, l; n)\}$ of resonant oscillation frequencies that characterize the composed reflecting-star-stationary-scalar-field configurations in the physical regime (17).

Substituting the composed radial function

$$\psi = rf^{1/2}(r)R \quad (18)$$

into the scalar radial equation (5) and neglecting terms of order $O(M/r^3)$ in the large- r regime (17), one obtains the characteristic Schrödinger-like differential equation

$$\frac{d^2\psi}{dr^2} + \left[\omega^2 - \mu^2 + \frac{2M(2\omega^2 - \mu^2)}{r} + \frac{4M^2(3\omega^2 - \mu^2) - l(l+1)}{r^2} \right] \psi = 0 \quad (19)$$

which determines the spatial behavior of the massive scalar field configurations.

It is convenient to use the dimensionless radial coordinate

$$x = 2\sqrt{\mu^2 - \omega^2}r, \quad (20)$$

in terms of which the scalar radial equation (19) takes the compact form

$$\frac{d^2\psi}{dx^2} + \left(-\frac{1}{4} + \frac{\kappa}{x} + \frac{\frac{1}{4} + \beta^2}{x^2} \right) \psi = 0, \quad (21)$$

where the dimensionless physical parameters $\{\kappa, \beta\}$, which characterize the composed star-field system, are given by

$$\kappa \equiv \frac{M(2\omega^2 - \mu^2)}{\sqrt{\mu^2 - \omega^2}} \quad (22)$$

¹¹ Taking cognizance of the analytically derived inequalities $\sqrt{1 - 2M/R_s} < \omega/\mu < 1$ [see Eq. (16)], one realizes that the assumed large-radii regime (17) corresponds to stationary bound-state massive scalar field configurations with $\omega \lesssim \mu$.

and

$$\beta^2 \equiv 4M^2 \left(3\omega^2 - \mu^2 \right) - \left(l + \frac{1}{2} \right)^2. \quad (23)$$

The general mathematical solution of the Whittaker differential equation (21) can be expressed in terms of the familiar confluent hypergeometric functions (see Eqs. 13.1.32 and 13.1.33 of [53]):

$$\psi(x) = e^{-\frac{1}{2}x} x^{\frac{1}{2}+i\beta} \left[A \cdot U \left(\frac{1}{2} + i\beta - \kappa, 1 + 2i\beta, x \right) + B \cdot M \left(\frac{1}{2} + i\beta - \kappa, 1 + 2i\beta, x \right) \right], \quad (24)$$

where $\{A, B\}$ are normalization constants.

The asymptotic spatial behavior of the radial function (24) is given by (see Eqs. 13.1.4 and 13.1.8 of [53])

$$\psi(x \rightarrow \infty) = A \cdot x^\kappa e^{-\frac{1}{2}x} + B \cdot \frac{\Gamma(1 + 2i\beta)}{\Gamma(\frac{1}{2} + i\beta - \kappa)} x^{-\kappa} e^{\frac{1}{2}x}. \quad (25)$$

Taking cognizance of the boundary condition (8), which characterizes the asymptotic spatial behavior of the stationary bound-state (normalizable) massive scalar field configurations, one deduces that the coefficient of the exponentially diverging term in the asymptotic radial equation (25) should vanish:

$$B = 0. \quad (26)$$

One therefore finds that, in the physical regime $R_s \gg M$ of weakly self-gravitating stars [see (17)], the supported bound-state configurations of the stationary massive scalar fields are characterized by the radial eigenfunction

$$\psi(x) = A \cdot e^{-\frac{1}{2}x} x^{\frac{1}{2}+i\beta} U \left(\frac{1}{2} + i\beta - \kappa, 1 + 2i\beta, x \right) \quad \text{for } R_s \gg M, \quad (27)$$

where $U(a, b, z)$ is the confluent hypergeometric function of the second kind [53].

Defining the dimensionless physical parameters

$$\alpha \equiv M\mu; \quad \ell \equiv l + \frac{1}{2}; \quad \gamma \equiv \mu R_s; \quad \varpi \equiv \frac{\omega}{\mu}, \quad (28)$$

and taking cognizance of the inner boundary condition (7) which characterizes the behavior of the massive scalar fields at the reflecting surface of the central compact star, one obtains from the radial solution (27) the non-linear resonance condition

Table 1 Resonant oscillation frequencies of the composed stationary compact-reflecting-star-linearized-massive-scalar-field configurations with $\gamma = 10$ and $l = 0$. We present the smallest resonant oscillation frequency $\varpi^{\min}(\alpha)$ of the stationary bound-state massive scalar fields for various values of the dimensionless star-field mass parameter $\alpha \equiv M\mu$.

