A 1 + 5-dimensional gravitational-wave solution: curvature singularity and spacetime singularity

Yu-Zhu Chen¹, Wen-Du Li^{1,2}, Wu-Sheng Dai^{2,3,a}

¹ Department of Physics, Tianjin University, Tianjin 300072, People's Republic of China

² Theoretical Physics Division, Chern Institute of Mathematics, Nankai University, Tianjin 300071, People's Republic of China

³ LiuHui Center for Applied Mathematics, Nankai University and Tianjin University, Tianjin 300072, People's Republic of China

Received: 18 September 2017 / Accepted: 30 October 2017 / Published online: 12 December 2017

© The Author(s) 2017. This article is an open access publication

Abstract We solve a 1 + 5-dimensional cylindrical gravitational-wave solution of the Einstein equation, in which there are two curvature singularities. Then we show that one of the curvature singularities can be removed by an extension of the spacetime. The result exemplifies that the curvature singularity is not always a spacetime singularity; in other words, the curvature singularity cannot serve as a criterion for spacetime singularities.

Contents

1 Introduction	1
2 1 + 5-dimensional cylindrical vacuum solution	2
3 Singularities of the spacetime	4
4 Extension of the spacetime: removing singularity	4
5 Singularity hypersurface	5
6 Gravitational wave: energy-momentum pseudotensor	5
7 Note on 1 + n-dimensional solution	6
8 Conclusions and outlook	7
References	7

1 Introduction

We find a 1+5-dimensional cylindrical vacuum gravitational-wave solution of the Einstein equation. Such a spacetime has both the curvature singularity, the singularity of the Riemann tensor components R_{ikl}^j , and the scalar polynomial curvature singularity, the singularity of the scalars formed out of the curvatures such as R , $R^{ij}R_{ij}$, and $R_{ikl}^j R_j^{ikl}$. In this 1 + 5-dimensional spacetime, there is a singularity hypersurface formed by singular points behaving like a horizon.

The spacetime singularity is an important problem in gravity theory. Nevertheless, even the definition of the spacetime singularity is still an open question. Many attempts have been made to seek a general criterion for the spacetime singularity.

There is no general criterion to judge whether a spacetime is singularity-free [1]. By intuition, the singularity should involve infinite curvatures. However, Geroch constructs some singular spacetimes whose Riemann curvatures are bounded everywhere; that is to say, the singularity cannot be determined by the Riemann curvature [2]. Some efforts are made to define a singular boundary which is coordinate independent, such as the “g-boundary” [3] and the “b-boundary” [4]. These prescriptions and other similar procedures, however, produce pathological topological boundaries in simple examples [1,5,6]. Hence, the singular boundary cannot be generally defined. Up to now, a generally accepted definition of the spacetime singularity is based on timelike- or null-geodesic incompleteness [7]. Based on geodesic incompleteness, Hawking and Penrose prove a series of singularity theorems [7]. Nevertheless, Geroch also constructs a singular spacetime whose geodesics are all complete, so that these singularities cannot be judged by the geodesic completeness [2]. In short, the reason why it is hard to present a general criterion for the singularity is that the spacetime may have an infinite variety of possible pathological behaviors [1].

On the contrary, one can easily judge that a spacetime has a singularity if the spacetime has one kind of pathological behaviors. It is a common impression that if the curvature is singular at some point, the spacetime is singular. It should be emphasized that an orthonormal frame is necessary. In an orthonormal frame, the metric of a 1 + n-dimensional spacetime reads $ds^2 = -(\theta^0)^2 + \sum_{i=1}^n (\theta^i)^2$ with θ^0 and θ^i the 1-form on the manifold, which are independent of the choice of coordinates. As a result, the curvature in an orthonormal frame is independent of the choice of the coordinates [8,9].

^a e-mail: daiwusheng@tju.edu.cn

In this paper, we exemplify by the 1 + 5-dimensional gravitational-wave solution obtained in the present paper that a curvature singularity is not always a spacetime singularity. Concretely, we show that the curvature singularity can be removed by an extension of the solution to the maximal manifold. This result tells us that the curvature singularity is not always a spacetime singularity and thus cannot serve as a criterion for spacetime singularities.

We show that this 1 + 5-dimensional cylindrical vacuum solution describes a gravitational wave, through calculating energy-momentum pseudotensors [10]. The gravitational wave is always an important problem in gravity theory [11]. The existence of the gravitational wave is confirmed experimentally [12, 13]. As the key source of the gravitational wave, the black-hole binaries are allowed stable, robust simulations of the merger process [14] based on the advance in numerical relativity [14]. New sources of the gravitational waves are discussed [15]. The application of the gravitational wave as a probe with the first order electroweak phase transition is also considered [16].

Moreover, we also give a brief discussion of the 1 + n -dimensional case.

Higher-dimensional gravity is an important issue in the theory of gravity. There are also some studies on the cylindrical spacetime, such as a cylindrical solution in the mimetic gravity theory [17] and higher-dimensional cylindrical or Kasner type electrovacuum solutions [18]. For higher-dimensional spacetime, there are some exact solutions of the Einstein equation, such as a rotating black ring solution in five dimensions [19], general Kerr–NUT–AdS metrics in arbitrary dimensions [20], and the exact solutions of Einstein–Maxwell gravity [21, 22]. In Refs. [23, 24], the property of the exact solution of the Einstein equation in higher dimensions are discussed. In Refs. [25, 26], an inverse scattering method for solving the vacuum solution of five dimensional Einstein equation are provided. In the brane world model, the four-dimensional world is regarded as a brane embedded in higher-dimensional spacetime [27–29]. Beyond general relativity, also some modified gravity theories are considered, such as the Lanczos–Lovelock model [30], and the Einstein–Gauss–Bonnet theory [31].

In Sect. 2, a 1 + 5-dimensional cylindrical vacuum solution of the Einstein equation is presented. In Sect. 3, the singularity of this 1 + 5-dimensional cylindrical spacetime is analyzed. In Sect. 4, we make an extension of this 1 + 5-dimensional cylindrical spacetime to remove the curvature singularity. In Sect. 5, we show that in this 1 + 5-dimensional cylindrical spacetime there exists a horizon-like singularity hypersurface. In Sect. 6, we show that the solution presented in this paper describes a gravitational wave. In Sect. 7, we give a brief discussion of 1 + n -dimensional cylindrical vacuum solutions. The conclusions are summarized in Sect. 8.