Dimensionless mass parameter $\alpha \equiv M\mu$	0.1	0.3	0.5	0.7	0.9	1.1
$\varpi^{\min}(\alpha; \gamma = 10, l = 0)$	0.9982	0.9909	0.9816	0.9711	0.9599	0.9483

Table 2 Resonant oscillation frequencies of the composed stationary compact-reflecting-star-linearized-massive-scalar-field configurations with $\alpha = 0.1$ and $l = 0$. We display the smallest resonant oscillation frequency $\varpi^{\min}(\gamma)$ of the stationary bound-state massive scalar fields for various values of the dimensionless mass-radius parameter $\gamma \equiv \mu R_s$.

Dimensionless mass-radius $\gamma \equiv \mu R_s$	1	2	3	4	5	6
$\varpi^{\min}(\gamma; \alpha = 0.1, l = 0)$	0.9959	0.9966	0.9970	0.9973	0.9975	0.9977

One finds that the characteristic resonant frequency $\varpi^{\min}(\alpha)$ of the composed star-field system is a monotonically decreasing function of the dimensionless mass parameter α . The resonant oscillation frequencies conform to the analytically derived bounds $\sqrt{1 - 2M/R_s} < \varpi < 1$ [see Eq. (16)]

One finds that the characteristic resonant frequency $\varpi^{\min}(\gamma)$ of the composed star-field system is a monotonically increasing function of the dimensionless mass-radius parameter γ . As expected, the resonant oscillation frequencies conform to the analytically derived bounds $\sqrt{1 - 2M/R_s} < \varpi < 1$

$$U\left(\frac{1}{2} + i\sqrt{4\alpha^2(3\varpi^2 - 1) - \ell^2} - \frac{\alpha(2\varpi^2 - 1)}{\sqrt{1 - \varpi^2}}, 1 + 2i\sqrt{4\alpha^2(3\varpi^2 - 1) - \ell^2}, 2\gamma\sqrt{1 - \varpi^2}\right) = 0 \quad (29)$$

for the composed spherically-symmetric-reflecting-star-linearized-massive-scalar-field configurations.

Interestingly, as we shall explicitly show below, the mathematically compact resonance equation (29), which is valid in the large-radii regime (17), determines the discrete family of resonant oscillation frequencies $\{\varpi(\alpha, \gamma, \ell; n)\}$ which characterize the stationary bound-state massive scalar field configurations in the curved spacetime of the spherically symmetric compact reflecting star.

5 The characteristic resonance spectrum of the stationary composed compact-reflecting-star-massive-scalar-field configurations

The analytically derived resonance condition (29), which determines the resonant oscillation spectrum of the stationary bound-state linearized massive scalar fields in the background of the compact reflecting star, can easily be solved numerically. Interestingly, one finds that a composed compact-reflecting-star-massive-scalar-field system with a given set of the dimensionless physical parameters $\{\alpha, \gamma, \ell\}$ is characterized by a discrete resonant oscillation spectrum of the form¹²

¹² As noted above, the radial scalar equation (5) is invariant under the symmetry transformation $\omega \rightarrow -\omega$. We shall assume that $\omega > 0$ without loss of generality.

$$\sqrt{1 - 2M/R_s} < \varpi^{\min} \equiv \varpi_1 < \varpi_2 < \varpi_3 < \dots < \varpi_\infty \quad (30)$$

with the asymptotic property $\varpi_\infty \rightarrow 1$ [see Eq. (34) below].

In Table 1 we display, for various values of the dimensionless star-field mass parameter $\alpha \equiv M\mu$, the value of the smallest resonant oscillation frequency $\varpi^{\min}(\alpha)$. Interestingly, one finds that the resonant oscillation frequency $\varpi^{\min}(\alpha)$, which characterizes the composed stationary compact-reflecting-star-massive-scalar-field configurations in the large-radii regime (17), is a monotonically decreasing function of the dimensionless physical parameter α . As a consistency check, we note that the numerically computed resonant oscillation frequencies $\varpi^{\min}(\alpha)$ of the composed star-field system conform to the analytically derived bounds $\sqrt{1 - 2M/R_s} < \varpi < 1$ [see Eq. (16)].