2 1 + 5-dimensional cylindrical vacuum solution

In this section, we find a 1 + 5-dimensional cylindrical vacuum solution of the Einstein equation.

Choose the metric of a 1 + 5-dimensional cylindrical spacetime as

$$ds^2 = e^{2\Phi(t,\rho)} (-dt^2 + d\rho^2) + e^{-2\psi(t,\rho)} \rho^2 d\Omega^2 + e^{2\psi(t,\rho)} dz^2, \quad (2.1)$$

where $d\Omega^2 = d\theta_1^2 + \sin^2 \theta_1 d\theta_2^2 + \sin^2 \theta_1 \sin^2 \theta_2 d\phi^2$.

With the metric (2.1), the vacuum Einstein equation $R_{ij} - \frac{1}{2} g_{ij} R = 0$ reads

$$-2 \frac{\partial \Phi}{\partial t} \frac{\partial \psi}{\partial t} - 2 \frac{\partial \Phi}{\partial \rho} \frac{\partial \psi}{\partial \rho} + \frac{3}{\rho} \frac{\partial \Phi}{\partial \rho} + 2 \frac{\partial^2 \psi}{\partial \rho^2} - 4 \left(\frac{\partial \psi}{\partial \rho} \right)^2 + \frac{9}{\rho} \frac{\partial \psi}{\partial \rho} + \frac{3}{\rho^2} \left[e^{2(\Phi+\psi)} - 1 \right] = 0, \quad (2.2)$$

$$-2 \frac{\partial \Phi}{\partial t} \frac{\partial \psi}{\partial \rho} + \frac{3}{\rho} \frac{\partial \Phi}{\partial t} - 2 \frac{\partial \Phi}{\partial \rho} \frac{\partial \psi}{\partial t} + 2 \frac{\partial^2 \psi}{\partial t \partial \rho} - 4 \frac{\partial \psi}{\partial t} \frac{\partial \psi}{\partial \rho} + \frac{3}{\rho} \frac{\partial \psi}{\partial t} = 0, \quad (2.3)$$

$$-2 \frac{\partial \Phi}{\partial t} \frac{\partial \psi}{\partial t} - 2 \frac{\partial \Phi}{\partial \rho} \frac{\partial \psi}{\partial \rho} + \frac{3}{\rho} \frac{\partial \Phi}{\partial \rho} + 2 \frac{\partial^2 \psi}{\partial t^2} - 4 \left(\frac{\partial \psi}{\partial t} \right)^2 - \frac{3}{\rho} \frac{\partial \psi}{\partial \rho} - \frac{3}{\rho^2} \left[e^{2(\Phi+\psi)} - 1 \right] = 0, \quad (2.4)$$

$$\frac{\partial^2 \Phi}{\partial \rho^2} - \frac{\partial^2 \Phi}{\partial t^2} + \frac{\partial^2 \psi}{\partial t^2} - \frac{\partial^2 \psi}{\partial \rho^2} + 2 \left(\frac{\partial \psi}{\partial \rho} \right)^2 - 2 \left(\frac{\partial \psi}{\partial t} \right)^2 - \frac{4}{\rho} \frac{\partial \psi}{\partial \rho} - \frac{1}{\rho^2} \left[e^{2(\Phi+\psi)} - 1 \right] = 0, \quad (2.5)$$

$$\frac{\partial^2 \Phi}{\partial \rho^2} - \frac{\partial^2 \Phi}{\partial t^2} + 3 \frac{\partial^2 \psi}{\partial t^2} - 3 \frac{\partial^2 \psi}{\partial \rho^2} + 6 \left(\frac{\partial \psi}{\partial \rho} \right)^2 - 6 \left(\frac{\partial \psi}{\partial t} \right)^2 - \frac{12}{\rho} \frac{\partial \psi}{\partial \rho} - \frac{3}{\rho^2} \left[e^{2(\Phi+\psi)} - 1 \right] = 0. \quad (2.6)$$

By light-cone coordinates

$$\begin{aligned} u &= t + \rho, \\ v &= t - \rho, \end{aligned} \quad (2.7)$$

Eqs. (2.2)–(2.6) can be simplified as

$$2 \left(-2 \frac{\partial \Phi}{\partial u} \frac{\partial \psi}{\partial u} + \frac{\partial^2 \psi}{\partial u^2} \right) + 3 \frac{1}{\rho} \left(\frac{\partial \Phi}{\partial u} + \frac{\partial \psi}{\partial u} \right) - 4 \left(\frac{\partial \psi}{\partial u} \right)^2 = 0, \quad (2.8)$$

$$2 \left(-2 \frac{\partial \Phi}{\partial v} \frac{\partial \psi}{\partial v} + \frac{\partial^2 \psi}{\partial v^2} \right) - 3 \frac{1}{\rho} \left(\frac{\partial \Phi}{\partial v} + \frac{\partial \psi}{\partial v} \right) - 4 \left(\frac{\partial \psi}{\partial v} \right)^2 = 0, \quad (2.9)$$

$$-4 \frac{\partial^2 \psi}{\partial u \partial v} + 8 \frac{\partial \psi}{\partial u} \frac{\partial \psi}{\partial v} + 3 \frac{1}{\rho} \frac{\partial \psi}{\partial \rho} = 0, \quad (2.10)$$

$$\frac{\partial^2 \Phi}{\partial u \partial v} = 0, \quad (2.11)$$

$$\frac{\partial \psi}{\partial \rho} + \frac{1}{\rho} \left[e^{2(\Phi+\psi)} - 1 \right] = 0. \quad (2.12)$$

Solving Eq. (2.12), we have

$$\psi = \ln \rho - \frac{1}{2} \ln \left(2 \int_1^\rho x e^{2\Phi} dx + c_1(t) \right), \quad (2.13)$$

where $c_1(t)$ is an arbitrary function of t . Substituting Eq. (2.13) into Eq. (2.10) gives

$$\begin{aligned} \frac{d^2 c_1(t)}{dt^2} + 2 \int_1^\rho 2x e^{2\Phi} \left[2 \left(\frac{\partial \Phi}{\partial t} \right)^2 + \frac{\partial^2 \Phi}{\partial t^2} \right] dx \\ - 4\rho e^{2\Phi} \frac{\partial \Phi}{\partial \rho} = 0. \end{aligned} \quad (2.14)$$

Taking the derivative with respect to ρ , we arrive at an equation of Φ ,

$$\rho \left(\frac{\partial^2 \Phi}{\partial t^2} - \frac{\partial^2 \Phi}{\partial \rho^2} \right) + 2\rho \left[\left(\frac{\partial \Phi}{\partial t} \right)^2 - \left(\frac{\partial \Phi}{\partial \rho} \right)^2 \right] - \frac{\partial \Phi}{\partial \rho} = 0. \quad (2.15)$$

Rewriting Eq. (2.15) in light-cone coordinates, we have

$$2(u-v) \frac{\partial^2 \Phi}{\partial u \partial v} + 4(u-v) \frac{\partial \Phi}{\partial u} \frac{\partial \Phi}{\partial v} - \frac{\partial \Phi}{\partial u} + \frac{\partial \Phi}{\partial v} = 0. \quad (2.16)$$

By Eq. (2.11), Φ can be expressed as

$$\Phi = f_1(u) + f_2(v). \quad (2.17)$$

Substituting Eq. (2.17) into Eq. (2.16) and separating variables, we have

$$\frac{1}{df_1(u)/du} + 4u = \frac{1}{df_2(v)/dv} + 4v = 4a, \quad (2.18)$$

where a is the separation constant. Then we arrive at

$$f_1(u) = -\frac{1}{4} \ln(u+a) = -\frac{1}{4} \ln(t+\rho+a), \quad (2.19)$$

$$f_2(v) = -\frac{1}{4} \ln(v+a) + C_0 = -\frac{1}{4} \ln(t-\rho+a) + C_0. \quad (2.20)$$

By Eq. (2.17) and then by Eq. (2.13), we obtain

$$\Phi = -\frac{1}{4} \ln(t+\rho+a) - \frac{1}{4} \ln(t-\rho+a) + C_0, \quad (2.21)$$

$$\psi = \ln \rho - \frac{1}{2} \ln \left(-2e^{2C_0} \sqrt{t+\rho+a} \sqrt{t-\rho+a} + c_1(t) \right). \quad (2.22)$$

$c_1(t)$ in Eq. (2.22) can be determined by substituting Eq. (2.22) into Eq. (2.10). This gives $d^2 c_1(t)/dt^2 = 0$ and then

$$c_1(t) = \alpha t + \beta, \quad (2.23)$$

where α and β are constants. Substituting Eqs. (2.21)–(2.23) into Eqs. (2.2) and (2.3) gives

$$\beta = a\alpha. \quad (2.24)$$

Then we obtain the solution

$$\Phi = -\frac{1}{4} \ln(t+\rho+a) - \frac{1}{4} \ln(t-\rho+a) + C_0, \quad (2.25)$$

$$\begin{aligned} \psi = \ln \rho - \frac{1}{2} \ln \left(-2e^{2C_0} \sqrt{t+\rho+a} \sqrt{t-\rho+a} \right. \\ \left. + \alpha t + a\alpha \right). \end{aligned} \quad (2.26)$$

Redefining the variables as $\alpha(t+a) \rightarrow t$, $\alpha\rho \rightarrow \rho$, $z/\alpha \rightarrow z$, and $1+M = 2e^{2C_0}/\alpha$, we have

$$\Phi = -\frac{1}{4} \ln(t^2 - \rho^2) + \frac{1}{2} \ln \frac{1+M}{2}, \quad (2.27)$$

$$\psi = \ln \rho - \frac{1}{2} \ln \left[t - (1+M) \sqrt{t^2 - \rho^2} \right]. \quad (2.28)$$

The metric then reads

$$\begin{aligned} ds^2 = \frac{1+M}{2\sqrt{t^2 - \rho^2}} \left(-dt^2 + d\rho^2 \right) \\ + \left[t - (1+M) \sqrt{t^2 - \rho^2} \right] d\Omega^2 \\ + \frac{\rho^2 dz^2}{t - (1+M) \sqrt{t^2 - \rho^2}}, \end{aligned} \quad (2.29)$$

where $t \geq \rho \geq 0$ and the parameter M is an arbitrary constant.

When $M = 0$, the spacetime described by the metric (2.29) reduces to a flat spacetime. Concretely, we perform a coordinate transformation

$$t = \frac{1}{2} \left(\chi^2 - \xi^2 + \varrho^2 \right), \quad (2.30)$$

$$\rho = \varrho \sqrt{\chi^2 - \xi^2}, \quad (2.31)$$

$$z = \frac{1}{2} \ln \frac{\chi + \xi}{\chi - \xi}. \quad (2.32)$$

The inverse transformation is $\chi = \sqrt{t + \sqrt{t^2 - \rho^2}} \cosh z$, $\xi = \sqrt{t + \sqrt{t^2 - \rho^2}} \sinh z$, and $\varrho = \sqrt{t - \sqrt{t^2 - \rho^2}}$. The metric (2.29) becomes

$$ds^2 = -\frac{1+M}{\chi^2 - \xi^2} (\chi d\chi - \xi d\xi)^2 + (1+M) d\varrho^2$$

$$\begin{aligned}
& + \left[\varrho^2 - \frac{M}{2} (\chi^2 - \xi^2 - \varrho^2) \right] d\Omega^2 \\
& + \frac{\varrho^2}{\left[\varrho^2 - \frac{M}{2} (\chi^2 - \xi^2 - \varrho^2) \right] (\chi^2 - \xi^2)} (\chi d\xi - \xi d\chi)^2.
\end{aligned} \quad (2.33)$$

Taking $M = 0$, it reduces to $ds^2 = -d\chi^2 + d\xi^2 + d\varrho^2 + \varrho^2 d\Omega^2$.

3 Singularities of the spacetime

In this section, we analyze the singularity of the spacetime described by the metric (2.29).