In Table 2 we present, for various values of the dimensionless mass-radius parameter $\gamma \equiv \mu R_s$ of the composed star-field system, the value of the smallest resonant oscillation frequency $\varpi^{\min}(\gamma)$ which characterizes the stationary bound-state field configurations. One finds that the characteristic resonant oscillation frequency $\varpi^{\min}(\gamma)$ of the stationary composed compact-reflecting-star-massive-scalar-field configurations is a monotonically increasing function of the dimensionless mass-radius parameter γ .

In Table 3 we display, for various values of the dimensionless angular harmonic index l of the stationary scalar field modes, the value of the smallest resonant oscillation frequency $\varpi^{\min}(l)$. Interestingly, one finds that the resonant oscillation frequency $\varpi^{\min}(l)$ of the stationary bound-state massive scalar field configurations is a monotonically increasing function of the spherical harmonic index l . Again, one finds that the numerically computed resonant oscillation frequencies $\varpi^{\min}(l)$ of the composed star-field system

Table 3 Resonant oscillation frequencies of the composed stationary compact-reflecting-star-linearized-massive-scalar-field configurations with $\alpha = 1$ and $\gamma = 10$. We present the smallest resonant oscillation frequency $\varpi^{\min}(l)$ which characterizes the stationary bound-state star-field configurations for various values of the dimensionless harmonic

Angular harmonic index l	0	1	2	3	4	5
$\varpi^{\min}(l; \alpha = 1, \gamma = 10)$	0.9541	0.9569	0.9622	0.9694	0.9773	0.9841

conform to the analytically derived bounds $\sqrt{1 - 2M/R_s} < \varpi < 1$.

6 Analytical treatment for marginally-bound stationary scalar field resonances

In the present section we shall explicitly show that the resonance equation (29), which determines the discrete family of resonant oscillation frequencies $\{\varpi(\alpha, \gamma, \ell; n)\}$ that characterize the stationary bound-state linearized massive scalar fields in the curved spacetime of the spherically symmetric compact reflecting star, can be solved analytically in the asymptotic regime¹³

$$\varpi^2 \rightarrow 1^- \quad (31)$$

Substituting the asymptotic equation (see Eq. 13.5.16 of [53])

$$U(a, b, x) \simeq \Gamma\left(\frac{1}{2}b - a + \frac{1}{4}\right) \pi^{-\frac{1}{2}} e^{\frac{1}{2}x} x^{\frac{1}{4} - \frac{1}{2}b} \cdot \cos\left[\sqrt{2(b-2a)x} + \pi\left(a - \frac{1}{2}b + \frac{1}{4}\right)\right] \text{ for } a \rightarrow -\infty \quad (32)$$

of the confluent hypergeometric function into the analytically derived resonance condition (29), one finds the characteristic equation

$$\cos\left(\sqrt{8\alpha\gamma} - \frac{\alpha\pi}{\sqrt{1-\varpi^2}} + \frac{1}{4}\pi\right) = 0 \quad (33)$$

for the marginally-bound ($\varpi^2 \rightarrow 1^-$) stationary scalar field configurations. The resonance condition (33) yields the compact expression¹⁴

$$\varpi^2 = 1 - \left(\frac{\alpha}{n - \frac{1}{4} + c}\right)^2 \quad \text{with } c \equiv \frac{\sqrt{8\alpha\gamma}}{\pi}; \quad n \gg 1 \quad (34)$$

¹³ Note that the asymptotic frequency regime (31) corresponds to the physically interesting case of marginally-bound stationary composed compact-reflecting-star-massive-scalar-field configurations.

¹⁴ Here the integer $n \gg 1$ is the resonance parameter which characterizes the stationary composed star-field configurations.

index l . One finds that the characteristic resonant frequency $\varpi^{\min}(l)$ of the composed star-field system is a monotonically increasing function of the angular harmonic index l . As a consistency check, we note that the numerically computed resonant oscillation frequencies conform to the analytically derived bounds $\sqrt{1 - 2M/R_s} < \varpi < 1$ [see Eq. (16)]

for the discrete spectrum of resonant oscillation frequencies which characterize the marginally-bound composed spherically-symmetric-compact-reflecting-star-stationary-massive-scalar-field configurations.

It is worth emphasizing the fact that, in accord with the exact (numerically computed) data presented in Tables 1 and 2, the analytically derived expression (34) for the asymptotic resonance spectrum implies that the resonant oscillation frequencies which characterize the stationary composed compact-reflecting-star-massive-scalar-field configurations are a monotonically decreasing function of the dimensionless star-field mass parameter $\alpha \equiv M\mu$ and a monotonically increasing function of the dimensionless mass-radius parameter $\gamma \equiv \mu R_s$.

7 Numerical confirmation

It is of physical interest to test the accuracy of the approximated (analytically derived) compact equation (34) for the asymptotic resonance spectrum which characterizes the stationary marginally-bound composed compact-reflecting-star-massive-scalar-field configurations.

In Table 4 we present the resonant oscillation frequencies $\varpi^{\text{analytical}}(n; \alpha, \gamma, \ell)$ of the stationary massive scalar fields as obtained from the approximated (analytical) resonance equation (34) in the asymptotic regime (31) of marginally-bound star-field configurations. Also displayed in Table 4 are the corresponding resonant oscillation frequencies $\varpi^{\text{numerical}}(n; \alpha, \gamma, \ell)$ of the stationary bound-state massive scalar field configurations as obtained by solving numerically the analytically derived resonance equation (29). In the physical regime (31) of marginally-bound star-field configurations, one finds a remarkably good agreement¹⁵ between the

¹⁵ It is interesting to note that the agreement between the analytically derived resonant oscillation frequencies of the stationary bound-state composed star-field configurations and the corresponding exact (numerically computed) resonant oscillation frequencies is quite good even for the fundamental (smallest) resonant modes with $n = O(1)$. This reasonably good analytical/numerical agreement in the $n = O(1)$ regime is quite surprising since the mathematically compact analytical equation (34) for the resonant frequencies of the stationary composed compact-reflecting-star-linearized-massive-scalar-field configurations is formally valid for marginally-bound scalar modes with $n \gg 1$.

Table 4 Resonant oscillation frequencies of the composed stationary compact-reflecting-star-linearized-massive-scalar-field configurations with $\alpha = 1$, $\gamma = 10$, and $l = 0$. We display the resonant oscillation frequencies $\varpi^{\text{analytical}}(n; \alpha, \gamma, \ell)$ of the stationary scalar fields as obtained from the analytically derived resonance equation (34) in the asymptotic regime (31) of marginally-bound star-field configurations. We also present the corresponding numerically computed resonant oscillation

Formula	$\varpi(n=1)$	$\varpi(n=2)$	$\varpi(n=3)$	$\varpi(n=4)$	$\varpi(n=5)$	$\varpi(n=6)$	$\varpi(n=7)$	$\varpi(n=8)$
Analytical [Eq. (34)]	0.96058	0.97605	0.98391	0.98845	0.99130	0.99321	0.99456	0.99554
Numerical [Eq. (29)]	0.95415	0.97312	0.98217	0.98730	0.99051	0.99264	0.99412	0.99521

exact resonant oscillation frequencies [as computed numerically directly from the resonance equation (29)] and the corresponding analytically derived resonant oscillation frequencies (34) which characterize the stationary composed compact-reflecting-star-massive-scalar-field configurations.

8 Summary and discussion

The elegant no-scalar-hair theorems of [1, 2] (see also [3–9] and the references therein) have revealed the interesting fact that asymptotically flat classical black-hole spacetimes with one-way membranes (absorbing event horizons) cannot support static configurations made of massive scalar fields. Intriguingly, it has recently been demonstrated explicitly that the presence of an absorbing boundary (an event horizon) is not a necessary condition for the validity of the no-hair property for compact gravitating objects in general relativity. In particular, it has been proved that asymptotically flat horizonless neutral reflecting (rather than absorbing) stars are ‘bald’ [40] in the sense that they cannot support static bound-state scalar (spin-0) fields, vector (spin-1) fields, and tensor (spin-2) fields [40–42].

One naturally wonders whether the no-scalar-hair behavior observed in [40] is a generic property of compact reflecting stars? In particular, in the present paper we have raised the following physically intriguing question: Can horizonless compact objects with reflecting boundary conditions (as opposed to the absorbing boundary conditions which characterize the horizons of classical black-hole spacetimes) support stationary (rather than static) massive scalar field configurations?

In order to address this physically interesting question, in the present paper we have studied analytically the characteristic Klein–Gordon wave equation for stationary linearized massive scalar field configurations in the curved spacetime of a spherically symmetric compact reflecting star. The main results derived in this paper and their physical implications are as follows:

1. It has been explicitly proved that the stationary composed compact-reflecting-star-linearized-massive-scalar-field

frequencies $\varpi^{\text{numerical}}(n; \alpha, \gamma, \ell)$ of the stationary composed star-field configurations. In the physical regime (31) of marginally-bound massive scalar field configurations, one finds a remarkably good agreement (see footnote 15) between the exact resonant field frequencies [as obtained numerically from the characteristic resonance condition (29)] and the corresponding analytically derived resonant oscillation frequencies (34) which characterize the stationary bound-state star-field configurations

configurations, if they exist, are restricted to the dimensionless frequency regime [see Eq. (16)] (see footnote 12)

$$\sqrt{1 - \frac{2M}{R_s}} < \frac{\omega}{\mu} < 1. \quad (35)$$

It is worth noting that the presence of a positive lower bound on resonant oscillation frequencies of the composed star-field configurations agrees with the no-scalar-hair theorem presented in [40], according to which spherically symmetric compact reflecting stars cannot support spatially regular static ($\omega = 0$) scalar field configurations.