The metric (2.29) is singular at

$$\rho = t, \quad (3.1)$$

$$\rho = \frac{\sqrt{M(M+2)}}{M+1} t. \quad (3.2)$$

In order to show the curvature singularity and the scalar polynomial curvature singularity, we here calculate the Riemann tensor R_{ikl}^j and the scalar $R_{ikl}^j R_j^{ikl}$ in the orthonormal frame.

An orthonormal frame $(e_0, e_1, e_2, \dots, e_m)$ and a natural frame $\left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \dots, \frac{\partial}{\partial x^m}\right)$ on a manifold are connected by

$$e_i = X_i^k \frac{\partial}{\partial x^k}, \quad (3.3)$$

where X_i^k is a nondegenerate matrix. Choosing the corresponding 1-form

$$\theta^i = X_j^{*i} dx^j \quad (3.4)$$

with X_j^{*i} the inverse matrix of X_i^k , we can express the metric of the manifold as

$$ds^2 = g_{ij} dx^i dx^j = -(\theta^0)^2 + \sum_{i=1}^n (\theta^i)^2. \quad (3.5)$$

In the orthonormal frame, the curvature components R_{ikl}^j can be obtained from the curvature components in the natural frame, $\mathcal{R}_{ikl}^j$, by

$$\begin{aligned}
R &= R_{ikl}^j \theta^i \otimes e_j \otimes \theta^k \otimes \theta^l \\
&= \mathcal{R}_{ikl}^j dx^i \otimes \frac{\partial}{\partial x^j} \otimes dx^k \otimes dx^l.
\end{aligned} \quad (3.6)$$

Then we arrive at [8]

$$R_{ikl}^j = X_q^{*j} X_i^p X_k^r X_l^s \mathcal{R}_{prs}^q. \quad (3.7)$$

The curvature components R_{ikl}^j are independent of the coordinates.

The components of the Riemann tensor R_{ikl}^j in the orthonormal frame can be calculated from the metric (2.29) directly:

$$\begin{aligned}
R_{220}^0 &= R_{330}^0 = R_{440}^0 = \rho^2 U(t, \rho) V(t, \rho), \\
R_{221}^1 &= R_{331}^1 = R_{441}^1 = -t^2 U(t, \rho) V(t, \rho), \\
R_{220}^1 &= R_{330}^1 = R_{440}^1 = t\rho U(t, \rho) V(t, \rho), \\
R_{550}^0 &= -3\rho^2 U(t, \rho) V(t, \rho), \\
R_{551}^1 &= 3t^2 U(t, \rho) V(t, \rho), \\
R_{550}^1 &= -3t\rho U(t, \rho) V(t, \rho), \\
R_{332}^2 &= R_{442}^2 = -R_{552}^2 = -R_{443}^3 = -R_{553}^3 = -R_{554}^4 \\
&= \sqrt{t^2 - \rho^2} U(t, \rho);
\end{aligned} \quad (3.8)$$

the scalar then becomes

$$R_{ikl}^j R_j^{ikl} = 72(t^2 - \rho^2) U^2(t, \rho). \quad (3.9)$$

Here, the singularities (3.1) and (3.2) are embodied in the factors

$$U(t, \rho) = \frac{M(M+2)}{2(M+1)} \frac{1}{\left[t - (1+M)\sqrt{t^2 - \rho^2}\right]^2}, \quad (3.10)$$

$$V(t, \rho) = \frac{1}{\sqrt{t^2 - \rho^2}}, \quad (3.11)$$

respectively.

Recall that the singularity of the components of the Riemann tensor R_{ikl}^j is the curvature singularity and the singularity of the scalar $R_{ikl}^j R_j^{ikl}$ is the scalar polynomial curvature singularity [1, 7]. Now we see that (1) the singularity $\rho = t$ in the metric are the curvature singularities; (2) the singularity $\rho = \frac{\sqrt{M(M+2)}}{M+1} t$ are both the curvature singularity and the scalar polynomial curvature singularity.

4 Extension of the spacetime: removing singularity

In this section, we make an extension of the spacetime described by the metric (2.29). It will be seen that the curvature singularity in the extended spacetime corresponding to $\rho = t$ in the metric (2.29) is no longer singular; while the scalar polynomial curvature singularity in the extended spacetime corresponding to $\rho = \frac{\sqrt{M(M+2)}}{M+1} t$ is still singular. That is to say, some curvature singularities can be removed.

Introduce new coordinates (η, r) by

$$t = \frac{1}{2} (\eta^2 + r^2),$$

$$\rho = \eta r, \quad (4.1)$$

where $\eta \geq r \geq 0$. In the coordinates (η, r) , the metric (2.29) becomes

$$ds^2 = (1 + M) \left(-d\eta^2 + dr^2 \right) + \frac{1}{2} \left[(2 + M) r^2 - M\eta^2 \right] d\Omega^2 + \frac{2\eta^2 r^2 dz^2}{(2 + M) r^2 - M\eta^2}. \quad (4.2)$$

The metric (4.2) with $\eta \geq 0$ and $r \geq 0$ describes a larger spacetime than the spacetime described by the metric (2.29). In fact, the spacetime described by the metric (2.29) is isometric to the area $\eta \geq r \geq 0$ in the metric (4.2). That is, the spacetime described by the metric (4.2) is an extension of the solution (2.29).

The metric (4.2) is singular at

$$r = \sqrt{\frac{M}{M+2}} \eta. \quad (4.3)$$

Similarly, we can calculate the Riemann tensor R_{ikl}^j and the scalar $R_{ikl}^j R_j^{ikl}$ from the metric (4.2) in the orthonormal frame.