2. We have then proved that spherically symmetric compact reflecting stars can support stationary (rather than static) bound-state linearized massive scalar fields. In particular, it has been shown that the resonant oscillation modes of the composed compact-reflecting-star-massive-scalar-field system depend on three dimensionless physical parameters: the star-field mass parameter $\alpha \equiv M\mu$, the mass-radius parameter $\gamma \equiv \mu R_s$, and the spherical harmonic index l of the field mode. Solving analytically the Klein–Gordon wave equation for the stationary massive scalar field modes in the large-radii regime $M/R_s \ll 1$, we have explicitly proved that the resonance equation [see Eqs. (28) and (29)]

$$U\left(\frac{1}{2} + i\sqrt{4M^2(3\omega^2 - \mu^2) - (l+1/2)^2} - \frac{M(2\omega^2 - \mu^2)}{\sqrt{\mu^2 - \omega^2}}, 1 + 2i\sqrt{4M^2(3\omega^2 - \mu^2) - (l+1/2)^2}, 2\sqrt{\mu^2 - \omega^2}R_s\right) = 0 \quad (36)$$

determines the discrete spectrum $\{\varpi(\alpha, \gamma, \ell; n)\}_{n=1}^{n=\infty}$ of dimensionless resonant oscillation frequencies which characterize the stationary bound-state scalar field modes in the spacetime of the spherically symmetric compact reflecting star.

3. Solving numerically the analytically derived resonance condition (36) for the composed star-field configurations, we have demonstrated that the smallest possible resonant oscillation frequency $\varpi^{\min}$ [see Eq. (30)] which characterizes the stationary bound-state scalar field configurations is a monotonically decreasing function of the dimensionless star-field mass parameter $M\mu$ and a monotonically increasing function of the dimensionless mass-radius parameter μR_s and the dimensionless angular harmonic index l (see Tables 1, 2, 3).
4. It has been explicitly proved that the large-radii resonance equation (36) for the composed star-field configurations is amenable to an analytical treatment in the physically interesting regime $(\omega/\mu)^2 \rightarrow 1^-$ of marginally-bound stationary massive scalar field configurations. In particular, we have derived the remarkably simple dimensionless analytical formula [see Eqs. (28) and (34)]

$$\frac{\omega}{\mu} = \sqrt{1 - \left(\frac{M\mu}{n - \frac{1}{4} + M\mu\sqrt{8R_s/M\pi^2}} \right)^2}; \quad n \gg 1 \quad (37)$$

for the discrete resonance spectrum which characterizes the stationary marginally-bound linearized massive scalar field configurations in the spacetime of the spherically symmetric horizonless reflecting star.

5. It has been explicitly demonstrated that the characteristic resonant frequencies of the stationary marginally-bound composed compact-reflecting-star-linearized-massive-scalar-field configurations, as deduced from the analytically derived resonance equation (37), agree remarkably well (see footnote 15) with the corresponding exact resonant oscillation frequencies of the stationary massive scalar fields as obtained numerically from the resonance condition (36) (see Table 4).

Finally, we would like to emphasize again that in the present exploration of the physical and mathematical properties of the composed star-field system we have made two simplifying assumptions: (1) The scalar field was treated at the linear level, and (2) the central supporting star was treated as a weakly self-gravitating object.¹⁶ As we have explicitly proved in this paper, using these two simplifying assumptions, the physical and mathematical properties of the stationary composed compact-reflecting-star-linearized-massive-scalar-field system can be studied analytically. As a next step, it would be interesting to use numerical techniques in order

to extend our results to the physically important regime of non-linear stationary bound-state scalar field configurations supported by strongly self-gravitating horizonless compact stars.

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¹⁶ It is worth emphasizing, however, that the analytically derived frequency bound (35), which characterizes the composed star-field configurations, is valid for generic values of the dimensionless mass-radius ratio M/R_s of the central supporting star.

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