The Riemann tensor components in the orthonormal frame are

$$\begin{aligned} R_{220}^0 &= R_{330}^0 = R_{440}^0 = r^2 W(\eta, r), \\ R_{220}^1 &= R_{330}^1 = R_{440}^1 = \eta r W(\eta, r), \\ R_{221}^1 &= R_{331}^1 = R_{441}^1 = -\eta^2 W(\eta, r), \\ R_{550}^0 &= -3r^2 W(\eta, r), \quad R_{551}^1 = 3\eta^2 W(\eta, r), \\ R_{551}^0 &= 3\eta r W(\eta, r), \\ R_{332}^2 &= R_{442}^2 = -R_{552}^2 = R_{443}^3 = -R_{553}^3 = -R_{554}^4 \\ &= (\eta^2 - r^2) W(\eta, r); \end{aligned} \quad (4.4)$$

the scalar is

$$R_{ikl}^j R_j^{ikl} = 72 (\eta^2 - r^2)^2 W^2(\eta, r). \quad (4.5)$$

After the extension, there is only one singularity (4.3) remaining in the curvature, embodied in the factor

$$W(\eta, r) = \frac{M(M+2)}{2(M+1)} \frac{1}{[(M+2)r^2 - M\eta^2]^2}. \quad (4.6)$$

Analyzing the components of the Riemann tensor R_{ikl}^j and the scalar $R_{ikl}^j R_j^{ikl}$, we see that $\eta = r$ corresponding to $\rho = t$ in the metric (2.29) is no longer singular, or, the curvature singularity at $\rho = t$ is removed by the extension. Concretely, we extend the 1 + 5-dimensional solution (2.29) to a larger spacetime through the coordinates transform (4.1). This extended spacetime is described by the metric (4.2).

The scalar $R_{ikl}^j R_j^{ikl}$ with the metric (4.2) at $r = \sqrt{\frac{M}{M+2}} \eta$ corresponding to $\rho = \frac{\sqrt{M(M+2)}}{M+1} t$ in the metric (2.29) is still singular.

5 Singularity hypersurface

The singularity in the spacetime described by the metric (2.29) is not zero-dimensional points but two hypersurfaces.

The singularity $\rho = \frac{\sqrt{M(M+2)}}{M+1} t$ is a spacetime singularity, since the components of Riemann tensor R_{ikl}^j and the scalar $R_{ikl}^j R_j^{ikl}$ blows up. When $\rho = \frac{\sqrt{M(M+2)}}{M+1} t$, the metric (2.29) describes a hypersurface.

It is worth mentioning that the points determined by $\rho = \frac{\sqrt{M(M+2)}}{M+1} t$ form a hypersurface behaving like a horizon. This can be seen from the following two facts.

1. The velocity of a light signal propagating in the z direction is

$$\frac{dz}{dt} = \pm \sqrt{\frac{1+M}{2}} \frac{\sqrt{|t - (1+M)\sqrt{t^2 - \rho^2}|}}{\rho(t^2 - \rho^2)^{1/4}}. \quad (5.1)$$

When $\rho = \frac{\sqrt{M(M+2)}}{M+1} t$, the coordinate velocity of the light signal vanishes. Similarly, the coordinate velocity of a light signal in the Schwarzschild spacetime in the radial direction is $dr/dt = \pm (1 - \frac{2M}{r})$, which also vanishes at $r = 2M$ [32].

2. In the region $0 < \rho < \frac{\sqrt{M(M+2)}}{M+1} t$, the signs of g_{ii} ($i = 0, 1, 2, 3, 4, 5$) become the opposite. Concretely, when $\frac{\sqrt{M(M+2)}}{M+1} t < \rho < t$, the coordinates z, θ_1, θ_2 , and ϕ are spacelike, while when $0 < \rho < \frac{\sqrt{M(M+2)}}{M+1} t$, the coordinates z, θ_1, θ_2 , and ϕ are timelike. In other words, in the region $\frac{\sqrt{M(M+2)}}{M+1} t < \rho < t$, the metric normal form is $\text{diag}(-, +, +, +, +, +)$, while in the region $0 < \rho < \frac{\sqrt{M(M+2)}}{M+1} t$, the metric normal form is $\text{diag}(-, +, -, -, -, -)$.

Nevertheless, this singularity hypersurface is not a horizon like that in the Schwarzschild spacetime, since it is not a one-way membrane.

6 Gravitational wave: energy-momentum pseudotensor

In this section, we show that the 1 + 5-dimensional solution (2.29) is a gravitational-wave solution.

In the metric (2.29), the hypersurface $\rho = t$ can be regarded as the boundary of the spacetime. It provides a pic-

ture that the scale of the spacetime expands with time going on. To illustrate that this solution describes a gravitational wave, we will show that the solution has a time-varying energy-momentum pseudotensor (energy-momentum complex). The energy-momentum pseudotensor is defined to describe the energy-momentum of the gravitational field. The reason why using pseudotensors is that according to the equivalence principle, the spacetime is flat at a given point in local geodesic coordinates; as a result the energy-momentum of a gravitational field in local geodesic coordinates vanishes. If a gravitational field has a time-varying energy-momentum pseudotensor, the solution may represent a gravitational wave [10,33,34]. More discussions on energy-momentum pseudotensors can be found in Ref. [35].

The definition of the energy-momentum pseudotensor is not unanimous. There are many different definitions for energy-momentum pseudotensors: the Einstein energy-momentum pseudotensor $\theta_i^j = \frac{1}{16\pi} H_i^{jk}$ with $H_i^{jk} = \frac{1}{\sqrt{-g}} g_{il} [-g(g^{jl} g^{km} - g^{kl} g^{jm})]_{,m}$ [34]; the Landau–Lifshitz energy-momentum pseudotensor $L^{mn} = \frac{1}{16\pi} S^{mjnk}$ with $S^{mjnk} = -g(g^{mn} g^{jk} - g^{mk} g^{jn})$ [10]; the Tolman energy-momentum pseudotensor $\mathcal{T}_k^i = \frac{1}{8\pi} U_k^{ij}$ with $U_k^{ij} = \sqrt{-g} [-g^{li} (-\Gamma_{kl}^i + \frac{1}{2} g_k^i \Gamma_{al}^a + \frac{1}{2} g_l^j \Gamma_{al}^a) + \frac{1}{2} g_k^i g^{lm} (-\Gamma_{lm}^j + \frac{1}{2} g_l^j \Gamma_{am}^a + \frac{1}{2} g_m^j \Gamma_{al}^a)]$ [33]; the Papapetrou energy-momentum pseudotensor $\Omega^{ij} = \frac{1}{16\pi} \mathcal{N}^{ikab}$ with $\mathcal{N}^{ikab} = \sqrt{-g} (g^{ik} \eta^{ab} - g^{ia} \eta^{kb} + g^{ab} \eta^{ik} - g^{kb} \eta^{ia})$ [36]; the Weinberg energy-momentum pseudotensor $t_{ij} = \frac{1}{8\pi} (R_{ij} - \frac{1}{2} g_{ij} R - R_{ij}^{(1)} + \frac{1}{2} \eta_{ij} R_k^{(1)})$ with $R_{ij}^{(1)} = \frac{1}{2} \eta^{kl} (\frac{\partial^2 h_{lk}}{\partial x^i \partial x^j} - \frac{\partial^2 h_{li}}{\partial x^k \partial x^j} - \frac{\partial^2 h_{lj}}{\partial x^k \partial x^i} + \frac{\partial^2 h_{ij}}{\partial x^k \partial x^l})$ and $g_{ij} = \eta_{ij} + h_{ij}$ [37]; the Møller energy-momentum pseudotensor $\mathcal{J}_j^i = \frac{1}{8\pi} \xi_j^{ik}$ with $\xi_j^{ik} = \sqrt{-g} (\frac{\partial g_{jl}}{\partial x^m} - \frac{\partial g_{im}}{\partial x^l}) g^{im} g^{kl}$ [38].

Direct calculation shows that, for the 1 + 5-dimensional solution (2.29), all the energy-momentum pseudotensors mentioned above in the “Cartesian coordinates” [33]

$$\begin{aligned} x_1 &= \rho \sin \theta_1 \sin \theta_2 \sin \phi, \\ x_2 &= \rho \sin \theta_1 \sin \theta_2 \cos \phi, \\ x_3 &= \rho \sin \theta_1 \cos \theta_2, \\ x_4 &= \rho \cos \theta_1, \end{aligned} \quad (6.1)$$

have time-varying tt -components and tx_1 -components.

Additionally, for the extended spacetime (4.2), these energy-momentum pseudotensors in “Cartesian coordinates”

$$\begin{aligned} y_1 &= r \sin \theta_1 \sin \theta_2 \sin \phi, \\ y_2 &= r \sin \theta_1 \sin \theta_2 \cos \phi, \\ y_3 &= r \sin \theta_1 \cos \theta_2, \\ y_4 &= r \cos \theta_1, \end{aligned} \quad (6.2)$$

except the Møller energy-momentum pseudotensor, also have time-varying $\eta\eta$ -components and ηy_1 -components. For the Møller energy-momentum pseudotensor, the $\eta\eta$ -component vanishes but the ηy_1 -component is also time-varying.

That is, the 1 + 5-dimensional solution (2.29) can be regarded as a gravitational-wave solution.

7 Note on 1 + n -dimensional solution

In the above, we consider a 1 + 5-dimensional solution. In this section, we wish here to add a few words to the 1 + n -dimensional solutions.

The 1 + n -dimensional cylindrical metric reads

$$\begin{aligned} ds^2 &= e^{2\Phi(t,\rho)} (-dt^2 + d\rho^2) \\ &+ e^{-2\psi(t,\rho)} \rho^2 \sum_{k=1}^{n-1} d\theta_k^2 \prod_{s=1}^{k-1} \sin^2 \theta_s + e^{2\psi(t,\rho)} dz^2. \end{aligned} \quad (7.1)$$

When $n=2$, the metric becomes $ds^2 = e^{2\Phi(t,\rho)} (-dt^2 + d\rho^2) + e^{2\psi(t,\rho)} dz^2$ and have only trivial solutions which can be transformed to a Minkowski solution. When $n \geq 3$, the Einstein equation of the metric (7.1) in the light-cone coordinates $u \equiv t + \rho$ and $v \equiv t - \rho$ reads

$$\begin{aligned} (n-3) \left(-2 \frac{\partial \Phi}{\partial u} \frac{\partial \psi}{\partial u} + \frac{\partial^2 \psi}{\partial u^2} \right) + (n-2) \frac{1}{\rho} \left(\frac{\partial \Phi}{\partial u} + \frac{\partial \psi}{\partial u} \right) \\ - (n-1) \left(\frac{\partial \psi}{\partial u} \right)^2 = 0, \end{aligned} \quad (7.2)$$

$$\begin{aligned} (n-3) \left(-2 \frac{\partial \Phi}{\partial v} \frac{\partial \psi}{\partial v} + \frac{\partial^2 \psi}{\partial v^2} \right) - (n-2) \frac{1}{\rho} \left(\frac{\partial \Phi}{\partial v} + \frac{\partial \psi}{\partial v} \right) \\ - (n-1) \left(\frac{\partial \psi}{\partial v} \right)^2 = 0, \end{aligned} \quad (7.3)$$

$$- \frac{\partial^2 \Phi}{\partial u \partial v} + \frac{(n-5)}{2} \left(\frac{\partial^2 \psi}{\partial u \partial v} + \frac{\partial \psi}{\partial u} \frac{\partial \psi}{\partial v} \right) = 0, \quad (7.4)$$

$$- 4 \frac{\partial^2 \psi}{\partial u \partial v} + 4(n-3) \frac{\partial \psi}{\partial u} \frac{\partial \psi}{\partial v} + (n-2) \frac{1}{\rho} \frac{\partial \psi}{\partial \rho} = 0, \quad (7.5)$$

$$\frac{1}{\rho} \frac{\partial \psi}{\partial \rho} + \frac{1}{\rho^2} [e^{2(\Phi+\psi)} - 1] = 0. \quad (7.6)$$

The 1 + 5-dimensional case is special. When $n = 5$, Eq. (7.4) becomes

$$\frac{\partial^2 \Phi}{\partial u \partial v} = 0. \quad (7.7)$$

The solution of Eq. (7.7) then reads, as expressed in Eq. (2.17), $\Phi = F_1(u) + F_2(v)$. It can be shown that once Eq. (7.7) is satisfied, the spacetime can be extended by introducing new light-cone coordinates by

$$\begin{aligned} d\tilde{u} &= e^{2F_1} du, \\ d\tilde{v} &= e^{2F_2} dv. \end{aligned} \quad (7.8)$$

Then in the new coordinates $(\tilde{u}, \tilde{v})$, the metric (7.1) becomes

$$ds^2 = -d\tilde{u}d\tilde{v} + e^{-2\psi} \rho^2 \sum_{k=1}^{n-1} d\theta_k^2 \prod_{s=1}^{k-1} \sin^2 \theta_s + e^{2\psi} dz^2. \quad (7.9)$$

The metric (7.9) is usually the extension of the metric (7.1).

8 Conclusions and outlook

In this paper, we show that the curvature singularity is not always a spacetime singularity. For exemplifying this, we first solve a vacuum solution of the Einstein equation, which is a $1+5$ -dimensional cylindrical gravitational-wave solution.

We first show that there are two singularities in the Riemann tensor R_{ikl}^j and the scalar $R^j_{ikl} R_j^{ikl}$ of the $1+5$ -dimensional gravitational-wave solution obtained in the present paper. In other words, there are two curvature singularities in this spacetime. Then we show that one of these two singularities can be removed by an extension of the spacetime. That is to say, the curvature singularity which can be removed by the extension is not a real spacetime singularity, or, the curvature singularity cannot be used as a criterion for spacetime singularities. We also show that there is a horizon-like singularity hypersurface.

Moreover, in the above, we only consider the case of $M > 0$. When $M < 0$, the situation is different. In this case, when $M = -2$, the metric (4.2) reduces to a Minkowski spacetime described by $ds^2 = d\eta^2 - dr^2 + \eta^2 d\Omega^2 + r^2 dz^2$.

The spacetime described by the metric (2.29) has only one curvature singularity at $\rho = t$. When $-1 < M < 0$, the metric normal form is $\text{diag}(-, +, +, +, +, +)$ and when $M < -1$ ($M \neq -2$) the metric normal form is $\text{diag}(+, -, +, +, +, +)$.

After the extension of the spacetime by the transformation of coordinates (4.1), when $-2 < M < 0$ ($M \neq -1$), there is no singularity. The metric normal form when $-1 < M < 0$ is $\text{diag}(-, +, +, +, +, +)$ and the metric normal form when $-2 < M < -1$ is $\text{diag}(+, -, +, +, +, +)$. When $M < -2$, there is a scalar polynomial curvature singularity at $r = \sqrt{\frac{M}{M+2}}\eta$. In the region $r > \sqrt{\frac{M}{M+2}}\eta$, the metric normal form is $\text{diag}(+, -, +, +, +, +)$ and in the region $r < \sqrt{\frac{M}{M+2}}\eta$, the metric normal form is $\text{diag}(+, -, -, -, -, -)$.

In future studies, we can start from the solution given in the present paper to consider the influence of the singularity to quantum effects by calculating, e.g., the partition function and the one-loop effective action based on the heat-kernel method [39, 40].

The singularity problem is an important problem in gravity theory. Some authors consider the singularity theorem under more general conditions. The Raychaudhuri equation and the singularity theorem in the Finsler spacetime are studied in Refs. [41, 42]. Some singularity theorems are proved in $\mathbf{C}^{1,1}$ metrics [43, 44]. There are also some analyses of the singularity starting from exact solutions of the Einstein equation [45–49]. It is shown that the central singularity can be replaced by a bounce by taking care of the quantum effects in the inhomogeneous dust collapse [50].

Acknowledgements We are very indebted to Dr. G. Zeitrauman for his encouragement. This work is supported in part by NSF of China under Grant no. 11575125 and no. 11675119.

Open Access This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (<http://creativecommons.org/licenses/by/4.0/>), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made. Funded by SCOAP³.

References

1. R.M. Wald, *General Relativity* (University of Chicago Press, Chicago, 2010)
2. R. Geroch, What is a singularity in general relativity? *Ann. Phys.* **48**(3), 526–540 (1968)
3. R. Geroch, Local characterization of singularities in general relativity. *J. Math. Phys.* **9**(3), 450–465 (1968)
4. B. Schmidt, A new definition of singular points in general relativity. *Gen. Relativ. Gravit.* **1**(3), 269–280 (1971)
5. R.A. Johnson, The bundle boundary in some special cases. *J. Math. Phys.* **18**(5), 898–902 (1977)
6. R. Geroch, L. Can-bin, R.M. Wald, Singular boundaries of spacetimes. *J. Math. Phys.* **23**(3), 432–435 (1982)
7. S.W. Hawking, G.F.R. Ellis, *The Large Scale Structure of Space-Time* (Cambridge University Press, Cambridge, 1973)
8. S.-S. Chern, W.-h. Chen, K.S. Lam, *Lectures on differential geometry* (World Scientific, Singapore, 1999)
9. C.W. Misner, K.S. Thorne, J.A. Wheeler, *Gravitation* (Princeton University Press, Princeton, 2017)
10. L.D. Landau, *The Classical Theory of Fields* (Elsevier, Amsterdam, 2013)
11. B.S. Sathyaprakash, B.F. Schutz, Physics, astrophysics and cosmology with gravitational waves. *Living Rev. Relativ.* **12**(1), 2 (2009)
12. B.P. Abbott, R. Abbott, T. Abbott, M. Abernathy, F. Acernese, K. Ackley, C. Adams, T. Adams, P. Addesso, R. Adhikari et al., Observation of gravitational waves from a binary black hole merger. *Phys. Rev. Lett.* **116**(6), 061102 (2016)
13. J. Aasi, J. Abadie, B. Abbott, R. Abbott, T. Abbott, M. Abernathy, T. Accadia, F. Acernese, C. Adams, T. Adams et al., Gravitational waves from known pulsars: results from the initial detector era. *Astrophys. J.* **785**(2), 119 (2014)
14. J. Centrella, J.G. Baker, B.J. Kelly, J.R. van Meter, Black-hole binaries, gravitational waves, and numerical relativity. *Rev. Mod. Phys.* **82**(4), 3069 (2010)

15. L. Senatore, E. Silverstein, M. Zaldarriaga, New sources of gravitational waves during inflation. *J Cosmol. Astropart. Phys.* **2014**(08), 016 (2014)
16. M. Kakizaki, S. Kanemura, Gravitational waves as a probe of extended scalar sectors with the first order electroweak phase transition. *Phys. Rev. D* **92**(11), 115007 (2015)
17. D. Momeni, K. Myrzakulov, R. Myrzakulov, M. Raza, Cylindrical solutions in mimetic gravity. *Eur. Phys. J. C* **76**(6), 301 (2016)
18. Ö. Delice, P. Kirezli, D.K. Çiftci, Higher dimensional cylindrical or kasner type electrovacuum solutions. *Gen. Relativ. Gravit.* **45**(11), 2251–2272 (2013)
19. R. Emparan, H.S. Reall, A rotating black ring solution in five dimensions. *Phys. Rev. Lett.* **88**(10), 101101 (2002)
20. W. Chen, H. Lü, C. Pope, General kerr–nut–ads metrics in all dimensions. *Class. Quantum Gravity* **23**(17), 5323 (2006)
21. F. Rahaman, M. Kalam, S. Chakraborty, Thin shell wormholes in higher dimensional Einstein–Maxwell theory. *Gen. Relativ. Gravit.* **38**(11), 1687–1695 (2006)
22. A. Aliev, Rotating black holes in higher dimensional Einstein–Maxwell gravity. *Phys. Rev. D* **74**(2), 024011 (2006)
23. S. Hollands, A. Ishibashi, R.M. Wald, A higher dimensional stationary rotating black hole must be axisymmetric. *Commun. Math. Phys.* **271**(3), 699–722 (2007)
24. S.I. Vacaru, On general solutions for field equations in Einstein and higher dimension gravity. *Int. J. Theor. Phys.* **49**(4), 884–913 (2010)
25. S. Tomizawa, Y. Morisawa, Y. Yasui, Vacuum solutions of five dimensional Einstein equations generated by inverse scattering method. *Phys. Rev. D* **73**(6), 064009 (2006)
26. S. Tomizawa, M. Nozawa, Vacuum solutions of five dimensional Einstein equations generated by inverse scattering method. II. Production of the black ring solution. *Phys. Rev. D* **73**(12), 124034 (2006)
27. T. Padmanabhan, *Gravitation: Foundations and Frontiers* (Cambridge University Press, Cambridge, 2010)
28. S. Mukohyama, Brane-world solutions, standard cosmology, and dark radiation. *Phys. Lett. B* **473**(3), 241–245 (2000)
29. T. Gherghetta, E. Roessl, M. Shaposhnikov, Living inside a hedgehog: higher-dimensional solutions that localize gravity. *Phys. Lett. B* **491**(3), 353–361 (2000)
30. T. Padmanabhan, D. Kothawala, Lanczos–Lovelock models of gravity. *Phys. Rep.* **531**(3), 115–171 (2013)
31. G. Dotti, J. Oliva, R. Troncoso, Static wormhole solution for higher-dimensional gravity in vacuum. *Phys. Rev. D* **75**(2), 024002 (2007)
32. H.C. Ohanian, R. Ruffini, *Gravitation and Spacetime* (Cambridge University Press, Cambridge, 2013)
33. K. Virbhadra, Energy and momentum of cylindrical gravitational waves-II. *Pramana* **45**(2), 215–219 (1995)
34. N. Rosen, K. Virbhadra, Energy and momentum of cylindrical gravitational waves. *Gen. Relativ. Gravit.* **25**(4), 429–433 (1993)
35. C.-C. Chang, J.M. Nester, C.-M. Chen, Pseudotensors and quasilo-cal energy–momentum. *Phys. Rev. Lett.* **83**(10), 1897 (1999)
36. A. Papapetrou, Static spherically symmetric solutions in the unitary field theory, in *Proceedings of the Royal Irish Academy. Section A: Mathematical and Physical Sciences*, vol. 52, pp. 69–86, JSTOR (1948)
37. S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*, vol. 1 (Wiley, New York, 1972)
38. C. Möller, On the localization of the energy of a physical system in the general theory of relativity. *Ann. Phys.* **4**(4), 347–371 (1958)
39. W.-S. Dai, M. Xie, The number of eigenstates: counting function and heat kernel. *J. High Energy Phys.* **2009**(02), 033 (2009)
40. W.-S. Dai, M. Xie, An approach for the calculation of one-loop effective actions, vacuum energies, and spectral counting functions. *J. High Energy Phys.* **2010**(6), 1–29 (2010)
41. A.B. Aazami, M.A. Javaloyes, Penrose’s singularity theorem in a Finsler spacetime. *Class. Quantum Gravity* **33**(2), 025003 (2015)
42. E. Minguzzi, Raychaudhuri equation and singularity theorems in Finsler spacetimes. *Class. Quantum Gravity* **32**(18), 185008 (2015)
43. M. Kunzinger, R. Steinbauer, M. Stojković, J.A. Vickers, Hawking’s singularity theorem for $C^{1,1}$ -metrics. *Class. Quantum Gravity* **32**(7), 075012 (2015)
44. M. Kunzinger, R. Steinbauer, J.A. Vickers, The Penrose singularity theorem in regularity $C^{1,1}$. *Class. Quantum Gravity* **32**(15), 155010 (2015)
45. D. Sarma, F. Ahmed, M. Patgiri, Axially symmetric, asymptotically flat vacuum metric with a naked singularity and closed timelike curves. *Adv High Energy Phys.* **2016**, 7943649 (2016)
46. F. Ahmed, Cylindrically symmetric, asymptotically flat, Petrov type D spacetime with a naked curvature singularity and matter collapse. *Adv. High Energy Phys.* **2017**, 2546186 (2017)
47. G.J. Olmo, D. Rubiera-Garcia, A. Sanchez-Puente, Geodesic completeness in a wormhole spacetime with horizons. *Phys. Rev. D* **92**(4), 044047 (2015)
48. M.D. Mkenyeleye, R. Goswami, S.D. Maharaj, Gravitational collapse of generalized Vaidya spacetime. *Phys. Rev. D* **90**(6), 064034 (2014)
49. O.C. Stoica, Schwarzschild’s singularity is semi-regularizable. *Eur. Phys. J. Plus* **127**(7), 1–8 (2012)
50. Y. Liu, D. Malafarina, L. Modesto, C. Bambi, Singularity avoidance in quantum-inspired inhomogeneous dust collapse. *Phys. Rev. D* **90**(4), 044040 (2014)